

Article

A Weighted Problem for p -Laplacian Discussed with a Result of Saddle Point Type

Irina Meghea ^{1,2}

¹ Department of Mathematical Methods and Models, Faculty of Applied Sciences, National University of Science and Technology POLITEHNICA Bucharest, 313 Splaiul Independentei, 060042 Bucharest, Romania; irina.meghea@upb.ro; Tel.: +40756575418

² Research Center for Environmental Protection and Eco-Friendly Technologies, National University of Science and Technology POLITEHNICA Bucharest, 1 Polizu Street, 011061 Bucharest, Romania

Abstract

Weighted problems for the p -Laplacian represent a wide range of models arising from real-world research. Solving such problems involves, in many situations, variational approaches that are various and a virtually inexhaustible resource. In this paper, starting from a weakened saddle point type result, an existence theorem for a special kind of Dirichlet weighted problem involving the p -Laplacian is proved using a broad range of additional results. The continuing importance of this type of result is supported by a discussion on the applicability of the model as well as by the possibility of constructing a concrete solution via an appropriate numerical method based on saddle point-type results.

Keywords: weighted problem; p -Laplacian; Dirichlet problem; real-world models

MSC: 47J30; 58E30; 58E50; 35P30; 35Q35

1. Introduction

Weighted problems involving the p -Laplacian represent an important tool for modeling special types of real-world phenomena, such as in population dynamics in heterogeneous media [1], autocatalytic chemical reactions with nonlinear diffusion [2], reaction-diffusion equations with structured source term [3], population biology and ecology with heterogeneous diffusion and growth, or flow in porous media and hydrology (with sources or sinks) [4].

There are very close matches that combine weighted p -Laplacian problems and minimax techniques, such as at Alziary et al. [5], where a resonant Cauchy problem is studied, or in the book [6], where “a weighted problem for the p -Laplacian” treats a weighted p -Laplacian boundary value problem as an application of abstract operator theory; they cast a weighted p -Laplacian eigenvalue problem into the framework of compact operators and variational eigenvalue problems, and this provides a rigorous functional-analytic underpinning for the spectral theory of the weighted p -Laplacian. While they do not go deeply into saddle point theorems there, the framework is exactly the same variational setting in which saddle-point and minimax methods are later applied.

Several papers study weighted eigenvalue problems for the p -Laplacian, often with variational and minimax tools, such as [7] which introduces an eigenvalue problem with possibly sign-changing weight m , or [8], where instruments are not always explicitly

Academic Editor: Wei-Shih Du

Received: 29 December 2025

Revised: 24 January 2026

Accepted: 29 January 2026

Published: 1 February 2026

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named “saddle point theorems”, but the structure (indefinite functional, constrained minimization, critical point theory) is the same variational toolkit.

More recently, weighted p -Laplacian problems using minimax-type methods have been considered [9,10] as a few examples where weighted p -Laplacian problems are attacked via critical point theory (mountain pass/three critical points, etc.).

Regarding the applications, they cover an extended field of subjects such as non-Newtonian fluids and porous media [11], nonlinear elasticity and plasticity [7,8], population dynamics and ecology [1], degenerate/singular diffusion and weighted Sobolev spaces [2].

In nonlinear functional analysis, saddle point type theorems giving critical points of functionals on infinite-dimensional Banach spaces have their origin in the result due to Rabinowitz in [12]. Its classical obtaining is via a minimax theorem deduced based on the original form of Ekeland variational principle [13], under the assumption of C^1 Fréchet class for a real function on a Banach space, followed by versions for Lipschitz functions [14,15]. The same variational principle is used to obtain, in the Finsler manifolds, three theorems of minimax type with the afferent saddle point theorems [16]. Other works can be mentioned, such as [17–19].

The weighted p -Laplacian treated using minimax variational methods represents a highly topical subject of particular importance, as demonstrated by recent studies [10,20–25].

In this paper, a variant of the saddle point theorem (SPT), obtained by the author in [26], is used to prove an existence result for a Dirichlet weighted problem involving the p -Laplacian, followed by concrete applications in describing solutions for some interesting real-world models.

2. Materials and Methods

2.1. Preliminaries for a Saddle Point Theorem

The main theoretical result used to characterize the solution of the problem studied in this paper is a particular variant of the saddle point theorem, previously obtained by the author in [26]. To introduce this theorem, several notions should first be defined.

Definition 1. Let X be a real normed space.

A nonempty set β of bounded parts of X , with the properties:

1° $\bigcup_{A \in \beta} A = X$,

2° $A \in \beta \Rightarrow -A \in \beta$ and $\lambda A \in \beta$ ($\lambda > 0$),

3° for every A, B in β there exists C in β such that $A \subset C$ and $B \subset C$,

is named *bornology* on X .

For instance, the set β_F of bounded symmetrical ($M = -M$) parts of X and the set β_G of finite symmetrical parts of X are bornologies on X (for the justification in the case β_F use the balls).

β_1, β_2 being bornologies on X , the property: $\forall A$ from $\beta_1 \exists B$ in β_2 such that $A \subset B$ will be designated by:

$$\beta_1 \preceq \beta_2.$$

We have:

$$\beta_G \preceq \beta \preceq \beta_F$$

for every bornology β on X (see, for instance, [27], I, §3, 3.7 for clarifications).

Definition 2. Let β be a bornology on X and $f: X \rightarrow \overline{\mathbf{R}}$ locally finite in the point a (i.e., there is a neighborhood of a on which f is finite). By definition, f is β -differentiable in a , if there exists φ in the dual X^* such that, for every S in β , we have:

$$\lim_{\substack{t \rightarrow 0 \\ h \in S}} \frac{f(a + th) - f(a)}{t} = \varphi(h)$$

(uniform limit on S for $t \rightarrow 0$).

φ is the β -derivative of f in a , and it is denoted:

$$\nabla_{\beta} f(a).$$

The function f is β -differentiable on a subset of X has the usual meaning.

Proposition 1. Let $f: X \rightarrow \overline{\mathbf{R}}$ be locally finite at a point a . f is β_F -differentiable in a respectively β_G -differentiable in a , iff f is Fréchet differentiable and Gâteaux differentiable, respectively, in a , and in this case

$$\nabla_{\beta_F} f(a) = f'(a) \text{ and } \nabla_{\beta_G} f(a) = f'_w(a)$$

respectively [27].

Proposition 2. Let β_1, β_2 be bornologies on X , $\beta_1 \prec \beta_2$. If $f: X \rightarrow \overline{\mathbf{R}}$ is β_2 -differentiable in a , then f is also β_1 -differentiable in a and

$$\nabla_{\beta_1} f(a) = \nabla_{\beta_2} f(a).$$

In particular, if f is β -differentiable in a , then f is Gâteaux differentiable in a and

$$\nabla_{\beta} f(a) = f'_w(a).$$

Proposition 3 (β -derivative of the composed function). Let $f: X \rightarrow \mathbf{R}$ β -differentiable in x_0 and $F: \mathbf{R} \rightarrow \mathbf{R}$ of C^1 class on J open interval with $f(x_0) \in J$. Then $g := F \circ f$ is β -differentiable in x_0 and

$$\nabla_{\beta} g(x_0) = F'(f(x_0)) \nabla_{\beta} f(x_0)$$

[27].

Definition 3. Let X be a real normed space, β a bornology on X and $\varphi: X \rightarrow \mathbf{R}$. Let c be in $\mathbf{R}$. φ verifies the Palais-Smale condition (respectively, the weak Palais-Smale condition) on the level c , $(PS)_c$, with respect to β , if $\forall (u_n)_{n \geq 1}$ a sequence of points in X such that

$$\lim_{n \rightarrow \infty} \varphi(u_n) = c \text{ and } \lim_{n \rightarrow \infty} \|\nabla_{\beta} \varphi(u_n)\| = 0,$$

this sequence has a convergent subsequence (respectively, c is a critical value for φ , that is $c = \varphi(u_0)$ and $\nabla_{\beta} \varphi(u_0) = 0$).

Let us now turn to the main foundational result. This is a variant of the saddle point theorem obtained in [26] via a minimax-type theorem derived using Ekeland variational principle.

Theorem 1 (Saddle Point Theorem). Let X be a Banach space and $\varphi: X \rightarrow \mathbf{R}$ continuous and Gâteaux differentiable with $\varphi'_w(\cdot)(v)$ upper semicontinuous $\forall v \in X$. Suppose $X = V \oplus W$, direct sum, with V, W closed subspaces and

$$\dim V < +\infty, \sup_{\sigma_R} \varphi < \inf_W \varphi, \sigma_R := \{v \in V: \|v\| = R\}.$$

Let be

$$S_R := \{v \in V: \|v\| \leq R\}, T := \{g \in C(S_R; X): g(t) = t \text{ on } \sigma_R\},$$

$$c := \inf_{g \in T} \sup_{S_R} (\varphi \circ g).$$

If φ verifies the weak $(PS)_c$ condition, then c is a critical value for φ .

Remark 1. The above theorem was obtained by Rabinowitz [12] under the assumption φ belongs to the C^1 -Fréchet class. The extension to continuous φ with $\varphi'_w(\cdot)(v)$ upper semicontinuous (u.s.c.) for all $v \in X$ was carried out by the author [26].

Remark 2. In the above statement, the Gâteaux derivative can be replaced by any β -derivative, where β denotes a bornology on X , as stated in paper [26].

2.2. The Weighted Problem with p -Laplacian

2.2.1. The Problem Under Study

Based on the above result, the solution of the following problem can be characterized as follows:

$$(P) \begin{cases} -\Delta_p u(x) = \lambda a(x)u^{p-1} + f(x, u(x)), & \text{on } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded smooth domain of $\mathbf{R}^N$, $f: \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is assumed to be a Carathéodory function (for details, see, for instance, Definition 2.9 from [28]) satisfying the growth condition:

$$|f(x, s)| \leq \beta |s|^{p-1} + \alpha(x), \quad \forall x \in \Omega \setminus A \text{ with } \mu(A) = 0 \quad \forall s \in \mathbf{R},$$

where μ is the Lebesgue measure, $\beta \geq 0$, $\alpha \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq p \leq \frac{2N}{N-2}$, when $N \geq 3$, $1 \leq p < +\infty$, if $N = 2$. In the above-presented problem (P), $a \in L^\infty(\Omega)$.

Clarification. Regarding the behavior of the nonlinear term, the function f does not intersect the spectrum of the weighted p -Laplacian; that is, there is no resonance either at infinity or at zero.

The energy functional φ associated with the problem (P) is $\varphi: W_0^{1,p}(\Omega) \rightarrow \mathbf{R}$, defined by:

$$\varphi(u) = \frac{1}{p} \|u\|_{1,p}^p - \int_\Omega \left[\frac{\lambda}{p} a(x) u^p(x) + F(x, u(x)) \right] dx, \quad (1)$$

where $F(x, t) = \int_0^t f(x, s) ds$, $\|u\|_{1,p} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$

which is equivalent to the norm $u \rightarrow \left(\|u\|_{L^p(\Omega)}^p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$ and it is also equivalent to the norm $u \rightarrow \|\nabla u\|_{L^p(\Omega)}$ (this equivalence is proven in [28]). The vector space $W_0^{1,p}(\Omega)$, endowed with one of these three equivalent norms, is a smooth Banach space (for more details, see, for instance, [28]). Regarding F , by taking into account the Carathéodory property of f , i.e., the growth condition, one immediately deduces:

$$|F(x, t)| = \left| \int_0^t f(x, s) ds \right| \leq \int_0^t |f(x, s)| ds \leq \int_0^t [\beta |s|^{p-1} + \alpha(x)] ds = \frac{\beta}{p} |t|^p + t\alpha(x), \quad (2)$$

$\forall x \in \Omega \setminus A$ with $\mu(A) = 0 \quad \forall t \in \mathbf{R}$.

Suppose, in addition, that f satisfies:

(f 1) $f(x, s) = o(|s|^{p-1})$, $s \rightarrow 0$, uniformly for $x \in \overline{\Omega}$.

(f 2) There exist constants $\mu > p$ and $r > 0$ such that, for $|s| \geq r$,

$$0 < \mu F(x, s) \leq s f(x, s), \quad \forall x \in \overline{\Omega}.$$

Property (f 1) implies [29]:

$$\lim_{\substack{s \rightarrow 0 \\ x \in \bar{\Omega}}} \frac{f(x, s)}{|s|^{p-1}} = 0 \Rightarrow \forall \varepsilon > 0 \exists \delta > 0, \text{ such that } |s| \leq \delta \Rightarrow \left| \frac{f(x, s)}{|s|^{p-1}} - 0 \right| \leq \varepsilon, \forall x \in \bar{\Omega}.$$

One integrates on $[0, t]$, $|t| \leq \delta$: $0 < |f(x, s)| \leq \varepsilon |s|^{p-1}$, for x fixed in $\bar{\Omega}$. Hence:

$$0 \leq F(x, t) \leq \frac{\varepsilon}{p} |t|^p. \quad (3)$$

Property (f 2) implies ([29]):

$$\frac{f(x, s)}{F(x, s)} \geq \frac{\mu}{s},$$

whence, by integration,

$$\ln |F(x, s)| \geq \mu \ln |s| + C_{\mu}$$

and hence, for $|s| \geq r$,

$$F(x, s) \geq C |s|^{\mu}. \quad (4)$$

2.2.2. The Eigenvalue Problem

The first eigenvalue of the associated homogeneous problem:

$$(EP) \begin{cases} -\Delta_p u(u) = \lambda a(x) u^{p-1}, & \text{on } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $a(x) \geq a_0 \geq 0$ a.e. on Ω , is (infimum of weighted Rayleigh quotient):

$$\lambda_1 = \inf_{u \neq 0} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} a(x) |u|^p dx} = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{W_0^{1,p}(\Omega)}^p}{\|u\|_{L_a^p(\Omega)}^p},$$

where the norm from the numerator has been defined above and the norm from the denominator is that weighted on $L^p(\Omega)$, i.e.,

$$\|u\|_{L_a^p(\Omega)} = \left(\int_{\Omega} a(x) |u(x)|^p dx \right)^{\frac{1}{p}},$$

while the usual norm on $L^p(\Omega)$ (and it is also used, as in the above relations can be seen) is:

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}}.$$

In [30], it is noted, in accordance with [31,32], that when the measure of the set $\{x \in \Omega: a(x) > 0\}$ is strictly greater than zero, the problem (EP) admits at least one sequence (λ_n) of eigenvalues tending to $+\infty$. A detailed presentation of such eigenvalues is also provided in [33].

Remark 3. The above condition imposed on the function a is motivated by the fact that, if $a(x) = 0$ on a set of positive Lebesgue measure, the eigenvalues may be excessively large or may fail to exist. Such a situation can arise in models involving equations with discontinuous coefficients, which may occur in composite materials or in models containing dead zones. Degenerate cases were investigated in [34].

3. Main Result

The central theoretical result of this paper is a generalization of Theorem 8.2 from [35], adapted to the weighted p -Laplacian Dirichlet problem. We formulate it as follows.

Theorem 2. Consider the above problem (P) under the conditions previously presented. Then, for any $\lambda \in \mathbf{R}$, the problem (P) has a nonzero solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$.

Clarification. For more information and details related to the definition of this type of solution, one can consult [28].

Proof. For the case $\lambda < \lambda_1(a)$, one may apply the version of the mountain pass theorem from [26] in a manner similar to that used in [29], involving differences too minor to warrant detailed discussion.

In what follows, we address the problem in the opposite case: $\lambda \geq \lambda_1(a)$. Taking into account the above considerations on the eigenvalues of the problem (EP), one can take $\lambda \in [\lambda_k(a), \lambda_{k+1}(a)]$. The energy functional φ defined above in (1) fulfills the conditions asked by the SPT result displayed in the previous section, i.e., it is continuous and Gâteaux differentiable with $\varphi'_w(\cdot)(v)$ upper semicontinuous $\forall v \in X$. For its first part containing the norm, the proof can be found in [28], while, for the second part, the proof of the needed properties can be found in [36]. Let us take the eigenvectors $v_1, \dots, v_k$, corresponding to the considered eigenvalues $\lambda_1(a), \dots, \lambda_k(a)$, and the vector subspace V of $W_0^{1,p}(\Omega)$ that they generate, $V = \langle v_1, \dots, v_k \rangle$, and W is the vector subspace of $W_0^{1,p}(\Omega)$ such that $V \oplus W = W_0^{1,p}(\Omega)$ (direct sum). Obviously, $\dim V < +\infty$. The choice of these eigenvectors generating V (which ensures its finite dimension) was made in order to guarantee that certain inequalities are satisfied in what follows.

For $u \in W$, we have

$$\|u\|_{1,p}^p - \lambda \int_{\Omega} a(x)u^p(x)dx \geq \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \|u\|_{1,p}^p,$$

since

$$\lambda_{k+1} \leq \frac{\|u\|_{1,p}^p}{\|u\|_{L_a^p(\Omega)}^p} \Rightarrow \int_{\Omega} a(x)|u(x)|^p dx \leq \frac{1}{\lambda_{k+1}} \|u\|_{1,p}^p.$$

So,

$$\begin{aligned} \varphi(u) &\geq \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \|u\|_{1,p}^p - \int_{\Omega} F(x, u(x))dx \geq \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \|u\|_{1,p}^p - \frac{\beta}{p} \|u\|_{1,p}^p - \int_{\Omega} \alpha(x)u(x)dx \geq \\ &\frac{1}{p} \left(1 - \frac{\lambda}{\lambda_{k+1}} - \beta\right) \|u\|_{1,p}^p - K \|u\|_{1,p}, \text{ since } \int_{\Omega} \alpha(x)u(x)dx \leq \|\alpha\|_{0,q'} \|u\|_{0,q} \leq K \|u\|_{1,p}. \end{aligned}$$

In the above relation, the inequality (2) was also involved in preparation of F :

$$\begin{aligned} \int_{\Omega} F(x, u(x))dx &\leq \int_{\Omega} |F(x, u(x))|dx \stackrel{(2)}{\leq} \frac{\beta}{p} \int_{\Omega} |u(x)|^p dx + \int_{\Omega} \alpha(x)u(x)dx = \\ &\frac{\beta}{p} \|u\|_{L^p(\Omega)}^p + \int_{\Omega} \alpha(x)u(x)dx \leq \frac{\beta}{p} \|u\|_{1,p}^p + \int_{\Omega} \alpha(x)u(x)dx. \end{aligned}$$

From the relation (3), one obtains:

$$\int_{\Omega} F(x, u(x)) dx \stackrel{(3)}{\leq} \frac{\varepsilon}{p} \int_{\Omega} |u(x)|^p dx = \frac{\varepsilon}{p} \|u\|_{0,p}^p \leq K \|u\|_{1,p}^p,$$

hence

$$\int_{\Omega} F(x, u(x)) dx = O\left(\|u\|_{1,p}^p\right) \text{ as } u \rightarrow 0.$$

On the other hand, when take u in V having $\|u\|_{1,p} = R$,

$$\begin{aligned} \varphi(u) &= \frac{1}{p} R^p - \frac{\lambda}{p} \int_{\Omega} a(x) u^p(x) dx - \int_{\Omega} F(x, u(x)) dx \stackrel{(4)}{\leq} \frac{R^p}{p} - \frac{a_0 \lambda_k}{p} \int_{\Omega} u^p(x) dx - C \int_{\Omega} |u(x)|^u dx \leq \\ &\frac{R^p}{p} - \left(\frac{a_0 \lambda_k}{p} + C \right) \|u\|_{0,p}^p \leq \frac{R^p}{p} - K' \left(\frac{a_0 \lambda_k}{p} + C \right) \|u\|_{1,p}^p = \frac{R^p}{p} - K' \left(\frac{a_0 \lambda_k}{p} + C \right) R^p < 0. \end{aligned}$$

Hence, R can be chosen so that the inequality required by the Saddle Point Theorem is satisfied. To prepare F in the above inequality, the relation (4) was employed.

It remains now only to verify the Palais-Smale-type condition $(PS)_c$ in order to allow the application of SPT and to complete the proof. To demonstrate that this property holds (one can consider $c = 0$), take $(u_n)_{n \geq 1}$ in $W_0^{1,p}(\Omega)$, endowed with the considered norm $\|\cdot\|_{1,p}$, i.e., satisfying: $\begin{cases} \lim_{n \rightarrow \infty} \varphi(u_n) = 0 \text{ and} \\ \lim_{n \rightarrow \infty} \|\varphi'_w(u_n)\|_{1,p'} = 0 \end{cases}$ respectively. $\lim_{n \rightarrow \infty} \varphi(u_n) = 0$ implies that the sequence $(\varphi(u_n))_{n \geq 1}$ is bounded, so $\exists \rho > 0$ such that $|\varphi(u_n)| \leq \rho \forall n$ from a rank on. $\lim_{n \rightarrow \infty} \|\varphi'_w(u_n)\|_{1,p'} = 0$ implies that the sequence $(\|\varphi'_w(u_n)\|)_{n \geq 1}$ is bounded, so $\exists M > 0$ such that $\|\varphi'_w(u_n)\|_{1,p'} \leq M \forall n$ from a rank on. Thus,

$$|\langle \varphi'_w(u_n), u_n \rangle| \leq \|\varphi'_w(u_n)\|_{1,p'} \|u_n\|_{1,p} \leq M \|u_n\|_{1,p}$$

from a rank on. Hence $\rho + \|u_n\|_{1,p} \geq \varphi(u_n) - \frac{1}{M} \langle \varphi'_w(u_n), u_n \rangle$ and, from this relation, we obtain that the sequence $(u_n)_{n \geq 1}$ is bounded. Evaluate now the last expression:

$$\begin{aligned} \varphi(u_n) - \frac{1}{M} \langle \varphi'_w(u_n), u_n \rangle &= \frac{1}{p} \|u_n\|_{1,p}^p - \frac{\lambda}{p} \int_{\Omega} a(x) u_n^p(x) dx - \int_{\Omega} F(x, u_n(x)) dx - \frac{1}{M} \int_{\Omega} (|\nabla u_n|^{p-2} |\nabla u_n|^2) dx + \\ &\frac{\lambda}{M} \int_{\Omega} a(x) u_n^p(x) dx + \frac{1}{M} \int_{\Omega} f(x, u_n(x)) u_n(x) dx \geq \frac{1}{p} \|u_n\|_{1,p}^p + \lambda \left(\frac{1}{M} - \frac{1}{p} \right) \int_{\Omega} a(x) u_n^p(x) dx - \frac{1}{M} \int_{\Omega} |\nabla u_n|^p dx + \\ &\left(\frac{\mu}{M} - 1 \right) \int_{\Omega} F(x, u_n(x)) dx \geq \frac{1}{p} \|u_n\|_{1,p}^p + \lambda a_0 \left(\frac{1}{M} - \frac{1}{p} \right) \int_{\Omega} u_n^p(x) dx - \frac{1}{M} \|u_n\|_{1,p}^p \geq \left(\frac{1}{p} - \frac{1}{M} \right) \|u_n\|_{1,p}^p. \end{aligned}$$

and one can take M such that $p < M < \mu$. So, we have:

$$\rho + \|u_n\|_{1,p} \geq \left(\frac{1}{p} - \frac{1}{M} \right) \|u_n\|_{1,p}^p,$$

i.e., the sequence $(u_n)_{n \geq 1}$ is bounded. Since $W_0^{1,p}(\Omega)$ endowed with the considered norm is complete [28], it is sufficient to prove that $(u_n)_{n \geq 1}$ has a Cauchy subsequence. To prove this last assertion, remark that $\varphi = \Psi - \psi_1 - \psi_2$, hence $\Psi = \varphi + \psi_1 + \psi_2$, where

$$\Psi(u) = \frac{1}{p} \|u_n\|_{1,p}^p, \psi_1(u) = \frac{\lambda}{p} \int_{\Omega} a(x) u^p(x) dx, \psi_2(u) = \int_{\Omega} F(x, u(x)) dx.$$

Their (weak) derivatives, respectively, are:

$$\langle \Psi'(u), h \rangle = \int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla h(x) dx, \forall h \in W_0^{1,p}(\Omega),$$

$$\langle \psi'_1(u), h \rangle = \lambda \int_{\Omega} a(x) u^{p-1}(x) h(x) dx, \forall h \in W_0^{1,p}(\Omega),$$

$$\langle \psi'_2(u), h \rangle = \int_{\Omega} f(x, u(x)) h(x) dx, \forall h \in W_0^{1,p}(\Omega).$$

For the next computations, the following inequalities [37] are necessary:

$$|\xi - \zeta|^p \leq A(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \text{ for } p \geq 2, \quad (5)$$

$$|\xi - \zeta|^2 \leq A(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}, \text{ for } 1 < p < 2. \quad (6)$$

Hence, in case $p \geq 2$, one obtains:

$$\|u_m - u_n\|_{1,p}^p \leq a \|\nabla(u_m - u_n)\|_{0,p}^p = a \int_{\Omega} |\nabla u_m - \nabla u_n|^p dx \stackrel{(5)}{\leq}$$

$$B \int_{\Omega} (|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n)(\nabla u_m - \nabla u_n) dx =$$

$$B \int_{\Omega} |\nabla u_m|^{p-2} \nabla u_m (\nabla u_m - \nabla u_n) dx - B \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n (\nabla u_m - \nabla u_n) dx =$$

$$B \langle \Psi'(u_m), u_m - u_n \rangle - B \langle \Psi'(u_n), u_m - u_n \rangle = B \langle \varphi'(u_m), u_m - u_n \rangle - B \langle \varphi'(u_n), u_m - u_n \rangle +$$

$$B \langle \psi'_1(u_m), u_m - u_n \rangle - B \langle \psi'_1(u_n), u_m - u_n \rangle + B \langle \psi'_2(u_m), u_m - u_n \rangle - B \langle \psi'_2(u_n), u_m - u_n \rangle \leq$$

$$B(|\varphi'(u_m)|_{1,p'} + |\varphi'(u_n)|_{1,p'}) \|u_m - u_n\|_{1,p}^p + B \|\psi'(u_m) - \psi'(u_n)\|_{1,p'} \|u_m - u_n\|_{1,p}^p,$$

where we used the notation $\psi = \psi_1 + \psi_2$, hence

$$\|u_m - u_n\|_{1,p}^p \leq B(|\varphi'(u_m)|_{1,p'} + |\varphi'(u_n)|_{1,p'} + \|\psi'(u_m) - \psi'(u_n)\|_{1,p'}) \|u_m - u_n\|_{1,p}^p.$$

Since $\varphi'(u_n) \rightarrow 0$ and $\psi'_2 = N_f$ (Nemytskii operator; for the proof of this property, one may consult [28]) is compact, we can assume, passing (if it is necessary) to a subsequence of $(u_n)_{n \geq 1}$, that it is convergent in $W_0^{1,p}(\Omega)$.

To prove the $(PS)_c$ condition for $1 < p < 2$,

$$\|u_m - u_n\|_{1,p}^2 \leq B' |\langle \Psi'(u_m), u_m - u_n \rangle - \langle \Psi'(u_n), u_m - u_n \rangle| (\|u_m\|_{1,p}^{2-p} + \|u_n\|_{1,p}^{2-p}). \quad (7)$$

To demonstrate the above relation, we use the statement: $\forall s \in (0, +\infty) \exists C_s > 0$ a constant such that

$$(x + y)^s \leq C_s (x^s + y^s), \forall x, y \in (0, +\infty) \quad (8)$$

and take:

$$\|u_m - u_n\|_{1,p}^2 \leq a \|\nabla(u_m - u_n)\|_{L^p(\Omega)}^{2-p} = a \left(\int_{\Omega} |\nabla u_m - \nabla u_n|^p dx \right)^{\frac{2}{p}}.$$

Continue with:

$$\begin{aligned}
& \int_{\Omega} |\nabla u_m - \nabla u_n|^p dx = \int_{\Omega} (|\nabla u_m - \nabla u_n|^2)^{\frac{p}{2}} dx \stackrel{(6)}{\leq} \\
& A^{\frac{p}{2}} \int_{\Omega} [(|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n) (\nabla u_m - \nabla u_n) (|\nabla u_m| + |\nabla u_n|)^{2-p}]^{\frac{p}{2}} dx = \\
& A^{\frac{p}{2}} \int_{\Omega} [(|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n) (\nabla u_m - \nabla u_n)]^{\frac{p}{2}} (|\nabla u_m| + |\nabla u_n|)^{\frac{p(2-p)}{2}} dx \leq \\
& A' \left[\int_{\Omega} (|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n) (\nabla u_m - \nabla u_n) dx \right]^{\frac{p}{2}} \left[\int_{\Omega} (|\nabla u_m| + |\nabla u_n|)^p dx \right]^{\frac{(2-p)}{2}} \stackrel{(8)}{\leq} \\
& A' \left[\int_{\Omega} (|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n) (\nabla u_m - \nabla u_n) dx \right]^{\frac{p}{2}} \left[C_p \left(\int_{\Omega} |\nabla u_m|^p dx + \int_{\Omega} |\nabla u_n|^p dx \right) \right]^{\frac{(2-p)}{2}} = \\
& A_p \left[\int_{\Omega} (|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n) (\nabla u_m - \nabla u_n) dx \right]^{\frac{p}{2}} \left[\left(\int_{\Omega} |\nabla u_m|^p dx \right)^{\frac{(2-p)}{2}} + \left(\int_{\Omega} |\nabla u_n|^p dx \right)^{\frac{(2-p)}{2}} \right] = \\
& A_p \left[\int_{\Omega} (|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n) (\nabla u_m - \nabla u_n) dx \right]^{\frac{p}{2}} \left(\|u_m\|_{1,p}^{\frac{(2-p)p}{2}} + \|u_n\|_{1,p}^{\frac{(2-p)p}{2}} \right), \\
& \text{so} \\
& \left(\int_{\Omega} |\nabla u_m - \nabla u_n|^p dx \right)^{\frac{2}{p}} \leq B'' \int_{\Omega} (|\nabla u_m|^{p-2} \nabla u_m - |\nabla u_n|^{p-2} \nabla u_n) (\nabla u_m - \nabla u_n) dx \left(\|u_m\|_{1,p}^{\frac{(2-p)p}{2}} + \|u_n\|_{1,p}^{\frac{(2-p)p}{2}} \right)^{\frac{2}{p}},
\end{aligned}$$

hence the inequality (7) holds true, and then the conclusion as in case $p \geq 2$, the necessary properties of Ψ and $W_0^{1,p}(\Omega)$ being kept also for $1 < p < 2$ [28]. $\square$

4. Discussion

The weighted p -Laplacian problem discussed through a saddle point-type result belongs to a class of boundary-value and eigenvalue problems for a weighted p -Laplacian operator, whose solutions are obtained via variational methods exhibiting a saddle-point structure. Roughly speaking, when the corresponding functional φ is not bounded from below on the entire X or fails to be coercive (for example, at resonance), saddle point-type results are often employed instead of simple minimization techniques. This corresponds to a saddle point-type result arising when a special geometry is present, namely when $\varphi|_{X_1}$ is low, while $\varphi|_{X_2}$ is high for a decomposition $X = X_1 \oplus X_2$; a minimax characterization is then used to obtain a critical value corresponding to a solution. Thus, the problems addressed in this work, together with the proof techniques we describe, are not fundamentally new; however, they differ from those encountered in the specialized literature. The aim of this section is to compare the problems and methods presented in this paper with related works in the literature, in order to highlight the differences, the novelty, their effectiveness, and, most importantly, to identify the fundamental applications of this type of problem in modeling real-world phenomena.

Closely related works combining weighted p -Laplacian problems with saddle point and minimax techniques can be found in [5], where the resonant Cauchy problem is studied:

$$-\Delta_p u(u) = \lambda_1 m(|x|) |u|^{p-2} u + f(x), \text{ on } \mathbf{R}^N, u \in D^{1,p}(\mathbf{R}^N),$$

with a positive radial weight $m(|x|)$ that decays at infinity. As key points, it is worked in weighted Sobolev and Lebesgue spaces $L^p(\mathbf{R}^N; m)$; the associated energy functional is *not* coercive at the first eigenvalue λ_1 , and they decompose the space along the first eigenfunction and show that the functional has *saddle point geometry* when $1 < p < 2$, and then apply a type of *saddle point theorem* to obtain existence of solutions. By comparing the above equation with that considered in our problem, we observe differences both in the

terms appearing on the right-hand side and in the domain where the solution is sought. In the present paper, the parameter λ is arbitrary, whereas in [5] only the first eigenvalue λ_1 is involved. Moreover, we employ a particular variant of the Saddle Point Theorem, and the problem is also solved for $p > 2$.

In the monograph [6], Section 2.3.2 “A weighted problem for the p -Laplacian”, addresses a weighted p -Laplacian boundary value problem as an application of abstract operator theory. The authors recast a weighted p -Laplacian eigenvalue problem within the framework of compact operators and variational eigenvalue problems. This gives a rigorous functional-analytic underpinning for the spectral theory of the weighted p -Laplacian. Although the authors do not explore saddle point theorems in depth, the framework is exactly the same variational setting in which saddle-point and minimax methods are later applied.

Several papers study weighted eigenvalue problems for the p -Laplacian, often with variational and minimax tools, as also in [7] where an eigenvalue problem is introduced:

$$-\Delta_p u(u) + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u,$$

with possibly sign-changing weight m . The associated functional is indefinite; Cuesta uses minimax arguments and the notion of a threshold quantity $\alpha(V, m)$ to ensure boundedness from below on a weighted unit sphere. In the case of the above equation, the nonlinear term f in our problem (P) has a special form, whereas the weight m is subject to more general conditions. Consequently, the resulting solutions are completely different.

In [8] is considered the equation:

$$-\Delta_p u(u) = \lambda g(x)|u|^{p-2}u,$$

with g in a weak $L^{N/p}$ space and an eigenvalue theory using variational techniques is developed. These are not always explicitly called “saddle point theorems”, but the structure (indefinite functional, constrained minimization, critical point theory) is the same variational toolkit. However, both the problem and the solving methods are different from those of this work.

Qualitative properties (nonexistence, Hardy-type inequalities, eigenvalues) for:

$$-\Delta_{p,w} u(u) := -\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) = f(x, u),$$

with weights $w \in A_p$ (Muckenhoupt) are studied in [9], which uses a weighted Picone identity and variational tools to derive eigenvalue properties. In the present work, a generalized form of the p -Laplacian is considered, and the analytical tools developed are completely different, as is also the case in the paper cited below.

In [10], a weighted p -Laplacian Dirichlet problem in a weighted Sobolev space $W_0^{1,p}(a; \Omega)$ is studied, and a three-critical-points theorem (Bonanno-Ricceri), a minimax/saddle point type result is applied to obtain at least three weak solutions. There is also a substantial body of literature on weighted eigenvalue problems and antimaximum principles for the p -Laplacian with Neumann or Dirichlet boundary conditions, again employing variational and spectral methods.

Interesting applications to real-world modeling involving the p -Laplacian and its weighted variants consider u as a state variable (such as temperature, concentration, displacement, population density, etc.), while the p -Laplacian encodes nonlinear diffusion or conductivity (with diffusion depending on $|\nabla u|$), and the weight describes heterogeneity of the medium (porosity, conductivity, density, geometry). Some concrete modeling contexts are discussed below.

In non-Newtonian fluid flow or filtration in porous media, the effective viscosity depends on the shear rate. In certain regimes, this leads to diffusion operators of p -Laplacian type. A weight $w(x)$ may represent spatially varying permeability or porosity of the

medium. General references on such quasilinear equations include Diaz's monograph [11], which repeatedly uses p -Laplacian-type operators in modeling diffusion and free-boundary phenomena.

The p -Laplacian also appears in models of nonlinear elasticity and plasticity, where the stored energy grows like $|\nabla u|^p$. In a heterogeneous material, a weight $w(x)$ can encode spatial variation in stiffness. Existence and multiplicity results for weighted p -Laplacian eigenvalue problems underpin stability or buckling analysis for such structures [7,8].

In reaction-diffusion models for population dynamics, one often writes:

$$\partial_t u - \operatorname{div}(D(x)|\nabla u|^{p-2}\nabla u) = f(x, u),$$

where $D(x)$ is a spatially varying diffusion coefficient, giving a weight in the elliptic stationary problem. Weights may model habitat quality, spatial "patchiness", or barriers; spectral properties of the weighted p -Laplacian then govern persistence and/or extinction thresholds [38–40].

Weights that vanish or blow up (e.g., $w(x) = |x|^\alpha$) model degeneracy near certain regions (interfaces, boundaries, singular points). Garain's work analyzes such degenerate weighted p -Laplacian equations, proving Hardy inequalities, comparison principles, and eigenvalue properties in the framework of weighted Sobolev spaces [9]. This is directly motivated by PDEs where the medium has "holes", highly conductive channels, or anisotropic structures.

There is also an active line of research on nonlocal and fractional (weighted) p -Laplacian operators (e.g., Ref. [41]), where saddle-point methods are used to obtain sign-changing or multiple solutions. These models appear, for example, in anomalous diffusion in heterogeneous media.

One may stress that the proposed solving method is limited to the case $\mu(\{x \in \Omega : a(x) > 0\}) > 0$ a.e., limitation reflected in possible applications in characterizing and solution construction of real-world models.

The importance of this type of result lies in the possibility of extending the analysis in order to obtain concrete solutions via appropriate variational methods based on saddle point-type results. Moreover, the relaxation of the conditions in SPT leads to analogous relaxations in the main theorem for the p -Laplacian problem. One may also note that the weakened assumptions represent an additional step toward a more realistic modeling framework.

5. Conclusions

In this paper, an existence result for a weighted Dirichlet problem involving the p -Laplacian, based on a saddle point-type theorem, has been established. The associated eigenvalue problem was also employed in the proof of this main result. The weakened conditions of the SPT result are reflected in the corresponding conditions required by the main theorem, leading to significant potential applications in modeling real-world phenomena.

In the Discussion section, published papers in the same field have been analyzed and compared in order to highlight the novelty of the present work and to emphasize the importance of this type of result for the development of solutions to concrete real-world problems. Such a structured approach leads toward a complete solution, which can be visualized after the application of an appropriate numerical method.

The present research is intended to be continued and extended through the development of appropriate numerical methods for obtaining and visualizing the solution using CFD software, the investigation of the existence of multiple solutions, the study of the von Neumann problem, and the possible introduction of fractional operators.

Funding: This work was supported by the National University of Science and Technology POLITEHNICA Bucharest.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares no conflicts of interest.

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