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A New Uniform Hyperbolic Polynomial B-Spline Computational Scheme for 3rd-Order Pseudo-Parabolic Problem

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Abstract

This work introduces a new UHPB-spline computation approach to approximate the pseudo-parabolic problem of order three, which exposes parabolic as well as hyperbolic physical appearance. The approach is used for the discretization of spatial derivatives, which contributes to better accuracy and flexibility. For the discretization of the time derivative, the FDM is employed, endorsing computational proficiency. The anticipated approach has significantly enhanced accuracy. Contrasting various existing approaches, the proposed process is tailored for handling the difficulty in the third-order problem while preserving accuracy and stability over extensive problems. A detailed von Neumann stability analysis confirms its unconditional stability, which makes it robust, especially for long-term simulations, while the numerical ROC demonstrates the second-order convergence both in space and in time. Two expressive examples are considered to determine the accuracy and usefulness of the projected process. Compared to existing techniques, the combination of UHPB-spline functions together with the Crank–Nicolson method and FDM is evidence of an influential and consistent tool to solve pseudo-parabolic problems of higher orders.

Keywords: third-order pseudo-parabolic problem; UHPB-spline functions; collocation approach; stability analysis; ROC

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1. Introduction

Scientists and engineers use partial differential equations (PDEs) to understand physical quantity fluctuations over space and time and predict system behavior in various natural environments. Electromagnetism, quantum physics, and fluid dynamics use PDEs the most. They offer critical knowledge about meteorological systems and oceanic currents by defining fluid dynamics and phase interactions.

Understanding differential equations and PDEs is necessary to forecast and understand thermodynamics, science, technology, and environmental changes. Pseudo-parabolic problems frequently arise in modeling phenomena where the typical parabolic behavior is inadequate, such as fluid dynamics, heat transfer, and population dynamics. A higher-order parabolic system of partial differential equations addresses soil moisture transport, as well as the transfer of fluid dynamics, moisture, temperature, and density, both in the presence of and absence from porous materials [1–4]. PDEs are crucial in biological modeling, especially for simulating intricate cell migration and diffusion, as they effectively represent behaviors [5].

The higher-order pseudo-parabolic equations play an important role in the mathematical modeling of moisture transfer, fluid filtration, and heat propagation [6]. Third-order pseudo-parabolic issues have been employed in modeling various phenomena, including wave processes, chemical reactions, engineering applications, diffusion, plasma physics, and heat conduction [7]. In [8], the authors applied a pseudo-parabolic model to texture recognition and a real-world application to plant species identification from leaf surface images. In [9], the higher-order pseudo-parabolic equation simulates turbulence and mass-transfer diffusion in fluids, especially long-wave propagation in nonlinear dispersive systems. Alternative analytical techniques such as characteristic-based methods have been successfully applied to higher-order dynamical systems [10].

A wide range of numerical techniques, including finite element, finite volume, and finite difference methods with differential quadrature [11–18] and the collocation technique [19–25], have been employed to tackle a wide range of models in the different science fields. In [26], the authors focused on identifying a problem that concerned the primary and the lower coefficients of a third-order parabolic equation. The authors in [27] constructed an unidentified potential coefficient in the pseudo-parabolic problem of order three. The authors of [28] considered the inverse problem of the pseudo-parabolic equation to determine the right side and used the Galerkin method to prove the existence of solutions in a local as well as global time. The authors of [29,30] constructed high-order difference methods to approximate, numerically, the delay pseudo-parabolic problem.

The pseudo-parabolic equation with a non-local reaction term under homogeneous Neumann boundary conditions was studied in [31], demonstrating its relevance for mass-conserving systems. The authors in [32] developed a numerical technique for reconstructing a time-dependent boundary function in a Dirichlet–Neumann two-dimensional pseudo-parabolic problem, building upon the method for the heat equation. Parabolic equations with involution are used in [33] to describe processes where the state at a location depends on its mirror-symmetric counterpart in heat conduction models for geometrically symmetric domains. The authors in [34] showed that the Neumann boundary condition on temperature creates a significant coupling between the Euler and Fourier components, requiring a pseudo-parabolic regularization of the temperature equation.

In this research, the aforementioned problem is approximated numerically using a new UHPB-spline functions-based computational approach. The advantages and the novelty of the present work are summarized in the following points.

- The projected approach is tailored for handling the difficulty in the third-order problem while preserving accuracy and stability above extensive problems.
- The anticipated approach is second-order convergent both in space and in time and has significantly enhanced the accuracy.
- The investigation addresses a class of PDEs that is pseudo-parabolic of order three; it shows both hyperbolic and parabolic properties.
- This approach aims to boost the efficiency and accuracy of numerical simulations in various applications, particularly those involving dynamic systems and complex geometries.

- Compared to existing techniques, the combination of UHPB-spline functions together with the Crank–Nicolson method and FDM evidence acts as an influential and consistent tool to solve pseudo-parabolic problems of higher orders.

The organization of this article is detailed as follows: In Section 2, the construction of the main considered model is introduced, with the complete discussion of the proposed numerical approach that supports the research. Section 3 is dedicated to an in-depth stability analysis, which thoroughly examines the method's performance and establishes that it is unconditionally stable under various conditions. Moving to Section 4, the practical application of the method is validated through a series of examples, where its precision and accuracy are meticulously evaluated and assessed. Finally, the concluding section sums up the key findings and insights derived from this study, providing a full summary of the research outcomes.

2. Implementation of the Methodology

In this part, we discourse the main problem, i.e., the pseudo-parabolic problem with the procedure of employed technology with Neumann boundary and non-local integral conditions. Now, let us define the main problem that will be investigated.

2.1. The Third-Order Pseudo-Parabolic Problem

The considered problem is as follows:

$$\frac{\partial w}{\partial t} - \sigma \frac{\partial^3 w}{\partial^2 \xi \partial t} - \delta(t) \frac{\partial^2 w}{\partial^2 \xi} = \chi(t)w(\xi, t) + \alpha(\xi, t), \quad \xi \in [0, 1], \quad t \in [0, T], \quad (1)$$

with the initial condition

$$w(\xi, 0) = \mu(\xi), \quad (2)$$

along with the Neumann boundary and the non-local integral conditions

$$w_\xi(0, t) = 0, \quad t \in [0, T], \quad (3)$$

$$\int_0^1 w(\xi, t) d\xi, \quad t \in [0, T], \quad (4)$$

where

$\sigma =$ positive number that influences the diffusion term.

$\frac{\partial^3 w}{\partial^2 \xi \partial t}$ = the diffusion term, the thermal diffusivity, and the conductivity represented by the diffusion term.

$\delta(t) \frac{\partial^2 w(\xi, t)}{\partial^2 \xi}$ = the wave propagation and wave dispersion represented by this term.

The nonlinearity term in Equation (1) is $\chi(t)w(\xi, t) + \alpha(\xi, t)$ which leads to the complex behaviors of such pattern formation.

The values of these functions $\mu(t)$, $\alpha(\xi, t)$, $\delta(t) > 0$, are provided as per the existence and uniqueness conditions [28].

2.2. Discretization of the Problem

In this part, we discuss the discretization of the considered problem. Discretization is a concept that converts a continuous mathematical model to a finite set of discrete elements by using Equations (1)–(4), where the parameters $\chi(t)$, $\delta(t)$, $\alpha(\xi, t)$, σ are known and $w(\xi, t)$ is unidentified. To solve the considered problem, we proceed with the collocation technique based on UHPB-spline functions. The space domain is divided into equal length segments, denoted as $\Delta\xi = \xi_{i+1} - \xi_i$, $i = 0, 1, \dots, M$. Here, the usage of the symbolizations are as follows: $\chi(t_j) = \chi^j$, $\delta(t_j) = \delta^j$, $\alpha(\xi_i, t_j) = \alpha_i^j$ and $w(\xi_i, t_j) = w_i^j$, where $t_j = j\Delta t$, $\xi_i = i\Delta\xi$, $\Delta t = \frac{T}{N}$, $i = 0, 1, \dots, M$, and $j = 0, 1, \dots, N$. The UHPB-spline functions $Hs_i(\xi)$, $i = -1, 0, \dots, M+1$ are given as follows [35,36]:

$$Hs_i(\xi) = \frac{1}{\wp} \begin{cases} \sinh(\xi - \xi_{i-2}) - (\xi - \xi_{i-2}), & \text{if } \xi \in [\xi_{i-2}, \xi_{i-1}) \\ -2\sinh(\xi - \xi_{i-1}) - \sinh(\xi - \xi_i) + \\ (2\cosh(\Delta\xi) + 1)(\xi - \xi_{i-1}) - \Delta\xi, & \text{if } \xi \in [\xi_{i-1}, \xi_i) \\ \sinh(\xi - \xi_i) + 2\sinh(\xi - \xi_{i+1}) - \\ (2\cosh(\Delta\xi) + 1)(\xi - \xi_i) + 2\Delta\xi \cosh(\Delta\xi), & \text{if } \xi \in [\xi_i, \xi_{i+1}) \\ -\sinh(\xi - \xi_{i+2}) + (\xi - \xi_{i+2}) & \text{if } \xi \in [\xi_{i+1}, \xi_{i+2}) \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

where $\wp = 2\Delta\xi(\cosh(\Delta\xi) - 1)$. Here, we choose $\{Hs_{-1}, Hs_0, Hs_1, \dots, Hs_{M+1}\}$ in such a procedure so that this resultant set forms a basis over the problem domain. The values of the above-mentioned functions and their derivatives at the knot points are given in Table 1.

Table 1. The $Hs_i(\xi)$ with derivatives at knot points.

	ξ_{i-2}	ξ_{i-1}	ξ_i	ξ_{i+1}	ξ_{i+2}
$Hs_i(\xi)$	0	ε_1	ε_2	ε_1	0
$Hs'_i(\xi)$	0	ε_3	0	ε_4	0
$Hs''_i(\xi)$	0	ε_5	ε_6	ε_5	0

$$\text{Where } \varepsilon_1 = \frac{\sinh(\Delta\xi) - \Delta\xi}{\wp}, \quad \varepsilon_2 = \frac{2(\Delta\xi \cosh(\Delta\xi) - \sinh(\Delta\xi))}{\wp}, \quad \varepsilon_3 = -\varepsilon_4 = \frac{1}{2\Delta\xi},$$

$$\varepsilon_5 = \frac{\sinh(\Delta\xi)}{\wp}, \text{ and } \varepsilon_6 = \frac{-2\sinh(\Delta\xi)}{\wp}.$$

It is assumed that approximate solution $w(\xi, t)$ at points (ξ, t_j) can be expressed as a linear combination of UHPB-spline functions and weighting coefficients given below:

$$w(\xi, t_j) \cong \sum_{k=-1}^{M+1} \Upsilon_k(t_j) Hs_k(\xi), \quad j = 0, 1, \dots, N, \quad (6)$$

where time extents are expressed as $\Upsilon_k(t_j)$. To start the approximation process, we required the initial values for the time-dependent quantity. In addition, since the UHPB-spline functions and its four principle derivatives vanish outside of $[\xi_{i-2}, \xi_{i+2}]$, the variation in the $w(\xi, t_j)$ over the component is taken as follows:

$$w(\xi, t_j) = \sum_{m=i-1}^{i+1} Hs_m(\xi) \Upsilon_m(t_j). \quad (7)$$

The $w(\xi, t)$ and evaluation of its derivatives with respect to ξ are accompanied by making use of Table 1 as follows:

$$w_i^j = \varepsilon_1 \Upsilon_{i-1}^j + \varepsilon_2 \Upsilon_i^j + \varepsilon_1 \Upsilon_{i+1}^j, \quad (8)$$

$$(w_\xi)_i^j = \varepsilon_3 \Upsilon_{i-1}^j + \varepsilon_4 \Upsilon_{i+1}^j, \quad (9)$$

$$(w_{\xi\xi})_i^j = \varepsilon_5 \Upsilon_{i-1}^j + \varepsilon_6 \Upsilon_i^j + \varepsilon_5 \Upsilon_{i+1}^j, \quad (10)$$

where $i = 0, 1, \dots, M$, $j = 0, 1, \dots, N$.

Now, we discretize Equation (1) using forward difference formula together with Crank–Nicolson scheme as follows:

$$\begin{aligned} \frac{w_i^{j+1} - w_i^j}{\Delta t} - \sigma \left(\frac{(w_{\xi\xi})_i^{j+1} - (w_{\xi\xi})_i^j}{\Delta t} \right) - \left(\frac{\delta^{j+1} (w_{\xi\xi})_i^{j+1} + \delta^j (w_{\xi\xi})_i^j}{2} \right) &= \frac{\chi^{j+1} (w)_i^{j+1} + \chi^j (w)_i^j}{2} \\ &+ \frac{\alpha_i^{j+1} + \alpha_i^j}{2}, \quad i = 0, 1, \dots, M, \quad j = 0, 1, \dots, N. \end{aligned} \quad (11)$$

By making simpler and reordering the terms in the above Equation (11), we achieve the following:

$$A^{j+1} w_i^{j+1} + B^{j+1} (w_{\xi\xi})_i^{j+1} = C^j (w)_i^j + D^j (w_{\xi\xi})_i^j + f_i^j, \quad (12)$$

where

$$\begin{aligned} A^{j+1} &= 1 - \frac{\Delta t}{2} \chi^{j+1}, \quad B^{j+1} = -\sigma - \frac{\Delta t}{2} \delta^{j+1}, \quad C^j = 1 + \frac{\Delta t}{2} \chi^j, \quad D^j = -\sigma + \frac{\Delta t}{2} \delta^j, \quad \text{and} \\ f_i^j &= \frac{\Delta t}{2} (\alpha_i^{j+1} + \alpha_i^j). \end{aligned}$$

In the above equation, we anticipate values of w_i^j and $(w_{\xi\xi})_i^j$ from Equations (8) and (10). Reordering terms in Equation (12), we obtain the following:

$$\bar{A}^{j+1} \Upsilon_{i-1}^{j+1} + \bar{B}^{j+1} \Upsilon_i^{j+1} + \bar{A}^{j+1} \Upsilon_{i+1}^{j+1} = \bar{C}^j \Upsilon_{i-1}^j + \bar{D}^j \Upsilon_i^j + \bar{C}^j \Upsilon_{i+1}^j + f_i^j, \quad i = 1, \dots, M-1, \quad j = 0, 1, \dots, N, \quad (13)$$

where

$$\bar{A}^{j+1} = A^{j+1} \varepsilon_1 + B^{j+1} \varepsilon_5, \quad \bar{B}^{j+1} = A^{j+1} \varepsilon_2 + B^{j+1} \varepsilon_6, \quad \bar{C}^j = C^j \varepsilon_1 + D^j \varepsilon_5, \quad \text{and} \quad \bar{D}^j = C^j \varepsilon_2 + D^j \varepsilon_6.$$

Now, we use the Neumann boundary condition $w_\xi(0, t) = 0$ and non-local integral condition $\int_0^1 w(\xi, t) d\xi = 0$ as follows:

$$\Upsilon_{-1}^j = -\frac{\varepsilon_3}{\varepsilon_4} \Upsilon_1^j, \quad j = 0, 1, \dots, N, \quad (14)$$

and discretization of the condition (4) using Simpson's 1/3 rule gives

$$\int_0^1 w(\xi, t) d\xi = \frac{h}{3} \left(w_0^j + w_M^j + 4 \sum_{i=1,3}^{M-1} w_i^j + 2 \sum_{i=2,4}^{M-2} w_i^j \right) = 0, \quad j = 0, 1, \dots, N. \quad (15)$$

From the above equation, we obtain the following:

$$\Upsilon_{M+1}^{j+1} = -\left(4 + \frac{\varepsilon_2}{\varepsilon_1} \right) \Upsilon_0^j - \left(3 + 4 \frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_3}{\varepsilon_4} \right) \Upsilon_1^j - 2 \left(4 + \frac{\varepsilon_2}{\varepsilon_1} \right) \sum_{i=2,4}^{M-2} \Upsilon_i^j - 4 \left(1 + \frac{\varepsilon_2}{\varepsilon_1} \right) \sum_{i=3,5}^{M-3} \Upsilon_i^j - \left(4 \frac{\varepsilon_2}{\varepsilon_1} + 3 \right) \Upsilon_{M-1}^j - \left(4 + \frac{\varepsilon_2}{\varepsilon_1} \right) \Upsilon_M^j. \quad (16)$$

For $i = 0$, using Equation (14) in Equation (13), we obtain the following:

$$\bar{B}^{j+1}\Upsilon_0^{j+1} + \left(1 - \frac{\varepsilon_3}{\varepsilon_4}\right)\bar{A}^{j+1}\Upsilon_1^{j+1} = \bar{D}^j\Upsilon_0^j + \left(1 - \frac{\varepsilon_3}{\varepsilon_4}\right)\bar{C}^j\Upsilon_1^j + f_0^j, \quad j = 0, 1, \dots, N. \quad (17)$$

For $i = M$, using Equation (16) in Equation (13), we obtain the following:

$$\begin{aligned} \hat{A}_1\Upsilon_0^{j+1} + \hat{A}_2\Upsilon_1^{j+1} + \hat{A}_3\sum_{i=2,4}^{m-2}\Upsilon_i^{j+1} + \hat{A}_4\sum_{i=3,5}^{m-3}\Upsilon_i^{j+1} + \hat{A}_5\Upsilon_{i-1}^{j+1} + \hat{A}_6\Upsilon_i^{j+1} = & \bar{C}^j\left(4 + \frac{\varepsilon_2}{\varepsilon_1}\right)\Upsilon_0^j - \bar{C}^j\left(3 + 4\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_3}{\varepsilon_4}\right)\Upsilon_1^j \\ & + \bar{C}^j\left(8 + 2\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{i=2,4}^{m-2}\Upsilon_i^j - 4\bar{C}^j\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right) - \bar{C}^j\left(2 + 4\frac{\varepsilon_2}{\varepsilon_1}\right)\Upsilon_{i-1}^j + \left(\bar{D}^j - \bar{C}^j\left(4 + \frac{\varepsilon_2}{\varepsilon_1}\right)\right)\Upsilon_i^j + f_M^j, \end{aligned} \quad (18)$$

where

$$\begin{aligned} \hat{A}_1 = -\left(4 + \frac{\varepsilon_2}{\varepsilon_1}\right)\bar{A}^{j+1}, \quad \hat{A}_2 = \left(-3 - 4\frac{\varepsilon_2}{\varepsilon_1} + \frac{\varepsilon_3}{\varepsilon_4}\right)\bar{A}^{j+1}, \quad \hat{A}_3 = -2\left(4 + \frac{\varepsilon_2}{\varepsilon_1}\right)\bar{A}^{j+1}, \\ \hat{A}_4 = -4\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right)\bar{A}^{j+1}, \quad \hat{A}_5 = \left(-2 - 4\frac{\varepsilon_2}{\varepsilon_1}\right)\bar{A}^{j+1}, \quad \hat{A}_6 = \bar{B}^{j+1} - \bar{A}^{j+1}\left(4 + \frac{\varepsilon_2}{\varepsilon_1}\right). \end{aligned}$$

At time step ζ_{j+1} , $j = 0, 1, \dots, N$, Equations (13), (17) and (18) form a $(M + 1) \times (M + 1)$ order linear system as follows:

$$\begin{bmatrix} \bar{B}^{j+1} & \left(1 - \frac{\varepsilon_3}{\varepsilon_4}\right)\bar{A}^{j+1} & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \bar{A}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \bar{A}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \bar{A}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \bar{A}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \bar{A}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \bar{A}^{j+1} & \bar{B}^{j+1} & \bar{A}^{j+1} \\ \hat{A}_1 & \hat{A}_2 & \hat{A}_3 & \dots & \hat{A}_5 & \hat{A}_4 & \hat{A}_3 & \hat{A}_5 & \hat{A}_6 \end{bmatrix} \begin{bmatrix} \Upsilon_0^{j+1} \\ \Upsilon_1^{j+1} \\ \Upsilon_2^{j+1} \\ \Upsilon_3^{j+1} \\ \vdots \\ \Upsilon_{M-3}^{j+1} \\ \Upsilon_{M-2}^{j+1} \\ \Upsilon_{M-1}^{j+1} \\ \Upsilon_M^{j+1} \end{bmatrix} = \begin{bmatrix} \hat{H}_0^j \\ \hat{H}_1^j \\ \hat{H}_2^j \\ \hat{H}_3^j \\ \vdots \\ \hat{H}_{M-3}^j \\ \hat{H}_{M-2}^j \\ \hat{H}_{M-1}^j \\ \hat{H}_M^j \end{bmatrix}, \quad (19)$$

where $\hat{H}_0^j = \bar{D}^j\Upsilon_0^j + \left(1 - \frac{\varepsilon_3}{\varepsilon_4}\right)\bar{C}^j\Upsilon_1^j + f_0^j$, $\hat{H}_i^j = \bar{C}^j\Upsilon_{i-1}^j + \bar{D}^j\Upsilon_i^j + \bar{C}^j\Upsilon_{i+1}^j + f_i^j$, $i = 1, \dots, M - 1$, and

$$\hat{H}_M^j = \bar{C}^j\left(4 + \frac{\varepsilon_2}{\varepsilon_1}\right)\Upsilon_0^j - \bar{C}^j\left(3 + 4\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_3}{\varepsilon_4}\right)\Upsilon_1^j + \bar{C}^j\left(8 + 2\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{i=2,4}^{m-2}\Upsilon_i^j - 4\bar{C}^j\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right) - \bar{C}^j\left(2 + 4\frac{\varepsilon_2}{\varepsilon_1}\right)\Upsilon_{i-1}^j + \left(\bar{D}^j - \bar{C}^j\left(4 + \frac{\varepsilon_2}{\varepsilon_1}\right)\right)\Upsilon_i^j + f_M^j.$$

This system of equations will start calculating for the next time solution. For beginning the iteration process, the system needs the initial vector $(\Upsilon_0^0, \Upsilon_1^0, \Upsilon_2^0, \dots, \Upsilon_M^0)$ which is evaluated by initial condition $w(\xi, 0) = \mu(\xi)$ and with the help of Neumann boundary and non-local integral conditions $(w_\xi)_0^0 = 0$ and $\int_0^1 w(\xi, 0) d\xi = 0$, respectively, as $w(\xi, 0) = \mu(\xi)$ gives the following:

$$\varepsilon_1\Upsilon_{i-1}^0 + \varepsilon_2\Upsilon_i^0 + \varepsilon_1\Upsilon_{i+1}^0 = \mu_i, \quad \text{for } i = 1, 2, \dots, M - 1. \quad (20)$$

For $i = 0$, using the discretized form of $(w_\xi)_0^0 = 0$ from Equation (14) in above equation, we obtain the following:

$$\varepsilon_2\Upsilon_0^0 + \left(\varepsilon_1 - \frac{\varepsilon_1\varepsilon_3}{\varepsilon_4}\right)\Upsilon_1^0 = \mu_0. \quad (21)$$

For $i=M$, using the discretized form of $\int_0^1 w(\xi, 0) d\xi = 0$ from Equation (16) in Equation (20), we obtain the following:

$$(-4\varepsilon_1 - \varepsilon_2)Y_0^0 - \left(3\varepsilon_1 + 4\varepsilon_2 - \frac{\varepsilon_1\varepsilon_3}{\varepsilon_4}\right)Y_1^0 - (8\varepsilon_1 + 2\varepsilon_2)\sum_{i=2,4}^{M-2} Y_i^j - 4(\varepsilon_1 + \varepsilon_2)\sum_{i=3,5}^{M-3} Y_i^j - (2\varepsilon_1 + 4\varepsilon_2)Y_{M-1}^j - 4\varepsilon_1 Y_M^j = \mu_M. \quad (22)$$

Equations (20)–(22) form $(M+1) \times (M+1)$ system as follows:

$$\begin{bmatrix} \varepsilon_2 & \bar{\eta} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varepsilon_1 & \varepsilon_2 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_1 \\ \hat{\eta}_1 & \hat{\eta}_2 & \hat{\eta}_3 & \cdots & \hat{\eta}_3 & \hat{\eta}_4 & \hat{\eta}_5 & \hat{\eta}_6 & \hat{\eta}_6 \end{bmatrix} \begin{bmatrix} Y_0^{j+1} \\ Y_1^{j+1} \\ Y_2^{j+1} \\ Y_3^{j+1} \\ \vdots \\ Y_{M-3}^{j+1} \\ Y_{M-2}^{j+1} \\ Y_{M-1}^{j+1} \\ Y_M^{j+1} \end{bmatrix} = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_{M-3} \\ \mu_{M-2} \\ \mu_{M-1} \\ \mu_M \end{bmatrix}, \quad (23)$$

where $\bar{\eta} = \varepsilon_1 - \frac{\varepsilon_1\varepsilon_3}{\varepsilon_4}$, $\hat{\eta}_1 = -4\varepsilon_1 - \varepsilon_2$, $\hat{\eta}_2 = 3\varepsilon_1 + 4\varepsilon_2 - \frac{\varepsilon_1\varepsilon_3}{\varepsilon_4}$, $\hat{\eta}_4 = -4(\varepsilon_1 + \varepsilon_2)$, $\hat{\eta}_5 = -(2\varepsilon_1 + 4\varepsilon_2)$, and $\hat{\eta}_6 = -4\varepsilon_1$.

3. Stability Analysis

The stability for the discretized system of the problem is analyzed by using the von Neumann procedure. According to Duhamels' principle [37], it is supposed that the stability analysis of an inhomogeneous system is an instantaneous result of the stability analysis for the consequent homogeneous system. Thus, for convenience and without loss of generality, we choose the forcing term $f = 0$, which is equal to zero. It is also assumed that $\chi^j = \chi^{j+1} = \hat{P}_1$, $\delta^j = \delta^{j+1} = \hat{P}_2$ as local constants. Equation (12) converts into the following:

$$\hat{X}w_i^{j+1} + \hat{Y}(w_{\xi\xi}^j)_i^{j+1} = \hat{R}(w)_i^j + \hat{S}(w_{\xi\xi}^j)_i^j, \quad (24)$$

where $\hat{X} = 1 - \frac{\Delta t}{2}\hat{P}_1$, $\hat{Y} = -\sigma - \frac{\Delta t}{2}\hat{P}_2$, $\hat{R} = 1 + \frac{\Delta t}{2}\hat{P}_1$, and $\hat{S} = -\sigma + \frac{\Delta t}{2}\hat{P}_2$.

Now, using Equations (8) and (10) in Equation (19), we obtain the following:

$$\bar{X}Y_{i-1}^{j+1} + \bar{Y}Y_i^{j+1} + \bar{X}Y_{i+1}^{j+1} = \bar{R}Y_{i-1}^j + \bar{S}Y_i^j + \bar{R}Y_{i+1}^j, \quad i = 0, 1, \dots, M, \quad j = 0, 1, \dots, N, \quad (25)$$

where $\bar{X} = \hat{X}\varepsilon_1 + \hat{Y}\varepsilon_3$, $\bar{Y} = \hat{X}\varepsilon_2 + \hat{Y}\varepsilon_6$, $\bar{R} = \hat{R}\varepsilon_1 + \hat{S}\varepsilon_3$, and $\bar{S} = \hat{R}\varepsilon_2 + \hat{S}\varepsilon_6$.

Next, taking Fourier mode for $Y_i^j = \lambda^j e^{i\kappa\eta}$ at ξ_i , where $\eta = \theta\Delta\xi$ and $\kappa = \sqrt{-1}$, we obtain the following:

$$\lambda = \frac{2\bar{R}\cos\eta + \bar{S}}{2\bar{X}\cos\eta + \bar{Y}}. \quad (26)$$

Using the values of $\bar{R}$, $\bar{S}$, $\bar{X}$, and $\bar{Y}$, we obtain the following:

$$\lambda = \frac{4\bar{R} \cos^2\left(\frac{\eta}{2}\right) + (\bar{S} - 2\bar{R})}{4\bar{X} \cos^2\left(\frac{\eta}{2}\right) + (\bar{Y} - 2\bar{X})}, \quad (27)$$

$$\text{where } \bar{R} = \varepsilon_1 \left(1 + \frac{\Delta t}{2}(1+t)\right) - \varepsilon_5 \left(1 + \frac{\Delta t}{2}(1+t)\right), \quad \bar{X} = \varepsilon_1 \left(1 - \frac{\Delta t}{2}(1+t)\right) - \varepsilon_5 \left(1 + \frac{\Delta t}{2}(1+t)\right),$$

$$\bar{S} - 2\bar{R} = (\varepsilon_1 + \varepsilon_2) \left(1 + \frac{\Delta t}{2}(1+t)\right) - (\varepsilon_5 + \varepsilon_6) \left(1 - \frac{\Delta t}{2}(1+t)\right), \quad \bar{Y} - 2\bar{X} = (\varepsilon_1 + \varepsilon_2) \left(1 - \frac{\Delta t}{2}(1+t)\right) - (\varepsilon_5 + \varepsilon_6) \left(1 + \frac{\Delta t}{2}(1+t)\right).$$

Since $\varepsilon_1, \varepsilon_2, \varepsilon_5 > 0$, $\varepsilon_6 < 0$, $\varepsilon_5 + \varepsilon_6 > 0$, it is recognized that $\bar{S} - 2\bar{R} < \bar{Y} - 2\bar{X}$ and $\bar{R} < \bar{X}$. Also, $0 \leq \cos^2\left(\frac{\eta}{2}\right) \leq 1$; therefore, $|\lambda| \leq 1$. So, the discretized system of the problem using UHPB-spline computation approach is unconditionally stable.

4. Results and Investigation for the Two Illustrative Examples

In this portion to evaluate the accuracy, effectiveness and convergence of the technique has been justified by using examples of the concern. The ROC is analyzed by the following:

$$ROC = \frac{\ln(e(\Delta\xi_1)/e(\Delta\xi_2))}{\ln(\Delta\xi_1/\Delta\xi_2)}, \text{ where } e(\Delta\xi_1) \text{ and } e(\Delta\xi_2) \text{ stand for errors with } \Delta\xi_1 \text{ and } \Delta\xi_2, \text{ respectively.}$$

Example 1. This illustration examines the outcomes of the problem (1) in $\xi \in [0, 1]$ and with conditions $w(\xi, 0) = \left(\frac{1}{3} - \xi^2\right)$, $w_\xi(0, t) = -2\xi e^t$, and integral condition $\int_0^1 w(\xi, t) d\xi = 0$, along with the exact one, $w(\xi, t) = \left(\frac{1}{3} - \xi^2\right) e^t$.

The values $\sigma = 1$, $\delta(t) = \chi(t) = 1 + t$, and $\alpha(\xi, t) = \left(\xi^2 - \frac{1}{3}\right) t e^t + 2(2+t) e^t$ have been chosen for Example 1.

Table 2 compares L_∞ and RMS errors for the obtained outcomes with the ones presented in Ref. [38] with $\Delta t = 0.0025$ for different space step grid sizes at $t = 1$. It is noticed that the error norms presented in Table 2 for obtained results are progressively decreasing and the results are second-order accurate for all space grid sizes, while the errors reported in Ref. [38] are suddenly decreased from grid size 0.1 to 0.05, and after that, they remain almost constant for the remaining space grid sizes. So, we can conclude that the present method offers better results and the method is second-order convergent in space variable. The error norms are quickly decreasing when the mesh sizes increase, signifying high accuracy and convergence within the anticipated context. The L_2 , L_∞ , and RMS error norms along with ROC for the time step mesh are exhibited in Table 3 for the obtained outcomes with $\Delta\xi = 0.002$ at $t = 1$. The above-mentioned tables prove that the order of convergence of the projected approach is $O(\xi^2, t^2)$. Next, Table 4 compares the results with those presented in [27] at $t = 1$ when space and time mesh sizes are equal. This table also shows that the obtained solutions are more accurate, and accuracy increases when the mesh sizes increase.

Table 2. The assessment of the obtained solutions from Ref. [38] in terms of L_∞ and RMS errors with $\Delta t = 0.0025$ at $t = 1$ and different space grid sizes, for Example 1.

$\Delta \xi$	Ref. [38]		Projected Method			
	L_∞	RMS	L_∞	ROC	RMS	ROC
0.1	3.5949×10^0	6.6999×10^0	1.382×10^{-3}	----	6.810×10^{-4}	---
0.025	2.047×10^{-5}	1.6194×10^{-6}	8.613×10^{-5}	3.66	3.949×10^{-5}	2.05
0.0125	2.1262×10^{-5}	1.139×10^{-6}	2.128×10^{-5}	2.01	9.628×10^{-6}	2.04
0.0025	2.1993×10^{-5}	5.0972×10^{-7}	5.208×10^{-7}	2.30	2.290×10^{-7}	2.32
0.00125	--	--	1.266×10^{-7}	2.04	6.104×10^{-8}	1.91

Table 3. The L_2 , L_∞ , and RMS error norms along with ROC with $\Delta \xi = 0.0205$ at various time steps, for Example 1.

t	L_2	ROC	L_∞	ROC	RMS	ROC
0.05	6.216×10^{-5}	--	1.341×10^{-4}	--	6.200×10^{-5}	--
0.025	1.437×10^{-5}	2.11	3.093×10^{-5}	2.12	1.434×10^{-5}	2.11
0.02	8.639×10^{-6}	2.28	1.855×10^{-5}	2.29	8.618×10^{-6}	2.28
0.0125	2.426×10^{-6}	2.70	5.139×10^{-6}	2.73	2.420×10^{-6}	2.70
0.001	9.924×10^{-7}	4.00	2.044×10^{-6}	4.13	9.899×10^{-7}	4.01

Table 4. The comparison of obtained outcomes with ones presented in Ref. [27] in terms of L_∞ , L_2 , and RMS error norms at $t = 1$, for Example 1.

$\Delta \xi = \Delta t =$	Present Method			Ref. [27]
	L_∞	L_2	RMS	RMS
0.2	3.337×10^{-3}	1.934×10^{-3}	1.765×10^{-3}	9.9×10^{-3}
0.1	8.333×10^{-4}	4.242×10^{-4}	4.044×10^{-4}	4.2×10^{-3}
0.025	5.208×10^{-5}	2.375×10^{-5}	2.346×10^{-5}	9.2×10^{-4}

Table 5 shows the L_2 , L_∞ , and RMS error norms together with ROC for the obtained solutions with $\Delta t = 0.001$ at different space grid sizes. It is initiated that error norms are very small, approximately $\sim (10^{-8} \text{ to } 10^{-6})$. The plot of the L_2 and L_∞ errors with $\Delta t = 0.002$ for space grids $M = 16, 50, 80, 100, 120, 140$, and 160 at $t = 1$ are presented in Figure 1. It can be perceived that the L_2 and L_∞ error norms are decreasing rapidly upon increasing the size of space grids. Figure 2. indicates a worthy covenant between the exact and approximate $w(\xi, t)$. Figures 3 and 4 display 3D surface plots of the exact and approx. $w(\xi, t)$ with $\Delta \xi = \Delta t = 0.05$ and abs. errors for $\Delta \xi = 0.02, 0.00625$ with $\Delta t = 0.01$, respectively. These plots exhibit highly accurate outcomes.

Table 5. The L_2 , L_∞ , and RMS errors, together with ROC of achieved outcomes with $\Delta t = 0.02$ at $t = 1$ and different space grid sizes, for Example 1.

M	L_2	ROC	L_∞	ROC	RMS	ROC
40	2.965×10^{-5}	---	6.446×10^{-5}	---	2.929×10^{-5}	---
50	1.508×10^{-5}	3.02	3.333×10^{-5}	2.95	1.493×10^{-5}	3.02
64	6.129×10^{-6}	3.65	1.398×10^{-5}	3.88	6.081×10^{-6}	3.64
88	2.176×10^{-6}	3.25	4.144×10^{-6}	3.82	2.164×10^{-6}	3.24

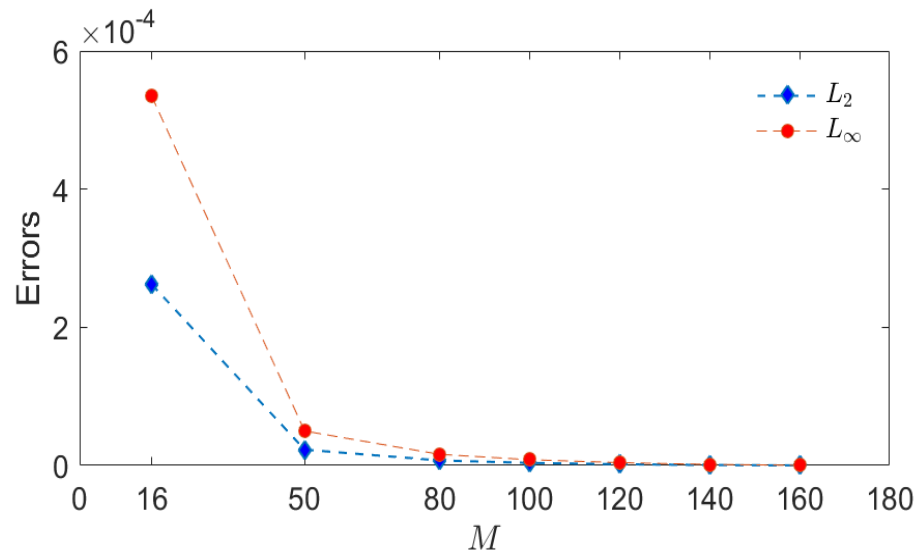


Figure 1. The L_2 and L_∞ errors with $\Delta t = 0.01$ at $t = 1$ for Example 1.

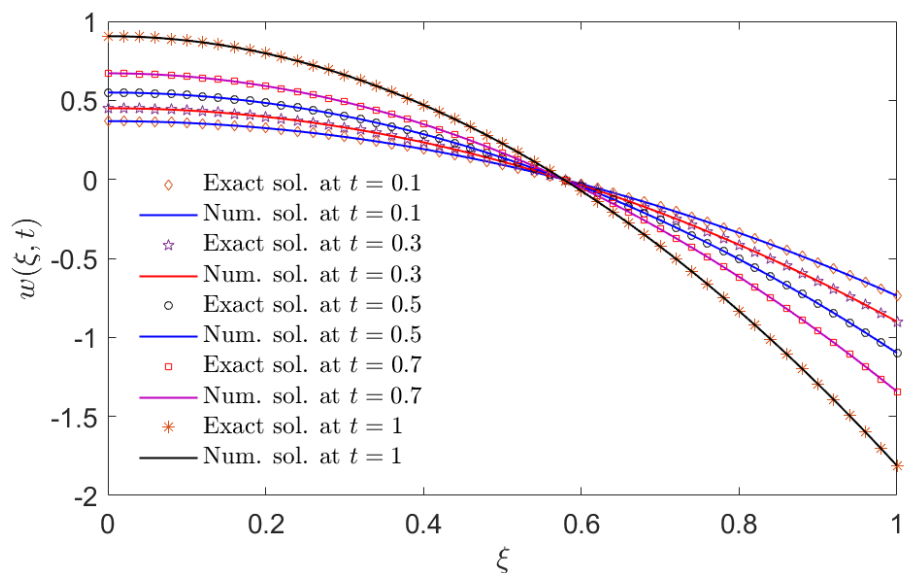


Figure 2. The comparison of the exact and numerical $w(\xi, t)$ with $\Delta \xi = 0.02$, and $\Delta t = 0.02$ at various t for Example 1.

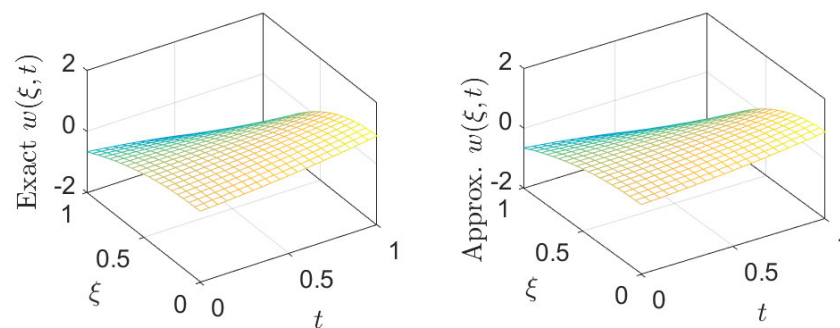


Figure 3. The 3D surface plots of the exact and approx. $w(\xi, t)$ with $\Delta \xi = \Delta t = 0.05$, $t \in [0, 1]$ for Example 1.

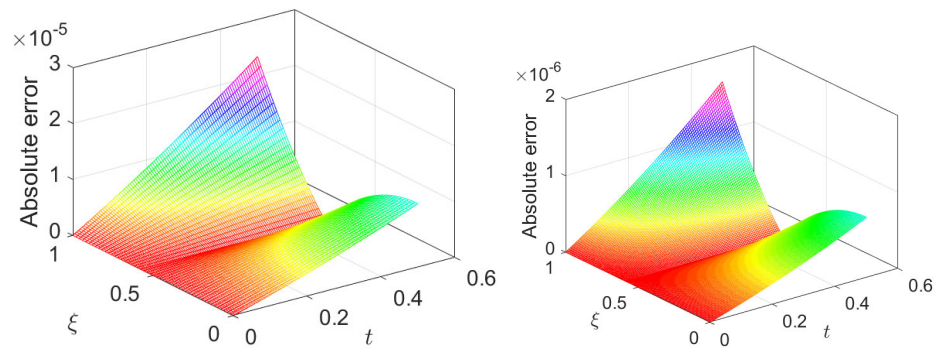


Figure 4. The abs. error plots $\Delta\xi = 0.02$ (left) and $\Delta\xi = 0.00625$ (right) with $\Delta t = 0.01$, $t \in [0, 0.5]$ for Example 1.

Example 2. This example approximates Equation (1) in $\xi \in [0, 1]$ with conditions $w(\xi, 0) = \cos(2\pi\xi)$, $w_\xi(0, t) = 0$, along with the exact one $w(\xi, t) = e^{-t} \cos(2\pi\xi)$.

First, the values $\sigma = 1$, $\delta(t) = 1$, $\chi(t) = -1 - 2t$, and $\alpha(\xi, t) = 2te^{-t} \cos(2\pi\xi)$ have been fixed for Example 2. Table 6 shows the obtained outcomes with $\Delta\xi = 0.01$ for $\Delta t = 0.025, 0.02, 0.0125, 0.01, 0.008$, and 0.00625 at $t = 1$. It can be seen that errors are decreasing as time intervals decrease. Table 7 shows the L_2 , L_∞ , and RMS error norms as well as ROC with $\Delta\xi = 0.01$ for some steps of time Δt at $t = 1$. It is apparent that errors are decreasing when the time mesh size increases. It can be seen that the convergent rate of the anticipated approach is two in time.

Table 6. The L_2 , L_∞ , and RMS error norms with $\Delta\xi = 0.01$ for different time intervals Δt at $t = 1$ for Example 2.

Δt	L_2	L_∞	RMS
0.025	1.346×10^{-5}	1.885×10^{-5}	1.339×10^{-5}
0.02	8.615×10^{-6}	1.206×10^{-5}	8.572×10^{-6}
0.0125	3.365×10^{-6}	4.712×10^{-6}	3.348×10^{-6}
0.01	2.154×10^{-6}	3.016×10^{-6}	2.143×10^{-6}
0.008	1.378×10^{-6}	1.930×10^{-6}	1.371×10^{-6}
0.00625	8.413×10^{-7}	1.178×10^{-6}	8.371×10^{-7}

Table 7. The L_2 , L_∞ , and RMS error norms together with ROC with $\Delta\xi = 0.01$ at $t = 1$ and different time steps Δt , for Example 2.

Δt	L_2	ROC	L_∞	ROC	RMS	ROC
0.05	5.386×10^{-5}	--	7.541×10^{-5}	--	5.359×10^{-5}	--
0.04	3.446×10^{-5}	2.001	4.826×10^{-5}	2.000	3.429×10^{-5}	2.001
0.025	1.346×10^{-5}	2.000	1.885×10^{-5}	2.000	1.339×10^{-5}	2.001
0.02	8.615×10^{-6}	2.000	1.206×10^{-5}	2.001	8.572×10^{-6}	1.998
0.0125	3.365×10^{-6}	2.000	4.712×10^{-6}	1.999	3.348×10^{-6}	2.000
0.01	2.154×10^{-6}	1.999	3.016×10^{-6}	1.999	2.143×10^{-6}	1.999
0.008	1.378×10^{-6}	2.002	1.930×10^{-6}	2.001	1.371×10^{-6}	2.002

Figure 5 compares the exact and approximate $w(\xi, t)$ with $\Delta\xi = \Delta t = 0.025$ at various t while Figure 6 exhibits the 3D surface plots of the exact and approx. $w(\xi, t)$ with

$\Delta\xi = 0.025$, $\Delta t = 0.1$, and $t \in [0,1]$. Figure 6 exemplifies showing a worthy pact between the exact and approximate $w(\xi, t)$. The abs. error plots with $\Delta\xi = \Delta t = 0.025$ and $\Delta\xi = \Delta t = 0.00625$ for $t \in [0,1]$ have been shown in Figure 7, which illustrates that abs. errors are very small, approximately equal to 10^{-6} . These investigations establish that the suggested approach produces highly accurate outcomes.

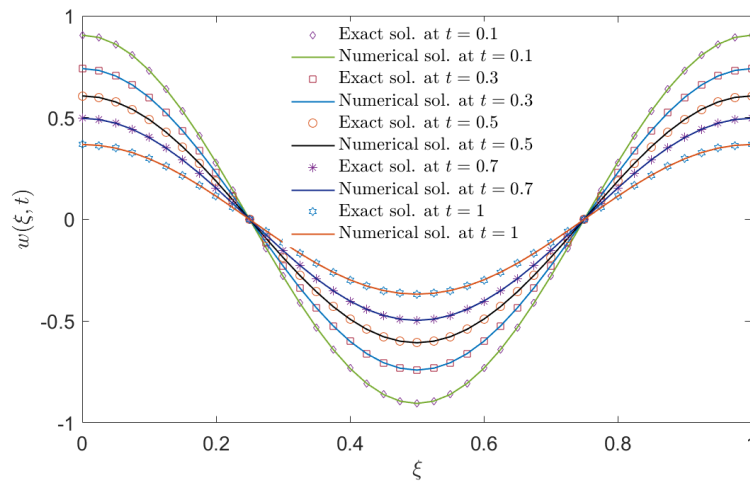


Figure 5. The assessment of the exact and approximated $w(\xi, t)$ with $\Delta\xi = \Delta t = 0.025$ at various t , for Example 2.

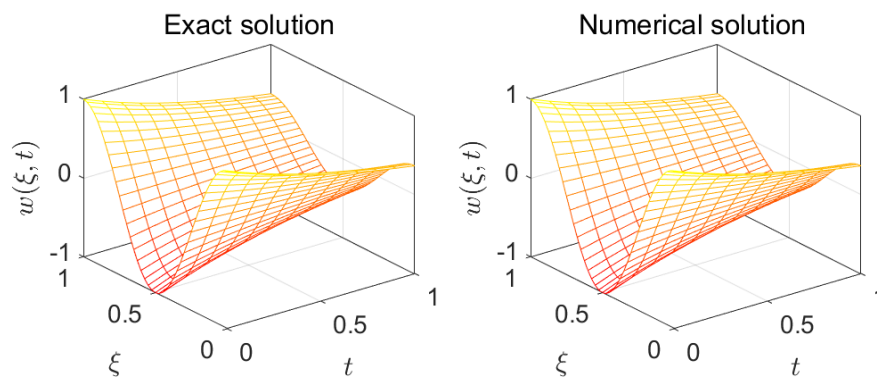


Figure 6. The 3D surface plots of the exact and approx. $w(\xi, t)$ with $\Delta\xi = 0.025$, $\Delta t = 0.1$, and $t \in [0,1]$ for Example 2.

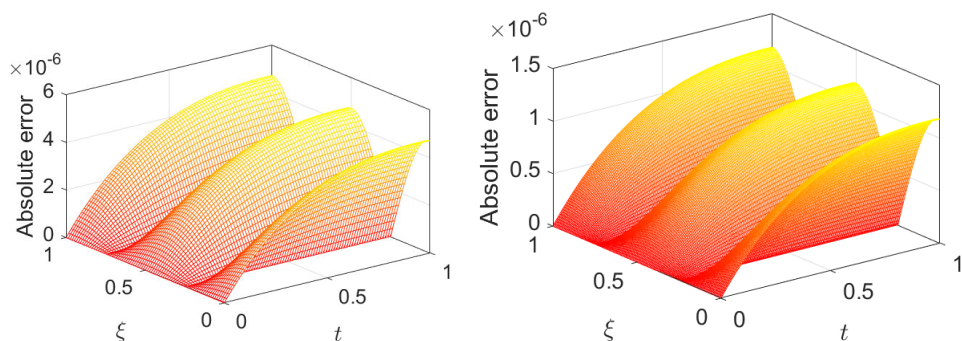


Figure 7. The abs. error plots $\Delta\xi = \Delta t = 0.025$ (left) and $\Delta\xi = \Delta t = 0.00625$ (right) with $t \in [0,1]$ for Example 2.

5. Conclusions

The pseudo-parabolic problem is numerically approximated by the collocation process based on the UHPB-spline functions along with the Crank–Nicolson method and FDM. The proposed method consistently produced more accurate and consistent results, signifying its efficiency while resulting in reduced errors compared to existing methodologies. The von Neumann stability analysis also exhibited stability without conditions, confirming that the approximated outcomes remained bounded across time. Furthermore, the second-order convergence, both in space and time, is demonstrated numerically.

Moreover, the computational efficiency and reduced costs associated with this process render it an economical choice for addressing large-scale problems. It is perceived that the projected process yields lower error norms than those found in Refs. [27,38]. The graphical conclusions are also presented to confirm the accuracy and efficiency of the proposed technique. One can perceive that Figures 2–6 have strong uniform representations between approximated and exact solutions, whereas Figures 1, 4, and 7 indicate that error norms are very small. The key advantage of the projected approach is that the numerical solution is achieved as a smooth piecewise continuous function, enabling us to approximate a highly accurate solution at any locality in the domain of concern.

In future work, the study can be extended to solve higher-dimensional problems with some modification in the UHPB-spline collocation method, which may considerably enhance its application.

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Abbreviations

The following abbreviations are used in this manuscript:

UHPB-spline	Uniform hyperbolic polynomial B-spline
ROC	Rate of convergence
FDM	Finite difference method
PDEs	Partial differential equations
RMS	Root mean square

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