

Article

Nonlinear Deformation Analysis of Sandwich Timoshenko Beams with Carbon Nanotube Reinforced Face Sheets and Re-Entrant Core Using GDQ Method

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Abstract

In this research, the nonlinear bending behavior of sandwich beams with auxetic re-entrant cores and carbon nanotube-reinforced (CNT) face sheets are investigated using the von Kármán strain theory and the Generalized Differential Quadrature (GDQ) method. The results demonstrate the high accuracy of the GDQ method in solving nonlinear problems with a minimal number of grid points. Validation is performed by comparing the obtained results with those reported in previous studies. The findings indicate that CNT-reinforced composite beams exhibit superior bending performance compared to sandwich beams with re-entrant cores and conventional composite face sheets. Furthermore, a parametric study on the core geometry reveals that optimal bending performance is achieved when the interior angle (θ) of the core is approximately 0 to 2 degrees, and the nondimensional auxetic parameter η_{-1} is minimized. This study highlights the significance of nonlinear analysis, particularly for long and slender beams.

Keywords: nonlinear deformation; sandwich beam; auxetic core; CNT-reinforced face sheet; GDQ method

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1. Introduction

Beams are simple structural elements. Due to the simplicity of their governing differential equations compared to other structures such as plates and shells, and the ease of conducting experiments on them, beams offer a more straightforward platform for research. Moreover, the results obtained from beam analysis can often be generalized to other structural forms. Sandwich beams with various core configurations present a modern and promising subject for investigation. In addition to enabling the study of sandwich structures themselves, they allow for comparative analysis of the performance of face-sheet and core materials. Numerous researchers have conducted studies on beams; among these researchers, the following works can be mentioned.

The use of various beam theories is the subject of several researchers [1–5]. Moreover, large deformation analyses of beam structures also draw significant attention [6–10]. Sokolinsky et al. [11] studied the four point bending of a sandwich beam with Aluminum face sheets and PVC foam core. They obtained results from both experimental tests and a higher-order sandwich theory, and compared the findings to evaluate the accuracy and applicability of the theoretical model. Lingaiah and Suryanarayana [12] conducted extensive four-point and three-point bending experiments on a wide range of sandwich beams with various face sheets and core materials. They analyzed the mechanical properties of the sandwich structures and compared the experimental results with existing analytical solutions. The development of new mathematical models, investigation of the behavior of less-studied materials, and introduction of novel numerical methods have always been central to the works of Rabczuk et al. [8,13–21]. Ghayesh [22] studied the nonlinear vibrations in an axially moving beam. Frostig and Baruch [23] explored the bending behavior of a sandwich beam with foam core under concentrated and distributed loads. They investigated the effect of different parameters on peeling. Simo addressed finite elasto-viscoplastic deformations of beams [24]. Hoff and Mautner [25] presented a number of useful formulas for the bending and buckling of sandwich beams, and demonstrated their efficiency by comparing them with experimental results. Li et al. [26], by conducting quasi-static three-point bending tests at different temperatures on sandwich beams with a closed-cell aluminum foam core and aluminum face sheets, showed that indentation and face sheet yielding are the most probable failure modes at various temperatures. Zeng et al. [27] studied the bending stiffness and failure mechanism of sandwich beams with corrugated truss core using experiment. Zenkour et al. [28] analyzed the bending response of sandwich beams resting on a Pasternak foundation, with viscoelastic face sheets and elastic cores. Saez et al. [29] studied the dynamic bending of sandwich beams with composite face sheets and foam core, using ABAQUS and crushable foam model. Demir et al. [30] investigated the static analysis of nano beams under Winkler foundation using finite element. Lin and Xiang [31] addressed the linear vibration of single walled carbon nanotube reinforced nanocomposite beams. Heshmati et al. [32] investigated the effects of carbon nanotube length, waviness, agglomeration and distribution on the vibration behavior of functionally graded nanocomposite beams reinforced by single walled carbon nanotubes (SWCNTs). They developed a micromechanical model for calculating the elastic properties of nano composites. Filippi and Carrera [33] studied the bending and vibration of laminated beams by introducing a new zigzag theory. Jam and Kiani [34] addressed the low velocity impact response of functionally graded CNT reinforced composite beams. The results revealed that increasing the volume fraction of CNTs increases the peak contact force and decreases the contact time. Reddy and Singh [35] studied the large deformation bending and vibration in thin and elastic curved beams using conventional and mixed finite elements.

A study in the literature revealed that the large deformation of composite beams with CNT reinforced and a re-entrant core has not been studied. In this research, using nonlinear von Kármán strain deformation relations and the generalized differential quadrature (GDQ) method, the governing differential equations of nonlinear bending of sandwich beams with re-entrant core and CNT-reinforced face sheets is investigated. The results reveal the importance of nonlinear analyzing, especially for long and slender beams.

The remainder of this work is organized as follows. In Section 2, the governing equations for Timoshenko beams with CNT-reinforced face sheet and auxetic core are derived. The generalized differential quadrature (GDQ) method is described in Section 3. Section 4 is allocated for validation and numerical results. Finally, concluding remarks are highlighted in Section 5.

2. Governing Equations

In this section, the governing differential equations of the Timoshenko beam, as well as the elasticity equations of the auxetic core and CNTRC face sheets, are investigated.

2.1. Differential Equations

In Timoshenko beam theory, the deformation fields are considered as follows.

$$U = u_0 - z\varphi_y(x) \quad (1)$$

$$W = w_0(x) \quad (2)$$

Here u_0 , w_0 and φ_y represent the axial deformation in the x-direction, the transverse deformation in the z direction, and the rotation about the y axis, respectively, at a point on the mid-surface of the beam. U and W denote the axial and transverse deformations in the x and z directions, respectively, at an arbitrary point within the beam. According to von Kármán theory, the nonzero strains are as follows:

$$\varepsilon_x = \frac{\partial U}{\partial x} + \frac{1}{2} \left(\frac{\partial W}{\partial x} \right)^2 = \frac{\partial u_0}{\partial x} - z \frac{\partial \varphi_y}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \quad (3)$$

$$\gamma_{xz} = \gamma_{zx} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = \frac{\partial w_0}{\partial x} - \varphi_y \quad (4)$$

Using the principle of virtual work, presented in Equation (5), the governing differential equations and boundary conditions can be derived. Here, U^{int} and W^{ext} represent the strain energy and the work done by external forces, respectively.

$$\delta(U^{int} - W^{ext}) = 0 \quad (5)$$

In which δU^{int} and δW^{ext} are defined as follows:

$$\delta U^{int} = \int_0^l \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \delta \varepsilon_x + k_s \tau_{xz} \delta \gamma_{xz} dz dx \quad (6)$$

$$\delta W^{ext} = \int_0^l q \delta w_0 dx \quad (7)$$

By defining N , M and Q as $\{N, M, Q\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma_x, z\sigma_x, k_s \tau_{xz}\} dz$, one can derive the differential equations and boundary conditions as given in Equations (8)–(13). Here k_s is the shear correction factor and it is assumed that $k_s = \frac{5}{6}$ in this research.

$$\frac{\partial N}{\partial x} = 0 \quad (8)$$

$$\frac{\partial}{\partial x} \left(N \frac{\partial w_0}{\partial x} \right) + \frac{\partial Q}{\partial x} = q \quad (9)$$

$$\frac{\partial M}{\partial x} - Q = 0 \quad (10)$$

$$S: \quad u_0 = w_0 = M = 0 \quad \rightarrow : N = N_0, M = Q = 0 \quad (11)$$

$$C: \quad u_0 = w_0 = \varphi_y = 0 \quad \downarrow : Q = Q_0, N = M = 0 \quad (12)$$

$$F: \quad N = M = Q = 0 \quad \curvearrowright : M = M_0, N = Q = 0 \quad (13)$$

Using Hooke's equation, the stresses can be expressed in terms of strains.

$$\begin{bmatrix} \sigma_x \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \gamma_{xz} \end{bmatrix} \quad (14)$$

By defining A, B, D and S as $\{A, B, D, S\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{Q_{11}, zQ_{11}, z^2Q_{11}, k_s Q_{55}\} dz$, the stress resultants can be rewritten as follows.

$$N = A \left(\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right) - B \frac{\partial \varphi_y}{\partial x} \quad (15)$$

$$M = B \left(\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right) - D \frac{\partial \varphi_y}{\partial x} \quad (16)$$

$$Q = S \left(\frac{\partial w_0}{\partial x} - \varphi_y \right) \quad (17)$$

By replacing the above equation in Equations (8)–(10) one can derive the differential equations as follows.

$$A \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} \right) - B \frac{\partial^2 \varphi_y}{\partial x^2} = 0 \quad (18)$$

$$\begin{aligned} & \left[A \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} \right) - B \frac{\partial^2 \varphi_y}{\partial x^2} \right] \frac{\partial w_0}{\partial x} + \\ & \left[A \left(\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right) - B \frac{\partial \varphi_y}{\partial x} \right] \frac{\partial^2 w_0}{\partial x^2} + \\ & S \left(\frac{\partial^2 w_0}{\partial x^2} - \frac{\partial \varphi_y}{\partial x} \right) = q \end{aligned} \quad (19)$$

$$B \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} \right) - D \frac{\partial^2 \varphi_y}{\partial x^2} - S \left(\frac{\partial w_0}{\partial x} - \varphi_y \right) = 0 \quad (20)$$

2.2. Mechanical Properties

In this section the mechanical properties of the face sheets and the core are presented. The face sheets are assumed to be CNTRC, and the core is considered to be re-entrant auxetic. In Figure 1, the geometrical parameters of the core, the shape of the sandwich beam with its coordinate system, and the deformation parameters are illustrated. The intervals $0 < x < L$, $-\frac{b}{2} < y < \frac{b}{2}$ and $-\frac{h}{2} < z < \frac{h}{2}$ define the geometry of the beam where L, h, and b represent the length, height, and width of the beam, respectively. Figure 1 also shows that in the global coordinate system the G_{13} and E_1 correspond to G_{12}^c and E_2^c for the core, respectively. The micromechanical equations of the core are presented below [36]. Here $\eta_1 = \frac{d}{l}$ and $\eta_3 = \frac{t}{l}$.

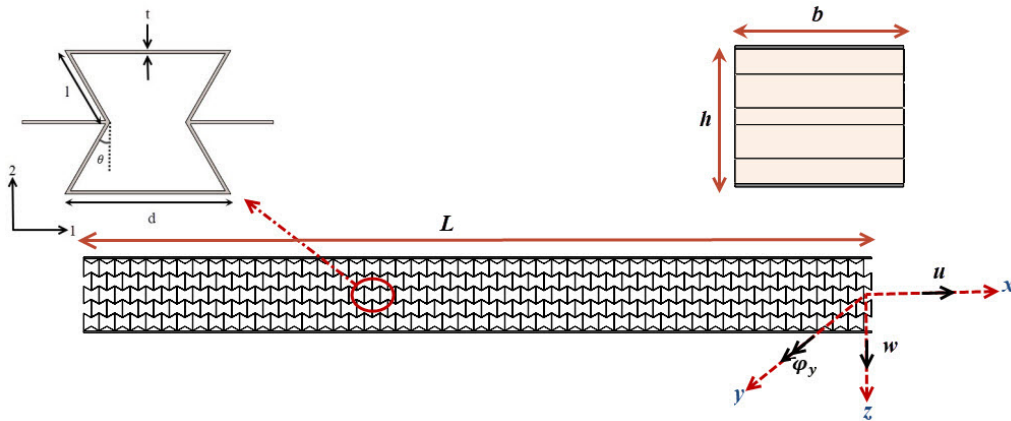


Figure 1. Sandwich beams with auxetic core and CNTRC face sheets.

$$E_2^c = E^c \frac{\eta_3^3}{\cos\theta(\eta_1 - \sin\theta)(\tan^2\theta + \eta_3^2)} \quad (21)$$

$$G_{12}^c = E^c \frac{\eta_3^3}{\eta_1(1 + 2\eta_1)\cos\theta} \quad (22)$$

$$\nu_{12}^c = \frac{-\sin\theta(1 - \eta_3^2)(\eta_1 - \sin\theta)}{\cos^2\theta(1 + (\tan^2\theta + \eta_1\sec^2\theta)\eta_3^2)} \quad (23)$$

$$\nu_{21}^c = \frac{-\sin\theta(1 - \eta_3^2)}{(\tan^2\theta + \eta_3^2)(\eta_1 - \sin\theta)} \quad (24)$$

In order to obtain the mechanical properties of face sheets, the rule of mixtures is used. Accordingly

$$E_1^f = E^m V_m + \eta_1 E_{11}^{CN} V_{CN} \quad (25)$$

$$\frac{\eta_2}{E_2^f} = \frac{V_m}{E_{22}^m} + \frac{V_{CN}}{E_{22}^{CN}} \quad (26)$$

$$\nu_{12}^f = V_{CN} \nu_{12}^{CN} + V_m \nu_m \quad (27)$$

$$\frac{\eta_3}{G_{13}^f} = \frac{V_m}{G^m} + \frac{V_{CN}}{G_{13}^{CN}} \quad (28)$$

Here η_1 , η_2 and η_3 are efficiency parameters, as presented in Table 1. m and CN refer to the matrix and CNT respectively. Under the assumption of orthotropy, one may write $\frac{\nu_{12}}{E_{11}} = \frac{\nu_{21}}{E_{22}}$. V_m and V_{CN} are the volume fractions of the matrix and CNT respectively. The matrix is assumed to be polymethyl methacrylate (PMMA), while the carbon nanotubes are considered as single-walled CNTs (SWCNTs) with a (10, 10) armchair configuration. The core is made of aluminum, and its mechanical properties are listed below. The mechanical properties of both the matrix and CNTs are also provided below [37].

$$E^m = 2.5 \text{ GPa}, \quad \nu^m = 0.3$$

$$E_{11}^{CN} = 600 \text{ GPa}, \quad E_{22}^{CN} = 10 \text{ GPa}, \quad G_{13}^{CN} = 17.2 \text{ GPa}, \quad \nu_{12}^{CN} = 0.17$$

$$E^c = 69 \text{ GPa}, \quad \nu^c = 0.33$$

Table 1. The efficiency parameters of CNTRC [38].

	η_1	η_2	η_3
$V_{CN} = 0.12$	0.137	1.002	0.715
$V_{CN} = 0.17$	0.142	1.626	1.138
$V_{CN} = 0.28$	0.141	1.585	1.109

The stiffness matrix elements are defined as follows

$$Q_{11}^c = \frac{E_2^c}{1 - v_{12}^c v_{21}^c} \quad Q_{55}^c = G_{12}^c \quad (29)$$

$$Q_{11}^f = \frac{E_1^f}{1 - v_{12}^f v_{21}^f} \quad Q_{55}^f = G_{13}^f \quad (30)$$

3. The GDQ Method

In the Generalized Differential Quadrature (GDQ) method, spatial differential equations can be discretized such that the derivatives of a function $f(x)$ are approximately represented by the weighted summation of the function values at all discrete points within the domain. The mathematical foundation of this method is thoroughly explained in [39]. Equation (31) presents the derivative approximation used in this method.

$$\frac{d^n f(x_i)}{dx} = \sum_{j=1}^N C_{ij}^n f(x_j) \quad n = 1, 2, \dots, N-1 \quad (31)$$

Here C_{ij}^n denotes the coefficients matrix for the n th order differentiation. The construction of this matrix is described below.

$$C_{ij}^1 = \frac{\prod_{k=1}^N (x_i - x_k)}{\prod_{k=1, k \neq j}^N (x_j - x_k)} \quad k \neq i, j; i \neq j; i, j = 1, 2, \dots, N \quad (32)$$

$$C_{ij}^1 = \sum_{k=1}^N \frac{1}{x_i - x_k} \quad k \neq i, i = j = 1, 2, \dots, N$$

Here C_{ij}^1 represents the coefficient matrix for the first derivative of the function. For higher-order derivatives, one can obtain the corresponding coefficient matrices by multiplying lower-order ones. For example, $C_{ij}^2 = C_{ij}^1 C_{ij}^1$ or $C_{ij}^5 = C_{ij}^4 C_{ij}^1$.

There are two ways to discretize the domain: one is to uniformly distribute the material points throughout the domain, and the other is to distribute them arbitrarily using a distribution function. In this work, the Gauss–Chebyshev–Lobatto formulation is employed, which is defined as follows.

$$x_i = \left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right) \frac{x_N - x_1}{2} + x_1 \quad (33)$$

To apply the boundary conditions to the differential equations, let $F(u) - b = 0$ represent the discretized system of nonlinear equations and $G(u) - d = 0$ represent the discretized system of nonlinear boundary conditions. Here, u is the vector of unknowns, while b and d are known vectors corresponding to the differential equations and boundary conditions, respectively. By introducing Lagrange multiplier λ and applying the calculus of variations, the following Newton–Raphson formulation can be derived [40]

$$H\Delta v = \Delta f \quad (34)$$

In which

$$\mathbf{H} = \begin{bmatrix} \frac{\partial F(u)}{\partial u} + \lambda^T \frac{\partial^2 G(u)}{\partial u^2} & \frac{\partial G(u)}{\partial u^T} \\ \frac{\partial G(u)}{\partial u} & 0 \end{bmatrix}, \quad \Delta \mathbf{v} = \begin{bmatrix} \Delta u \\ \Delta \lambda \end{bmatrix}, \quad \Delta \mathbf{f} = \begin{bmatrix} \Delta b \\ \Delta d \end{bmatrix} \quad (35)$$

The calculations of $\frac{\partial G(u)}{\partial u}$ for CC and CF beams are described in the Appendix. Finally, the following recurrence equation can be used in each iteration.

$$\mathbf{v}^{(n+1)} = \mathbf{v}^{(n)} + \Delta \mathbf{v}^{(n+1)} \quad (36)$$

To explain more about the procedure of the Newton-Raphson method, it is assumed that there is a residual vector $\mathbf{R}$ which is obtained as:

$$\mathbf{H} \Delta \mathbf{v} = -\mathbf{R}, \quad \Delta \mathbf{f} = -\mathbf{R} \quad (37)$$

Hence, any increment n of it will lead to zero as follows:

$$\mathbf{R}^{(n+1)} = \mathbf{R}^{(n)} + \Delta \mathbf{R} = \mathbf{0} \quad (38)$$

And this iterative procedure continues until the convergence criterion is satisfied. There are several forms of convergence criterion. In this work, the following convergence criterion is selected:

$$\left\| \frac{\Delta \mathbf{v}}{\mathbf{v}} \right\| < \varepsilon \quad (39)$$

4. Results

The results of this research include a convergence study of the GDQ method, validation against the existing literature, deformation analysis of sandwich beams under various boundary conditions and loading scenarios, and a parametric investigation of core geometry specifically the parameters θ and η_1 as well as the effects of face-sheet thickness to core thickness and length to thickness ratios.

4.1. Convergence Study

To perform the parametric study using the GDQ method, a [0/90/0] laminated composite is considered with the following properties $\frac{L}{h} = 10$, $b = 1$, $E_1 = 25E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$ and $\nu_{12} = 0.25$. Table 2 presents the results of the nondimensional deformation under various boundary conditions and a uniformly distributed load, evaluated for different numbers of material points. It is observed that this numerical method produces accurate results even with a small number of discretization points. The results indicate that convergence occurs with as few as nine nodes. However, in this study, a more conservative approach is adopted by using $N = 27$.

Table 2. Convergence study on the number of material points $\left(\bar{w} = \frac{10^2 E_2 w_0 \left(\frac{L}{2} \right) h^3}{q L^4} \right)$.

	N	3	5	7	9	11	13	15	17	19
$\bar{w}$	CC	0.3750	0.5043	0.5043	0.5043	0.5043	0.5043	0.5043	0.5043	0.5043
	CS	0.5222	0.6931	0.6931	0.6931	0.6931	0.6931	0.6931	0.6931	0.6931
	SS	0.3750	1.0215	1.0214	1.0214	1.0214	1.0214	1.0214	1.0214	1.0214
	CF	9.4673	3.3191	3.3214	3.3213	3.3213	3.3213	3.3213	3.3213	3.3213

4.2. Validation of Results

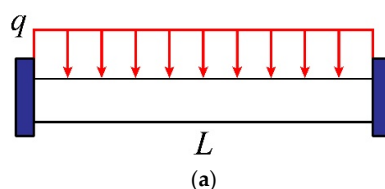
In this section, several examples are presented to validate the results. Table 3 compares the deformation results with those reported in previous studies. The mechanical properties, geometry, and laminate configuration are assumed to be identical to those

described in Sections 1–4. The consistency between the present results and those from the literature is clearly evident. Tables 2 and 3 show that, for several beams under identical conditions, the absolute values of maximum deformation correspond respectively to the boundary conditions CC, CS, SS, and CF. In CC boundary condition, the beam cannot move in x and y -direction at both ends. Moreover, there is not any rotation in both ends (see Equation (12)). In CS boundary condition, the beam does not have any displacement in x and y -directions at both ends. In addition, in the left end which is clamped (C), the beam is not allowed to rotate. In the SS boundary condition, only displacements in x and y -directions are restricted (see Equation (11)). Finally, in CF beam, rotation and displacements in x and y -directions in the clamped side (C) are restricted, but the beam can move without limitation in the free side (F).

To validate the nonlinear bending results presented in the following sections, an example is provided in this part. In this example, an isotropic beam with CC boundary conditions and uniform loading is considered. The mechanical properties are defined as $E = 1.2 \times 10^4 \frac{lb}{in^2}$, $\nu = 0.2$ and the geometry is given by $L = 10in$ and $b = h = 1in$. It is assumed that $p^* = \frac{qL^3}{EI}$, where q is the distributed load applied to the beam and I is the moment of inertia, as displayed in Figure 2a. The results are compared with those reported in [41]. In Figure 2b, the central deflection w of the beam is presented, and the comparison is illustrated. Based on this figure, it can be concluded that the present results demonstrate high accuracy. At the end, the deformed profiles of the beam at different load steps $p^* = \{0, 50, 100, 150, 200\}$ are displayed in Figure 2c.

Table 3. Validation of nondimensional deformation results $\left(\bar{w} = \frac{10^2 E_2 w_0 \left(\frac{L}{2}\right) h^3}{q L^4}\right)$.

$\frac{L}{h}$	Reference	CC	CS	SS	CF
5	Present	1.6293	1.9216	2.1464	6.6982
	[42]	1.6293	1.9216	2.1464	6.6978
	[43]	-	-	2.1480	-
	[44]	1.5369	-	2.4124	6.8240
	[45]	1.629	1.922	2.146	6.698
10	Present	0.5043	0.6931	1.0214	3.3228
	[42]	0.5043	0.6931	1.0214	3.3228
	[43]	-	-	1.0230	-
	[44]	0.5315	-	1.0963	3.4554
	[45]	0.504	0.693	1.021	3.323



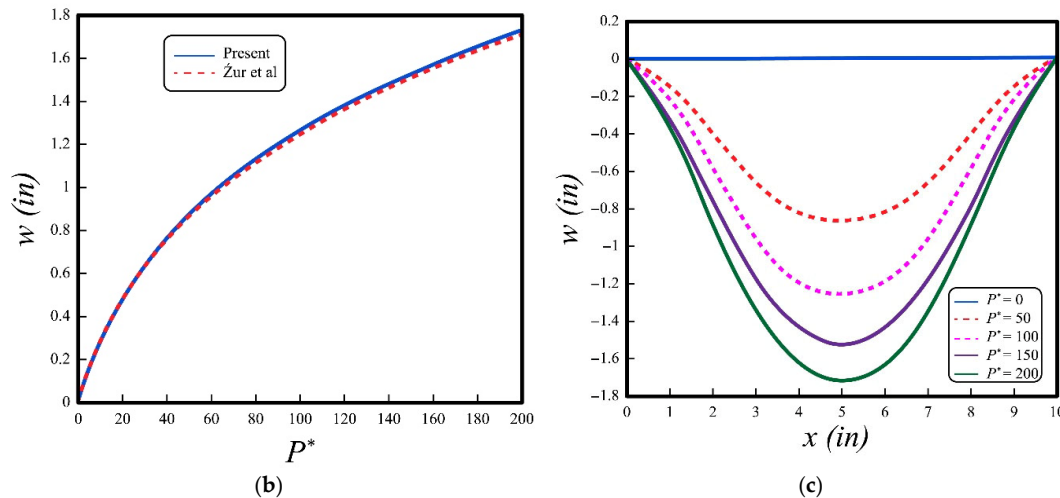


Figure 2. (a) Initial configuration of isotropic CC beam subjected to dead load, (b) comparison of the central deformation of the beam with the results of [41], and (c) deformed shapes of CC beam under distributed load for $p^* = \{0, 50, 100, 150, 200\}$.

4.3. Numerical Results

The study of bending behavior in sandwich beams composed of CNT reinforced composite face sheets and a re-entrant core is of significant importance, as the use of both these structural components is becoming more common. Therefore, this section presents numerical results for sandwich beams under various loadings and boundary conditions. Tables 4 and 5 report the nondimensional deformation of the beam for different length to thickness ratios and face sheet h_f to total thickness ratios h . Subsequently, nondimensionalization is defined as $\bar{w} = \frac{100E^m w_0 (\frac{L}{2}) h^3}{qL^4}$. It is assumed that for the face sheets $V_{CN} = 0.12$ and for the core $\eta_1 = 3$, $\eta_3 = 0.01$ and $\theta = 45^\circ$. Based on Tables 4 and 5, it can be inferred that increasing the thickness of the face sheets in a beam with constant overall thickness leads to a reduction in deformation. This indicates that the composite beam performs better in bending compared to the sandwich beam.

Table 4. Nondimensional deformation ($\bar{w}$) in sandwich beams with $\frac{h_f}{h} = 0.2$.

$\frac{L}{h}$	CC	CS	SS	CF
5	5.6465	7.1169	8.8865	28.2794
10	2.0191	3.0365	5.2591	17.3972
20	1.1123	1.9776	4.3522	14.6728
50	0.8583	1.6776	4.0983	10.6417

Table 5. Nondimensional deformation ($\bar{w}$) in sandwich beams with $\frac{h_f}{h} = 0.5$.

$\frac{L}{h}$	CC	CS	SS	CF
5	2.5703	3.5107	5.1128	16.6097
10	1.1193	1.8415	3.6618	12.2568
20	0.7566	1.4146	3.2991	11.1649
50	0.6550	1.2944	3.1976	8.0197

In Figures 3–6, the load–displacement diagrams are plotted under boundary conditions SS and CF, respectively, for the following loading scenarios: uniform loading, concentrated moment at the free end, concentrated lateral force at the free end, and uniform

loading. In plotting these diagrams, the following parameters are considered: $\frac{h_f}{h} = 0.5$, $L = 10$ m, $b = 1$ m and $h = 0.1$. This means the beam is modeled as a composite structure, with the sandwich core thickness assumed to be zero. The schematic of initial configuration of simply supported beam subjected to the dead load is displayed in Figure 3a. Moreover, the load-deflection curves of the center of the beam for three values of CNT volume fraction $V_{CN} = \{0.12, 0.17, 0.28\}$ are shown in Figure 3b. The effect of nonlinearity is obvious in this figure. In fact, by using linear theory, the deflection of the beam could be analyzed until load $q = 1$ kN/m; however, by considering nonlinear theory, the deflection of the beam can be predicted until $q = 80$ kN/m. Moreover, the deformed profiles of the beam for mentioned CNT volume fraction are depicted in Figure 3c. It is observed that all curves demonstrate nonlinear deformations; however, the larger volume fraction will lead to the smaller deformation and larger stiffness of the beam. Next, the clamped-free beam under the end moment is considered, as shown in Figure 4a. The load-deflection curve of the free end of the beam for CNT fraction $V_{CN} = \{0.12, 0.17, 0.28\}$ are depicted in Figure 4b. Moreover, the deformed profiles corresponding to these fractions are portrayed in Figure 4c. It can be observed that for larger values of CNT fraction, the beam shows stiffer behavior.

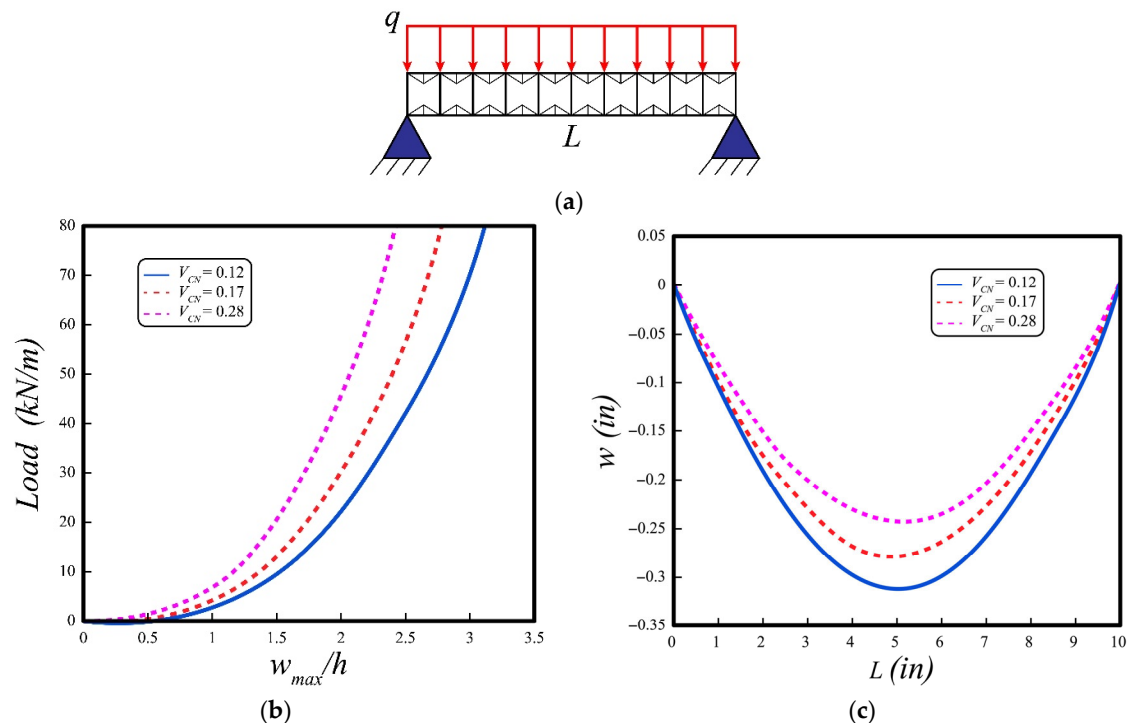
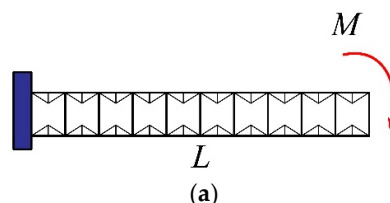


Figure 3. (a) Schematic of initial configuration of the SS CNT reinforced composite face sheets and a re-entrant core beam under dead load. (b) Load deformation diagram with uniformly distributed load and SS boundary conditions; (c) deformed shapes of the CF beams for different values of V_{CN} .



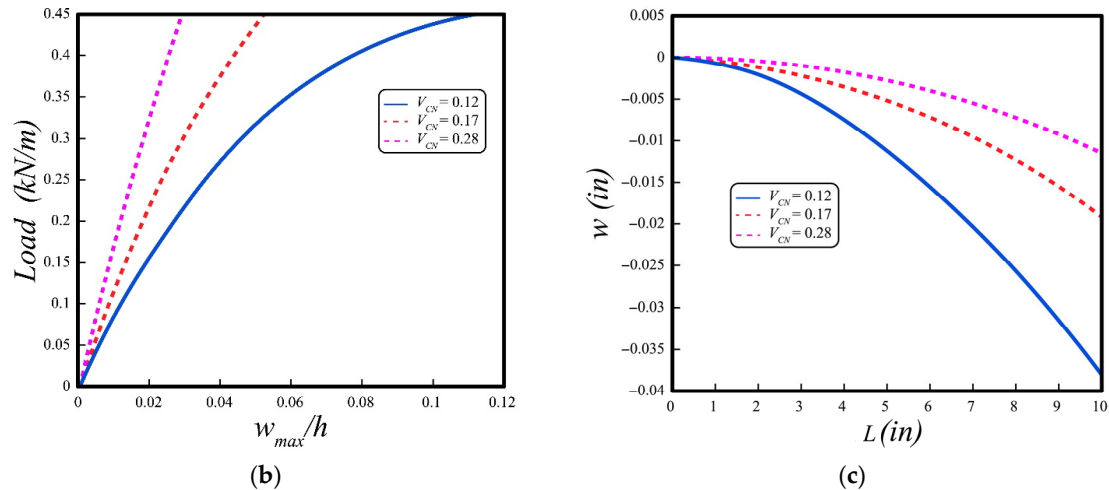


Figure 4. (a) Schematic of initial configuration of the CF CNT reinforced composite face sheets and a re-entrant core beam under end moment. (b) Load deformation diagram with concentrated moment at one end and CF boundary conditions; (c) deformed shapes of the CF beams for different values of V_{CN} .

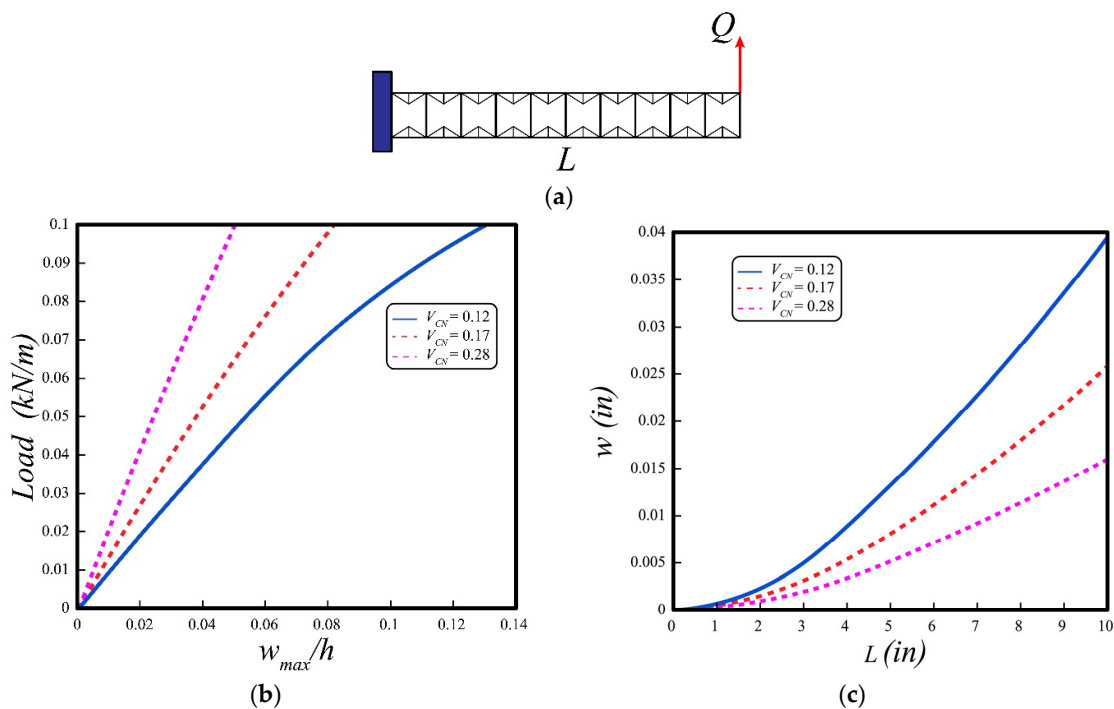


Figure 5. (a) Schematic of initial configuration of the CF CNT reinforced composite face sheets and a re-entrant core beam subjected to end force. (b) Load deformation diagram with concentrated load at one end and CF boundary conditions; (c) deformed shapes of the CF beams for different values of V_{CN} .

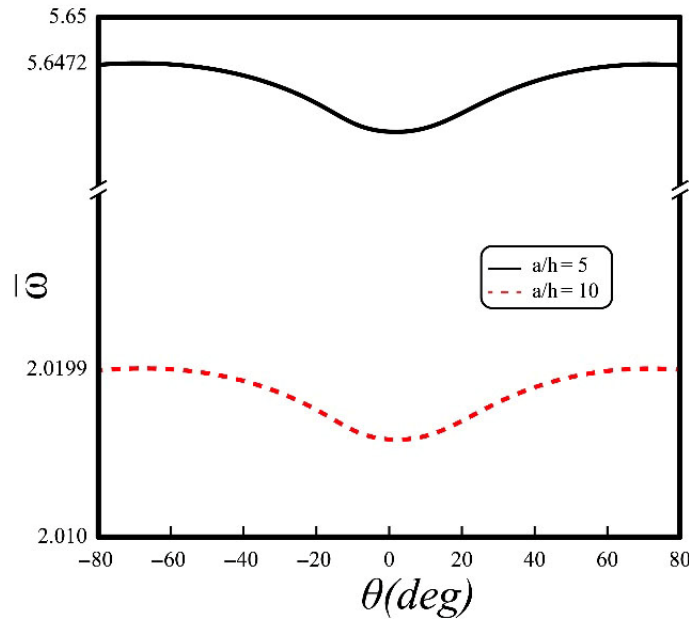


Figure 6. The effect of core angle on nondimensional deformation with CC boundary conditions.

The clamped-free beam subjected to the end force is considered for the next problem, as shown in Figure 5a. The load-deflection curves plotted in Figure 5b demonstrate that the behavior of the beam is nonlinear, and increasing the CNT volume fraction leads to increasing the stiffness of the beam. Finally, the deformed profiles of the beam are shown in Figure 5c. According to these diagrams, the nonlinear behavior of the beam becomes evident at larger deformations. For smaller deformations, the load-displacement curves remain linear. The diagrams also reveal that the nonlinear effects become more pronounced with increasing beam thickness. In all diagrams, it is observed that increasing the volume fraction of carbon nanotubes (CNT) enhances the bending resistance, resulting in smaller deformations under a given load. Therefore, carbon nanotubes serve as an effective reinforcement. These diagrams also highlight the importance of nonlinear analysis, especially under CF boundary conditions, where even small deformations amounting to a fraction of the beam thickness leads to noticeable deviations from linear bending behavior.

4.4. Parametric Study

In this section a parametric study is conducted on the geometrical parameters of the core and the length to thickness ratio of the beam. Figures 6–13 illustrate the effect of varying the parameters θ and η_1 on the dimensionless deformation under different boundary conditions. These diagrams are plotted for $\frac{h_f}{h} = 0.2$.

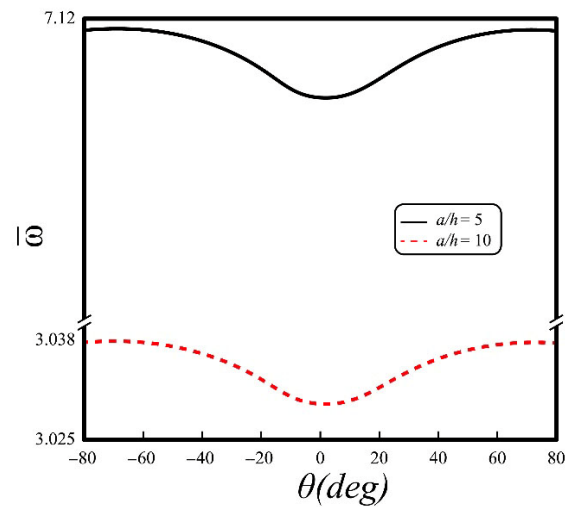


Figure 7. The effect of core angle on nondimensional deformation with CS boundary conditions.

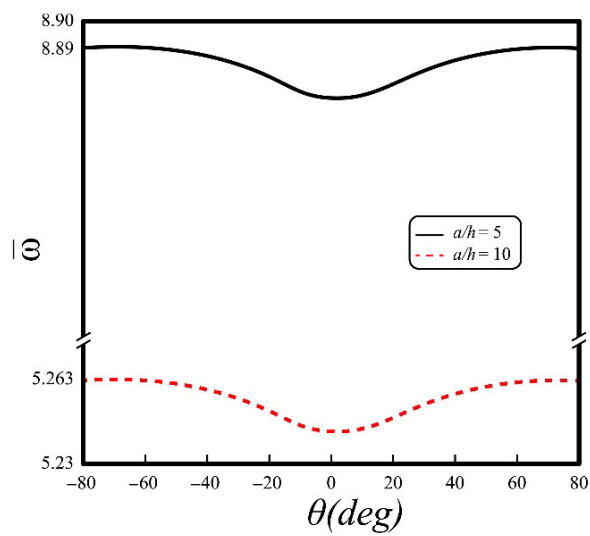


Figure 8. The effect of core angle on nondimensional deformation with SS boundary conditions.

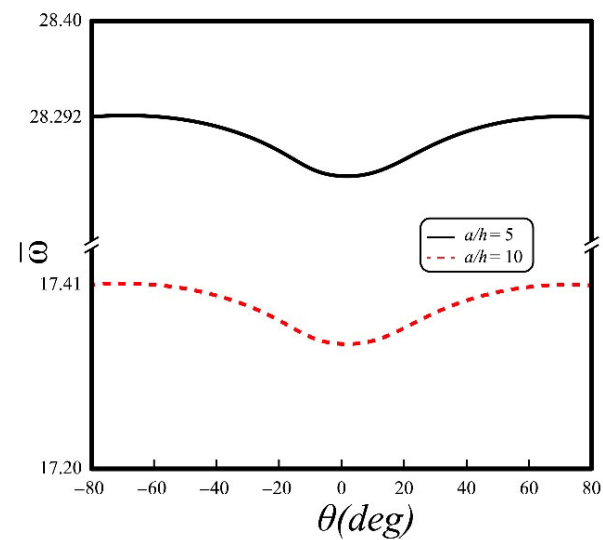


Figure 9. The effect of core angle on nondimensional deformation with CF boundary conditions.

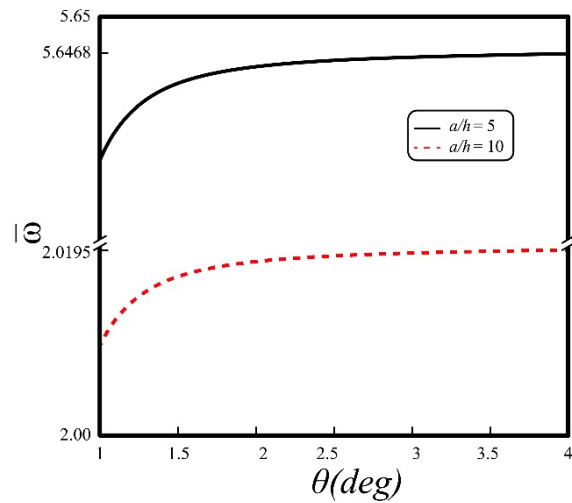


Figure 10. The effect of η_1 on nondimensional deformation with CC boundary conditions.

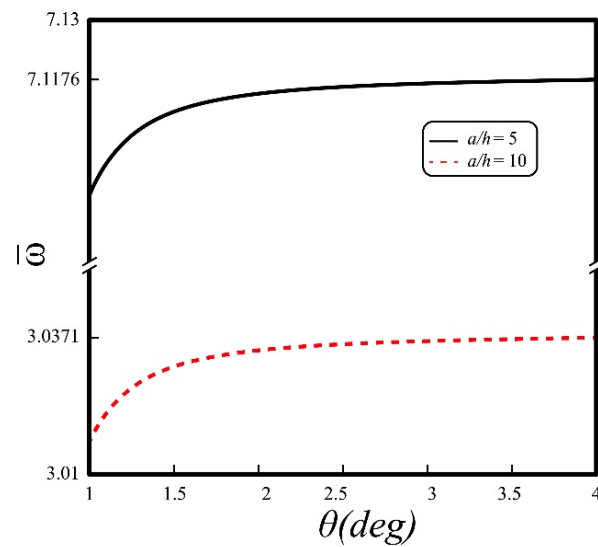


Figure 11. The effect of η_1 on nondimensional deformation with CS boundary conditions.

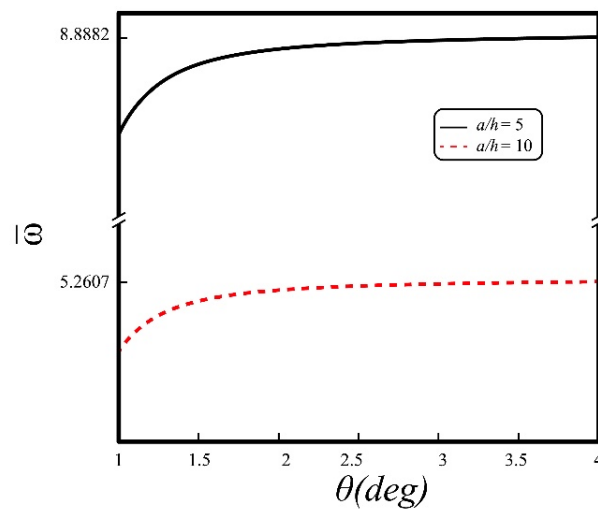


Figure 12. The effect of η_1 on nondimensional deformation with SS boundary conditions.

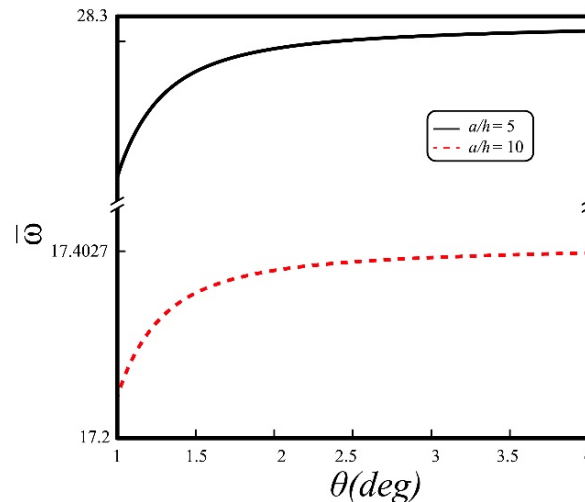


Figure 13. The effect of η_1 on nondimensional deformation with CF boundary conditions.

Figures 6–10 show that under various boundary conditions, the minimum deformation occurs at a core angle of approximately $\theta = 2^\circ$. Figures 10–13 indicate that, for different boundary conditions, the minimum beam deformation is obtained at the lowest value of η_1 . The purpose of these figures is to examine the influence of boundary conditions on the structural performance with respect to different core parameters. This analysis reveals that boundary conditions do not affect the optimal values of θ and η_1 . However, in two-dimensional structures such as plates or shells, boundary conditions may have a more significant influence. Another important point highlighted by this parametric study is the variation in behavior among different structural types. In other words, the optimal values for beams may differ from those for plates and shells. By comparing these figures, it can be observed that, for a constant thickness, the maximum deformation corresponds respectively to the boundary conditions CC, CS, SS, and CF.

5. Conclusions

In this study, nonlinear bending behavior of sandwich beams with auxetic cores and carbon nanotube-reinforced composite (CNTRC) face sheets was investigated using von Kármán nonlinear strain relations and the mesh-free Generalized Differential Quadrature (GDQ) method. To validate the results, comparisons were made with findings from other researchers, confirming the accuracy of the proposed model.

The key findings of this research are summarized below:

- The carbon nanotube reinforced composite beam shows better bending behavior compared to the sandwich beam with a re-entrant auxetic core and CNTRC face sheets.
- Core analysis reveals that the optimal bending behavior, regardless of boundary conditions, occurs at core angles approximately between 0° and 2° .
- The core study also shows that the best bending performance, independent of boundary conditions, is achieved at the lowest values of η_1 .
- The minimum deformations correspond respectively to the CC, CS, SS and CF boundary conditions.
- Linear bending analysis is not valid for long beams with high length to thickness ratios—nonlinear analysis must be employed in such cases.

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investigation, K.K.A., N.F. and P.P.; resources, A.G.H. and N.F.; data curation, N.F.; writing—original draft preparation, N.F.; writing—review and editing, Y.A.R. and P.P.; visualization, N.F.; supervision, A.G.H., K.K.A., Y.A.R. and P.P.; project administration, N.F.; funding acquisition, A.G.H., K.K.A. and Y.A.R. All authors have read and agreed to the published version of the manuscript.

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