

Article

Sum of Squares Decompositions and Rank Bounds for Biquadratic Forms

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Abstract

We study positive semi-definite (PSD) biquadratic forms and their sum-of-squares (SOS) representations. For the class of partially symmetric biquadratic forms, we establish necessary and sufficient conditions for positive semi-definiteness and prove that every PSD partially symmetric biquadratic form is an SOS. This extends the known result for fully symmetric biquadratic forms. Furthermore, we describe an efficient computational procedure for constructing SOS decompositions, exploiting the Kronecker-product structure of the associated matrix representation. We introduce simple biquadratic forms. For $m \geq 2$, we provide an explicit example to show the lower bound for sos rank of $m \times 2$ biquadratic forms is $m + 1$, and show that previously proved results indicating that a 2×2 PSD biquadratic form can be expressed as the sum of three squares and a 3×2 PSD biquadratic form can be expressed as the sum of four squares are tight. We also present an 3×3 SOS biquadratic form, which can be expressed as the sum of six squares, but not the sum of five squares. Moreover, we establish a universal upper bound $mn - 1$ for any $m \times n$ SOS biquadratic form, which improves the trivial bound mn .

Keywords: biquadratic forms; sum-of-squares; positive semi-definiteness; sos rank; M-eigenvalues; partially symmetric biquadratic forms

MSC: 11E25; 12D15; 14P10; 15A69; 90C23



Academic Editor: Alexander Tikhomirov

Received: 11 January 2026

Revised: 5 February 2026

Accepted: 9 February 2026

Published: 11 February 2026

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1. Introduction

Denote $\{1, \dots, m\}$ as $[m]$. Let $\mathcal{A} = (a_{ijkl})$, where $a_{ijkl} \in \mathbb{R}$ for $i, k \in [m], j, l \in [n]$. Then $\mathcal{A}$ is called an $m \times n$ biquadratic tensor. Biquadratic tensors arise in solid mechanics [1,2], statistics [3], spectral graph theory [4], and polynomial theory [5]. If $a_{ijkl} = a_{kjil} = a_{klij}$, for $i, k \in [m], j, l \in [n]$, then $\mathcal{A}$ is called a symmetric biquadratic tensor. Denote the set of all $m \times n$ biquadratic tensors by $BQ(m, n)$ and the set of all $m \times n$ symmetric biquadratic tensors by $SBQ(m, n)$.

With $\mathcal{A} = (a_{ijkl}) \in BQ(m, n)$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, we may define a homogeneous polynomial P by $P(\mathbf{x}, \mathbf{y}) = \sum_{i,k=1}^m \sum_{j,l=1}^n a_{ijkl} x_i x_k y_j y_l$. Then, we call P a biquadratic form. A PSD (positive semi-definite) biquadratic form is one for which $P(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y}$. It is SOS (sum of squares) if it can be written as a finite sum of squares

of bilinear forms. If the biquadratic form P is PSD or SOS respectively, then the biquadratic tensor $\mathcal{A}$ is also said to be PSD or SOS respectively. Note that while a biquadratic tensor uniquely defines a biquadratic form, a biquadratic form corresponds to infinitely many biquadratic tensors but uniquely corresponds to one symmetric biquadratic tensor.

A PSD polynomial is nonnegative everywhere. A very important problem in algebra and optimization is to identify structured PSD tensors and forms. As Hilbert [6] proved in 1888, a PSD form may not be SOS in general. On the other hand, the existence of an SOS decomposition makes the problem of verifying nonnegativity computationally tractable via semi-definite programming. Thus, it is also very important to identify whether a given structured PSD tensor or form class is SOS or not.

Biquadratic forms arise naturally in several areas of applied mathematics and engineering [7]. In elasticity theory, the strong ellipticity condition for isotropic materials is governed by a biquadratic form whose positivity guarantees wave propagation. In statistics and machine learning, biquadratic kernels appear in higher-order learning models and tensor-based data analysis. Moreover, verifying nonnegativity of biquadratic forms is a fundamental subproblem in polynomial optimization and semi-definite programming relaxations [8,9]. Thus, understanding when a positive semi-definite biquadratic form admits an SOS decomposition not only advances classical real algebraic geometry but also provides practical tools for certifying positivity in these applications.

Very recently, several structured PSD biquadratic tensor classes have been identified. In [7], three PSD biquadratic tensor classes, diagonally dominated symmetric biquadratic tensors, symmetric M-biquadratic tensors, and symmetric B_0 -biquadratic tensors were identified. In [10], completely positive biquadratic tensors were introduced. All completely positive biquadratic tensors are SOS. Two subclasses of completely positive biquadratic tensors, biquadratic Cauchy tensors, and biquadratic Pascal tensors, were identified there. In [11], it was shown that diagonally dominated symmetric biquadratic tensors are SOS tensors.

A PSD biquadratic form may not be SOS. In 1975, Choi [12] presented a 3×3 PSD biquadratic form which is not SOS. In [11], it was shown that such a problem is caused by non-symmetry. Symmetric biquadratic forms were introduced in [11], and it was shown that all PSD symmetric biquadratic forms are SOS, regardless of how big m and n are. A biquadratic form is said to be symmetric if it remains invariant under permutations of the indices in x -variables and y -variables respectively.

In this paper, we consider the intermediate class of partially symmetric biquadratic forms, which are symmetric only either in the x variables or the y variables. We establish necessary and sufficient conditions for positive semi-definiteness of monic partially symmetric biquadratic forms and, more importantly, prove that every PSD partially symmetric biquadratic form is SOS. This resolves an open question about the SOS property under partial symmetry and extends the known result for fully symmetric forms.

Furthermore, we describe an efficient computational procedure for constructing SOS decompositions, exploiting the Kronecker-product structure of the associated matrix representation.

We introduce simple biquadratic forms. For $m \geq 2$, we present a $m \times 2$ PSD biquadratic form and show that it can be expressed as the sum of $m + 1$ squares but cannot be expressed as the sum of m squares. This provides a lower bound for an sos rank of $m \times 2$ biquadratic forms and shows that previously proved results that indicate a 2×2 PSD biquadratic form can be expressed as the sum of three squares and a 3×2 PSD biquadratic form can be expressed as the sum of four squares are tight. We also present an 3×3 SOS biquadratic form, which can be expressed as the sum of six squares, but not the sum of five squares.

In addition, we derive a universal upper bound $\text{SOS-rank}(P) \leq mn - 1$ for any SOS biquadratic form, which improves the trivial bound mn .

2. Monic Partially Symmetric Biquadratic Forms

Let $\mathcal{A} = (a_{ijkl}) \in SBQ(m, n)$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. Then $\mathcal{A} \cdot \mathbf{yxy} \in \mathbb{R}^m$ and $(\mathcal{A} \cdot \mathbf{yxy})_i = \sum_{k=1}^m \sum_{j,l=1}^n a_{ijkl} y_j x_k y_l$ for $i \in [m]$, and $\mathcal{A}\mathbf{x} \cdot \mathbf{xy} \in \mathbb{R}^n$ with $(\mathcal{A}\mathbf{x} \cdot \mathbf{xy})_j = \sum_{i,k=1}^m \sum_{l=1}^n a_{ijkl} x_i x_k y_l$ for $j \in [n]$. If there are $\lambda \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^m$, $\|\mathbf{x}\|_2 = 1$, $\mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{y}\|_2 = 1$, such that

$$\mathcal{A} \cdot \mathbf{yxy} = \lambda \mathbf{x}, \quad \mathcal{A}\mathbf{x} \cdot \mathbf{xy} = \lambda \mathbf{y},$$

then λ is called an M-eigenvalue of $\mathcal{A}$, with $\mathbf{x}$ and $\mathbf{y}$ as its M-eigenvectors [13]. For other works on biquadratic forms and biquadratic tensors, see [5,14–17]

Suppose that $\mathcal{A} = (a_{ijkl})$, where $a_{ijkl} \in \mathbb{R}$ for $i, k \in [m]$, $j, l \in [n]$, is an $m \times n$ biquadratic tensor. Let

$$P(\mathbf{x}, \mathbf{y}) = \sum_{i,k=1}^m \sum_{j,l=1}^n a_{ijkl} x_i y_j x_k y_l, \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. Then P is called a biquadratic form. If $P(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, then P is called positive semi-definite (PSD). If

$$P(\mathbf{x}, \mathbf{y}) = \sum_{p=1}^r f_p(\mathbf{x}, \mathbf{y})^2,$$

where f_p for $p \in [r]$ are bilinear forms, then we can say that P is sum of squares (SOS). The smallest r is called the SOS rank of P . Clearly, P is PSD or SOS if and only if $\mathcal{A}$ is PSD or SOS, respectively.

While a biquadratic form P may be constructed from a biquadratic tensor $\mathcal{A}$, the tensor $\mathcal{A}$ is not unique to P . However, there is a unique symmetric biquadratic tensor $\mathcal{A}$ associated with P .

The following definition extends the definition of symmetric forms [18].

Definition 1. Suppose $P(\mathbf{x}, \mathbf{y}) = P(x_1, \dots, x_m, y_1, \dots, y_n)$. If

$$P(x_1, \dots, x_m, y_1, \dots, y_n) = P(x_{\sigma(1)}, \dots, x_{\sigma(m)}, y_1, \dots, y_n)$$

for any permutation σ , then P is called x -symmetric. Similarly, we define y -symmetric biquadratic forms. If P is both x -symmetric and y -symmetric, we call P a symmetric biquadratic form. If P is either x -symmetric or y -symmetric, we call P a partially symmetric biquadratic form.

Note that a symmetric biquadratic form P has only four free coefficients. If we fix the diagonal coefficients as 1, we may write such a form as [11]

$$P(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^m \sum_{j=1}^n x_i^2 y_j^2 + a \sum_{i \neq k} \sum_{j=1}^n x_i x_k y_j^2 + b \sum_{i=1}^m \sum_{j \neq l} x_i^2 y_j y_l + c \sum_{i \neq k} \sum_{j \neq l} x_i y_j x_k y_l.$$

As proven in [11], P is PSD if and only if the parameters a , b , and c satisfy:

- (i) $1 - b + (m - 1)(a - c) \geq 0$;
- (ii) $1 - a + (n - 1)(b - c) \geq 0$;
- (iii) $1 - a - b + c \geq 0$;
- (iv) $1 + (m - 1)a + (n - 1)b + (m - 1)(n - 1)c \geq 0$.

Now the question is whether a given PSD x -symmetric biquadratic form is SOS or not. Note that an x -symmetric $m \times n$ biquadratic form P has $2n^2$ free coefficients. If we fix the diagonal coefficients as 1, we may write such a form as

$$\begin{aligned} P(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^m \sum_{j=1}^n x_i^2 y_j^2 + \sum_{i \neq k} \sum_{j,l=1}^n a_{jl} x_i x_k y_j y_l + \sum_{i=1}^m \sum_{j \neq l} b_{jl} x_i^2 y_j y_l \\ &= (\mathbf{x}^\top \mathbf{x})(\mathbf{y}^\top \mathbf{y}) + \left((\mathbf{1}_m^\top \mathbf{x})^2 - \mathbf{x}^\top \mathbf{x} \right) (\mathbf{y}^\top \mathbf{A} \mathbf{y}) + (\mathbf{x}^\top \mathbf{x}) (\mathbf{y}^\top \mathbf{B} \mathbf{y}), \end{aligned} \quad (2)$$

where $A = (a_{jl})$ and $B = (b_{jl})$ are $n \times n$ symmetric matrices, B has zero diagonal (i.e., $b_{jj} = 0$ for all j), and $\mathbf{1}_m$ is the all-ones vector of length m . We call such a form a monic x -symmetric biquadratic form.

Remark 1. Similarly, we can define monic y -symmetric $m \times n$ biquadratic form as follows:

$$\begin{aligned} P(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^m \sum_{j=1}^n x_i^2 y_j^2 + \sum_{i,k=1}^m \sum_{j \neq l} a_{ik} x_i x_k y_j y_l + \sum_{i \neq k} \sum_{j=1}^n b_{ik} x_i x_k y_j^2 \\ &= (\mathbf{x}^\top \mathbf{x})(\mathbf{y}^\top \mathbf{y}) + (\mathbf{x}^\top \mathbf{A} \mathbf{x}) \left((\mathbf{1}_n^\top \mathbf{y})^2 - \mathbf{y}^\top \mathbf{y} \right) + (\mathbf{x}^\top \mathbf{B} \mathbf{x}) (\mathbf{y}^\top \mathbf{y}), \end{aligned}$$

where $A = (a_{ik})$ and $B = (b_{ik})$ are $m \times m$ symmetric matrices, and B has zero diagonal (i.e., $b_i = 0$ for all i). Our results on x -symmetric in this paper may be naturally extended to y -symmetric biquadratic forms.

2.1. Positive Semi-Definiteness of Monic x -Symmetric Forms

Theorem 1. Let P be an $m \times n$ monic x -symmetric biquadratic form as in Equation (2), and let $\mathcal{A} \in \text{SBQ}(m, n)$ be the corresponding symmetric biquadratic tensor. Let $A = (a_{jl})$ and $B = (b_{jl})$ be the $n \times n$ symmetric matrices of coefficients. Then P is PSD if and only if the following two matrix inequalities hold:

$$I + B - A \succeq 0, \quad I + B + (m-1)A \succeq 0, \quad (3)$$

where I denotes the $n \times n$ identity matrix.

Proof. P is positive semi-definite if and only if all M-eigenvalues are nonnegative. A constant λ is an M-eigenvalue of $\mathcal{A}$ if there exists unit vectors $\mathbf{x}$ and $\mathbf{y}$ such that

$$[(\mathbf{y}^\top (I + B - A) \mathbf{y}) I_m + (\mathbf{y}^\top \mathbf{A} \mathbf{y}) \mathbf{1}_m \mathbf{1}_m^\top] \mathbf{x} = \lambda \mathbf{x}, \quad (4)$$

and

$$[(I + B - A) + (\mathbf{x}^\top \mathbf{1}_m)^2 A] \mathbf{y} = \lambda \mathbf{y}. \quad (5)$$

Here, I_m denotes the $m \times m$ identity matrix, $\mathbf{1}_m$ is the all-ones vector of length m , and the second equality follows from $\mathbf{x}^\top \mathbf{x} = 1$.

Equation (4) prompts us to consider the following two cases with respect to the eigenvectors $\mathbf{x}$ and $\mathbf{y}$:

Case (i) $\mathbf{x}^\top \mathbf{1}_m = 0$ or $\mathbf{A} \mathbf{y} = 0$. In this case, Equations (4) and (5) reduce to

$$(\mathbf{y}^\top (I + B - A) \mathbf{y}) \mathbf{x} = \lambda_1 \mathbf{x} \text{ and } (I + B - A) \mathbf{y} = \lambda_1 \mathbf{y}.$$

Consequently, λ_1 is an M-eigenvalue of $\mathcal{A}$ if and only if

$$(I + B - A)\mathbf{y} = \lambda_1 \mathbf{y} \text{ and } \lambda_1 = \mathbf{y}^\top (I + B - A)\mathbf{y}.$$

Thus, all such M-eigenvalues λ_1 are nonnegative if and only if $I + B - A \succeq 0$.

Case (ii) $\mathbf{x}^\top \mathbf{1}_m \neq 0$ and $A\mathbf{y} \neq 0$. Equation (4) implies that $\mathbf{x}$ is parallel to $\mathbf{1}_m$. Together with the condition that $\mathbf{x}$ is a unit vector, this forces $\mathbf{x} = \frac{1}{\sqrt{m}} \mathbf{1}_m$. Substituting this expression for $\mathbf{x}$ into (4) and (5) yields

$$(I + B + (m - 1)A)\mathbf{y} = \lambda \mathbf{y} \text{ and } \lambda_2 = \mathbf{y}^\top (I + B + (m - 1)A)\mathbf{y}.$$

Consequently, all M-eigenvalues λ_2 are nonnegative if and only if $\mathbf{y}^\top (I + B + (m - 1)A)\mathbf{y} \geq 0$ for any $A\mathbf{y} \neq 0$.

Suppose that P is PSD, then it follows from Case (i) that $I + B - A \succeq 0$. Now, let $\mathbf{y}_i$ be an orthonormal eigenbasis of $I + B + (m - 1)A$. For each eigenvector $\mathbf{y}_i$, we distinguish two cases: If $A\mathbf{y}_i \neq 0$, Case (ii) directly gives $\mathbf{y}_i^\top (I + B + (m - 1)A)\mathbf{y}_i \geq 0$. Otherwise, if $A\mathbf{y}_i = 0$, then

$$\mathbf{y}_i^\top (I + B + (m - 1)A)\mathbf{y}_i = \mathbf{y}_i^\top (I + B - A)\mathbf{y}_i \geq 0.$$

where the inequality follows from $I + B - A \succeq 0$. Since $\{\mathbf{y}_i\}$ is an orthonormal basis of $\mathbb{R}^n$, it follows that $I + B + (m - 1)A \succeq 0$.

On the other hand, suppose that Equation (3) holds, the M-eigenvalues in Cases (i) and (ii) are both nonnegative, and hence P is PSD. This completes the proof. $\square$

Remark 2. When P is also y -symmetric, the matrices A and B reduce to

$$A = (a - c)I_n + c\mathbf{1}_n\mathbf{1}_n^\top, \quad B = b(\mathbf{1}_n\mathbf{1}_n^\top - I_n),$$

for three constants $a, b, c \in \mathbb{R}$. Then, the two matrix inequalities in Equation (3) reproduce the four linear inequalities (i)–(iv) of Theorem 4.2 in [11]. Thus Theorem 1 extends the earlier result to the more general x -symmetric setting.

2.2. SOS Property of Monic x -Symmetric Forms

We now prove that every PSD x -symmetric biquadratic form is SOS. This resolves the open question raised in the introduction.

Theorem 2. Let P be an $m \times n$ monic x -symmetric biquadratic form as in Equation (2). If P is positive semi-definite, then P is a sum of squares of bilinear forms.

Proof. Let A and B be the $n \times n$ symmetric matrices as in Theorem 1. Since P is PSD, Theorem 1 yields

$$Q := I_n + B - A \succeq 0, \quad R := I_n + B + (m - 1)A \succeq 0.$$

Denote $\mathcal{A}$ by the unique symmetric biquadratic tensor corresponding to P , and define the $mn \times mn$ matrix M by

$$M_{(i,j),(k,l)} = a_{ijkl}, \quad \text{for } i, k \in [m], j, l \in [n],$$

where a_{ijkl} are the entries of $\mathcal{A}$. Then M can be written in block form as

$$M = I_m \otimes (I_n + B - A) + (\mathbf{1}_m \mathbf{1}_m^\top) \otimes A = I_m \otimes Q + \frac{1}{m} (\mathbf{1}_m \mathbf{1}_m^\top) \otimes (R - Q).$$

For any vectors $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, let $\mathbf{z} = \mathbf{x} \otimes \mathbf{y}$. Then

$$P(\mathbf{x}, \mathbf{y}) = \sum_{i,k=1}^m \sum_{j,l=1}^n a_{ijkl} x_i x_k y_j y_l = \mathbf{z}^\top M \mathbf{z}.$$

We now show that M is positive semi-definite. Let $\mathbf{z} \in \mathbb{R}^{mn}$ be partitioned as $\mathbf{z} = (\mathbf{z}_1^\top, \dots, \mathbf{z}_m^\top)^\top$ with $\mathbf{z}_i \in \mathbb{R}^n$, and set $\mathbf{s} = \sum_{i=1}^m \mathbf{z}_i$. Then,

$$\begin{aligned} \mathbf{z}^\top M \mathbf{z} &= \sum_{i=1}^m \mathbf{z}_i^\top Q \mathbf{z}_i + \mathbf{s}^\top A \mathbf{s} \\ &= \sum_{i=1}^m \mathbf{z}_i^\top Q \mathbf{z}_i + \frac{1}{m} \mathbf{s}^\top (R - Q) \mathbf{s} \\ &= \sum_{i=1}^m \mathbf{z}_i^\top Q \mathbf{z}_i - \frac{1}{m} \mathbf{s}^\top Q \mathbf{s} + \frac{1}{m} \mathbf{s}^\top R \mathbf{s}. \end{aligned}$$

Observe that

$$\sum_{i=1}^m \mathbf{z}_i^\top Q \mathbf{z}_i - \frac{1}{m} \mathbf{s}^\top Q \mathbf{s} = \sum_{i=1}^m \left(\mathbf{z}_i - \frac{1}{m} \mathbf{s} \right)^\top Q \left(\mathbf{z}_i - \frac{1}{m} \mathbf{s} \right).$$

Hence,

$$\mathbf{z}^\top M \mathbf{z} = \sum_{i=1}^m \left(\mathbf{z}_i - \frac{1}{m} \mathbf{s} \right)^\top Q \left(\mathbf{z}_i - \frac{1}{m} \mathbf{s} \right) + \frac{1}{m} \mathbf{s}^\top R \mathbf{s}.$$

Since $Q \succeq 0$ and $R \succeq 0$, both terms are nonnegative for every $\mathbf{z}$. Therefore $M \succeq 0$.

Because M is positive semi-definite, it can be decomposed as $M = \sum_{p=1}^{mn} \mathbf{w}_p \mathbf{w}_p^\top$ (for example, by its spectral decomposition). Then

$$P(\mathbf{x}, \mathbf{y}) = \mathbf{z}^\top M \mathbf{z} = \sum_{p=1}^{mn} (\mathbf{w}_p^\top \mathbf{z})^2.$$

Each inner product $\mathbf{w}_p^\top \mathbf{z}$ is a bilinear form in $\mathbf{x}$ and $\mathbf{y}$ because $\mathbf{z} = \mathbf{x} \otimes \mathbf{y}$ and $\mathbf{w}_p \in \mathbb{R}^{mn}$ corresponds to a matrix $W_p \in \mathbb{R}^{m \times n}$ such that $\mathbf{w}_p^\top \mathbf{z} = \mathbf{x}^\top W_p \mathbf{y}$. Hence P is a sum of squares of bilinear forms. $\square$

Remark 3. Theorem 2, together with the result of [11] for fully symmetric forms, shows that for two natural symmetry classes (fully symmetric and x -symmetric) every PSD biquadratic form is SOS. This provides a partial answer to the open problem of characterizing which structured PSD biquadratic forms are SOS.

2.3. Bounding the SOS Rank

For a PSD monic x -symmetric biquadratic form P , the constructive proof of Theorem 2 yields an explicit upper bound on the minimum number of squares required in an SOS decomposition.

Theorem 3 (SOS rank bound). Let P be an $m \times n$ monic x -symmetric biquadratic form as in Equation (2) that is positive semi-definite, and let

$$Q = I_n + B - A, \quad R = I_n + B + (m - 1)A,$$

where A, B are the symmetric coefficient matrices from Equation (2). Then $Q \succeq 0$ and $R \succeq 0$, and the SOS rank of P satisfies

$$\text{SOS-rank}(P) \leq \text{rank}(R) + (m - 1) \cdot \text{rank}(Q). \quad (6)$$

Here, $\text{SOS-Rank}(P)$ denotes the minimal number of squares of bilinear forms needed to represent P as a sum of squares.

Proof. Because P is PSD, Theorem 1 guarantees $Q \succeq 0$ and $R \succeq 0$. Let $\mathcal{A}$ be the unique symmetric biquadratic tensor corresponding to P and define the symmetric matrix

$$M = I_m \otimes Q + \frac{1}{m}(\mathbf{1}_m \mathbf{1}_m^\top) \otimes (R - Q) \in \mathbb{R}^{mn \times mn}, \quad (7)$$

as in the proof of Theorem 2. For any $\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n$ set $\mathbf{z} = \mathbf{x} \otimes \mathbf{y}$,

$$P(\mathbf{x}, \mathbf{y}) = \mathbf{z}^\top M \mathbf{z}. \quad (8)$$

The matrix M is positive semi-definite and, via the orthogonal transformation U , whose first column is $\mathbf{1}_m / \sqrt{m}$ and remaining columns form an orthonormal basis of $\mathbf{1}_m^\perp$, one obtains the block diagonal form

$$(U^\top \otimes I_n) M (U \otimes I_n) = \text{diag}(R, Q, \dots, Q), \quad (9)$$

where R appears once, and Q appears $m - 1$ times. Consequently

$$\text{rank}(M) = \text{rank}(R) + (m - 1) \text{rank}(Q). \quad (10)$$

Suppose that $M = \sum_{p=1}^{\text{rank}(M)} \mathbf{w}_p \mathbf{w}_p^\top$. Then

$$P(\mathbf{x}, \mathbf{y}) = \sum_{p=1}^{\text{rank}(M)} (\mathbf{w}_p^\top (\mathbf{x} \otimes \mathbf{y}))^2 = \sum_{p=1}^{\text{rank}(M)} (\mathbf{x}^\top W_p \mathbf{y})^2.$$

Here, $W_p \in \mathbb{R}^{m \times n}$ is the matricization of $w_p \in \mathbb{R}^{mn}$. This combined with Equation (10) shows that P is a sum of $\text{rank}(M)$ squares of bilinear forms, proving Equation (6). $\square$

Remark 4. Relationship between Theorem 3 and Theorem 4 of [5]. Theorem 3 provides a refined, structure-dependent bound for x -symmetric forms: $\text{SOS-rank}(P) \leq \text{rank}(R) + (m - 1) \text{rank}(Q)$. For the completely squared form $P_{m,n}$ with $A = B = O$, we have $\text{rank}(Q) = \text{rank}(R) = n$, so Theorem 3 yields the same mn bound as that of Theorem 4 of [5]. However, for many x -symmetric forms, the matrices Q and R have rank much smaller than n , making Theorem 3 significantly sharper than the universal mn bound. The value of Theorem 3 lies in its ability to exploit the x -symmetry to obtain tighter bounds and to guide the efficient computational procedure of Section 3.

Moreover, an SOS decomposition whose rank equals the upper bound in (6) is attained in Section 3. Specifically, the explicit spectral construction of Section 3 produces exactly this many squares (one for each positive eigenvalue of R and $m - 1$ copies of each positive eigenvalue of Q), so the bound is attainable by that construction.

3. Computational Method for SOS Decomposition

Theorem 2 provides a constructive proof that every PSD x -symmetric biquadratic form admits an SOS decomposition. Here we outline a concrete numerical procedure to compute such a decomposition.

We may rewrite the biquadratic form in Equation (2) as follows:

$$P(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^\top \mathbf{x})(\mathbf{y}^\top (I_n + B - A)\mathbf{y}) + (\mathbf{1}_m^\top \mathbf{x})^2 (\mathbf{y}^\top A \mathbf{y}).$$

Suppose that P is PSD. The construction of its SOS expression proceeds in three steps.

1. Form the matrix M . Let $Q = I_n + B - A \in \mathbb{R}^{n \times n}$ and $R = I_n + B + (m-1)A \in \mathbb{R}^{n \times n}$. Then the $mn \times mn$ symmetric matrix M is defined by

$$M = I_m \otimes Q + \frac{1}{m}(\mathbf{1}_m \mathbf{1}_m^\top) \otimes (R - Q).$$

2. Compute a positive semi-definite factorization of M . It follows from the proof in Theorem 2 that $M \succeq 0$. Thus, we can obtain vectors $\mathbf{w}_1, \dots, \mathbf{w}_r \in \mathbb{R}^{mn}$ such that

$$M = \sum_{p=1}^r \mathbf{w}_p \mathbf{w}_p^\top.$$

A convenient choice is the Cholesky decomposition $M = LL^\top$ (after a suitable permutation if M is singular), in which case the $\mathbf{w}_p$'s are the columns of L . Alternatively, one may use the spectral decomposition $M = U\Lambda U^\top$ and set $\mathbf{w}_p = \sqrt{\lambda_p} \mathbf{u}_p$ (where λ_p are the nonzero eigenvalues and $\mathbf{u}_p$ the corresponding eigenvectors), and then $r = \text{rank}(M)$.

3. Extract the bilinear forms. Each vector $\mathbf{w}_p$ can be reshaped into an $m \times n$ matrix W_p by partitioning it into m consecutive blocks of length n , i.e.,

$$\mathbf{w}_p = \begin{pmatrix} \mathbf{w}_p^{(1)} \\ \vdots \\ \mathbf{w}_p^{(m)} \end{pmatrix}, \quad W_p = [\mathbf{w}_p^{(1)} \cdots \mathbf{w}_p^{(m)}]^\top.$$

Then the bilinear form f_p is given by $f_p(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top W_p \mathbf{y}$, and we have

$$P(\mathbf{x}, \mathbf{y}) = \sum_{p=1}^r f_p(\mathbf{x}, \mathbf{y})^2.$$

If the original x -symmetric form is not monic, we first apply the reduction described in the proof of Theorem 4: for each j with $d_j > 0$, set $z_j = \sqrt{d_j} y_j$, thereby obtaining a monic form in the variables $\mathbf{x}$ and $\mathbf{z}$. After constructing the SOS decomposition for the monic form, we substitute back $y_j = z_j / \sqrt{d_j}$ to obtain an SOS decomposition in the original variables.

Complexity and structure. The matrix M has size mn , which can be large when m and n are big. However, its special Kronecker-product structure can be exploited to accelerate the factorization. Observe that M is block circulant with respect to the $\mathbf{x}$ indices. By applying an orthogonal transformation that diagonalizes the matrix $\mathbf{1}_m \mathbf{1}_m^\top$, one can block diagonalize M into m independent $n \times n$ matrices. Specifically, let U be an $m \times m$ orthogonal matrix whose first column is $\mathbf{1}_m / \sqrt{m}$ and whose remaining columns are any orthonormal basis of $\mathbf{1}_m^\perp$. Then

$$(U^\top \otimes I_n) M (U \otimes I_n) = \begin{bmatrix} R & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix},$$

where the first block is R , and the remaining $m-1$ blocks are each Q . Consequently, the eigenvalues of M are precisely the eigenvalues of R (with multiplicity 1) together with the eigenvalues of Q (with multiplicity $m-1$). Moreover, the eigenvectors of M can be obtained from those of Q and R via the same transformation. Using this structured approach, one may directly compute the eigenvalues and eigenvectors of Q and R and then assemble the

vectors $\mathbf{w}_p$ without ever forming the full matrix M . We summarize the efficient procedure as follows.

1. Compute the spectral decompositions $Q = U_Q \Lambda_Q U_Q^\top$ and $R = U_R \Lambda_R U_R^\top$.
2. For each positive eigenvalue λ of Q with eigenvector $\mathbf{u}$, produce $m - 1$ vectors $\mathbf{w}$ of length mn by setting, for $k = 2, \dots, m$,

$$\mathbf{w} = \mathbf{v}_k \otimes \sqrt{\lambda} \mathbf{u},$$

where $\{\mathbf{v}_k\}_{k=2}^m$ is any orthonormal basis of $\mathbf{1}_m^\perp$ (e.g., the last $m - 1$ columns of the $m \times m$ discrete cosine transform matrix).

3. For each positive eigenvalue μ of R with eigenvector $\mathbf{u}$, produce one vector

$$\mathbf{w} = \frac{1}{\sqrt{m}} \mathbf{1}_m \otimes \sqrt{\mu} \mathbf{u}.$$

4. Each such $\mathbf{w}$ gives a bilinear form as described above.

The use of spectral decomposition significantly reduces the computational cost of calculating the eigenpairs of M from $O(m^3 n^3)$ to $O(n^3 + mn^2)$, which is a substantial saving when m is large. Here, the first step requires $O(n^3)$ operations, while the second and third steps require $O(mn^2)$ and $O(n^2)$ operations, respectively.

The number of squares obtained in this way equals $\text{rank}(Q) \cdot (m - 1) + \text{rank}(R)$, which is at most mn and often much smaller when Q and R are low rank.

4. Extension to General Partially Symmetric Forms

For general x -symmetric biquadratic forms, the diagonal coefficients (coefficients of $x_i^2 y_j^2$) are independent of i . We denote them by d_j for $j = 1, \dots, n$. Then,

$$P(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^m \sum_{j=1}^n d_j x_i^2 y_j^2 + \sum_{i \neq k} \sum_{j,l=1}^n a_{jl} x_i x_k y_j y_l + \sum_{i=1}^m \sum_{j \neq l} b_{jl} x_i^2 y_j y_l. \quad (11)$$

The SOS result for monic x -symmetric forms extends immediately to all general x -symmetric PSD biquadratic forms, as the following theorem shows.

Theorem 4. Let P be an x -symmetric biquadratic form given by Equation (11). If P is a PSD tensor, then P is an SOS.

Proof. Let $\mathbf{e}_i \in \mathbb{R}^m$ and $\mathbf{e}_j \in \mathbb{R}^n$ be standard basis vectors. Then we have $P(\mathbf{e}_i, \mathbf{e}_j) = d_j$ since the last two terms in Equation (11) become zero. This combined with P is PSD shows $d_j \geq 0$ for all j . Let $J = \{j \in [n] : d_j > 0\}$ and $J_0 = \{j \in [n] : d_j = 0\}$. We consider two cases.

Case 1: $J_0 = \emptyset$ (all $d_j > 0$). Define a linear change of variables in $\mathbf{y}$ by $z_j = \sqrt{d_j} y_j$ for $j = 1, \dots, n$. Then

$$P(\mathbf{x}, \mathbf{y}) = \tilde{P}(\mathbf{x}, \mathbf{z}) := \sum_{j=1}^n \left(\sum_{i=1}^m x_i^2 \right) z_j^2 + \sum_{i \neq k} \sum_{j,l=1}^n \tilde{a}_{jl} x_i x_k z_j z_l + \sum_{i=1}^m \sum_{j \neq l} \tilde{b}_{jl} x_i^2 z_j z_l,$$

where $\tilde{a}_{jl} = a_{jl} / \sqrt{d_j d_l}$ and $\tilde{b}_{jl} = b_{jl} / \sqrt{d_j d_l}$. Then $\tilde{P}$ is a monic x -symmetric biquadratic form with all diagonal coefficients equal to 1. Moreover, $\tilde{P}$ is PSD because P is PSD and the transformation is invertible. By Theorem 2, $\tilde{P}$ is SOS, i.e., there exist bilinear forms $f_p(\mathbf{x}, \mathbf{z})$ such that $\tilde{P} = \sum_p f_p^2$. Substituting back $z_j = \sqrt{d_j} y_j$, each $f_p(\mathbf{x}, \mathbf{z})$ becomes a bilinear form in $\mathbf{x}$ and $\mathbf{y}$ (since it is linear in $\mathbf{z}$, and $\mathbf{z}$ is linear in $\mathbf{y}$). Hence P is SOS.

Case 2: $J_0 \neq \emptyset$ (some $d_j = 0$). For any $j_0 \in J_0$, we have $d_{j_0} = 0$. Fix an arbitrary index $i_0 \in [m]$. Taking $\mathbf{x} = \mathbf{e}_{i_0}$ and $\mathbf{y} = \mathbf{e}_{j_0}$ yields $P(\mathbf{e}_{i_0}, \mathbf{e}_{j_0}) = d_{j_0} = 0$. Since P is PSD, the quadratic form in $\mathbf{y}$ given by $P(\mathbf{e}_{i_0}, \mathbf{y}) = \sum_{j=1}^n d_j y_j^2 + \sum_{j \neq l} b_{jl} y_j y_l$ is nonnegative and vanishes at $\mathbf{y} = \mathbf{e}_{j_0}$. Therefore, the gradient with respect to $\mathbf{y}$ at $\mathbf{e}_{j_0}$ must be zero, i.e.,

$$\nabla_{\mathbf{y}} P(\mathbf{e}_{i_0}, \mathbf{e}_{j_0}) = 2B\mathbf{e}_{j_0} = 2 \begin{bmatrix} b_{1j_0} & \cdots & b_{j_0-1j_0} & 0 & b_{(j_0+1)j_0} & \cdots & b_{nj_0} \end{bmatrix}^\top = \mathbf{0}_n.$$

The notation $\nabla_{\mathbf{y}}$ denotes the gradient with respect to the variable vector $\mathbf{y}$. Hence, we get $b_{jj_0} = 0$ for all j .

Similarly, the quadratic form in $\mathbf{x}$ given by $P(\mathbf{x}, \mathbf{e}_{j_0}) = a_{j_0j_0} \sum_{i \neq k} x_i x_k$ is nonnegative and vanishes at $\mathbf{x} = \mathbf{e}_{i_0}$. Thus, it follows from

$$\nabla_{\mathbf{x}} P(\mathbf{e}_{i_0}, \mathbf{e}_{j_0}) = 2a_{j_0j_0}(\mathbf{1}_m - \mathbf{e}_{i_0}) = \mathbf{0}_m$$

that $a_{j_0j_0} = 0$. Furthermore, the quadratic form in $\mathbf{y}$ given by

$$P\left(\frac{1}{\sqrt{m}}\mathbf{1}_m, \mathbf{y}\right) = \sum_{j=1}^n d_j y_j^2 + (m-1)\mathbf{y}^\top A\mathbf{y} + \mathbf{y}^\top B\mathbf{y}$$

is also nonnegative. By $d_{j_0} = 0$, $a_{j_0j_0} = 0$ and $b_{j_0j_0} = 0$, we have $P(\frac{1}{\sqrt{m}}\mathbf{1}_m, \mathbf{e}_{j_0}) = 0$. This combined with $b_{jj_0} = 0$ for all j derives

$$\nabla_{\mathbf{y}} P\left(\frac{1}{\sqrt{m}}\mathbf{1}_m, \mathbf{e}_{j_0}\right) = 2(m-1) \begin{bmatrix} a_{1j_0} & \cdots & a_{j_0-1j_0} & 0 & a_{(j_0+1)j_0} & \cdots & a_{nj_0} \end{bmatrix}^\top = \mathbf{0}_n,$$

which implies $a_{jj_0} = 0$ for all j . This combined with the fact both A and B are symmetric matrices implies that the variable y_{j_0} does not appear in any term of P namely,

$$P(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^m \sum_{j \in J} d_j x_i^2 y_j^2 + \sum_{i \neq k} \sum_{j, l \in J} a_{jl} x_i x_k y_j y_l + \sum_{i=1}^m \sum_{j \neq l, j, l \in J} b_{jl} x_i^2 y_j y_l$$

does not depend on y_{j_0} for any $j_0 \in J_0$. Consequently, P can be viewed as an x -symmetric biquadratic form in the variables $\mathbf{x}$ and $\mathbf{y}' = (y_j)_{j \in J}$. If J is nonempty, this reduces to Case 1. If J is empty, then P does not depend on $\mathbf{y}$ at all and is identically zero, which is trivially SOS. This completes the proof. $\square$

Remark 5. Theorem 4 completely resolves the question for the x -symmetric class: every PSD x -symmetric biquadratic form, regardless of its diagonal values, admits an SOS decomposition.

Remark 6. When the original x -symmetric form is not monic, the scaling argument of Theorem 4 reduces the problem to the monic case, possibly with a smaller number n' of active y -variables. Applying Theorem 3 to the reduced form yields the same type of bound, now involving the ranks of the reduced matrices $\tilde{Q}$ and $\tilde{R}$.

Remark 7. By swapping the roles of $\mathbf{x}$ and $\mathbf{y}$, the same result holds for y -symmetric biquadratic forms: every PSD y -symmetric biquadratic form is SOS. Thus, every PSD partially symmetric biquadratic form is SOS.

5. Lower Bounds for SOS Rank in Small Dimensions

In this section, we use simple biquadratic forms as a tool to present some lower bounds for SOS rank in small dimensions.

5.1. Simple Biquadratic Forms

Let $m \geq n$. We say a biquadratic form is a simple biquadratic form if it contains only distinct terms of the type $x_i^2 y_j^2$. For each $s = 1, 2, \dots, mn$, the form $P_{m,n,s}$ contains exactly s distinct terms. We assume that the terms in $P_{m,n,s}$ are taken in the following order. For $k = 0, 1, \dots, mn - 1$, write $k = pm + q$ with integers $0 \leq p < n$ and $0 \leq q < m$ (so $p = \lfloor k/m \rfloor$ and $q = k \bmod m$). Define

$$i_k = q + 1, \quad j_k = (p + q) \bmod n + 1. \quad (12)$$

Then, the sequence of index pairs is $(i_0, j_0), (i_1, j_1), \dots, (i_{mn-1}, j_{mn-1})$. We set

$$P_{m,n,s}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{s-1} x_{i_k}^2 y_{j_k}^2.$$

Thus, $P_{m,n,s}$ has exactly s distinct square terms, and $P_{m,n,mn}$ contains every term $x_i^2 y_j^2$ ($i = 1, \dots, m, j = 1, \dots, n$) exactly once.

For the small dimensions used in our theorems, the resulting forms are as follows. When $m = 2, n = 2$, we have

$$\begin{aligned} P_{2,2,1} &= x_1^2 y_1^2, \\ P_{2,2,2} &= x_1^2 y_1^2 + x_2^2 y_2^2, \\ P_{2,2,3} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_1^2 y_2^2, \\ P_{2,2,4} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2. \end{aligned}$$

When $m = 3, n = 2$, we have

$$\begin{aligned} P_{3,2,1} &= x_1^2 y_1^2, \\ P_{3,2,2} &= x_1^2 y_1^2 + x_2^2 y_2^2, \\ P_{3,2,3} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_1^2, \\ P_{3,2,4} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_1^2 + x_1^2 y_2^2, \\ P_{3,2,5} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2, \\ P_{3,2,6} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + x_3^2 y_2^2. \end{aligned}$$

When $m = 3, n = 3$, we have

$$\begin{aligned} P_{3,3,1} &= x_1^2 y_1^2, \\ P_{3,3,2} &= x_1^2 y_1^2 + x_2^2 y_2^2, \\ P_{3,3,3} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2, \\ P_{3,3,4} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + x_1^2 y_2^2, \\ P_{3,3,5} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + x_1^2 y_2^2 + x_2^2 y_3^2, \\ P_{3,3,6} &= x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_1^2. \end{aligned}$$

5.2. Tightness Examples and a General Lower Bound

Theorem 1 of [5] shows that a 2×2 PSD biquadratic form can always be expressed as the sum of three squares. Theorem 2 of [19] shows that a 3×2 PSD biquadratic form can always be expressed as the sum of four squares. The following theorem shows that these two bounds are tight, and present a general lower bound for $m \geq n = 2$.

Theorem 5 (A general lower bound for $m \times 2$ forms). *Let $m \geq 2$. The simple biquadratic form*

$$P_{m,2,m+1}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^m x_{i_k}^2 y_{j_k}^2, \quad (13)$$

where (i_k, j_k) are defined by the ordering rule in Equation (12), is positive semi-definite and satisfies

$$\text{SOS-rank}(P_{m,2,m+1}) = m + 1.$$

Proof. The nonnegativity and the upper bound $\text{SOS-rank}(P_{m,2,m+1}) \leq m + 1$ are immediate from the decomposition in Equation (13).

For the lower bound, assume for contradiction that $P_{m,2,m+1} = \sum_{t=1}^m L_t^2$ with each L_t bilinear. Write $L_t = \sum_{i=1}^m \sum_{j=1}^2 c_{ij}^{(t)} x_i y_j$. Let $A_i = (c_{i1}^{(t)})_{t=1}^m \in \mathbb{R}^m$ and $B_i = (c_{i2}^{(t)})_{t=1}^m \in \mathbb{R}^m$. From the square terms present in $P_{m,2,m+1}$ one obtains the following conditions: For the index $i_0 = 1$, both $x_1^2 y_1^2$ and $x_1^2 y_2^2$ appear; hence $\|A_1\|^2 = 1$ and $\|B_1\|^2 = 1$. For each $i = 2, \dots, m$, exactly one of $x_i^2 y_1^2$ or $x_i^2 y_2^2$ appears, depending on the parity of i in the ordering. Consequently, for each such i , either $\|A_i\|^2 = 1$ and $B_i = 0$, or $\|B_i\|^2 = 1$ and $A_i = 0$. The form $P_{m,2,m+1}$ contains no mixed terms $x_p x_q y_r y_s$ with $p \neq q$ or $r \neq s$. Setting the coefficients of all such mixed monomials in $\sum_t L_t^2$ to zero yields the orthogonality relations

$$A_p \cdot A_q = 0 \ (p \neq q), \quad B_p \cdot B_q = 0 \ (p \neq q), \quad A_p \cdot B_q = 0 \ (\text{all } p, q).$$

Thus the nonzero vectors among $\{A_1, B_1, A_2, B_2, \dots, A_m, B_m\}$ are pairwise orthogonal. Because of the square term conditions, exactly $m + 1$ of these vectors are nonzero and have norm 1 (namely A_1, B_1 , and one vector for each $i = 2, \dots, m$). We therefore have $m + 1$ mutually orthogonal unit vectors in $\mathbb{R}^m$, which is impossible. Hence no m -square decomposition exists, and $\text{SOS-rank}(P_{m,2,m+1}) \geq m + 1$. $\square$

Remark 8. Fang and Huang informed us that they obtained the same lower bound as the above theorem but with a different method [20].

5.3. A Lower Bound for SOS Rank of 3×3 Biquadratic Forms

Denote the SOS rank of a bilinear form by $\text{sos}(P)$.

Theorem 6 (A 3×3 form requiring six squares). *Let*

$$P'(\mathbf{x}, \mathbf{y}) = P_{3,3,6}(\mathbf{x}, \mathbf{y}) = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_1^2,$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Then

$$\text{sos}(P') = 6.$$

Proof. The upper bound $\text{sos}(P') \leq 6$ is immediate by taking

$$L_1 = x_1 y_1, \ L_2 = x_2 y_2, \ L_3 = x_3 y_3, \ L_4 = x_1 y_2, \ L_5 = x_2 y_3, \ L_6 = x_3 y_1,$$

which yields $P' = \sum_{i=1}^6 L_i^2$.

For the lower bound, assume for contradiction that $P' = \sum_{i=1}^5 L_i^2$ with each L_i bilinear. Write $L_i = \sum_{a,b=1}^3 c_{ab}^{(i)} x_a y_b$. Because P' contains exactly the six square terms $x_a^2 y_b^2$ for

$$(a, b) \in S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (3, 1)\}$$

with coefficient 1 and no other square terms, we must have

$$\sum_{i=1}^5 (c_{ab}^{(i)})^2 = 1 \quad ((a, b) \in S), \quad \sum_{i=1}^5 (c_{ab}^{(i)})^2 = 0 \quad ((a, b) \notin S).$$

Hence each L_i involves only the six coefficient types corresponding to S :

$$L_i = a_i x_1 y_1 + b_i x_1 y_2 + c_i x_2 y_2 + d_i x_2 y_3 + e_i x_3 y_3 + f_i x_3 y_1.$$

Define vectors $A, B, C, D, E, F \in \mathbb{R}^5$ by $A = (a_1, \dots, a_5)$, etc. From the square-term conditions, we have $\|A\| = \|B\| = \dots = \|F\| = 1$.

Since P' contains no mixed terms $x_p x_q y_r y_s$ with $p \neq q$ or $r \neq s$, all cross terms in $\sum_{i=1}^5 L_i^2$ must vanish. This yields the orthogonality conditions

$$A \cdot B = A \cdot C = \dots = E \cdot F = 0,$$

i.e., the six vectors $A, \dots, F$ in $\mathbb{R}^5$ are pairwise orthogonal, and each has norm 1. This is impossible because $\mathbb{R}^5$ can contain at most five nonzero mutually orthogonal vectors. Therefore no such decomposition with five squares exists, and $\text{sos}(P') \geq 6$. $\square$

6. An Improved Universal SOS Rank Bound for General Biquadratic Forms

The next theorem presents an explicit expression for the SOS Rank of a biquadratic form. Firstly, recall that [5] any $m \times n$ biquadratic polynomial can be written as follows:

$$P(\mathbf{x}, \mathbf{y}) = \sum_{i,k=1}^m \sum_{j,l=1}^n a_{ijkl} x_i y_j x_k y_l = \mathbf{z}^\top (B + P(\Gamma)) \mathbf{z} := \mathbf{z}^\top M(\Gamma) \mathbf{z}, \quad (14)$$

where $\mathbf{z} = \mathbf{x} \otimes \mathbf{y}$, $B_{(ij)(kl)} = a_{ijkl} \in \mathbb{R}^{mn \times mn}$, $\Gamma \in \mathbb{R}^{\binom{m}{2} \times \binom{n}{2}}$ is a symmetric parameter matrix that captures the degrees of freedom in expressing the full cross terms with $i \neq k$ and $j \neq l$, and $P(\Gamma)$ is detailed in [5].

Theorem 7. Suppose that the $m \times n$ biquadratic form in Equation (14) is SOS. Then we have

$$\text{SOS-rank}(P) = \min_{\Gamma: M(\Gamma) \succeq 0} \text{rank}(M(\Gamma)). \quad (15)$$

Proof. Suppose that P admits an SOS decomposition $\sum_{r=1}^R f_r^2$. Then, it holds that $\text{supp}(f_r) \subseteq \frac{1}{2} \text{New}(P)$ (Ref. [21], Theorem 1) (Ref. [5], Theorem 4). Here, $\text{New}(P)$ is the Newton polytope of P . From this we may derive that the SOS decomposition can only be the SOS of bilinear forms, i.e., $\sum_r (\mathbf{c}_r^\top \mathbf{z})^2$. Thus, the SOS rank of P equals the minimum possible rank of $M(\Gamma)$ satisfying $M(\Gamma) \succeq 0$. $\square$

While the previous sections focused on x -symmetric forms, the tool provided by Theorem 6.1 allows us to derive a universal upper bound on the SOS rank that holds for any SOS biquadratic form, regardless of symmetry.

Theorem 8. For any integers $m, n \geq 2$, every SOS biquadratic form $P \in \mathbb{R}[x_1, \dots, x_m, y_1, \dots, y_n]$ satisfies

$$\text{SOS-rank}(P) \leq mn - 1.$$

Proof. Let P be SOS. By Theorem 6.1,

$$\text{SOS-rank}(P) = \min_{\Gamma: M(\Gamma) \succeq 0} \text{rank}(M(\Gamma)),$$

where $M(\Gamma)$ is an $mn \times mn$ symmetric matrix representing P as $P(\mathbf{x}, \mathbf{y}) = \mathbf{z}^\top M(\Gamma) \mathbf{z}$ with $\mathbf{z} = \mathbf{x} \otimes \mathbf{y}$. The set $\mathcal{M}_P = \{M(\Gamma) : \Gamma\}$ is an affine subspace of symmetric matrices of dimension $d = \binom{m}{2} \binom{n}{2} \geq 1$ (see [5]). Let $\mathcal{M}_P^+ = \{M \in \mathcal{M}_P : M \succeq 0\}$; since P is SOS, $\mathcal{M}_P^+ \neq \emptyset$.

Take any $M_0 \in \mathcal{M}_P^+$. If $\text{rank}(M_0) \leq mn - 1$, we are done. If $\text{rank}(M_0) = mn$, then M_0 is positive definite. Because $\dim \mathcal{M}_P \geq 1$, we can choose a nonzero direction $\Delta \in \mathcal{M}_P - M_0$ and consider the line $M(t) = M_0 + t\Delta$. Since the PSD cone is pointed, there exists $t^* > 0$ such that $M(t) \succeq 0$ for $t \in [0, t^*]$ and $M(t^*)$ lies on the boundary of the PSD cone, i.e., $\text{rank}(M(t^*)) \leq mn - 1$. Thus $M(t^*) \in \mathcal{M}_P^+$ and has rank at most of $mn - 1$. Hence the minimum rank over $\mathcal{M}_P^+$ is at most $mn - 1$, and according to Theorem 5.1, this minimum equals $\text{SOS-rank}(P)$. $\square$

Remark 9. The bound $mn - 1$ improves the trivial bound mn obtained by the vectorization $\mathbf{z} = \mathbf{x} \otimes \mathbf{y}$. For structured classes such as x -symmetric forms, tighter bounds are given by Theorem 3; for example, when Q and R are low rank, the SOS rank can be much smaller than $mn - 1$.

To quantify the worst-case SOS rank across all biquadratic forms, we introduce the following quantity:

Definition 2. Let $m, n \geq 2$. Let $\text{BSR}(m, n)$ be the maximum sos rank of $m \times n$ SOS biquadratic forms.

Then from the discussion in the last section and this section, we have the following theorem.

Theorem 9. We have the following conclusions.

1. $\text{BSR}(m, n) = \text{BSR}(n, m) \leq mn - 1$ for all $m, n \geq 2$.
2. $\text{BSR}(2, 2) = 3$.
3. $\text{BSR}(3, 2) = 4$.
4. $\text{BSR}(m_1, n_1) \leq \text{BSR}(m_2, n_2)$ if $m_1 \leq m_2$ and $n_1 \leq n_2$.

Corollary 1. For $m = n = 3$, we have

$$6 \leq \text{BSR}(3, 3) \leq 8,$$

where the lower bound follows from Theorem 6.

Remark 10. Determining the exact value of $\text{BSR}(3, 3)$, whether it equals 6, 7 or 8, remains an open problem.

7. Concluding Remarks

In this paper, we have studied the sum-of-squares property for partially symmetric biquadratic forms. Our main results can be summarized as follows:

- We established necessary and sufficient conditions for positive semi-definiteness of monic x -symmetric biquadratic forms (Theorem 1).

- We proved that every PSD x -symmetric biquadratic form is SOS (Theorem 2), extending the known result for fully symmetric forms [11] to the partially symmetric setting.
- We derived an explicit upper bound on the SOS rank for such forms, expressed in terms of the ranks of two associated matrices (Theorem 3).
- We provided an efficient computational procedure for constructing SOS decompositions, exploiting the Kronecker-product structure to reduce the cost from $O(m^3n^3)$ to $O(n^3 + mn^2)$ (Section 3).
- We demonstrated the tightness of the known SOS rank bounds for 2×2 and 3×2 biquadratic forms by exhibiting an $m \times 2$ form that requires exactly $m + 1$ squares (Theorem 5). This also establishes a general lower bound $m + 1$ for the sos rank of a $m \times 2$ PSD biquadratic form.
- We present an 3×3 SOS biquadratic form, which can be expressed as the sum of six squares, but not the sum of five squares (Theorem 6).
- We proved an improved universal upper bound $\text{SOS-rank}(P) \leq mn - 1$ for any SOS biquadratic form (Theorem 8).

These results completely resolve the SOS question for the class of partially symmetric biquadratic forms, showing that partial symmetry either in the x or y variables is sufficient to guarantee an SOS decomposition for all PSD forms in the class.

Open problems and future work. Several natural questions remain:

1. What is the exact value of $\text{BSR}(3,3)$? Theorem 6 shows $\text{BSR}(3,3) \geq 6$, while Corollary 1 gives the upper bound 8. Closing the gap $6 \leq \text{BSR}(3,3) \leq 8$ is a natural next step.
2. What is the maximal possible SOS rank for an $m \times n$ PSD x -symmetric biquadratic form? Theorem 3 gives an upper bound $\text{rank}(R) + (m - 1)\text{rank}(Q)$, which can be as large as mn . However, the universal bound in Theorem 8 shows that $\text{SOS-rank}(P) \leq mn - 1$ for any SOS biquadratic form. It remains unclear whether $mn - 1$ can be attained by an x -symmetric form and how the worst-case growth with m and n behaves under the x -symmetry constraint.
3. Can similar SOS guarantees be established for biquadratic forms with other types of symmetry, such as block symmetry or cyclic symmetry?
4. How do these results extend to higher-degree multiquadratic forms (e.g., trilinear forms raised to the fourth power)?
5. Are there practical applications in optimization or engineering where the efficient SOS construction of Section 3 can be deployed at scale?

We hope that the techniques developed here, especially the use of the Kronecker-product representation and the block-diagonalization trick, will be useful in tackling these and related questions in the future.

Author Contributions: Conceptualization, L.Q., C.C., and Y.X.; methodology, L.Q.; validation, L.Q.; formal analysis, L.Q., C.C. and Y.X.; investigation, L.Q.; writing—original draft, L.Q., C.C., and Y.X.; writing—review and editing, L.Q., C.C. and Y.X.; supervision, L.Q.; project administration, L.Q.; funding acquisition, L.Q., C.C., and Y.X. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially supported by Research Center for Intelligent Operations Research, Hong Kong Polytechnic University (4-ZZT8); the National Natural Science Foundation of China (nos. 12471282 and 12131004); and the Jiangsu Provincial Scientific Research Center of Applied Mathematics (grant no. BK20233002).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: We are grateful to the editor and the anonymous reviewers for their insightful comments and suggestions, which have improved our manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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