

## Article

# Solving of a Variational Inequality Problem Under the Presence of Computational Errors

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## Abstract

W. Takahashi and M. Toyoda (2003) proved weak convergence of an iteration process of solving a variational inequality problem for an inverse strongly-monotone mapping. In our recent work we showed that, for the same iterative process, most of its exact iterates are approximate solutions of the variational inequality. In this paper, we show that the iteration process for solving a variational inequality problem for an inverse strongly monotone mapping generates an approximate solution in the presence of small computational errors. We also estimate a number of iterates needed in order to obtain such an approximate solution.

**Keywords:** Hilbert space; iteration; monotone operator; variational inequality

**MSC:** 47H05; 47H09

## 1. Introduction

In their work [1] W. Takahashi and M. Toyoda analyzed an algorithm which finds a common element of the collection of fixed points of a nonexpansive operator and the collection of solutions of a variational inequality problem for an inverse strongly-monotone operator. They established the weak convergence of that algorithm. Such weak convergence results are interesting from a point of view of the theory but practically are not useful because in practice one can do only a finite number of iterates. Hence, it is important to know that the algorithm produces an approximate solution and how many iterations are needed in order to reach such an approximate solution. This is the goal of our recent work [2], where we proved that, for the algorithm of [1], most of its exact iterates are approximate common solutions of a variational inequality corresponding to an inverse strongly-monotone operator. In this work, we prove that the iteration process for solving the variational inequality problem corresponding to an inverse strongly-monotone operator generates an approximate solution in the presence of small computational errors and estimates a number of iterates needed in order to reach such approximate solutions. Our paper has three main results, Theorems 1–3. Theorem 1 shows that our iteration process generates approximate solutions under the presence of summable errors. Such results are very important in the superiorization theory [3–5]. Theorem 2 shows that our iteration process generates approximate solutions under the presence of nonsummable computational errors. It has a prototype in [6] obtained for a variational inequality with a Lipschitz operator but for another iteration process based on the extragradient method [7]. Each iteration of that process has two steps. In the present paper, for the variational inequality with an inverse strongly-monotone mapping, we use the algorithm of [1] for which each iteration has only one step. Theorem 3 is an extension of Theorem 2. In Theorems 1 and 2, we assume that



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the variational inequality has a solution, while in Theorem 3, only an approximate solution is needed.

Suppose that  $(H, \langle \cdot, \cdot \rangle)$  is a real Hilbert space endowed with an inner product which induces the Euclidean norm

$$\|z\| = \langle z, z \rangle^{1/2}, \quad z \in H,$$

$K \subset H$  is a nonempty, closed, convex set,  $0 < \alpha < 1$ ,  $A : K \rightarrow H$  and that for every pair of points  $x, y \in K$ ,

$$\langle A(x) - A(y), x - y \rangle \geq \alpha \|A(x) - A(y)\|^2.$$

It means that the operator  $A$  is  $\alpha$ -inverse strongly monotone [1,8,9].

For every point  $z \in H$  and every positive number  $r$ , let

$$B(z, r) = \{u \in H : \|u - z\| \leq r\}.$$

Define  $I(z) = z$ ,  $z \in H$  and

$$VI(K, A) = \{\xi \in K : \langle A(\xi), v - \xi \rangle \geq 0, v \in K\}.$$

Some examples of inverse strongly-monotone mappings were considered in [1]. In particular, if an operator  $T : K \rightarrow K$  is nonexpansive, then the operator  $A = I - T$  is  $(1/2)$ -inverse strongly monotone, and  $VI(K, A)$  is the collection of fixed points of  $T$ .

We say that an operator  $A : K \rightarrow H$  is strongly monotone if there is  $\eta > 0$  for which

$$\langle A(u) - A(v), u - v \rangle \geq \eta \|u - v\|^2.$$

In this case, the operator  $A$  is called  $\eta$  strongly monotone. It is not difficult to see every  $\eta$  strongly monotone and  $\kappa$  Lipschitz continuous operator is  $(\eta/\kappa^2)$  inverse strongly monotone. It was shown in [1] that, if  $H = \mathbb{R}^1$ ,  $K = [0, 1]$  and

$$A(x) = x^2(1 + x), \quad x \in [0, 1],$$

then  $A$  is inverse strongly monotone but not strongly monotone.

In the sequel, we use the following result [6,10,11].

**Lemma 1.** Assume that a set  $D \subset H$  is nonempty, convex, and closed. Then for any  $x \in H$  there is a unique element  $P_D(x) \in D$  for which

$$\|x - P_D(x)\| = \inf\{\|x - v\| : v \in D\}.$$

Moreover, the following assertions hold.

1.

$$\|P_D(x) - P_D(y)\| \leq \|x - y\| \text{ for all } x, y \in H$$

and for every  $x \in H$  and every  $z \in D$ ,

$$\langle z - P_D(x), x - P_D(x) \rangle \leq 0,$$

$$\|z - P_D(x)\|^2 + \|x - P_D(x)\|^2 \leq \|z - x\|^2.$$

2. Assume that  $x \in H$ ,  $\xi \in D$  and that for every  $z \in D$ ,

$$\langle z - \xi, x - \xi \rangle \leq 0.$$

Then  $\xi = P_D(x)$ .

3. For every  $x, y \in H$ ,

$$\|P_D(x) - P_D(y)\|^2 + \|(I - P_D)(x) - (I - P_D)(y)\|^2 \leq \|x - y\|^2,$$

$$\|P_D(x) - P_D(y)\|^2 \leq \langle P_D(x) - P_D(y), x - y \rangle.$$

For the proof of the next result, see [2].

**Proposition 1.** Assume that  $\lambda$  is a positive number,  $u \in K$ . Then  $u \in VI(K, A)$  if and only if

$$P_K(u - \lambda A(u)) = u.$$

Let

$$0 < a \leq b < 2\alpha.$$

The following method was considered in [1].

Let  $\{\lambda_n\}_{n=0}^\infty \subset [a, b]$ .

1. Initialization. Fix  $x_0 \in H$ .
2. Iterative step. For every nonnegative integer  $t$  given  $x_t \in H$ , calculate

$$x_{t+1} = P_K(x_t - \lambda_t A(x_t)).$$

It was proved in [1] that the iterates of the method weakly converge to an element of  $VI(K, A)$ .

## 2. Preliminaries and Notation

In the previous section, we used the framework of [1] with the operator  $A$  defined on the set  $K$ . Clearly, all the exact iterates of the iterative process associated with our problem [1] also belong to  $K$ . Since in the present paper we consider inexact iterates, they do not necessarily belong to  $K$ . Therefore, from now on we assume that  $U \subset H$  is a nonempty convex set,  $K \subset U$  is a nonempty, closed, convex set,  $0 < \alpha < 1$ , and that

$$A : U \rightarrow H$$

satisfies for every pair of points  $\xi, \eta \in U$ ,

$$\langle A(\xi) - A(\eta), \xi - \eta \rangle \geq \alpha \|A(\xi) - A(\eta)\|^2. \quad (1)$$

Our inexact iterates will belong to the set  $U$ . In particular, it is possible that  $U = H$ , but we prefer the most general case. Thus, the operator  $A$  is inverse strongly-monotone on the set  $U$ , but we solve the variational inequality on the set  $K$ . Set

$$S = \{z \in K : \langle A(z), u - z \rangle \geq 0, u \in K\}. \quad (2)$$

For every positive number  $\epsilon$ , define

$$S_\epsilon = \{z \in U : B(z, \epsilon) \cap K \neq \emptyset \text{ and}$$

$$\langle A(z), \xi - z \rangle \geq -\epsilon \|\xi - z\| - \epsilon \text{ for all } \xi \in K\}. \quad (3)$$

Evidently,  $S$  is the collection of solutions of the variational inequality, while  $S_\epsilon$  is the collection of its  $\epsilon$ -approximate solutions.

Fix

$$0 < a \leq b < 2\alpha, \quad (4)$$

In the iterative process, for any iteration, we should choose a coefficient  $\lambda_t$ . As usual, the sequence of these coefficients is bounded from above and separated from zero. In the sequel, we choose  $\lambda_t \in [a, b]$ ,  $t = 0, 1, \dots$ . In our study, the nonexpansiveness of the mapping  $I - \lambda_t A$  is needed. By Proposition 2, this is true if  $b < 2\alpha$ .

### 3. Auxiliary Results

In [2], it was shown that, for the iteration process presented in Section 1, most of its exact iterates are approximate solutions of our variational inequality. The study of [2] are based on auxiliary results proved there, and these results are stated below. They will also be used in this paper.

**Proposition 2.** Assume that  $\xi, \eta \in U$  and  $\lambda$  is a positive number. Then

$$\|(I - \lambda A)(\xi) - (I - \lambda A)(\eta)\|^2 \leq \|\xi - \eta\|^2 + \lambda(\lambda - 2\alpha)\|A(\xi) - A(\eta)\|^2.$$

If  $\lambda \leq 2\alpha$ , then  $I - \lambda A$  is a nonexpansive operator.

The nonexpansivity of  $I - \lambda A$  follows from the inequality above.

**Lemma 2.** Assume that

$$x \in U, \lambda \in [a, b], u \in K,$$

$$u = P_K(u - \lambda A(u)), y = P_K(x - \lambda A(x)).$$

Then

$$\|y - u\|^2 \leq \|x - u\|^2 + \lambda(\lambda - 2\alpha)\|A(u) - A(x)\|^2,$$

$$\|y - u\|^2 \leq \|x - u\|^2 - \|y - x + \lambda(A(x) - A(u))\|^2$$

and

$$\|y - u\|^2 \leq \|x - u\|^2 - 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\}.$$

Lemma 2 is deduced from its assumptions, Lemma 1 and Proposition 2.

**Lemma 3.** Assume that  $\epsilon_0 > 0$ ,  $x \in U$ ,  $\lambda \in [a, b]$  and

$$\|x - P_K(x - \lambda A(x))\| \leq \epsilon_0.$$

Then, for each  $\xi \in K$ ,

$$\langle Ax, \xi - x \rangle \geq -\lambda^{-1}\epsilon_0\|\xi - x\| - \lambda^{-1}\epsilon_0^2 - \|A(x)\|\epsilon_0.$$

Lemma 3 is deduced from its assumptions and Lemma 1.

Lemma 2 is an important ingredient in the study of exact iterates of our algorithm. Now we prove the following lemma will play an important role in the study of inexact iterates of our algorithm, where computational errors are taken into account.

**Lemma 4.** Assume that  $\delta \in (0, 1)$ ,  $M_0 > 0$ ,  $M \geq 1$ ,

$$\lambda \in [a, b],$$

$u \in K$ ,  $x \in U$  satisfy

$$\|u\| \leq M, u = P_K(x - \lambda A(u)), \quad (5)$$

$$\|x\| \leq M, \quad (6)$$

$$\|A(u)\| \leq M_0, \|A(x)\| \leq M_0, \quad (7)$$

$$y \in B(P_K(x - \lambda A(x)), \delta). \quad (8)$$

Then

$$\begin{aligned} \|y - u\|^2 &\leq \|x - u\|^2 - \lambda(2\alpha - \lambda)\|A(u) - A(x)\|^2 + \delta(4M + 1), \\ \|y - u\|^2 &\leq \|x - u\|^2 - 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\} + 2\delta(8M + 2), \\ \|y - u\|^2 &\leq \|x - u\|^2 - \|y - x + \lambda(A(x) - A(u))\|^2 + \delta(12M + 2 + 2\lambda M_0). \end{aligned}$$

**Proof.** Set

$$\hat{y} = P_K(x - \lambda A(x)). \quad (9)$$

Lemma 2, (5) and (9) imply that

$$\|\hat{y} - u\|^2 \leq \|x - u\|^2 - \|\hat{y} - x + \lambda(A(x) - A(u))\|^2, \quad (10)$$

$$\|\hat{y} - u\|^2 \leq \|x - u\|^2 - \lambda(2\alpha - \lambda)\|A(u) - A(x)\|^2, \quad (11)$$

$$\|\hat{y} - u\|^2 \leq \|x - u\|^2 - 4^{-1}\|\hat{y} - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\}. \quad (12)$$

By (11),

$$\|\hat{y} - u\| \leq \|x - u\|. \quad (13)$$

Equations (1), (5), (6), (8) and (9) imply that

$$\begin{aligned} &\|y - u\|^2 - \|\hat{y} - u\|^2 \\ &= \|y - u - \hat{y} + u\|(\|y - u\| + \|\hat{y} - u\|) \\ &\leq \|y - \hat{y}\|(2\|\hat{y} - u\| + \|y - \hat{y}\|) \leq \delta(4M + \delta). \end{aligned} \quad (14)$$

By (11) and (14),

$$\begin{aligned} \|y - u\|^2 &\leq \|u - \hat{y}\|^2 + \delta(4M + 1) \\ &\leq \|x - u\|^2 + \lambda(\lambda - 2\alpha)\|A(u) - A(x)\|^2 + \delta(4M + 1). \end{aligned} \quad (15)$$

It follows from (5), (6), (8), (9) and (13) that

$$\begin{aligned} &\|y - x\|^2 - \|\hat{y} - x\|^2 \\ &= \|y - x - \hat{y} + x\|(\|y - x\| + \|\hat{y} - x\|) \\ &\leq \|y - \hat{y}\|(2\|\hat{y} - x\| + \|y - \hat{y}\|) \leq \delta(8M + \delta). \end{aligned} \quad (16)$$

Equations (12), (14) and (16) imply that

$$\begin{aligned} \|y - u\|^2 &\leq \|\hat{y} - u\|^2 + \delta(4M + 1) \\ &\leq \|x - u\|^2 - 4^{-1}\|\hat{y} - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\} + \delta(4M + 1) \\ &\leq \|x - u\|^2 - 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\} + 2\delta(8M + 2). \end{aligned} \quad (17)$$

By (5)–(9) and (13),

$$\|\hat{y} - x + \lambda(A(x) - A(u))\|^2 - \|y - x + \lambda(A(x) - A(u))\|^2$$

$$\begin{aligned}
&= \| \hat{y} - x + \lambda(A(x) - A(u)) \| - \| y - x + \lambda(A(x) - A(u)) \| \\
&\times (\| \hat{y} - x + \lambda(A(x) - A(u)) \| + \| y - x + \lambda(A(x) - A(u)) \|) \\
&\leq \| \hat{y} - y \| (2\| \hat{y} - x + \lambda(A(x) - A(u)) \| + \| \hat{y} - y \|) \\
&\leq \delta(2\| \hat{y} - x \| + \lambda(\| A(x) \| + \| A(u) \|) + \delta) \\
&\leq \delta(8M + 2\lambda M_0 + 1).
\end{aligned} \tag{18}$$

It follows from (10), (14) and (18) that

$$\begin{aligned}
\| y - u \|^2 &\leq \| u - \hat{y} \|^2 + \delta(4M + 1) \\
&\leq \| x - u \|^2 - \| \hat{y} - x + \lambda(A(x) - A(u)) \|^2 + \delta(4M + 1) \\
&\leq \| x - u \|^2 - \| y - x + \lambda(A(x) - A(u)) \|^2 + \delta(8M + 2\lambda M_0 + 1) + \delta(4M + 1).
\end{aligned}$$

Lemma 4 is proved.  $\square$

#### 4. Inexact Iterates with Summable Errors

By (1), for each  $u, v \in U$ ,

$$\| A(u) - A(v) \| \leq \alpha^{-1} \| u - v \|. \tag{19}$$

The following theorem shows that, under the presence of summable computational errors, most of inexact iterates are approximate solutions. The amount of iterates which are not  $\epsilon$ -approximate solutions does not exceed a constant depending on  $a, b, \alpha$  and  $\epsilon$ . This result is proved in the following way. First, using the summability of computational errors, we show that the sequence of inexact iterates  $\{x_t\}_{t=0}^\infty$  is bounded. Then using by induction Lemma 4, we obtain an estimation for

$$\| x_t - u_* \|^2 - \| x_{t+1} - u_* \|^2,$$

where  $u_*$  is a fixed solution of the variational inequality. Summing this estimation from 0 to  $Q$ , where  $Q$  is an arbitrary natural number, we obtained an estimation of the cardinality of the iterates which are not  $\epsilon$ -approximate solutions.

**Theorem 1.** Assume that  $\{\Delta_i\}_{i=0}^\infty$  satisfies

$$\Delta := \sum_{i=0}^\infty \Delta_i < \infty, \tag{20}$$

$\epsilon \in (0, 1)$ ,  $M_0 \geq 1$ ,  $\epsilon_0 > 0$ ,

$$S \cap B(0, M_0) \cap A^{-1}(B(0, M_0)) \neq \emptyset, \tag{21}$$

$$\epsilon_0 < \min\{\epsilon/4, 4^{-1}\epsilon a, 4^{-1}(M_0 + 1)^{-1}(2\alpha^{-1} + 1)^{-1}\}, \tag{22}$$

$\{x_t\}_{t=0}^\infty \subset U$ ,

$$\{\lambda_t\}_{t=0}^\infty \subset [a, b], \tag{23}$$

$$\| x_0 \| \leq M. \tag{24}$$

For each integer  $t \geq 0$ ,

$$\| x_{t+1} - P_K(x_t - \lambda x_t A(x_t)) \| \leq \Delta_t. \tag{25}$$

A natural number  $n_0$  satisfies for each integer  $t \geq n_0$ ,

$$\Delta_t \leq \epsilon_0/2. \quad (26)$$

Then

$$\|x_t\| \leq 3M_0 + \Delta, \quad t = 0, 1, \dots$$

and

$$\begin{aligned} & \text{Card}(\{t \in \{n_0, n_0 + 1, \dots\} : \|x_t - x_{t+1}\| > \epsilon_0\}) \\ & \leq \epsilon_0^{-2}((2M_0 + \Delta)^2 + 4\Delta(24M_0 + 8\Delta + 2) \min\{1, b^{-2}a(2\alpha - b)\}^{-1}). \end{aligned}$$

Moreover, if  $t \geq n_0$  is an integer such that

$$\|x_t - x_{t+1}\| \leq \epsilon_0,$$

then

$$x_t \in S_\epsilon.$$

**Proof.** By (21), the following exists:

$$u_* \in S \cap B(0, M_0) \cap A^{-1}(B(0, M_0)). \quad (27)$$

Proposition 1 and (27) imply that

$$u_* = P_K(u_* - \lambda A(u_*)), \quad \lambda \in [a, b]. \quad (28)$$

Proposition 2, (4), (23) and (28) imply that, for each integer  $t \geq 0$ ,

$$\|P_K(x_t - \lambda_t A(x_t)) - u_*\| \leq \|x_t - u_*\|. \quad (29)$$

Equations (25) and (29) imply that, for each integer  $t \geq 0$ ,

$$\begin{aligned} \|x_{t+1} - u_*\| & \leq \|x_{t+1} - P_k(x_t - \lambda_t A(x_t))\| + \|P_k(x_t - \lambda_t A(x_t)) - u_*\| \\ & \leq \Delta_t + \|x_t - u_*\|. \end{aligned} \quad (30)$$

In view of (24) and (27),

$$\|x_0 - u_*\| \leq 2M_0. \quad (31)$$

By (20), (30) and (31), for each integer  $\tau \geq 0$ ,

$$\begin{aligned} \|x_{\tau+1} - u_*\| & \leq \|x_0 - u_*\| + \sum_{t=0}^{\tau} \Delta_t \\ & \leq 2M_0 + \sum_{t=0}^{\tau} \Delta_t \leq 2M_0 + \Delta. \end{aligned} \quad (32)$$

It follows from (27) and (32) that, for each integer  $t \geq 0$ ,

$$\|x_t - u_*\| \leq 2M_0 + \Delta, \quad \|x_t\| \leq 3M_0 + \Delta. \quad (33)$$

In view of (19), for every nonnegative integer  $t$ ,

$$\|A(x_t) - A(u_*)\| \leq \alpha^{-1}\|u_* - x_t\|. \quad (34)$$

Equations (27), (31) and (34) imply that, for each integer  $t \geq 0$ ,

$$\|x_t\| \leq 2\alpha^{-1}M_0 + M_0. \quad (35)$$

Set

$$M_1 = 2\alpha^{-1}M_0 + M_0. \quad (36)$$

Let  $t \geq 0$  be an integer. By (23), (25), (27), (28), (33), (35), (36) and Lemma 4 applied with

$$\delta = \Delta_t, u = u_*, M = 3M_0 + \Delta, M_0 = M_1, x = x_t, y = x_{t+1}, \lambda = \lambda_t$$

we have

$$\begin{aligned} & \|x_{t+1} - u_*\|^2 \\ & \leq \|x_t - u_*\|^2 - 4^{-1}\|x_t - x_{t+1}\|^2 \min\{1, b^{-2}a(2\alpha - b)\} + \Delta_t(26M_0 + 8\Delta + 2). \end{aligned} \quad (37)$$

Set

$$E = \{t \in \{n_0, n_0 + 1, \dots\} : \|x_t - x_{t+1}\| > \epsilon_0\}. \quad (38)$$

Let  $Q > n_0$  be an integer. By (33), (37) and (38),

$$\begin{aligned} (2M_0 + \Delta)^2 & \geq \|x_{n_0} - u_*\|^2 \\ & \geq \|x_{n_0} - u_*\|^2 - \|x_Q - u_*\|^2 \\ & = \sum_{t=n_0}^{Q-1} (\|x_t - u_*\|^2 - \|x_{t+1} - u_*\|^2) \\ & \geq 4^{-1} \left( \sum_{t=n_0}^{Q-1} \|x_t - x_{t+1}\|^2 \right) \min\{1, b^{-2}a(2\alpha - b)\} \\ & \quad - (24M_0 + 8\Delta + 2) \sum_{t=n_0}^{Q-1} \Delta_t. \end{aligned}$$

This implies that

$$\begin{aligned} & 4((2M_0 + \Delta)^2 + \Delta(24M_0 + 8\Delta + 2)) \min\{1, b^{-2}a(2\alpha - b)\}^{-1} \\ & \geq \sum_{t=n_0}^{Q-1} \|x_t - x_{t+1}\|^2 \geq \epsilon_0^2 \text{Card}(E \cap \{0, \dots, Q-1\}) \end{aligned}$$

and

$$\begin{aligned} & \text{Card}(E \cap \{0, \dots, Q-1\}) \\ & \leq 4\epsilon_0^{-2}((2M_0 + \Delta)^2 + \Delta(24M_0 + 8\Delta + 2)) \min\{1, b^{-2}a(2\alpha - b)\}^{-1}. \end{aligned}$$

Since  $Q$  is any natural number larger than  $n_0$ , we conclude that

$$\begin{aligned} & \text{Card}(\{t \in \{n_0, n_0 + 1, \dots\} : \|x_t - x_{t+1}\| > \epsilon_0\}) \\ & \leq 4\epsilon_0^{-2}((2M_0 + \Delta)^2 + \Delta(24M_0 + 8\Delta + 2)) \min\{1, b^{-2}a(2\alpha - b)\}^{-1}. \end{aligned}$$

Let an integer  $t \geq n_0$  satisfy

$$\|x_t - x_{t+1}\| \leq \epsilon_0. \quad (39)$$

In view of (25), (26) and (39),

$$\begin{aligned} & \|x_t - P_K(x_t - \lambda_t A(x_t))\| \\ & \leq \|x_t - x_{t+1}\| + \|x_{t+1} - P_K(x_t - \lambda_t A(x_t))\| \leq \epsilon_0 + \Delta_t \leq 2\epsilon_0. \end{aligned} \quad (40)$$

By (22), (23), (35), (40) and Lemma 3 applied with  $x = x_t$ ,  $\lambda = \lambda_t$  and  $\epsilon_0$  replaced by  $2\epsilon_0$ , for each  $\xi \in K$  we have

$$\begin{aligned} \langle Ax_t, \xi - x_t \rangle & \geq -2\lambda_t^{-1}\epsilon_0\|\xi - x_t\| - 4\lambda^{-1}\epsilon_0^2 - 2\|A(x_t)\|\epsilon_0. \\ & \geq -2a^{-1}\epsilon_0\|\xi - x_t\| - 4a^{-1}\epsilon_0^2 - 2\epsilon_0 M_0(2a^{-1} + 1) \\ & \geq \epsilon\|\xi - x_t\| - \epsilon \end{aligned}$$

and  $x_t \in S_\epsilon$ . Theorem 1 is proved.  $\square$

## 5. Inexact Iterates with Nonsummable Errors

The following theorem shows that, under the presence of nonsummable computational errors, our iterative process generates  $\epsilon$ -approximate solutions, where  $\epsilon$  is a small positive number depending on  $a, b, \alpha$  and on a computational error  $\delta$  which occurs at every iteration. We obtain the relation between  $\delta$  and  $\epsilon$  as well as the number of iterates needed in order to obtain an  $\epsilon$ -approximate solution. This result is proved in the following way. We assume that  $k_0$  is a natural number and that, for each integer  $t \in \{0, \dots, k_0\}$ , the iterate  $x_t$  is not an  $\epsilon$ -approximate solutions. Then applying Lemma 4, we obtain an estimation for

$$\|x_t - u_*\|^2 - \|x_{t+1} - u_*\|^2, \quad t = 0, \dots, k_0$$

where  $u_*$  is a fixed solution of the variational inequality. Summing this estimation from 0 to  $k_0$ , we obtained an estimation of  $k_0$ .

**Theorem 2.** Assume that  $\epsilon \in (0, 1)$ ,  $M_0 > 1$ ,  $\epsilon_0 > 0$ ,

$$S \cap B(0, M_0) \cap A^{-1}(B(0, M_0)) \neq \emptyset. \quad (41)$$

$$\epsilon_0 < \min\{4^{-1}\epsilon a, 4^{-1}(M_0 + 1)^{-1}(4\alpha^{-1} + 1)^{-1}\epsilon\}, \quad (42)$$

$\delta \in (0, 1)$  satisfies

$$\delta \leq 8^{-1}(12M_0 + 1)^{-1}\epsilon_0^2 \min\{1, b^{-2}a(2\alpha - b)\}^{-1}, \quad (43)$$

a natural number

$$k_0 > 32M_0^2\epsilon_0^{-2} \min\{1, b^{-2}a(2\alpha - b)\}^{-1}. \quad (44)$$

Assume that  $\{x_t\}_{t=0}^\infty \subset U$ ,

$$\{\lambda_t\}_{t=0}^\infty \subset [a, b], \quad (45)$$

$$\|x_0\| \leq M \quad (46)$$

and that, for every nonnegative integer  $t$ ,

$$\|x_{t+1} - P_K(x_t - \lambda x_t A(x_t))\| \leq \delta. \quad (47)$$

Then there is a  $j \in \{1, \dots, k_0\}$  such that

$$\|x_j - x_{j+1}\| \leq \epsilon_0,$$

$$\|x_i - x_{i+1}\| > \epsilon_0$$

for each integer  $i$  satisfying  $0 \leq i < j$  and that

$$\|x_i\| \leq 3M_0, \quad i = 0, \dots, j.$$

Moreover, if  $j \geq 0$  is an integer and

$$\|x_j\| \leq 3M_0, \quad \|x_j - x_{j+1}\| \leq \epsilon_0,$$

then

$$x_j \in S_\epsilon.$$

**Proof.** By (41), there is a

$$u_* \in S \cap B(0, M_0) \cap A^{-1}(B(0, M_0)). \quad (48)$$

Assume that an integer  $i \geq 0$  satisfies

$$\|x_i - u_*\| \leq 2M_0. \quad (49)$$

(In view of (46) and (48), Equation (49) holds for  $i = 0$ ). By Proposition 1, (47)–(49) and Lemma 4 applied with

$$u = u_*, \quad M = 3M_0, \quad x = x_i, \quad y = x_{i+1}, \quad \lambda = \lambda_i$$

we have

$$\begin{aligned} & \|x_{i+1} - u_*\|^2 \\ & \leq \|x_i - u_*\|^2 - 4^{-1}\|x_i - x_{i+1}\|^2 \min\{1, b^{-2}a(2\alpha - b)\} + \delta(12M_0 + 1). \end{aligned} \quad (50)$$

Assume that

$$\|x_i - x_{i+1}\| > \epsilon_0. \quad (51)$$

It follows from (43), (50) and (51) that

$$\begin{aligned} \|x_{i+1} - u_*\|^2 & \leq \|x_i - u_*\|^2 - 4^{-1}\epsilon_0^2 \min\{1, b^{-2}a(2\alpha - b)\} + \delta(12M_0 + 1) \\ & \leq \|x_i - u_*\|^2 - 8^{-1}\epsilon_0^2 \min\{1, b^{-2}a(2\alpha - b)\}. \end{aligned}$$

Thus, the following property is true:

(P1) if  $i \geq 0$  is an integer and (49) and (51) hold, then

$$\|x_{i+1} - u_*\|^2 \leq \|x_i - u_*\|^2 - 8^{-1}\epsilon_0^2 \min\{1, b^{-2}a(2\alpha - b)\}. \quad (52)$$

Assume that, for each integer  $i \in \{0, \dots, k_0\}$ , the relation (51) holds. Property (P1) applied by induction imply that (49) and (52) hold for all  $i = 0, \dots, k_0$ . This implies that

$$\begin{aligned}
4M_0^2 &\geq \|x_0 - u_*\|^2 \\
&\geq \|x_0 - u_*\|^2 - \|x_{k_0} - u_*\|^2 \\
&= \sum_{i=0}^{k_0-1} (\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2) \\
&\geq 8^{-1}\epsilon_0^2 \min\{1, b^{-2}a(2\alpha - b)\}k_0
\end{aligned}$$

and

$$k_0 \leq 32M_0^2\epsilon_0^{-2} \min\{1, b^{-2}a(2\alpha - b)\}^{-1}.$$

This contradicts (44). The contradiction we have reached proves that there is  $j \in \{1, \dots, k_0\}$  for which

$$\|x_j - x_{j+1}\| \leq \epsilon_0,$$

for each integer  $i$  satisfying  $0 \leq i < j$ ,

$$\|x_i - x_{i+1}\| \geq \epsilon_0$$

for each integer  $i = 0, \dots, j$ ,

$$\|x_i - u_*\| \leq 2M_0, \|x_i\| \leq 3M_0.$$

Assume that  $j \geq 0$  is an integer and

$$\|x_j\| \leq 3M_0, \|x_j - x_{j+1}\| \leq \epsilon_0. \quad (53)$$

In view of (47) and (53),

$$\begin{aligned}
&\|x_j - P_K(x_j - \lambda_j A(x_j))\| \\
&\leq \|x_j - x_{j+1}\| + \|x_{j+1} - P_K(x_j - \lambda_j A(x_j))\| \leq \epsilon_0 + \delta \leq 2\epsilon_0.
\end{aligned} \quad (54)$$

By (45), (54) and Lemma 3 applied with  $x = x_j$ ,  $\lambda = \lambda_j$  and  $\epsilon_0$  replaced by  $2\epsilon_0$ , for each  $\xi \in K$ , we have

$$\begin{aligned}
\langle Ax_j, \xi - x_j \rangle &\geq -2\lambda_j^{-1}\epsilon_0\|\xi - x_t\| - 4\lambda_j^{-1}\epsilon_0^2 - 2\|A(x_j)\|\epsilon_0. \\
&\geq -4a^{-1}\epsilon_0\|\xi - x_j\| - 4a^{-1}\epsilon_0^2 - 2\epsilon_0\|A(x_j)\|.
\end{aligned} \quad (55)$$

In view of (48) and (53),

$$\|x_j - u_*\| \leq 4M_0. \quad (56)$$

It follows from (19), (48) and (56) that

$$\begin{aligned}
\|A(x_j)\| &\leq \|A(x_j) - A(u_*)\| + \|A(u_*)\| \\
&\leq \alpha^{-1}\|x_j - u_*\| + M_0 \leq 4M_0\alpha^{-1} + M_0.
\end{aligned} \quad (57)$$

By (42), (55) and (57),

$$\begin{aligned}
\langle Ax_j, \xi - x_j \rangle &\geq -4a^{-1}\epsilon_0\|\xi - x_j\| - 4a^{-1}\epsilon_0^2 - 2\epsilon_0M_0(4\alpha^{-1} + 1) \\
&\geq -\epsilon\|\xi - x_t\| - \epsilon
\end{aligned}$$

and  $x_j \in S_\epsilon$ . Theorem 2 is proved.  $\square$

The problem of the existence of  $u \in S$  is well-studied in the literature on variational inequalities. See [1] and the references mentioned therein. In particular, it exists if the set  $K$  is bounded. Clearly, if  $S \neq \emptyset$ , then (41) holds for some constant  $M_0$ . If the operator  $A$  is given analytically, this constant can be found in principle.

## 6. An Extension of Theorem 2

In Theorem 2, we assume the existence of the point  $u_* \in S$ . By Proposition 1, this inclusion is equivalent to the equation

$$u_* = P_K(u_* - \lambda A(u_*))$$

with  $\lambda > 0$ . The existence of such point  $u_*$  is guaranteed only if the set  $K$  is bounded. In the next theorem, we prove the extension of Theorem 2 assuming the existence of a point  $u_*$  such that

$$\|u_* - P_K(u_* - \lambda A(u_*))\|$$

is small. Note that in this extension all the parameters  $\lambda_t, t = 0, 1, \dots$  are the same.

**Theorem 3.** Let  $\epsilon \in (0, 1)$ ,  $M_0 > 1$ ,  $\epsilon_0 > 0$ ,

$$\lambda \in [a, b], \quad (58)$$

$$\epsilon_0 < \min\{4^{-1}\epsilon a, 8^{-1}(M_0 + 1)^{-1}(4\alpha^{-1} + 1)^{-1}\epsilon\}, \quad (59)$$

$\delta, \delta_0 \in (0, 1)$  satisfy

$$\delta, \delta_0 \leq 16^{-1}(8M_0 + 4)^{-1}\epsilon_0^2(5 + b\alpha^{-1})^{-1} \min\{1, b^{-2}a(2\alpha - b)\}, \quad (60)$$

$$u_* \in U \cap B(0, M_0) \cap A^{-1}(B(0, M_0)), \quad (61)$$

$$\|u_* - P_K(u_* - \lambda A(u_*))\| \leq \delta_0, \quad (62)$$

and let a natural number

$$k_0 > 32M_0^2\epsilon_0^{-2} \min\{1, b^{-2}a(2\alpha - b)\}^{-1}. \quad (63)$$

Assume that  $\{x_t\}_{t=0}^\infty \subset U$ ,

$$\|x_0\| \leq M, \quad (64)$$

and that for every nonnegative integer  $t$ ,

$$\|x_{t+1} - P_K(x_t - \lambda A(x_t))\| \leq \delta. \quad (65)$$

Then there is a  $j \in \{1, \dots, k_0\}$  such that

$$\|x_j - x_{j+1}\| \leq \epsilon_0,$$

$$\|x_i - x_{i+1}\| > \epsilon_0$$

for each integer  $i$  satisfying  $0 \leq i < j$  and that

$$\|x_i\| \leq 3M_0, \quad i = 0, \dots, j.$$

Moreover, if  $j \geq 0$  is an integer and

$$\|x_j\| \leq 3M_0, \|x_j - x_{j+1}\| \leq \epsilon_0,$$

then

$$x_j \in S_\epsilon.$$

In the sequel, we use the next lemma.

**Lemma 5.** Assume that  $\delta, \delta_0 \in (0, 1)$ ,  $M_0 > 1$ ,

$$\lambda \in [a, b], \quad (66)$$

$$u_* \in U \cap B(0, M_0) \cap A^{-1}(B(0, M_0)), \quad (67)$$

$$\|u_* - P_K(u_* - \lambda A(u_*))\| \leq \delta_0, \quad (68)$$

$$x \in U \cap B(0, M_0), \quad (69)$$

$$y \in B(P_K(x - \lambda A(x)), \delta). \quad (70)$$

Then

$$\begin{aligned} \|y - u_*\|^2 &\leq \|x - u_*\|^2 - 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\} \\ &\quad + (\delta + \delta_0)(5 + \lambda\alpha^{-1})(4M_0 + 1). \end{aligned}$$

**Proof.** By (4) and (66),

$$\lambda \leq b < 2\alpha. \quad (71)$$

It follows from Proposition 2 that, for all  $v_1, v_2 \in U$ ,

$$\|(I - \lambda A)v_1 - (I - \lambda A)v_2\|^2 \leq \|v_1 - v_2\|^2 + \lambda(\lambda - 2\alpha)\|A(v_1) - A(v_2)\|^2. \quad (72)$$

Equations (68) and (70)–(72) imply that

$$\begin{aligned} &\|y - u_*\| \\ &\leq \|y - P_K(x - \lambda A(x))\| + \|P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*))\| \\ &\quad + \|P_K(u_* - \lambda A(u_*)) - u_*\| \\ &\leq \delta + \delta_0 + \|x - \lambda A(x) - (u_* - \lambda A(u_*))\|. \end{aligned} \quad (73)$$

By (72),

$$\|x - \lambda A(x) - (u_* - \lambda A(u_*))\|^2 \leq \|x - u_*\|^2 - \lambda(-\lambda + 2\alpha)\|A(u_*) - A(x)\|^2. \quad (74)$$

It follows from (66), (69), (71) and (74) that

$$\|x - \lambda A(x) - (u_* - \lambda A(u_*))\| \leq \|x - u_*\| \leq 2M_0. \quad (75)$$

Equations (66), (73), (74) and (75) imply that

$$\begin{aligned}
\|y - u_*\|^2 &\leq (\delta + \delta_0 + \|x - \lambda A(x) - (u_* - \lambda A(u_*))\|)^2 \\
&\leq (\delta + \delta_0)^2 + 4(\delta + \delta_0)M_0 + \|x - \lambda A(x) - (u_* - \lambda A(u_*))\|^2 \\
&\leq (\delta + \delta_0)(4M_0 + 1) + \|x - u_*\|^2 - \lambda(2\alpha - \lambda)\|A(u_*) - A(x)\|^2 \\
&\leq \|x - u_*\|^2 - a(2\alpha - b)\|A(u_*) - A(x)\|^2 + (\delta + \delta_0)(4M_0 + 1).
\end{aligned} \tag{76}$$

In view of (68) and (70),

$$\|y - u_*\| \leq \delta + \delta_0 + \|P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*))\|. \tag{77}$$

Lemma 1 and (68), (70) and (74) imply that

$$\begin{aligned}
&\|P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*))\|^2 \\
&\leq \langle P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*)), (x - \lambda A(x)) - (u_* - \lambda A(u_*)) \rangle \\
&\leq \langle y - u_*, x - \lambda A(x) - (u_* - \lambda A(u_*)) \rangle \\
&+ \langle P_K(x - \lambda A(x)) - y - (P_K(u_* - \lambda A(u_*)) - u_*), x - \lambda A(x) - (u_* - \lambda A(u_*)) \rangle \\
&\leq 2^{-1}[\|y - u_*\|^2 + \|(x - \lambda A(x)) - (u_* - \lambda A(u_*))\|^2 \\
&\quad - \|(y - u_*) - (x - \lambda A(x) - (u_* - \lambda A(u_*)))\|^2 \\
&\quad + (\delta + \delta_0)\|(x - \lambda A(x)) - (u_* - \lambda A(u_*))\|] \\
&\leq 2^{-1}[\|y - u_*\|^2 + \|x - u_*\|^2 - \|y - x + \lambda(A(x) - A(u_*))\|^2] \\
&\quad + (\delta + \delta_0)\|(x - \lambda A(x)) - (u_* - \lambda A(u_*))\|.
\end{aligned} \tag{78}$$

By (19), (67) and (69),

$$\|A(x) - A(u_*)\| \leq \alpha^{-1}\|x - u_*\| \leq 2M_0\alpha^{-1}. \tag{79}$$

Equations (67), (69), (78) and (79) imply that

$$\begin{aligned}
&\|P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*))\|^2 \\
&\leq 2^{-1}[\|y - u_*\|^2 + \|x - u_*\|^2 - \|y - x + \lambda(A(x) - A(u_*))\|^2] \\
&\quad + 2M_0(\delta + \delta_0)(1 + \lambda\alpha^{-1}).
\end{aligned} \tag{80}$$

It follows from (66), (67), (69) and (72) that

$$\|P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*))\| \leq \|u_* - x\| \leq 2M_0. \tag{81}$$

In view of (68), (70) and (81),

$$\begin{aligned}
&|\|y - x_*\|^2 - \|P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*))\|^2| \\
&\leq |\|y - u_*\| - \|P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*))\|| \\
&\quad \times (2\|P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*))\| \\
&\quad + \|y - u_* - (P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*)))\|) \\
&\leq (\|y - P_K(x - \lambda A(x))\| + \|u_* - P_K(u_* - \lambda A(u_*))\|) \\
&\quad \times (4M_0 + \|y - P_K(x - \lambda A(x))\| + \|u_* - P_K(u_* - \lambda A(u_*))\|)
\end{aligned}$$

$$\leq (\delta + \delta_0)(4M_0 + \delta + \delta_0). \quad (82)$$

By (80) and (82),

$$\begin{aligned} \|y - u_*\|^2 &\leq \|P_K(x - \lambda A(x)) - P_K(u_* - \lambda A(u_*))\|^2 + (\delta + \delta_0)(4M_0 + 2) \\ &\leq 2^{-1}[\|y - u_*\|^2 + \|x - u_*\|^2 - \|y - x + \lambda(A(x) - A(u_*))\|^2] \\ &\quad + (\delta + \delta_0)(2M_0(3 + \lambda\alpha^{-1}) + 2). \end{aligned} \quad (83)$$

It follows from (83) that

$$\begin{aligned} \|y - u_*\|^2 &\leq \|x - u_*\|^2 - \|y - x + \lambda(A(x) - A(u_*))\|^2 \\ &\quad + 4M_0(\delta + \delta_0)(3 + \lambda\alpha^{-1} + 2). \end{aligned} \quad (84)$$

Equations (76) and (84) imply that

$$\begin{aligned} \|y - u_*\|^2 &\leq \|x - u_*\|^2 \\ &\quad - \max\{a(2\alpha - b)\|A(u_*) - A(x)\|^2, \|y - x + \lambda(A(x) - A(u_*))\|^2\} \\ &\quad + (\delta + \delta_0)(4M_0 + 1)(5 + \lambda\alpha^{-1}). \end{aligned} \quad (85)$$

By the Cauchy–Schwartz inequality and (66),

$$\begin{aligned} \|y - x\|^2 &= \|y - x + \lambda(A(x) - \lambda A(u_*)) + \lambda(A(u_*) - A(x))\|^2 \\ &\leq 2\|y - x + \lambda(A(x) - \lambda A(u_*))\|^2 + 2\lambda^2\|A(u_*) - A(x)\|^2 \\ &\leq 2\|y - x + \lambda(A(x) - \lambda A(u_*))\|^2 + 2b^2\|A(u_*) - A(x)\|^2 \\ &\leq 4\max\{\|y - x + \lambda(A(x) - \lambda A(u_*))\|^2, b^2\|A(u_*) - A(x)\|^2\}. \end{aligned} \quad (86)$$

By (86),

$$\begin{aligned} &\max\{\|y - x + \lambda(A(x) - \lambda A(u_*))\|^2, a(2\alpha - b)\|A(u_*) - A(x)\|^2\} \\ &\geq \max\{\|y - x + \lambda(A(x) - \lambda A(u_*))\|^2, b^2\|A(u_*) - A(x)\|^2\} \\ &\quad \times \min\{1, b^{-2}a(2\alpha - b)\} \geq 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\}. \end{aligned} \quad (87)$$

It follows from (85) and (87) that

$$\begin{aligned} \|y - u_*\|^2 &\leq \|x - u_*\|^2 \\ &\quad - 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\} + (\delta + \delta_0)(4M_0 + 1)(5 + \lambda\alpha^{-1}). \end{aligned}$$

Lemma 5 is proved.  $\square$

Proof of Theorem 3 By (61) and (64),

$$\|x_0 - u_*\| \leq 2M_0. \quad (88)$$

Assume that an integer  $i \geq 0$  satisfies

$$\|x_i - u_*\| \leq 2M_0. \quad (89)$$

By (61), (62), (65), (89) and Lemma 5 applied with  $M_0$  replaced by  $3M_0$ ,

$$x = x_i, \quad y = x_{i+1}$$

we have

$$\begin{aligned} & \|x_{i+1} - u_*\|^2 \\ \leq & \|x_i - u_*\|^2 - 4^{-1} \|x_i - x_{i+1}\|^2 \min\{1, b^{-2}a(2\alpha - b)\} + (\delta + \delta_0)(5 + \lambda\alpha^{-1})(8M_0 + 1). \end{aligned} \quad (90)$$

Assume that

$$\|x_i - x_{i+1}\| > \epsilon_0. \quad (91)$$

It follows from (54), (90) and (91) that

$$\begin{aligned} & \|x_{i+1} - u_*\|^2 \leq \|x_i - u_*\|^2 \\ & - 4^{-1} \epsilon_0^2 \min\{1, b^{-2}a(2\alpha - b)\} + (\delta + \delta_0)(5 + b\alpha^{-1})(8M_0 + 1) \\ & \leq \|x_i - u_*\|^2 - 8^{-1} \epsilon_0^2 \min\{1, b^{-2}a(2\alpha - b)\}. \end{aligned}$$

Thus, the following property is true:

(P1) if  $i \geq 0$  is an integer and (89) holds, then

$$\|x_{i+1} - u_*\|^2 \leq \|x_i - u_*\|^2 - 8^{-1} \epsilon_0^2 \min\{1, b^{-2}a(2\alpha - b)\}. \quad (92)$$

Assume that, for each integer  $i \in \{0, \dots, k_0\}$ , relation (91) holds. By (88), (91) and (P1) applied by induction, relation (89) holds for all  $i = 0, \dots, k_0 + 1$  and inequality (92) holds for all  $i = 0, \dots, k_0$ . It follows from (89) and (92) that

$$\begin{aligned} 4M_0^2 & \geq \|x_0 - u_*\|^2 \\ & \geq \|x_0 - u_*\|^2 - \|x_{k_0} - u_*\|^2 \\ & = \sum_{i=0}^{k_0-1} (\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2) \\ & \geq 8^{-1} \epsilon_0^2 k_0 \min\{1, b^{-2}a(2\alpha - b)\} \end{aligned}$$

and

$$k_0 \leq 32M_0^2 \epsilon_0^{-2} \min\{1, b^{-2}a(2\alpha - b)\}^{-1}.$$

This contradicts (63). The contradiction we have reached proves that there is a  $j \in \{1, \dots, k_0\}$  such that

$$\|x_j - x_{j+1}\| \leq \epsilon_0,$$

for each integer  $i$  satisfying  $0 \leq i < j$ ,

$$\|x_i - x_{i+1}\| > \epsilon_0$$

$$\|x_{i+1} - u_*\| \leq 2M_0, \quad \|x_{i+1}\| \leq 3M_0.$$

Assume that  $j \geq 0$  is an integer and

$$\|x_j\| \leq 3M_0, \quad \|x_j - x_{j+1}\| \leq \epsilon_0. \quad (93)$$

In view of (60), (65) and (93),

$$\begin{aligned} & \|x_j - P_K(x_j - \lambda A(x_j))\| \\ & \leq \|x_j - x_{j+1}\| + \|x_{j+1} - P_K(x_j - \lambda A(x_j))\| \leq \epsilon_0 + \delta \leq 2\epsilon_0. \end{aligned} \quad (94)$$

By (58), (94) and Lemma 3 applied with  $\epsilon_0$  replaced by  $2\epsilon_0$ , for each  $\xi \in K$  we have

$$\begin{aligned} \langle Ax_j, \xi - x_j \rangle & \geq -2\lambda^{-1}\epsilon_0\|\xi - x_t\| - 4\lambda^{-1}\epsilon_0^2 - 2\|A(x_j)\|\epsilon_0 \\ & \geq -2a^{-1}\epsilon_0\|\xi - x_j\| - 4a^{-1}\epsilon_0^2 - 2\epsilon_0\|A(x_j)\|. \end{aligned} \quad (95)$$

In view of (61) and (93),

$$\|x_j - u_*\| \leq 4M_0. \quad (96)$$

It follows from (61) and (96) that

$$\begin{aligned} \|A(x_j)\| & \leq \|A(x_j) - A(u_*)\| + \|A(u_*)\| \\ & \leq \alpha^{-1}\|x_j - u_*\| + M_0 \leq 4M_0\alpha^{-1} + M_0. \end{aligned} \quad (97)$$

By (59), (95) and (97), for each  $\xi \in K$ ,

$$\begin{aligned} \langle Ax_j, \xi - x_j \rangle & \geq -2a^{-1}\epsilon_0\|\xi - x_j\| - 4a^{-1}\epsilon_0^2 - 2\epsilon_0M_0(4\alpha^{-1} + 1) \\ & \geq -\epsilon\|\xi - x_t\| - \epsilon \end{aligned}$$

and  $x_j \in S_\epsilon$ . Theorem 3 is proved.

## 7. An Example

It should be mentioned that Theorem 2 provides the evaluation of  $\delta$  and  $k_0$ , which follow from (42)–(44). Here  $\delta$  is the computational error produced by our computer system, while  $k_0$  is the number of iterates needed in order to reach an element of  $S_\epsilon$  which is an  $\epsilon$ -approximate solution of our variational inequality. It is not difficult to see that  $\delta = c_1\epsilon^2$  and  $k = c_2\epsilon^{-2}$ , where  $c_1$  and  $c_2$  are positive constants depending on  $M_0, a, b, \alpha$ .

Let us consider the following particular example.

Assume that  $A(x) = x$  for all  $x \in H$  and  $K = U = H$ . Then  $S = \{0\}$ . Let  $\epsilon \in (0, 1/2)$  and  $M_0 = 10^2$ . Clearly, in this case  $\alpha = 1$  and we can choose  $a = b = 1$ . Let  $\epsilon \in (0, 1/2)$  and  $M_0 = 10^2$ . Then

$$S_\epsilon \subset \{y \in X : \|y\| \leq 2\epsilon\}$$

and the assertion of Theorem 2 holds with

$$\epsilon_0 = 4^{-1} \cdot 10^{-3}\epsilon,$$

$$\delta = 64^{-1} \cdot 10^{-5}\epsilon^2$$

and with  $k_0$ , which is the smallest integer larger than  $64 \cdot 10^4 \cdot \epsilon_0^{-2}$ .

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