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Periodic Solutions and Exponential Stability of Nonlinear Neutral Differential Systems with State-Dependent Delays via Matrix Measure Approach

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Abstract

This paper investigates a class of nonlinear neutral functional differential systems with multiple state-dependent delays. By combining the matrix measure framework with Banach's and Krasnoselskii's fixed point principles, we derive new explicit and verifiable criteria for the existence and uniqueness of T -periodic solutions. Furthermore, we establish sufficient conditions for the exponential stability of the unique periodic solution, showing how the strength of the linear dissipation competes with the sensitivity of the state-dependent delays. The proposed results extend and unify several earlier contributions on systems with constant or time-varying delays, and rigorously characterize the influence of nonlinear feedback-dependent delay terms on periodic dynamics. Three detailed numerical examples are presented to verify the sharpness and applicability of the obtained conditions.

Keywords: Arzela–Ascoli's theorem; fixed point theorem; matrix measure; state-dependent delay; neutral differential equations; periodic solution

MSC: 34K13; 34K40; 47H10; 34A34

1. Introduction

The study of periodic dynamics in neutral functional differential systems has been a central topic in nonlinear analysis due to its wide range of applications in biological rhythms, population dynamics, neural activity, engineering control processes, and economic cycles, where oscillatory behaviors naturally arise and require strict regulation (see, e.g., [1–12]). Over the last few decades, many analytical techniques have been introduced to establish the existence, uniqueness, and stability of periodic solutions in delayed dynamical systems.

Recent years have seen growing interest in nonlinear differential systems with complex delay structures driven by internal feedback mechanisms. Notably, Abbas et al. [13] analyzed the stability and exponential lag-synchronization of neural networks with state-dependent and distributed delays, while related studies addressed adaptive delays in population dynamics [14], control systems [15], and hybrid dynamics with nonlinear feedback delays [16].



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Among the available analytical tools, the matrix measure approach is efficient in quantifying dissipativity and its ability to counterbalance delay effects. Although well developed for constant and time-dependent delays, it has not been systematically extended to *state-dependent neutral systems*. In this context, we use the term *feedback-dependent delay* for state-dependent delays that act within the neutral term and directly affect the derivative, whereas delays in the forcing term are referred to as state-dependent delays (see [17]). This distinction captures an additional nonlinear mechanism influencing periodicity and stability.

Earlier contributions have primarily focused on special cases of neutral delay differential equations. For instance, using Krasnoselskii's fixed point theorem, Luo [18] investigated the existence of positive periodic solutions for equations of the form

$$(\phi(t) - c\phi(t - \tau(t)))' = -b(t)\phi(t) + f(t, \phi(t - \tau(t))), \quad (1)$$

under periodicity assumptions on the involved functions. Such models naturally arise in hematopoiesis [19,20], Nicholson-type population systems [21,22], and other biological feedback processes [23], highlighting their relevance in modeling delayed biological interactions.

Subsequent studies [18,24–26] extended this analysis to multi-delay neutral structures. However, in all these works, the delays remain exogenous and independent of the system state. For example, Guo et al. [24] considered the system

$$(\phi(r) - c\phi(r - \tau))' = A(r, \phi(r))x(r) + p(r, \phi(r - \sigma_1(r)), \dots, \phi(r - \sigma_m(r))), \quad (2)$$

under conditions on the matrix measure with periodic coefficients. In [26], the author employed an integrable dichotomy approach to establish periodic solutions for the same type of system when the matrix A depends on r only.

More recently, Mesmouli et al. [25] generalized and improved the previous results to fully nonlinear multi-delay systems of the form

$$(\phi(r) - g(r, \phi(r - \tau(r))))' = A(r)\phi(r) + p(r, \phi(r - \sigma_1(r)), \dots, \phi(r - \sigma_m(r))), \quad (3)$$

thus extending the theory to systems where nonlinearity appears in both the neutral term and the multi-delay forcing term, while still assuming the delays are independent of the state.

Motivation and Contribution. Motivated by the lack of development in this direction, we consider a class of nonlinear neutral differential systems with multiple state-dependent delays of the form

$$\begin{aligned} &(\phi(t) - g(t, \phi(t - \tau(t, \phi(t))))') \\ &= B(t)\phi(t) + f(t, \phi(t - \sigma_1(t, \phi(t))), \dots, \phi(t - \sigma_m(t, \phi(t)))), \end{aligned} \quad (4)$$

where $B(t)$ is a T -periodic matrix and the state-dependent delays $\tau(\cdot, \cdot)$ and $\sigma_i(\cdot, \cdot)$ encode nonlinear feedback in the system.

The main contributions of this work are summarized as follows:

- We develop new explicit and verifiable conditions that guarantee the existence of at least one T -periodic solution of (4) by applying Krasnoselskii's fixed point theorem combined with matrix measure estimates.
- We show that the obtained periodic solution is unique whenever the combined influence of state-dependent delays satisfies a contraction-type constraint.

- Most importantly, we establish *exponential stability* of the unique periodic orbit by leveraging uniform matrix dissipation and the bounded sensitivity of the state-dependent delays, thereby quantifying how feedback mechanisms may enhance or destroy periodic stability.

Thus, the present paper not only extends previous studies on neutral systems with delay-independent or time-dependent delays, but also provides a unified framework capable of treating systems with complex nonlinear delay structures arising from real-world feedback processes.

The remainder of this paper is organized as follows. In Section 2, we recall basic matrix measure preliminaries and auxiliary lemmas. In Section 3, we develop our main results on the existence, uniqueness, and exponential stability of T -periodic solutions to (4), including illustrative examples that verify the applicability of the proposed conditions. Finally, Section 4 provides concluding remarks and perspectives on future research.

2. Preliminaries

In this section, we recall several auxiliary notions and mathematical tools that will be used throughout the paper. We begin with the fundamental properties of matrix measures governing the linear dissipative structure of periodic systems. Consider the linear differential system

$$\phi'(t) = B(t)\phi(t), \quad (5)$$

where $B(t)$ is a continuous T -periodic $n \times n$ matrix function on $\mathbb{R}$. Let $\mathcal{K}(t, s)$ denote the fundamental matrix solution of (5) satisfying $\mathcal{K}(s, s) = I$. The classical identities hold:

$$\mathcal{K}(t, s)\mathcal{K}(s, w) = \mathcal{K}(t, w), \quad \mathcal{K}^{-1}(t, w) = \mathcal{K}(w, t), \quad t, s, w \in \mathbb{R}.$$

2.1. Vector Norms and Matrix Measure

For $\phi = (\phi_1, \dots, \phi_n)^T \in \mathbb{R}^n$, we use the induced 1-norm

$$\|\phi\|_1 = \sum_{i=1}^n |\phi_i|,$$

and for $B = (b_{ij}) \in \mathbb{R}^{n \times n}$, we consider the induced matrix norm

$$\|B\| = \max_{1 \leq j \leq n} \sum_{i=1}^n |b_{ij}|.$$

The logarithmic matrix norm (matrix measure) associated with $\|\cdot\|_1$ is defined (see [27]) by

$$\mu(B) = \lim_{\varepsilon \rightarrow 0^+} \frac{\|I + \varepsilon B\| - 1}{\varepsilon} = \max_{1 \leq j \leq n} \left(b_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^n |b_{ij}| \right).$$

The quantity $\mu(B(t))$ measures the instantaneous expansion or contraction rate of the linear dynamics generated by (5). In particular, negative values yield exponential-decay estimates.

2.2. Auxiliary Lemmas

Lemma 1 ([24]). *The fundamental matrix solution of (5) satisfies*

$$\|\mathcal{K}(t, w)\| \leq \exp\left(\int_w^t \mu(B(\xi)) d\xi\right), \quad t \geq w.$$

Lemma 2 ([27]). If $\phi(t)$ is a solution of (5), then

$$\|\phi(t)\|_1 \leq \|\phi(t_0)\|_1 \exp\left(\int_{t_0}^t \mu(B(w)) dw\right), \quad t \geq t_0.$$

Lemma 3 ([24]). If

$$\exp\left(\int_0^T \mu(B(w)) dw\right) < 1,$$

then the linear periodic system (5) admits no nontrivial T -periodic solution.

Lemma 4 ([28]). If system (5) has no nontrivial T -periodic solution, then for any continuous T -periodic function $V(t)$, the nonhomogeneous system

$$\phi'(t) = B(t)\phi(t) + V(t)$$

admits a unique T -periodic solution given by

$$\phi(t) = \mathcal{K}(t, t_0)\phi(t_0) + \int_{t_0}^t \mathcal{K}(t, w)V(w) dw, \quad t \in \mathbb{R}.$$

Lemma 5. Let

$$\theta := \exp\left(\int_0^T \mu(B(w)) dw\right) < 1. \quad (6)$$

Then every T -periodic solution $\phi(t)$ of system (4) satisfies the equivalent integral equation

$$\begin{aligned} \phi(t) = & g(t, \phi(t - \tau(t, \phi(t)))) \\ & + (I - \mathcal{K}(t + T, t))^{-1} \int_t^{t+T} \mathcal{K}(t + T, w) \left[B(w) g(w, \phi(w - \tau(w, \phi(w)))) \right. \\ & \left. + f(w, \phi(w - \sigma_1(w, \phi(w))), \dots, \phi(w - \sigma_m(w, \phi(w)))) \right] dw. \end{aligned} \quad (7)$$

Moreover, under (6), the operator $(I - \mathcal{K}(t + T, t))^{-1}$ is well defined and uniformly bounded for all $t \in \mathbb{R}$.

Proof. Let $\phi(t)$ be a T -periodic solution of system (4), i.e.,

$$\left(\phi(t) - g(t, \phi(t - \tau(t, \phi(t)))) \right)' = B(t)\phi(t) + f(t, \phi(t - \sigma_1(t, \phi(t))), \dots, \phi(t - \sigma_m(t, \phi(t)))). \quad (8)$$

We proceed in several steps.

Step 1. Reduction to a linear nonhomogeneous system.

Define the auxiliary function

$$y(t) := \phi(t) - g(t, \phi(t - \tau(t, \phi(t)))). \quad (9)$$

By (8), we have

$$y'(t) = B(t)\phi(t) + f(t, \phi(t - \sigma_1(t, \phi(t))), \dots, \phi(t - \sigma_m(t, \phi(t)))). \quad (10)$$

Since

$$\phi(t) = y(t) + g(t, \phi(t - \tau(t, \phi(t)))),$$

we substitute this into (10) to obtain

$$\begin{aligned} y'(t) = & B(t)[y(t) + g(t, \phi(t - \tau(t, \phi(t))))] \\ & + f(t, \phi(t - \sigma_1(t, \phi(t))), \dots, \phi(t - \sigma_m(t, \phi(t)))). \end{aligned} \quad (11)$$

Thus $y(t)$ satisfies the linear nonhomogeneous system

$$y'(t) = B(t)y(t) + V(t), \quad (12)$$

where the “forcing” term $V(t)$ is defined by

$$V(t) := B(t)g(t, \phi(t - \tau(t, \phi(t)))) + f(t, \phi(t - \sigma_1(t, \phi(t))), \dots, \phi(t - \sigma_m(t, \phi(t)))). \quad (13)$$

Step 2. T -periodicity of y and V .

By assumption, $\phi(t)$ is T -periodic:

$$\phi(t + T) = \phi(t) \quad \forall t \in \mathbb{R}.$$

The functions g, f, τ , and σ_i are T -periodic in the first argument and continuous in all arguments (hypotheses of the main system). Therefore, from (9)–(13), we obtain

$$y(t + T) = y(t), \quad V(t + T) = V(t) \quad \forall t \in \mathbb{R},$$

i.e., both y and V are T -periodic and continuous.

Step 3. Variation-of-constants formula for y .

Let $\mathcal{K}(t, s)$ be the fundamental matrix of the homogeneous linear system

$$\psi'(t) = B(t)\psi(t), \quad \mathcal{K}(s, s) = I, \quad (14)$$

as introduced in Section 2. By the standard variation-of-constants formula (see Lemma 4), every solution of (12) can be written as

$$y(t) = \mathcal{K}(t, t_0)y(t_0) + \int_{t_0}^t \mathcal{K}(t, w)V(w)dw, \quad t \in \mathbb{R}, \quad (15)$$

for any fixed $t_0 \in \mathbb{R}$.

Now apply (15) with $t_0 = t$ and t replaced by $t + T$:

$$y(t + T) = \mathcal{K}(t + T, t)y(t) + \int_t^{t+T} \mathcal{K}(t + T, w)V(w)dw. \quad (16)$$

Using the T -periodicity of $y(t)$ established above, we have

$$y(t + T) = y(t),$$

so (16) becomes

$$y(t) = \mathcal{K}(t + T, t)y(t) + \int_t^{t+T} \mathcal{K}(t + T, w)V(w)dw. \quad (17)$$

Rearranging terms,

$$(I - \mathcal{K}(t + T, t))y(t) = \int_t^{t+T} \mathcal{K}(t + T, w)V(w)dw. \quad (18)$$

Step 4. Invertibility of $I - \mathcal{K}(t + T, t)$ and uniform bound.

We now show that under condition (6),

$$\theta := \exp\left(\int_0^T \mu(B(w))dw\right) < 1,$$

the operator $I - \mathcal{K}(t + T, t)$ is invertible for all t and its inverse is uniformly bounded in t . Recall from Lemma 1 that

$$\|\mathcal{K}(t, w)\| \leq \exp\left(\int_w^t \mu(B(s)) ds\right), \quad t \geq w.$$

Fix $t \in \mathbb{R}$. Then

$$\|\mathcal{K}(t + T, t)\| \leq \exp\left(\int_t^{t+T} \mu(B(s)) ds\right).$$

Since B and hence $\mu(B(\cdot))$ are T -periodic, we have

$$\int_t^{t+T} \mu(B(s)) ds = \int_0^T \mu(B(w)) dw,$$

and therefore

$$\|\mathcal{K}(t + T, t)\| \leq \exp\left(\int_0^T \mu(B(w)) dw\right) = \theta < 1, \quad \forall t \in \mathbb{R}. \quad (19)$$

Hence the operator norm of $\mathcal{K}(t + T, t)$ is strictly less than 1, so the Neumann series

$$\sum_{n=0}^{\infty} (\mathcal{K}(t + T, t))^n$$

converges in the operator norm on $\mathbb{R}^n$. Consequently, $I - \mathcal{K}(t + T, t)$ is invertible and

$$(I - \mathcal{K}(t + T, t))^{-1} = \sum_{n=0}^{\infty} (\mathcal{K}(t + T, t))^n, \quad (20)$$

with the uniform bound

$$\|(I - \mathcal{K}(t + T, t))^{-1}\| \leq \sum_{n=0}^{\infty} \|\mathcal{K}(t + T, t)\|^n \leq \sum_{n=0}^{\infty} \theta^n = \frac{1}{1 - \theta}, \quad (21)$$

for all $t \in \mathbb{R}$. This proves both the existence and uniform boundedness of $(I - \mathcal{K}(t + T, t))^{-1}$.

Step 5. Explicit integral representation for y and ϕ .

Since $(I - \mathcal{K}(t + T, t))^{-1}$ exists, we can solve (18) for $y(t)$:

$$y(t) = (I - \mathcal{K}(t + T, t))^{-1} \int_t^{t+T} \mathcal{K}(t + T, w) V(w) dw. \quad (22)$$

Recalling the definition of $V(w)$ in (13), we obtain

$$y(t) = (I - \mathcal{K}(t + T, t))^{-1} \int_t^{t+T} \mathcal{K}(t + T, w) \left[B(w) g(w, \phi(w - \tau(w, \phi(w)))) \right. \\ \left. + f(w, \phi(w - \sigma_1(w, \phi(w))), \dots, \phi(w - \sigma_m(w, \phi(w)))) \right] dw. \quad (23)$$

Finally, using the relation (9),

$$\phi(t) = y(t) + g(t, \phi(t - \tau(t, \phi(t)))),$$

and substituting (23) for $y(t)$, we arrive at (7).

This completes the proof of the lemma. $\square$

2.3. Fixed Point Principles

We recall two classical fixed point results which will be used in the proofs of the existence and uniqueness. The existence of a fixed point is guaranteed by Krasnoselskii's fixed point theorem, which is applicable to operators decomposed as the sum of a contraction and a compact map. This approach is well-suited for neutral systems with state-dependent delays, where standard contraction arguments alone may fail.

Theorem 1 (Krasnoselskii's fixed point theorem [29]). *Let Π be a nonempty, closed, bounded, and convex subset of a Banach space S . Assume $\mathcal{P}_1, \mathcal{P}_2 : \Pi \rightarrow S$ satisfy the following:*

- $\mathcal{P}_1$ is a contraction;
- $\mathcal{P}_2$ is completely continuous;
- $\mathcal{P}_1(a) + \mathcal{P}_2(b) \in \Pi$ for all $a, b \in \Pi$.

Then the equation $\phi = \mathcal{P}_1\phi + \mathcal{P}_2\phi$ has at least one fixed point in Π .

Theorem 2 (Banach contraction principle [30]). *If (S, ρ) is a complete metric space and $\mathcal{P} : S \rightarrow S$ satisfies*

$$\rho(\mathcal{P}(a), \mathcal{P}(b)) \leq \gamma \rho(a, b), \quad \forall a, b \in S, \gamma \in (0, 1),$$

then $\mathcal{P}$ admits a unique fixed point in S .

3. Main Results

In this section, we establish rigorous criteria under which system (4) admits a T -periodic solution, such a solution is unique, and it is exponentially stable. Our approach fully incorporates the effect of state-dependent delays and the dissipative properties induced by the matrix measure of $B(t)$.

3.1. Functional Setting and Hypotheses

Let $BC(\mathbb{R}, \mathbb{R}^n)$ denote the Banach space of bounded and continuous functions $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$, equipped with the norm

$$\|\phi\| := \max_{t \in [0, T]} \|\phi(t)\|_1.$$

We define the closed, convex, and bounded subset of T -periodic functions

$$\Pi := \{ \phi \in BC(\mathbb{R}, \mathbb{R}^n) : \phi(t + T) = \phi(t) \forall t \in \mathbb{R}, \|\phi\| \leq E \}, \quad (24)$$

where $E > 0$ is a prescribed constant to be fixed later. Since constant functions belong to Π , the set is nonempty. Moreover, periodicity guarantees that Π inherits completeness from $BC(\mathbb{R}, \mathbb{R}^n)$.

To proceed with the operator analysis, we impose the following structural assumptions.

(H1) There exists $L_g \in (0, 1)$ such that

$$\|g(t, z) - g(t, y)\|_1 \leq L_g \|z - y\|_1, \quad \forall z, y \in \mathbb{R}^n, \forall t \in \mathbb{R}. \quad (25)$$

(H2) There exists $L_f > 0$ such that for all $z_1, \dots, z_m, y_1, \dots, y_m \in \mathbb{R}^n$,

$$\|f(t, z_1, \dots, z_m) - f(t, y_1, \dots, y_m)\|_1 \leq L_f \sum_{i=1}^m \|z_i - y_i\|_1, \quad \forall t \in \mathbb{R}. \quad (26)$$

The following bounds will be used repeatedly:

$$\psi_g := \sup_{t \in [0, T]} \|g(t, 0)\|_1, \quad \psi_f := \sup_{t \in [0, T]} \|f(t, 0, \dots, 0)\|_1, \quad \lambda := \sup_{t \in [0, T]} \|B(t)\|.$$

(H3) There exists a continuous and T -periodic function κ such that

$$\mu(B(t)) \leq \kappa(t), \quad t \in [0, T],$$

and

$$\mu_0 := \sup_{0 \leq w \leq t \leq T} \int_t^{t+T} \exp\left(\int_w^{t+T} \kappa(s) ds\right) dw < \infty. \quad (27)$$

Moreover, the exponential bound

$$\theta := \exp\left(\int_0^T \mu(B(w)) dw\right) < 1, \quad (28)$$

ensures exponential decay over one period for the homogeneous linear system associated with (5).

(H4) The parameters satisfy

$$(L_g E + \psi_g) + \frac{\lambda(L_g E + \psi_g) + L_f m E + \psi_f}{1 - \theta} \mu_0 \leq E. \quad (29)$$

This ensures that the nonlinear operator $(\mathcal{P}_1 + \mathcal{P}_2)$ maps Π into itself.

(H5) There exist constants $C_\tau \geq 1$ and $C_\sigma \geq 1$ such that for all $z, y \in \Pi$ and all $t \in \mathbb{R}$,

$$\|z(t - \tau(t, z(t))) - y(t - \tau(t, y(t)))\|_1 \leq C_\tau \|z - y\|, \quad (30)$$

$$\sum_{i=1}^m \|z(t - \sigma_i(t, z(t))) - y(t - \sigma_i(t, y(t)))\|_1 \leq C_\sigma \|z - y\|. \quad (31)$$

These constants quantify delay sensitivity and play a key role in the contraction analysis for uniqueness.

Based on Lemma 5, we introduce the operators $\mathcal{P}_1, \mathcal{P}_2 : \Pi \rightarrow BC(\mathbb{R}, \mathbb{R}^n)$ defined by

$$(\mathcal{P}_1 \phi)(t) = g(t, \phi(t - \tau(t, \phi(t)))), \quad (32)$$

$$\begin{aligned} (\mathcal{P}_2 \phi)(t) = & (I - \mathcal{K}(t + T, t))^{-1} \int_t^{t+T} \mathcal{K}(t + T, w) \left[B(w) g(w, \phi(w - \tau(w, \phi(w)))) \right. \\ & \left. + f(w, \phi(w - \sigma_1(w, \phi(w))), \dots, \phi(w - \sigma_m(w, \phi(w)))) \right] dw, \end{aligned} \quad (33)$$

where $\mathcal{K}(t, s)$ is the fundamental matrix of the associated homogeneous linear system and $(I - \mathcal{K}(t + T, t))^{-1}$ is well defined and uniformly bounded by Lemma 5 and condition (28).

In this subsection, we show that under assumptions (H1)–(H4), the neutral system (4) admits at least one T -periodic solution. The strategy is to use the functional setting introduced in the previous subsection and apply Krasnoselskii's fixed point theorem (Theorem 1) to the operator

$$\mathcal{P} := \mathcal{P}_1 + \mathcal{P}_2$$

on the closed, convex, and bounded subset $\Pi \subset BC(\mathbb{R}, \mathbb{R}^n)$.

Lemma 6. Under assumptions (H1)–(H4), the operators $\mathcal{P}_1, \mathcal{P}_2 : \Pi \rightarrow BC(\mathbb{R}, \mathbb{R}^n)$ are well defined. Moreover, for every $\phi \in \Pi$ the functions $\mathcal{P}_1 \phi$ and $\mathcal{P}_2 \phi$ are T -periodic and

$$\|\mathcal{P}_1 \phi\| \leq L_g E + \psi_g, \quad \|\mathcal{P}_2 \phi\| \leq \frac{\lambda(L_g E + \psi_g) + L_f m E + \psi_f}{1 - \theta} \mu_0.$$

In particular, if (H4) holds, then

$$\|\mathcal{P}_1 \phi + \mathcal{P}_2 \phi\| \leq E,$$

so that $\mathcal{P}_1\phi + \mathcal{P}_2\phi \in \Pi$ for all $\phi \in \Pi$.

Proof. Let $\phi \in \Pi$ be arbitrary.

- (a) Boundedness of $\mathcal{P}_2\phi$. Using the Lipschitz condition (H1) with $y = 0$, we obtain for each t :

$$\begin{aligned} \|(\mathcal{P}_1\phi)(t)\|_1 &= \|g(t, \phi(t - \tau(t, \phi(t))))\|_1 \\ &= \|g(t, \phi(t - \tau(t, \phi(t)))) - g(t, 0) + g(t, 0)\|_1 \\ &\leq \|g(t, \phi(t - \tau(t, \phi(t)))) - g(t, 0)\|_1 + \|g(t, 0)\|_1 \\ &\leq L_g \|\phi(t - \tau(t, \phi(t))) - 0\|_1 + \|g(t, 0)\|_1 \\ &\leq L_g \|\phi\| + \psi_g, \end{aligned}$$

since $\|\phi(t - \tau(t, \phi(t)))\|_1 \leq \|\phi\|$ for all t , and $\psi_g = \sup_{t \in [0, T]} \|g(t, 0)\|_1$. Taking the maximum over $t \in [0, T]$ yields

$$\|\mathcal{P}_1\phi\| = \max_{t \in [0, T]} \|(\mathcal{P}_1\phi)(t)\|_1 \leq L_g E + \psi_g.$$

- (b) Periodicity of $\mathcal{P}_1\phi$. Since $\phi \in \Pi$ is T -periodic and both g and τ are T -periodic in the first argument, we have, $(\mathcal{P}_1\phi)(t + T) = (\mathcal{P}_1\phi)(t)$ for all t , i.e., $\mathcal{P}_1\phi$ is T -periodic.
- (c) Boundedness of $\mathcal{P}_2\phi$. We now estimate $\|(\mathcal{P}_2\phi)(t)\|_1$. By Lemma 5 and (28), we have the uniform bound

$$\|(I - \mathcal{K}(t + T, t))^{-1}\| \leq \frac{1}{1 - \theta}, \quad \forall t \in \mathbb{R}.$$

Next, Lemma 1 and assumption (H3) yield

$$\|\mathcal{K}(t + T, w)\| \leq \exp\left(\int_w^{t+T} \mu(B(s)) ds\right) \leq \exp\left(\int_w^{t+T} \kappa(s) ds\right), \quad w \leq t + T.$$

Using again the Lipschitz bounds (H1)–(H2) together with the definitions of ψ_g, ψ_f, λ and the fact that $\|\phi\| \leq E$, we find for all w ,

$$\begin{aligned} \|g(w, \phi(w - \tau(w, \phi(w))))\|_1 &\leq L_g \|\phi(w - \tau(w, \phi(w)))\|_1 + \|g(w, 0)\|_1 \leq L_g E + \psi_g, \\ \|f(w, \phi(w - \sigma_1), \dots, \phi(w - \sigma_m))\|_1 &\leq L_f \sum_{i=1}^m \|\phi(w - \sigma_i(w, \phi(w)))\|_1 + \|f(w, 0, \dots, 0)\|_1 \\ &\leq L_f m E + \psi_f, \\ \|B(w)\| &\leq \lambda. \end{aligned}$$

Substituting these estimates into the bound for $\|(\mathcal{P}_2\phi)(t)\|_1$, we obtain

$$\begin{aligned} \|(\mathcal{P}_2\phi)(t)\|_1 &\leq \frac{1}{1 - \theta} \int_t^{t+T} \exp\left(\int_w^{t+T} \kappa(s) ds\right) \left(\lambda(L_g E + \psi_g) + (L_f m E + \psi_f)\right) dw \\ &= \frac{\lambda(L_g E + \psi_g) + L_f m E + \psi_f}{1 - \theta} \int_t^{t+T} \exp\left(\int_w^{t+T} \kappa(s) ds\right) dw. \end{aligned}$$

By definition of μ_0 in (27),

$$\int_t^{t+T} \exp\left(\int_w^{t+T} \kappa(s) ds\right) dw \leq \mu_0,$$

for all $t \in [0, T]$, so

$$\|(\mathcal{P}_2\phi)(t)\|_1 \leq \frac{\lambda(L_g E + \psi_g) + L_f m E + \psi_f}{1 - \theta} \mu_0, \quad \forall t \in [0, T].$$

Taking the maximum over $t \in [0, T]$ yields

$$\|\mathcal{P}_2\phi\| \leq \frac{\lambda(L_g E + \psi_g) + L_f m E + \psi_f}{1 - \theta} \mu_0.$$

- (d) Periodicity of $\mathcal{P}_2\phi$. The integrand in the definition of $(\mathcal{P}_2\phi)(t)$ is T -periodic in t since B, g, f , the delays, and ϕ are all T -periodic in t , and $\mathcal{K}(t, s)$ inherits T -periodicity from B . A direct change of variables then shows that $(\mathcal{P}_2\phi)(t + T) = (\mathcal{P}_2\phi)(t)$, so $\mathcal{P}_2\phi$ is T -periodic.
- (e) Mapping into Π . Combining the estimates, for each $\phi \in \Pi$, we have

$$\|\mathcal{P}_1\phi\| \leq L_g E + \psi_g, \quad \|\mathcal{P}_2\phi\| \leq \frac{\lambda(L_g E + \psi_g) + L_f m E + \psi_f}{1 - \theta} \mu_0.$$

Therefore,

$$\|\mathcal{P}_1\phi + \mathcal{P}_2\phi\| \leq (L_g E + \psi_g) + \frac{\lambda(L_g E + \psi_g) + L_f m E + \psi_f}{1 - \theta} \mu_0.$$

If condition (H4) holds, then the right-hand side is bounded by E , so $\mathcal{P}_1\phi + \mathcal{P}_2\phi$ is T -periodic and satisfies $\|\mathcal{P}_1\phi + \mathcal{P}_2\phi\| \leq E$, i.e., $\mathcal{P}_1\phi + \mathcal{P}_2\phi \in \Pi$.

□

Lemma 7. Under assumptions (H1) and (H5), the operator $\mathcal{P}_1$ is a contraction on Π provided $L_g C_\tau < 1$.

Proof. Let $z, y \in \Pi$ be arbitrary and fix $t \in \mathbb{R}$. We estimate

$$\|(\mathcal{P}_1 z)(t) - (\mathcal{P}_1 y)(t)\|_1 = \|g(t, z(t - \tau(t, z(t)))) - g(t, y(t - \tau(t, y(t))))\|_1.$$

Using the Lipschitz continuity of g in (H1), we obtain

$$\|(\mathcal{P}_1 z)(t) - (\mathcal{P}_1 y)(t)\|_1 \leq L_g \|z(t - \tau(t, z(t))) - y(t - \tau(t, y(t)))\|_1.$$

Now applying the delay sensitivity estimate (30), we get

$$\|z(t - \tau(t, z(t))) - y(t - \tau(t, y(t)))\|_1 \leq C_\tau \|z - y\|.$$

Combining these inequalities yields

$$\|(\mathcal{P}_1 z)(t) - (\mathcal{P}_1 y)(t)\|_1 \leq L_g C_\tau \|z - y\|.$$

Taking the maximum over $t \in [0, T]$, we conclude that

$$\|\mathcal{P}_1 z - \mathcal{P}_1 y\| \leq L_g C_\tau \|z - y\|.$$

If $L_g C_\tau < 1$, then $\mathcal{P}_1$ is a contraction on Π . □

Lemma 8. Under assumptions (H1)–(H4), the operator $\mathcal{P}_2 : \Pi \rightarrow BC(\mathbb{R}, \mathbb{R}^n)$ is completely continuous.

Proof. We must show that $\mathcal{P}_2$ is continuous and maps bounded sets into relatively compact sets (in the norm of $BC(\mathbb{R}, \mathbb{R}^n)$).

- (a) Continuity. Let (ϕ_n) be a sequence in Π such that $\phi_n \rightarrow \phi$ in Π , i.e., $\|\phi_n - \phi\| \rightarrow 0$ as $n \rightarrow \infty$. We estimate

$$\|(\mathcal{P}_2\phi_n)(t) - (\mathcal{P}_2\phi)(t)\|_1$$

for fixed $t \in [0, T]$. Taking norms and using the same bounds as in Lemma 6, we obtain

$$\begin{aligned} \|(\mathcal{P}_2\phi_n)(t) - (\mathcal{P}_2\phi)(t)\|_1 &\leq \|(I - \mathcal{K}(t+T, t))^{-1}\| \int_t^{t+T} \|\mathcal{K}(t+T, w)\| \\ &\quad \times \left(\|B(w)\| \|g(w, \phi_n(w - \tau(w, \phi_n(w)))) - g(w, \phi(w - \tau(w, \phi(w))))\|_1 \right. \\ &\quad \left. + \|f(\cdots \phi_n \cdots) - f(\cdots \phi \cdots)\|_1 \right) dw. \end{aligned}$$

From (28), Lemma 1, and (H3), we have the uniform bounds

$$\|(I - \mathcal{K}(t+T, t))^{-1}\| \leq \frac{1}{1-\theta}, \quad \|\mathcal{K}(t+T, w)\| \leq \exp\left(\int_w^{t+T} \kappa(s) ds\right).$$

Using (H1)–(H2) and the delay-sensitivity estimates analogous to (30)–(31), we obtain

$$\|g(w, \phi_n(w - \tau(w, \phi_n(w)))) - g(w, \phi(w - \tau(w, \phi(w))))\|_1 \leq L_g C_\tau \|\phi_n - \phi\|,$$

and similarly

$$\|f(\cdots \phi_n \cdots) - f(\cdots \phi \cdots)\|_1 \leq L_f C_\sigma \|\phi_n - \phi\|.$$

Thus,

$$\begin{aligned} \|(\mathcal{P}_2\phi_n)(t) - (\mathcal{P}_2\phi)(t)\|_1 &\leq \frac{1}{1-\theta} \int_t^{t+T} \exp\left(\int_w^{t+T} \kappa(s) ds\right) [\lambda L_g C_\tau + L_f C_\sigma] \|\phi_n - \phi\| dw \\ &\leq \frac{\lambda L_g C_\tau + L_f C_\sigma}{1-\theta} \mu_0 \|\phi_n - \phi\|. \end{aligned}$$

The right-hand side tends to 0 as $n \rightarrow \infty$ because $\|\phi_n - \phi\| \rightarrow 0$. Since the bound is uniform in $t \in [0, T]$, we conclude

$$\|\mathcal{P}_2\phi_n - \mathcal{P}_2\phi\| \rightarrow 0,$$

so $\mathcal{P}_2$ is continuous.

- (b) Relative compactness of $\mathcal{P}_2(\Pi)$. By Lemma 6, for all $\phi \in \Pi$ we have the uniform bound

$$\|\mathcal{P}_2\phi\| \leq C_0 := \frac{\lambda(L_g E + \psi_g) + L_f m E + \psi_f}{1-\theta} \mu_0,$$

so the family $\{\mathcal{P}_2\phi : \phi \in \Pi\}$ is uniformly bounded in $BC(\mathbb{R}, \mathbb{R}^n)$.

Next we show that $\{\mathcal{P}_2\phi : \phi \in \Pi\}$ is equicontinuous on $[0, T]$. For $\phi \in \Pi$ fixed and $t \in [0, T]$, by formula (33) set

$$H(w, \phi) := B(w) g(w, \phi(w - \tau(w, \phi(w)))) + f(w, \phi(w - \sigma_1(w, \phi(w))), \dots, \phi(w - \sigma_m(w, \phi(w)))) ,$$

and perform the change of variables $s = w - t$ (so that $w = t + s$ and $s \in [0, T]$). We then obtain

$$(\mathcal{P}_2\phi)(t) = \int_0^T \phi(t, s, \phi) ds,$$

where

$$G(t, s, \phi) := (I - \mathcal{K}(t + T, t))^{-1} \mathcal{K}(t + T, t + s) H(t + s, \phi), \quad t \in [0, T], s \in [0, T], \phi \in \Pi.$$

Since Π is bounded, there exists $E > 0$ such that $\|\phi\|_1 \leq E$ for all $\phi \in \Pi$. Hence all arguments of $B, g, f, \mathcal{K}$ and of the delay functions remain in a compact set. By the continuity of $B, g, f, \mathcal{K}$ and the delays, it follows that H is bounded on $[0, T] \times \Pi$, and $\mathcal{K}$ and $(I - \mathcal{K}(t + T, t))^{-1}$ are bounded on $[0, T]$. Therefore there exists $M > 0$ such that

$$\|G(t, s, \phi)\|_1 \leq M, \quad \forall t, s \in [0, T], \forall \phi \in \Pi.$$

Moreover, the mappings

$$(t, s, \phi) \mapsto (I - \mathcal{K}(t + T, t))^{-1}, \quad (t, s, \phi) \mapsto \mathcal{K}(t + T, t + s), \quad (t, s, \phi) \mapsto H(t + s, \phi)$$

are continuous on the compact set $[0, T] \times [0, T] \times \Pi$; hence G is continuous on this compact set and thus *uniformly* continuous there. Consequently, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|t_2 - t_1| < \delta \implies \|G(t_2, s, \phi) - G(t_1, s, \phi)\|_1 \leq \frac{\varepsilon}{T},$$

for all $s \in [0, T]$ and all $\phi \in \Pi$. Now let $t_1, t_2 \in [0, T]$ with $|t_2 - t_1| < \delta$ and fix $\phi \in \Pi$. Using the representation above, we have

$$(\mathcal{P}_2\phi)(t_2) - (\mathcal{P}_2\phi)(t_1) = \int_0^T (G(t_2, s, \phi) - G(t_1, s, \phi)) ds.$$

Taking norms and applying the previous estimate yields

$$\|(\mathcal{P}_2\phi)(t_2) - (\mathcal{P}_2\phi)(t_1)\|_1 \leq \int_0^T \|G(t_2, s, \phi) - G(t_1, s, \phi)\|_1 ds \leq \int_0^T \frac{\varepsilon}{T} ds = \varepsilon.$$

Defining

$$\omega(h) := \sup\{\|(\mathcal{P}_2\phi)(t + h) - (\mathcal{P}_2\phi)(t)\|_1 : \phi \in \Pi, t, t + h \in [0, T]\},$$

the previous inequality shows that $\omega(h) \rightarrow 0$ as $|h| \rightarrow 0$, and

$$\|(\mathcal{P}_2\phi)(t_2) - (\mathcal{P}_2\phi)(t_1)\|_1 \leq \omega(|t_2 - t_1|), \quad \forall \phi \in \Pi, t_1, t_2 \in [0, T].$$

Thus the family $\{\mathcal{P}_2\phi : \phi \in \Pi\}$ is equicontinuous on $[0, T]$. By the Arzela–Ascoli theorem, any bounded and equicontinuous subset of $C([0, T], \mathbb{R}^n)$ is relatively compact. Identifying $BC(\mathbb{R}, \mathbb{R}^n)$ with the space of T -periodic continuous functions restricted to $[0, T]$, we conclude that $\mathcal{P}_2(\Pi)$ is relatively compact; hence $\mathcal{P}_2$ is completely continuous. $\square$

We are now in a position to state and prove the main existence result.

Theorem 3 (Existence of periodic solutions). *Suppose that assumptions (H1)–(H4) hold and that the constants in (H5) satisfy $L_g C_\tau < 1$. Then the neutral functional differential system (4) admits at least one T -periodic solution.*

Proof. We apply Krasnoselskii's fixed point theorem (Theorem 1) to the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ defined on the nonempty, closed, bounded, and convex subset Π . From Lemma 7, $\mathcal{P}_1$ is

a contraction on Π because $L_g C_\tau < 1$. From Lemma 8, $\mathcal{P}_2$ is completely continuous on Π . From Lemma 6, under (H4), we have

$$\mathcal{P}_1 z + \mathcal{P}_2 y \in \Pi, \quad \forall z, y \in \Pi.$$

Therefore, all hypotheses of Krasnoselskii's fixed point theorem are satisfied on Π with operators $\mathcal{P}_1$ and $\mathcal{P}_2$. It follows that there exists $\phi^* \in \Pi$ such that

$$\phi^* = \mathcal{P}_1 \phi^* + \mathcal{P}_2 \phi^*.$$

Finally, by Lemma 5, any fixed point of $\mathcal{P}_1 + \mathcal{P}_2$ in Π corresponds to a T -periodic solution ϕ of system (4), and conversely any T -periodic solution arises as such a fixed point. Hence system (4) admits at least one T -periodic solution. $\square$

Example 1 (Existence of periodic solutions). Consider system (4) with $n = 2$, $m = 2$, and $T = 2\pi$. Let

$$B(t) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \sin t \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad \phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix},$$

$$g(t, \phi(t - \tau)) = 10^{-4} \sin t \begin{pmatrix} \phi_2(t - \tau) + 1 \\ \phi_1(t - \tau) \end{pmatrix},$$

$$f(t, \phi(t - \sigma_1), \phi(t - \sigma_2)) = 10^{-5} \cos t \begin{pmatrix} \phi_1(t - \sigma_1) + \phi_2(t - \sigma_2) \\ \phi_2(t - \sigma_1) + \phi_1(t - \sigma_2) \end{pmatrix},$$

and

$$\tau(t, \phi) = 0.1 + 0.01 \tanh(\|\phi\|), \quad \sigma_1(t, \phi) = 10^{-2} \|\phi\|, \quad \sigma_2(t, \phi) = 10^{-3} \sin(\|\phi\|).$$

Then, system (4) admits at least one nontrivial 2π -periodic solution.

Proof. By simple calculations, we have

$$L_g = 10^{-4}, \quad L_f = 10^{-5}, \quad \psi_g = 10^{-4}, \quad \psi_f = 0.$$

and

$$\mu(B(t)) \leq -\frac{1}{4} \Rightarrow \theta = \exp\left(\int_0^{2\pi} \mu(B(s)) ds\right) = e^{-\pi/2} < 1.$$

From (27):

$$\mu_0 = 4(1 - e^{-\pi/2}) \approx 3.16 < \infty.$$

Choosing any $E \geq 1$, condition (28) holds automatically due to very small coefficients. Therefore, (H1)–(H4) are satisfied and by Theorem 3, system (4) admits at least one nontrivial 2π -periodic solution. $\square$

3.2. Uniqueness of the Periodic Solution

We now show that under an additional contractivity condition, the T -periodic solution established in Theorem 3 is unique. The proof combines the delay-sensitivity bounds and the matrix measure estimates derived earlier to demonstrate that the full operator $\mathcal{P} := \mathcal{P}_1 + \mathcal{P}_2$ becomes a strict contraction on Π .

Theorem 4 (Uniqueness of periodic solutions). Assume that (H1)–(H5) hold and in addition,

$$L_g C_\tau + \frac{\lambda L_g C_\tau + L_f C_\sigma}{1 - \theta} \mu_0 < 1. \quad (34)$$

Then the neutral system (4) admits a unique T -periodic solution in Π .

Proof. Let $\phi_1, \phi_2 \in \Pi$ and fix $t \in [0, T]$. We estimate the difference

$$(\mathcal{P}\phi_1)(t) - (\mathcal{P}\phi_2)(t) = (\mathcal{P}_1\phi_1 - \mathcal{P}_1\phi_2)(t) + (\mathcal{P}_2\phi_1 - \mathcal{P}_2\phi_2)(t),$$

we have

$$\begin{aligned} \|(\mathcal{P}_1\phi_1)(t) - (\mathcal{P}_1\phi_2)(t)\|_1 &\leq L_g \|\phi_1(t - \tau(t, \phi_1(t))) - \phi_2(t - \tau(t, \phi_2(t)))\|_1 \\ &\leq L_g C_\tau \|\phi_1 - \phi_2\|, \end{aligned} \quad (35)$$

and

$$\begin{aligned} \|(\mathcal{P}_2\phi_1)(t) - (\mathcal{P}_2\phi_2)(t)\|_1 &\leq \frac{\lambda L_g C_\tau + L_f C_\sigma}{1 - \theta} \|\phi_1 - \phi_2\| \int_t^{t+T} \exp\left(\int_w^{t+T} \kappa(s) ds\right) dw \\ &\leq \frac{\lambda L_g C_\tau + L_f C_\sigma}{1 - \theta} \mu_0 \|\phi_1 - \phi_2\|. \end{aligned} \quad (36)$$

Combining (35) and (36), for all t ,

$$\|(\mathcal{P}\phi_1)(t) - (\mathcal{P}\phi_2)(t)\|_1 \leq \left(L_g C_\tau + \frac{\lambda L_g C_\tau + L_f C_\sigma}{1 - \theta} \mu_0 \right) \|\phi_1 - \phi_2\|.$$

Taking the maximum over $t \in [0, T]$ gives

$$\|\mathcal{P}\phi_1 - \mathcal{P}\phi_2\| \leq q \|\phi_1 - \phi_2\|,$$

where

$$q := L_g C_\tau + \frac{\lambda L_g C_\tau + L_f C_\sigma}{1 - \theta} \mu_0.$$

By condition (34), we have $q < 1$, so $\mathcal{P}$ is a strict contraction on the complete metric space Π . By Banach's fixed point theorem, the unique fixed point of $\mathcal{P}$ corresponds to the unique T -periodic solution of (4). $\square$

Example 2 (Uniqueness of the periodic solution). Consider system (4) with $n = 2$, $m = 2$, and period $T = 2\pi$. Let

$$\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} -\frac{1}{4} & \frac{1}{8} \cos t \\ \frac{1}{4} & -4 \end{pmatrix}.$$

The neutral feedback term is defined by

$$g(t, \phi(t - \tau)) = 10^{-3} \cos(t) \begin{pmatrix} \phi_2(t - \tau) + 3 \\ \phi_1(t - \tau) \end{pmatrix},$$

and the functional source term is

$$f(t, \phi(t - \sigma_1), \phi(t - \sigma_2)) = 10^{-7} \sin(t) \begin{pmatrix} \phi_1(t - \sigma_1) + \phi_2(t - \sigma_2) \\ \phi_2(t - \sigma_1) + \phi_1(t - \sigma_2) \end{pmatrix}.$$

The state-dependent delays are given by

$$\tau(t, \phi) = 10^{-3} \sin t \tanh(\|\phi(t)\|), \quad \sigma_1(t, \phi) = 10^{-5} \sin(\|\phi(t)\|), \quad \sigma_2(t, \phi) = \frac{1}{15} \cos t \|\phi(t)\|.$$

Proof. We first estimate the logarithmic norm:

$$\mu(B(t)) = \max\left\{-\frac{1}{4} + \frac{1}{8} |\cos t|, -\frac{1}{4} + \frac{1}{4}\right\} \leq -\frac{1}{8}.$$

Hence,

$$\theta = e^{\int_0^{2\pi} \mu(B(w)) dw} = e^{-\pi/4} < 1, \quad \mu_0 = 8(1 - e^{-\pi/4}) \approx 4.35.$$

Verification of hypothesis (H5). Since $|\sin t| \leq 1$ and $\tanh(\cdot)$ is globally Lipschitz with constant 1, for any $\phi_1, \phi_2 \in \Pi$ we have

$$|\tau(t, \phi_1) - \tau(t, \phi_2)| \leq 10^{-3} \|\phi_1(t) - \phi_2(t)\|.$$

Using the delay-sensitivity estimate (30), this yields

$$\|\phi_1(t - \tau(t, \phi_1)) - \phi_2(t - \tau(t, \phi_2))\| \leq (1 + 10^{-3}) \|\phi_1 - \phi_2\|,$$

and thus we may take

$$C_\tau = 1 + 10^{-3} \approx 1.001.$$

Similarly, since $\sin(\cdot)$ is globally Lipschitz with constant 1 and $|\cos t| \leq 1$, we obtain

$$|\sigma_1(t, \phi_1) - \sigma_1(t, \phi_2)| \leq 10^{-5} \|\phi_1(t) - \phi_2(t)\|, \quad |\sigma_2(t, \phi_1) - \sigma_2(t, \phi_2)| \leq \frac{1}{15} \|\phi_1(t) - \phi_2(t)\|.$$

Applying (31), we deduce

$$\|\phi_1(t - \sigma_i(t, \phi_1)) - \phi_2(t - \sigma_i(t, \phi_2))\| \leq C_\sigma \|\phi_1 - \phi_2\|,$$

with $C_\sigma = 1 + \max\left\{10^{-5}, \frac{1}{15}\right\} \approx 1.07$. Finally, taking

$$L_g = 10^{-3}, \quad L_f = 10^{-7},$$

the contraction factor in Theorem 4 becomes

$$L_g C_\tau + \frac{\lambda L_g C_\tau + L_f C_\sigma}{1 - \theta} \mu_0 \approx 0.00172 < 1.$$

Therefore, system (4) admits a unique 2π -periodic solution. $\square$

3.3. Exponential Stability of the Periodic Solution

In this subsection, we derive sufficient conditions ensuring that the unique T -periodic solution $\phi^*(t)$ obtained in Theorem 4 is *exponentially stable*. The argument combines the matrix measure estimate for the linear part with the Lipschitz-type control of the state-dependent delays.

Definition 1. A T -periodic solution $\phi^*(t)$ of system (4) is said to be exponentially stable if there exist constants $M \geq 1$ and $\beta > 0$ such that for some $t_0 \in \mathbb{R}$ and for any solution $\phi(t)$ of (4) with $\|\phi(t_0) - \phi^*(t_0)\|_1$ that is sufficiently small, one has

$$\|\phi(t) - \phi^*(t)\|_1 \leq M e^{-\beta(t-t_0)} \|\phi(t_0) - \phi^*(t_0)\|_1, \quad t \geq t_0.$$

In particular, choosing $t_0 = 0$ without loss of generality, we obtain

$$\|\phi(t) - \phi^*(t)\|_1 \leq M e^{-\beta t} \|\phi(0) - \phi^*(0)\|_1, \quad t \geq 0.$$

To obtain this stability property, we impose an additional *dissipation-type* assumption on the matrix measure of the linear part.

(H6) There exists a constant $\alpha > 0$ such that

$$\mu(B(t)) \leq -\alpha < 0, \quad \forall t \in [0, T]. \quad (37)$$

Note that (H6) is stronger than the integral condition (6), and implies $\theta = \exp\left(\int_0^T \mu(B(w)) dw\right) \leq e^{-\alpha T} < 1$.

Theorem 5 (Exponential stability). *Assume that conditions (H1)–(H6) hold, and suppose in addition that*

$$\alpha > L_g C_\tau + L_f C_\sigma. \quad (38)$$

Then the unique T -periodic solution $\phi^(t)$ of (4) is exponentially stable.*

Proof. Let $\phi(t)$ be any solution of system (4), and let $\phi^*(t)$ denote the unique T -periodic solution ensured by Theorem 4. We define the deviation

$$\eta(t) := \phi(t) - \phi^*(t).$$

Our goal is to derive a differential inequality for $\|\eta(t)\|_1$ which shows exponential decay.

Step 1: Derivation of the perturbation system.

Recall that $\phi(t)$ and $\phi^*(t)$ satisfy

$$(\phi(t) - g(t, \phi(t - \tau(t, \phi(t))))') = B(t)\phi(t) + f(t, \phi(t - \sigma_1(t, \phi(t))), \dots, \phi(t - \sigma_m(t, \phi(t)))),$$

and

$$(\phi^*(t) - g(t, \phi^*(t - \tau(t, \phi^*(t))))' = B(t)\phi^*(t) + f(t, \phi^*(t - \sigma_1(t, \phi^*(t))), \dots, \phi^*(t - \sigma_m(t, \phi^*(t)))).$$

Subtracting these two relations gives

$$\begin{aligned} & \frac{d}{dt} \left(\phi(t) - \phi^*(t) - g(t, \phi(t - \tau(t, \phi(t)))) + g(t, \phi^*(t - \tau(t, \phi^*(t)))) \right) \\ &= B(t)(\phi(t) - \phi^*(t)) \\ & \quad + f(t, \phi(t - \sigma_1(\cdot)), \dots) - f(t, \phi^*(t - \sigma_1(\cdot)), \dots). \end{aligned}$$

Setting $\eta(t) = \phi(t) - \phi^*(t)$ and introducing

$$\Delta_g(t) := g(t, \phi(t - \tau(t, \phi(t)))) - g(t, \phi^*(t - \tau(t, \phi^*(t)))), \quad (39)$$

$$\begin{aligned} \Delta_f(t) &:= f(t, \phi(t - \sigma_1(t, \phi(t))), \dots, \phi(t - \sigma_m(t, \phi(t)))) \\ & \quad - f(t, \phi^*(t - \sigma_1(t, \phi^*(t))), \dots, \phi^*(t - \sigma_m(t, \phi^*(t)))). \end{aligned} \quad (40)$$

we can rewrite the above equation as

$$\eta'(t) = B(t)\eta(t) + \Delta_g(t) + \Delta_f(t). \quad (41)$$

This is the evolution equation for the deviation $\eta(t)$.

Step 2: Lipschitz control of the nonlinear terms. We estimate $\Delta_g(t)$ and $\Delta_f(t)$ in terms of the deviation $\eta(t) := \phi(t) - \phi^*(t)$. Using (H1) and the delay sensitivity estimate (30), we obtain

$$\begin{aligned}\|\Delta_g(t)\|_1 &= \left\| g(t, \phi(t - \tau(t, \phi(t)))) - g(t, \phi^*(t - \tau(t, \phi^*(t)))) \right\|_1 \\ &\leq L_g \|\phi(t - \tau(t, \phi(t))) - \phi^*(t - \tau(t, \phi^*(t)))\|_1 \\ &\leq L_g C_\tau \|\phi(t) - \phi^*(t)\|_1 \\ &\leq L_g C_\tau \|\eta(t)\|_1,\end{aligned}\tag{42}$$

Similarly, by (H2) and the delay-sensitivity bound (31),

$$\begin{aligned}\|\Delta_f(t)\|_1 &\leq L_f \sum_{i=1}^m \|\phi(t - \sigma_i(t, \phi(t))) - \phi^*(t - \sigma_i(t, \phi^*(t)))\|_1 \\ &\leq L_f C_\sigma \|\eta(t)\|_1,\end{aligned}\tag{43}$$

Step 3: Gronwall inequality and exponential decay.

From the perturbation system (41), (H6) and the bounds (42)–(43), one has

$$\begin{aligned}\frac{d}{dt} \|\eta(t)\|_1 &\leq \mu(B(t)) \|\eta(t)\|_1 + \|\Delta_g(t)\|_1 + \|\Delta_f(t)\|_1 \\ &\leq (-\alpha + L_g C_\tau + L_f C_\sigma) \|\eta(t)\|_1.\end{aligned}$$

By assumption (38), we define

$$\beta := \alpha - (L_g C_\tau + L_f C_\sigma) > 0,$$

so that the previous inequality becomes

$$\frac{d}{dt} \|\eta(t)\|_1 \leq -\beta \|\eta(t)\|_1.$$

Integrating from t_0 to $t \geq t_0$ yields

$$\|\eta(t)\|_1 \leq \|\eta(t_0)\|_1 e^{-\beta(t-t_0)}.$$

In particular, taking $t_0 = 0$ and recalling that $\eta(t) = \phi(t) - \phi^*(t)$, we obtain

$$\|\phi(t) - \phi^*(t)\|_1 \leq \|\phi(0) - \phi^*(0)\|_1 e^{-\beta t}, \quad t \geq 0.$$

Thus, the periodic solution $\phi^*(t)$ is exponentially stable with $M = 1$ and exponent $\beta > 0$ in the sense of the above definition. $\square$

Remark 1. The stability margin

$$\beta = \alpha - (L_g C_\tau + L_f C_\sigma)$$

quantifies how strongly the dissipative effect of the linear part (measured by the negative matrix measure bound $\mu(B(t)) \leq -\alpha$) dominates the possible destabilizing influence of the nonlinear, state-dependent delays. If the sensitivity constants C_τ and C_σ become too large, the quantity β may vanish or become negative, leading to a loss of exponential stability even though a T -periodic solution may still exist.

Example 3 (Exponential stability of the periodic solution). Consider system (4) with $n = 2$, $m = 2$, and $T = 2\pi$, where

$$\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} -1 & \frac{1}{10} \sin t \\ 0 & -1 \end{pmatrix}.$$

The neutral feedback function is chosen as

$$g(t, \phi(t)) = 8 \times 10^{-4} \begin{pmatrix} \phi_2(t - \tau) 2 \sin t \\ \phi_1(t - \tau) \end{pmatrix}.$$

The nonlinear forcing term is

$$f(t, \phi(t - \sigma_1), \phi(t - \sigma_2)) = 5 \times 10^{-5} \begin{pmatrix} \phi_1(t - \sigma_1) + \phi_2(t - \sigma_2) \\ \phi_2(t - \sigma_1) + \phi_1(t - \sigma_2) \end{pmatrix}.$$

The state-dependent delays are

$$\tau(t, \phi) = 10^{-3} |\sin t| \|\phi(t)\|, \quad \sigma_1(t, \phi) = 10^{-4} \|\phi(t)\|, \quad \sigma_2(t, \phi) = 10^{-4} |\cos t| \|\phi(t)\|.$$

Proof. Matrix measure:

$$\mu(B(t)) = \max(-1, -1) + \left| \frac{1}{10} \sin t \right| \leq -1 + 0.1 = -0.9 = -\alpha,$$

and thus hypothesis (H6) holds with $\alpha = 0.9$. Lipschitz bounds from construction:

$$L_g = 8 \times 10^{-4}, \quad L_f = 5 \times 10^{-5}.$$

Delay-sensitivity on II:

$$C_\tau = C_\sigma = 1.$$

Then Theorem 5 condition (38) gives

$$L_g C_\tau + L_f C_\sigma = 8.5 \times 10^{-4} < \alpha = 0.9.$$

Since (H1)–(H6) and (38) are satisfied, Theorem 5 applies, and we conclude that the unique 2π -periodic solution of (4) is exponentially stable. $\square$

4. Conclusions

In this work, we developed a unified analytical framework for nonlinear neutral functional differential systems with multiple state-dependent delays. Using matrix measure estimates together with Banach's and Krasnoselskii's fixed point principles, we established explicit and easily verifiable criteria ensuring the existence and uniqueness of T -periodic solutions. Furthermore, by imposing a strengthened dissipativity requirement on the matrix measure, we proved that the obtained periodic solution is exponentially stable.

A central feature of the proposed model is that state-dependent delays appear both in the neutral term and in the nonlinear forcing term, producing a richer feedback structure than in classical neutral delay systems. The operator approach adopted here, based on a contraction-compact decomposition, allows us to treat these nonlinear feedback effects in a systematic and quantitative way through explicit constants and inequalities.

Relation to previous equations. The general system studied in this paper, Equation (4) encompasses several previously studied models as special cases:

- Luo [18]: Choosing $g(t, z) = cz$, $\tau(t, \phi(t)) = \tau(t)$, $B(t) = -b(t)$, and $f(t, \dots) = f(t, \phi(t - \tau(t)))$ recovers their neutral equation with state-independent delays.
- Guo et al. [24]: Setting $g(t, z) = cz$, $\tau(t, \phi(t)) = \tau$, $\sigma_i(t, \phi(t)) = \sigma_i(t)$, $B(t) = A(t, \phi(t))$, and $f = p$ retrieves the multi-delay neutral system with time-dependent delays only.
- Sa [26]: By further restricting $B(t) = A(t)$ depending only on time, the system reduces to their framework under integrable dichotomy conditions.
- Mesmouli et al. [25]: Choosing $\tau(t, \phi(t)) = \tau(t)$ and $\sigma_i(t, \phi(t)) = \sigma_i(t)$ with specific forms of g and f recovers their totally nonlinear multi-delay system, but with state-independent delays.

Thus, the present framework not only reproduces these classical models as particular cases but also extends them to fully nonlinear systems with multiple state-dependent delays, providing a more general and unified theoretical setting.

Discussion. The presented results provide several fundamental insights:

- The matrix measure bounds guarantee that the linear component induces a net contraction over each period, thereby preventing the accumulation of deviations in the solution trajectory and replacing the classical dichotomy requirements by directly verifiable integral estimates.
- State-dependent delays enter the dynamics as a nonlinear perturbation mechanism, whose influence is fully quantified by the sensitivity constants C_τ and C_σ . Conditions (34) and (38) provide explicit admissible bounds, ensuring uniqueness and exponential stability.
- The structural assumptions play complementary roles in the operator analysis: (H1) controls the neutral nonlinearity via a strict Lipschitz constant, (H2) bounds the delayed nonlinear term, (H3) ensures exponential decay of the associated linear periodic system, (H4) guarantees invariance of the solution ball, (H5) controls state-dependent delay deviations, and (H6) strengthens dissipativity to obtain stability.
- The results reveal a clear hierarchy of dissipativity:
 - *Weak dissipativity* (via (H3) and (H4)) guarantees the existence of periodic solutions;
 - *Strong dissipativity* (via (H6) and (38)) ensures uniqueness and exponential stability.
- Classical periodic results for neutral equations with constant or time-dependent delays are recovered as special cases when the delays are state independent, confirming that the present conditions form a genuine generalization of earlier theories.

The illustrative examples confirm that the proposed criteria are computationally tractable and capable of detecting the transition between stable and potentially unstable periodic oscillatory behavior. The framework therefore bridges theoretical rigor with practical modeling needs in systems where delays are driven by internal feedback mechanisms.

Future Directions. Possible extensions of the present study include the following:

- Robustness analysis under parameter uncertainty;
- Extension to stochastic, fractional, or impulsive systems with state-dependent delays;
- Design of stabilizing controllers using matrix measure feedback;
- Applications to biological and engineering models where feedback-dependent delays arise naturally.

These directions are expected to further broaden the applicability and impact of matrix measure techniques in the analysis of complex delayed dynamical systems.

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