

Article

A 3D Contact Model with Shoulder-to-Shoulder Paraboloidal Asperities for Rough Surfaces

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Abstract

To address the limitations of existing two-dimensional asperity shoulder-to-shoulder contact models and to investigate the normal and tangential contact mechanisms of rough surfaces, this study proposes a three-dimensional elastic contact model based on shoulder-to-shoulder paraboloidal asperities within the framework of Hertzian contact theory. The proposed model extends the conventional two-dimensional elastic shoulder-to-shoulder asperity contact model developed by Sepehri by explicitly incorporating the effects of principal surface curvatures, thereby enabling the accurate representation of three-dimensional contact conditions. Analytical relationships among contact load, elastic interference, contact area, and contact pressure are derived. The predictive performance of the proposed model is evaluated through systematic comparisons with Sepehri's model and finite element analysis (FEA) results. The results demonstrate that the proposed model exhibits significantly improved agreement with the FEA predictions. The maximum relative errors for contact area, maximum contact pressure, and contact interference are 5.74%, 5.96%, and 2.77%, respectively. Furthermore, parametric studies examine the effects of relative offset r and pressing amount Δd on contact displacement, contact area, and maximum contact pressure. Furthermore, parametric studies are conducted to examine the effects of the relative offset r the pressing amount Δd on contact displacement, contact area, and maximum contact pressure. The results indicate that the contact area increases with both r and Δd , whereas the maximum contact pressure and contact displacement exhibit similar trends, increasing with Δd but decreasing with increasing r . These findings validate the accuracy and applicability of the proposed model under varying contact conditions, providing a reliable theoretical foundation for refining asperity contact models and supporting practical engineering contact analyses.

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MSC: 74M15

1. Introduction

The contact behavior of mechanical joint surfaces plays a crucial role in determining the static, dynamic, thermal, and electrical performance of mechanical systems [1,2]. To elucidate the complex contact mechanisms governing joint interfaces, extensive research on surface contact theory has been conducted over several decades. Since the 1970s, a wide variety of asperity contact models have been proposed to characterize the micromechanical interactions between rough surfaces. The pioneering work by Greenwood and Williamson [3] postulated that mechanical joint surfaces could be represented by a collection of hemi-spherically tipped asperities with uniform radii of curvature. They also assumed a Gaussian distribution for these asperities and described the contact between two mechanical joint surfaces as a series of micro-contacts between individual asperities. Equations governing surface separation and contact load were derived based on these assumptions. Subsequently, following Greenwood and Williamson's pioneering work, numerous researchers have furthered the study of asperity contact mechanisms [4,5], expanding the models from elastic to elastic–plastic contact situations [6,7] and from hemi-spherically tipped to non-hemi-spherically tipped asperities [8–13], as well as from Gaussian height distributions to non-Gaussian distributions [14–17].

Despite these developments, most existing asperity contact models primarily focus on the relationship between normal contact load and normal interference at asperity summits, while tangential contact behaviors—such as tangential contact stiffness and tangential contact damping—are often neglected [18]. In practical engineering applications, however, asperity contacts rarely occur strictly at summit-to-summit locations. Instead, contact commonly takes place on the shoulders of asperities, leading to oblique contact forces and coupled normal–tangential interactions. To account for such shoulder-to-shoulder contact phenomena, Sepehri et al. [19–21] investigated 2-D elastic shoulder–shoulder contact problems and integrated their findings into various models, addressing challenges from elastic–plastic interference of surfaces [22] to contact damping calculations. Attempts were also made to extend the 2-D formulation to 3-D contact problems. However, in Sepehri's model, the equivalent combined contact curvature is determined solely from the curvature of a planar curve, which cannot accurately represent the combined curvature characteristics of 3-D surfaces. As a result, the Sepehri model is inherently limited when applied to genuine 3-D shoulder-to-shoulder surface contact scenarios. In 3-D surface contact, unlike the curvature of a planar curve, surface curvature varies with direction at a given point and is characterized by two principal curvatures. Accurately capturing the combined curvature of contacting asperities therefore requires explicit consideration of these principal curvatures. Consequently, to establish a physically consistent and mathematically rigorous shoulder-to-shoulder contact model applicable to 3-D rough surfaces, it is necessary to reformulate the contact problem by incorporating surface principal curvatures into the contact framework.

Motivated by these considerations, this study proposes a 3-D elastic shoulder-to-shoulder contact model based on paraboloidal asperities. The model extends Sepehri's 2-D formulation by introducing the principal curvatures of contacting surfaces and employing Hertzian elastic contact theory to derive analytical relationships among contact load, contact interference, contact area, and contact pressure. Finite element analysis (FEA) is subsequently employed to validate the proposed model through systematic comparisons with both Sepehri's model and numerical results, thereby demonstrating its improved accuracy and applicability for three-dimensional asperity contact analysis.

2. Theoretical Model

2.1. 2-D Contact Model

In the 2-D contact model, the interaction between two nominally flat rough surfaces primarily arises from the contact between individual asperities present on the mating surfaces, as illustrated in Figure 1.

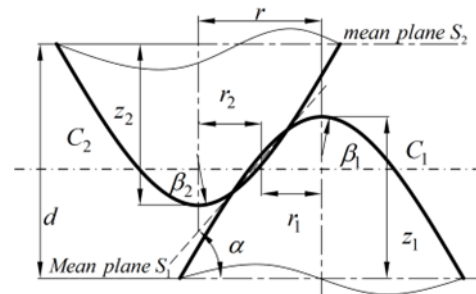


Figure 1. 2-D parabolic asperity contact model.

An asperity denoted as C_1 on surface S_1 interacts with another asperity labeled as C_2 on surface S_2 . The Hertz contact force, denoted as f_a , acts along the common normal direction of the two surfaces at the center of the contact area and is not orthogonal to the mean planes of the two surfaces. The components of f_a along the normal and tangential directions of surface S_1 are denoted as f_N and f_t , respectively. The angle between f_a and f_N is denoted as α .

Sepehri proposed the application of Hertz's hemisphere contact theory to address 2-D asperity shoulder–shoulder contact issues. The relationship between the normal contact force f_a and the normal interference w between these asperities, depicted in Figure 2, can be mathematically expressed as follows:

$$f_a = \frac{4}{3} E' \beta(r)^{1/2} w^{3/2} \quad (1)$$

According to Hertz's hemisphere contact theory, the contact area and maximum contact pressure are as follows:

$$\begin{cases} s = \pi \left(\frac{3\pi E' f_a \beta(r)}{4} \right)^{2/3} \\ p = \frac{3f_a}{2s} \end{cases} \quad (2)$$

where E' is the combined Young's modulus for the two mating surfaces. $\beta(r)$ is the combined radius of curvature at the contact point when the relative offset of two asperities is r along the horizontal direction. $\beta(r)$ is derived from a parabolic equation. w is the contact interference along the normal direction of the contact region.

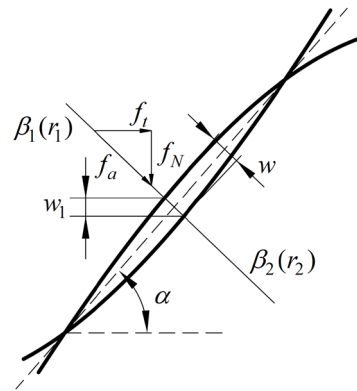


Figure 2. Overlap region showing normal and oblique interferences.

In the context of the 2-D asperity shoulder–shoulder contact depicted in Figure 1, locating the contact point of two asperities necessitates identifying the midpoint within the contact region. Here, r_1 , and r_2 denote the distances along the x -direction from the peaks of the respective asperities to the midpoint of the intersecting area. β_p represents the equivalent combined contact radius of curvature of the asperities' summits, while β_s stands for the sum of the radii of curvature of the two mating asperities. Mathematically, r_1 , r_2 , β_s , and β_p can be expressed as follows:

$$r_1 = \frac{\beta_1}{\beta_s} r; r_2 = \frac{\beta_2}{\beta_s} r \quad (3)$$

$$\beta_s = \beta_1 + \beta_2; \beta_p = \frac{\beta_1 \beta_2}{\beta_1 + \beta_2} \quad (4)$$

where r is the relative distance between the two asperities' summits and $\beta(r)$ can be given by

$$\beta(r) = \beta_p \left(1 + \frac{r^2}{\beta_s^2} \right)^{3/2} \quad (5)$$

The 2-D contact model establishes the equivalent combined contact curvature $\beta(r)$ through a 2-D parabolic contact derivation. Nevertheless, practical scenarios involve three-dimensional surface contacts between mechanical surfaces. Unlike the situation where the curvature of a two-dimensional curve holds a singular value at a given point, surface curvature presents numerous values in various directions at any given point. For instance, at the contact point of asperity C_1 depicted in Figure 1, the radius of curvature ($\beta_1(r_1)$) in the plane is

$$\beta(r_1) = \beta_1 \left(1 + \frac{r_1^2}{\beta_1^2} \right)^{3/2} \quad (6)$$

However, if the asperity were paraboloidal, as depicted in another sectional plane (indicated by the dash and dot line passing through the contact point in Figure 1), parallel to the mean plane, the radius of curvature of this section is denoted as r_1 . This value significantly deviates from the value derived from Equation (6) presented earlier. Hence, replacing the radius of surface curvature solely with the radius of curvature of one sectional curve is inadequate. Establishing a shoulder-to-shoulder contact model that accommodates three-dimensional scenarios is imperative.

2.2. 3-D Paraboloid Contact Model

2.2.1. Model Assumptions

This paper introduces a 3-D elastic contact model for paraboloid asperities, which accounts for shoulder-to-shoulder contact. The model is established on the following assumptions:

- Asperities on mechanical surfaces could be described as paraboloid.
- Contact between asperities occurs not only at their summits but also extends to their shoulders. This contact is characterized as surface contact rather than the curve contact delineated in Sepehri's model.
- Asperities possess isotropic material properties. To simplify this model, plastic deformation and friction are omitted.
- The contact mechanism adheres to Hertz's elastic contact theory.

2.2.2. Geometrical Model

In this study, we assume that the asperities present on the mechanical joint surfaces exhibit a paraboloid shape, as illustrated in Figure 3. We establish the origin of the coordinate system at the bottom center of each asperity, with the contact point situated on the xoz plane. The corresponding paraboloid describing the mating surfaces can be mathematically defined by the following equation:

$$C = -\frac{1}{2\beta}x^2 - \frac{1}{2\beta}y^2 + z \quad (7)$$

where z denotes the height of the paraboloid summit concerning the mean plane depicted in Figure 1 and β signifies the radius of curvature at the paraboloid summit. C denotes the formula representing the surface of the paraboloid.

The curvatures of surfaces at any given point are defined as the curvatures of curves that lie on the surface and pass through that point. The principal curvatures of a surface refer to the maximum and minimum curvatures of the curves resulting from the intersection of the surface with its normal planes in the vicinity of the point of interest, as depicted in Figure 3. These curvature values are vital for characterizing the shape of the surface, particularly in understanding the normal space of the contact region when the normal contact interference, denoted as w , is zero.

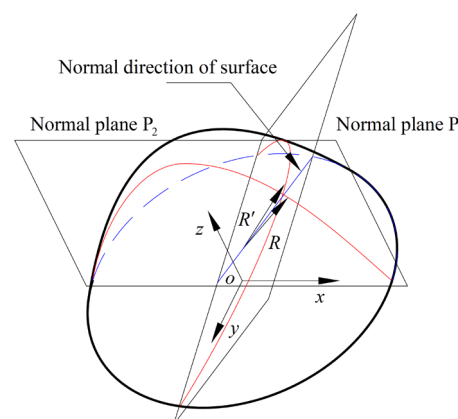


Figure 3. Geometric model of paraboloid asperity.

The paraboloid model, presented herein, delineates the principle curvatures through two distinct curves. The first curve, denoted as τ_1 , corresponds to the intersection of the xoz plane (depicted as P_1 in Figure 3) and the asperity surface. The second curve, designated as τ_2 , represents the intersection of the normal plane (illustrated as P_2 in Figure 3) at

the contact point. This plane is orthogonal to both the preceding plane, P_1 , and the asperity surface in proximity to the contact point.

The first principal curvature, denoted as $1/R$, represents the curvature of curve τ_1 at its point of contact. The sectional perspective of P_1 corresponds to that illustrated in Figure 1. Thus, the equation for curvature is provided below, with its directional vector being $(0, 1, 0)$.

$$\frac{1}{R} = \beta^{-1} \left(1 + \frac{r_c^2}{\beta^2} \right)^{-3/2} \quad (8)$$

where r_c is the relative distance between the contact point and the asperity summit in the x direction.

The existence of the second principal curvature $1/R'$ is evident in curve τ_2 , as depicted in Figure 3. The parametric equation governing curve τ_2 is as follows:

$$x = \sqrt{G - t^2} - \frac{\beta^2}{r_c}; y = t; z = \frac{1}{2\beta} t^2 + \frac{1}{2\beta} \left(\sqrt{G - t^2} - \frac{\beta^2}{r_c} \right) \quad (9)$$

where t represents the independent variable in this curve and G denotes a constant value, determined when the summit curvature β and relative distance r_c are established. This constant is expressed as

$$G = \frac{\beta^4}{r_c^2} + 2\beta^2 + r_c^2 \quad (10)$$

At the contact point, the independent variable $t = 0$. For this situation, the final curvature of curve τ_2 at the contact point is

$$\frac{1}{R'} = (r_c^2 + \beta^2)^{-1/2} \quad (11)$$

The combined curvature radius of two contacting paraboloid surfaces at the contact point can be defined as follows [6]:

$$\frac{1}{R_m} = \frac{1}{R_1} + \frac{1}{R'_1} + \frac{1}{R_2} + \frac{1}{R'_2} \quad (12)$$

By substituting Equations (8) and (11) into Equation (12), the equation representing the combined curvature radius of two contacting surfaces can be written as

$$R_m = \beta_d \left(1 + r^2 / \beta_s^2 \right)^{3/2} / \left(2 + r^2 / \beta_s^2 \right) \quad (13)$$

where β_d is the equivalent combined contact radius of the asperity summits' curvature.

By employing an appropriate coordinate transformation, as depicted in Figure 4, wherein it is assumed that the tangential plane at the point of contact between the two surfaces forms the xoy plane, and the contact point prior to deformation under load serves as the origin of the coordinate system, a novel Cartesian coordinate system can be established. This transformation enables the conversion of any surface contact issue into a normal contact problem. At this juncture, the two contacting surfaces are envisioned as smooth and continuous entities, amenable to representation through the Taylor series expansion of a binary function. Given the negligible impact of higher-order terms in Taylor's formula on the computation of contact stress and displacement, and the computational burden incurred by considering high-order terms in contact deformation calculations, it is judicious to omit higher-order terms in the ultimate fitting formula, retaining solely those within the quadratic term. Thus, when the interference w equals zero, the contacting

surfaces in close proximity to the contact point can be expressed by the subsequent equations [18]:

$$C_1 = a_1 \bar{x}^2 + b_1 \bar{y}^2 + c_1 \bar{x}\bar{y} \quad (14)$$

$$C_2 = a_2 \bar{x}^2 + b_2 \bar{y}^2 + c_2 \bar{x}\bar{y} \quad (15)$$

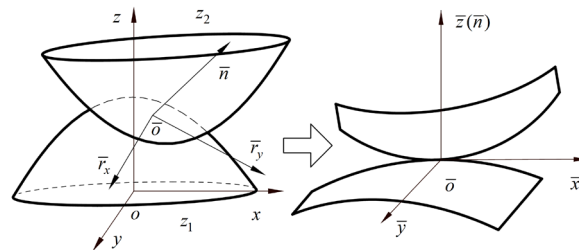


Figure 4. Transformation of coordinates of the contact surfaces.

Following a coordinate transformation, the normal contact space, wherein normal interference ($w = 0$), is denoted as

$$d_n = A\bar{x}^2 + B\bar{y}^2 \quad (16)$$

Equations (14)–(16) represent elliptic paraboloid surface contact equations. As per Hertz contact theory [18], the relationship between surface principal curvatures and parameters A and B in Equation (16) can be expressed as

$$\begin{cases} A + B = \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R'_1} + \frac{1}{R_2} + \frac{1}{R'_2} \right) \\ B - A = \frac{1}{2} \left(\left(\frac{1}{R_1} - \frac{1}{R'_1} \right)^2 + \left(\frac{1}{R_2} - \frac{1}{R'_2} \right)^2 + 2 \left(\frac{1}{R_1} - \frac{1}{R'_1} \right) \left(\frac{1}{R_2} - \frac{1}{R'_2} \right) \cos 2\omega \right) \end{cases} \quad (17)$$

where R_1 , R_2 , R'_1 , and R'_2 denote the principal radii of curvatures at the contact point. The angle ω represents the deviation between the two principal radii of curvature, R_1 and R_2 . Since the principal curvatures R_1 and R_2 share the same direction $(0, 1, 0)$, in this parabolic model, $\omega = 0$. Utilizing Equations (16) and (17), the normal contact space can be determined. Ultimately, the contact issue between paraboloids at the shoulder–shoulder interface is transformed into an elliptic contact problem resembling the Hertz contact problem in proximity to the contact area.

2.2.3. Equations of Contact Load and Contact Interference

According to Hertz's contact theory, arbitrary surface contact problems may be approximated by elliptical contact problems. The relationships between the semi-axis, contact area, and contact load of elliptical contact areas are as follows:

$$a = \alpha \left(\frac{3f_a R_m}{2E} \right)^{1/3}; b = \beta \left(\frac{3f_a R_m}{2E} \right)^{1/3} \quad (18)$$

The contact area and maximum contact pressure are as follows:

$$s = \pi ab; p = \frac{3f_a}{2s} \quad (19)$$

The relationship between normal contact interference w and normal contact load f_a at the contact region can be given as

$$w = \gamma \left(\frac{9f_a^2}{32E^2R_m} \right)^{1/3} \quad (20)$$

where γ is the coefficient related to elliptic integrals [19].

In addition, the relationship between normal contact interference w and surface approach d between the planes of surfaces can be expressed by [12]

$$w = \left(z_1 + z_2 - d - \frac{r^2}{2\beta_s} \right) \left(1 + \frac{r^2}{\beta_s^2} \right)^{-1/2} \quad (21)$$

Finally, by integrating Equations (20) and (21), we derive the relationship between the normal contact force f_a and the contact interference w as follows:

$$f_a = \frac{4\sqrt{2}}{3} E \beta_d^{1/2} \left(z_1 + z_2 - d - \frac{r^2}{2\beta_s} \right)^{3/2} \left(2 + \frac{r^2}{\beta_s^2} \right)^{-1/2} \gamma^{-3} \quad (22)$$

The normal contact force f_a is aligned with the normal direction of the contact area. This force can be decomposed into its normal and tangential components along the average plane, as illustrated in Figure 2. In this scenario, the relationship between the surface approach d and the normal/tangential force can be expressed as follows:

$$f_N = \frac{4\sqrt{2}}{3} E R_m^{1/2} \left(z - d - \frac{r^2}{2\beta_s^2} \right)^{3/2} \left(1 + \frac{r^2}{\beta_s^2} \right)^{-5/4} \gamma^{-3} \quad (23)$$

$$f_t = \frac{4\sqrt{2}}{3} E R_m^{1/2} \left(z - d - \frac{r^2}{2\beta_s^2} \right)^{3/2} \left(1 + \frac{r^2}{\beta_s^2} \right)^{-5/4} \frac{r}{\beta_s} \gamma^{-3} \quad (24)$$

where z denotes the total summit height resulting from the sum of two individual asperities, z and z_2 . f_N and f_t are the normal and tangential components of the contact force between the asperities, respectively.

3. FE Model

In order to assess the efficacy of the model proposed herein, a comparative analysis is conducted between the present model, Sepehri's model, and the finite element model. FEM has been extensively employed in previous studies on micro-asperity contact modeling to capture three-dimensional contact behavior, stress distributions, and deformation characteristics of asperities with various geometries and is therefore considered a well-established and reliable benchmark method [22–27]. Here, finite element analysis (FEA) is employed as a numerical verification tool, providing an independent solution to the same governing equations under linear elastic assumptions. Key parameters including normal contact interference, contact area, and maximum contact pressure are analyzed carefully across the various models.

The FE model used to simulate the shoulder-to-shoulder contact of two asperities was developed using the general-purpose finite element analysis software ANSYS 18.2. Each asperity (1 and 2) is represented as a rotating paraboloid with a peak top curvature radius (β) of 2.5 mm and a peak height (h) of 5 mm. The appropriateness of the asperities' size selection is affirmed by prior literature [20].

Figure 5 shows the shoulder-to-shoulder FE contact model. The finite element simulations solve the three-dimensional linear elasticity equations under small deformation assumptions. A standard low-order finite element method based on eight-node

hexahedral elements is employed. The element length is set to 0.4 mm in the non-contact area. In the contact area, after local refinement, the smallest contact element size is approximately 0.05 mm. The FE model includes 296,076 elements and 424,896 nodes. The mesh is refined based on a convergence criterion of the normal load change of less than 1%. Contact and target elements are used to model the contact surfaces between the asperities. The friction coefficient on the contact surfaces is taken as zero. The x and y displacement of the upper surface at asperity 1 is constrained, while full constraints are imposed on the bottom surface of asperity 2.

In this study, asperities possess isotropic material properties, and plastic deformation and friction are omitted. Their Young's modulus and Poisson's ratio are 210 GPa and 0.3, respectively. Within the shoulder-to-shoulder contact analysis, the offset (r) of the central axis of the two asperities varies from 0 to 4 mm in increments of 0.5 mm.

The contact status was defined as the initial state when the normal interference w was zero. Subsequently, the upper asperity was depressed by 0.2 mm based on the initial state to assess contact interference, contact area, and maximum contact pressure. Figure 6 shows the contact pressure of two asperities when the offset (r) is equal to 2 mm. It can be seen that the maximum contact pressure is 26,009 MPa, and the maximum contact pressure area mainly appears in the center of the contact surface and presents an elliptical shape.

While alternative numerical techniques, such as the Boundary Element Method (BEM), are known to offer improved computational efficiency for surface-dominated contact problems by eliminating volumetric meshing [28–32], FEM is retained in this study to maintain consistency with existing micro-asperity contact literature and to facilitate direct comparison with previously reported results. The application of BEM to the present contact configuration is considered a promising topic for future investigation.

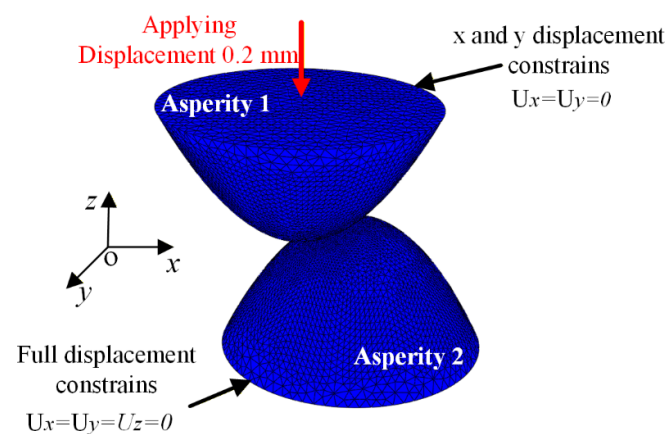


Figure 5. FE model of paraboloid asperity contact.

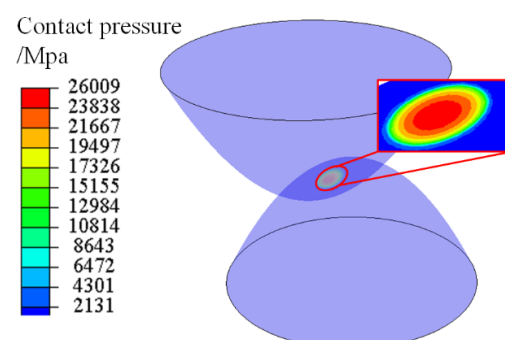


Figure 6. Contact pressure of the two asperities under $r = 2$ mm.

As shown in Figure 7, the comparison of contact areas reveals a minor discrepancy. In the case of peak-to-peak contact, where the relative offset r between the centers of the mating asperities is zero, the calculated error is 5.74%. The predicted results align for both the present model and Sepehr's model under these conditions. However, as the offset (r) in the x -direction increases, Sepehr's model demonstrates a more rapid increase in the contact area compared to both the FE model and the model proposed in this study. Specifically, when $r = 4$ mm, the error between the FE model and Sepehr's model rises to 31.21%. It is noteworthy that the contact area of the proposed model shows good agreement with the FE model, with the percentage error decreasing from 5.74% to 2.11% as the offset r varies from 0 to 4 mm.

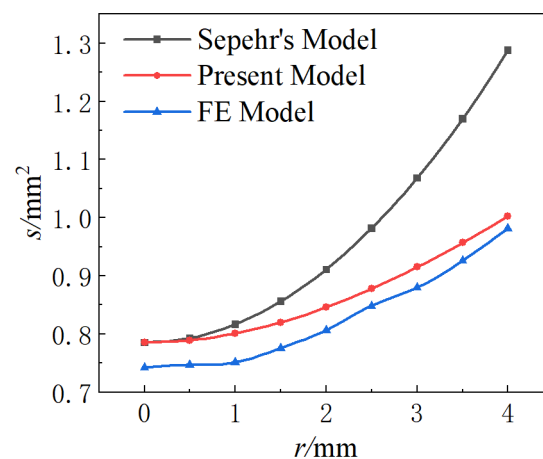


Figure 7. Contact area comparison.

The maximum contact pressure of various models is illustrated in Figure 8. The model proposed in this paper closely aligns with the finite element analysis results. The maximum percentage error, occurring at the peak contact scenario, is 5.96%. However, Sepehr's model exhibits a significant deviation from the finite element results, with a maximum error of 9.15%.

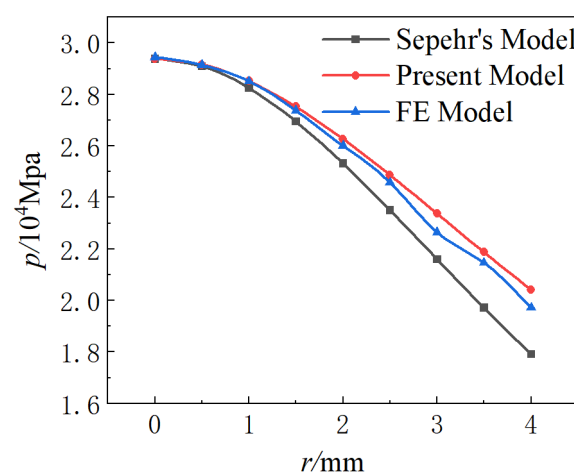


Figure 8. Contact stress comparison.

Contact interference among different models is depicted in Figure 9. Since the calculation of contact displacement in this model and Sepehr's model employs the same method, the contact displacement curves of the two models overlap. Compared with the

finite element calculation results, the maximum error is 2.77%, and the minimum error is 1.56%.

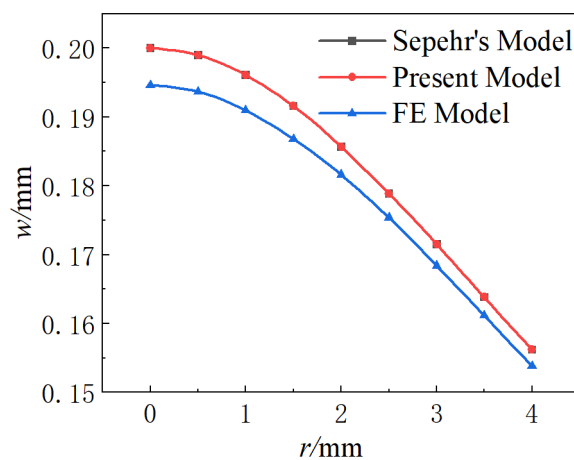


Figure 9. Contact interference comparison.

Based on the comparison results of the present model, Sepehri's model, and FE model, it can be observed that the errors of the model concerning contact interference, contact area, and maximum contact pressure are all smaller than those of Sepehri's model, demonstrating the predictive accuracy of the proposed model under controlled numerical conditions for three-dimensional side contact situations.

4. Discussion

Based on the evaluation of the model's efficacy, this study further analyzes the effects of different relative offsets r (ranging from 0 to 4 mm, with an increment of 0.5 mm each time) and the pressing amount Δd (starting from 0.025 mm, with an increment of 0.025 mm each time, up to 0.2 mm) on contact interference, contact area, and maximum contact pressure.

Figure 10 illustrates the variation in the contact area as functions of the offset r and the pressing amount Δd . It is observed that the contact area increases monotonically with both parameters. This behavior can be attributed to the combined effects of enhanced geometric overlap between the two asperities and increased normal compression. Moreover, the growth rate of the contact area becomes more pronounced at larger values of r and Δd , indicating a nonlinear amplification of the contact region under combined lateral offset and normal loading.

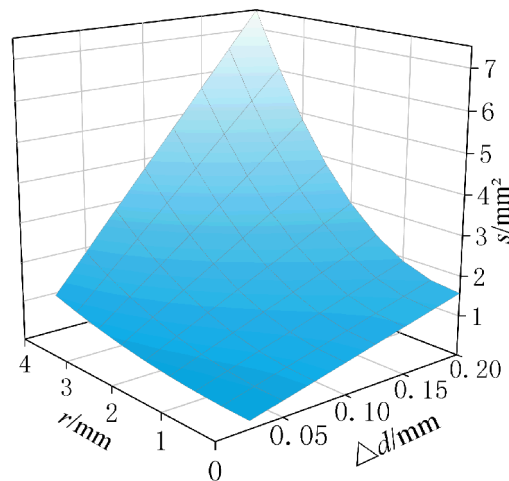


Figure 10. The variation in contact area with the offset r and the pressing amount Δd .

The variation in the maximum contact pressure with respect to r and Δd is presented in Figure 11. As expected, the maximum contact pressure increases with increasing pressing amount Δd . In contrast, its dependence on the offset r exhibits a saturation-like behavior. Specifically, at larger offsets, the rate of increase in the maximum contact pressure gradually decreases. This trend is consistent with the corresponding evolution of the contact displacement, suggesting that the stress concentration at the contact interface becomes less sensitive to further lateral offset once a certain geometric configuration is reached.

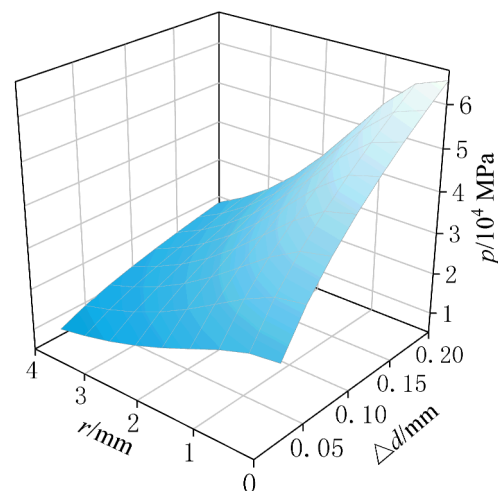


Figure 11. The variation in maximum contact pressure with the offset r and the pressing amount Δd .

Figure 12 shows the variation in contact displacement as functions of r and Δd . The results indicate that the contact displacement increases with the pressing amount Δd , reflecting the dominant role of normal loading in driving deformation. However, for larger offsets r , the increase in contact displacement becomes progressively less significant. This behavior implies that the effective compliance of the contact system decreases as the asperities become laterally misaligned, thereby limiting additional deformation under further normal loading.

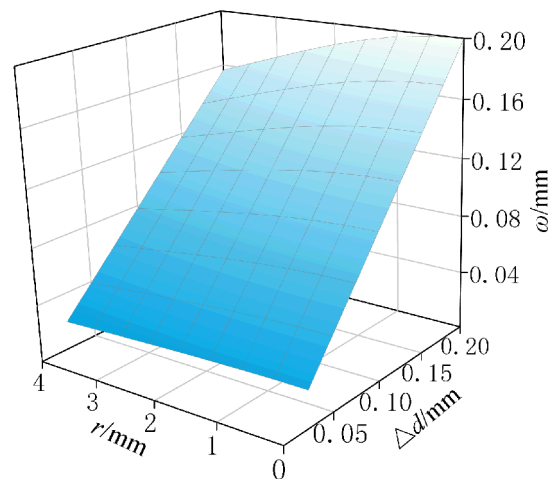


Figure 12. The variation in contact displacement with the offset r and the pressing amount Δd .

In summary, the coupled influences of the relative offset r and the pressing amount Δd on the contact area, maximum contact pressure, and contact displacement reveal the inherently nonlinear and three-dimensional nature of lateral asperity contact. The trends observed in Figures 10–12 demonstrate that the proposed model is capable of capturing the complex geometric and mechanical interactions governing three-dimensional contact behavior under combined normal and lateral effects, thereby confirming its applicability over a wide range of contact conditions.

5. Conclusions

To expand upon Sepehri's 2-D contact model within a 3-D context, a shoulder–shoulder contact model employing paraboloid asperities was introduced. Principal curvatures of the surfaces were determined using curvature computation formulas. The normal contact space within the contact area was derived utilizing the principal curvature of the surfaces, and the contact problem was approximated through Hertz's elliptic contact theory in the proximity of the contact region.

Hertz's elliptic elastic contact theory was employed to ascertain the normal interference between two contacting asperities, yielding equations for contact force, contact area, and maximum contact pressure. Ultimately, the model proposed in this paper extends Sepehri's model to encompass a 3-D shoulder–shoulder contact scenario.

Furthermore, a parabolic finite element contact model was utilized for numerical verification of the proposed model. The results demonstrate that the proposed model aligns more closely with finite element outcomes compared to Sepehri's model. The maximum errors in contact area, contact pressure, and contact interference are 5.74%, 5.96%, and 2.77%, respectively, supporting the accuracy and reliability of the proposed model under controlled numerical conditions.

Finally, the variations in contact area, maximum contact pressure, and contact displacement under different offsets r and compression amounts Δd were analyzed. These research results provide strong support for further optimization of the asperity contact model and offer important references for analyzing contact issues in practical engineering applications.

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