

Variational Mechanics for Mining Infrastructure Design: A Systematic Review from Hamilton's Principle to Physics-Constrained Optimization and Digital Twins

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Abstract

This article presents a systematic synthesis of variationally grounded approaches for the design and optimization of mining structural infrastructure. This study is motivated by the critical need to ensure stiffness, reliability, and operational availability under severe loading, mass constraints, and aggressive environmental conditions. Methodologically, the study situates structural modeling and synthesis within the continuity of the principle of stationary action. It demonstrates that, in the quasi-static regime, structural equilibrium is obtained as the stationarity of the total potential energy; consequently, the finite element method (FEM) arises naturally as a Ritz–Galerkin approximation of this underlying variational statement. On this basis, topology optimization is interpreted as a physics-constrained optimization problem wherein the design is posed as an outer optimality level acting over an energetically defined state. It is worth noting that SIMP-based formulations require explicit regularization to define the effective problem being solved. Emphasis is placed on the traceability between physical assumptions, discretization choices, regularization, and the resulting structural interpretations. The core contribution of this paper is a systematic literature review that consolidates evidence across variational mechanics, FEM-based optimization, and industrial applications, identifying recurrent methodological patterns and gaps that currently limit transfer to mining practice. Furthermore, a fully specified illustrative case is included to demonstrate reporting discipline and methodological consistency, rather than as a validation of a new optimization method. The conclusions highlight that a variational reading provides a coherent theoretical backbone for structural analysis, synthesis, simulation, and physics-based digital twins, while also clarifying the extensions required for industrial deployment, such as stability constraints, manufacturability, and multiphysics coupling within Mining 4.0 workflows.



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1. Introduction

Mining-related structural infrastructure, including conveyor galleries and bridges, transfer gantries, equipment support frames, walkways, and confinement and maintenance systems, is operated under demanding conditions characterized by long spans and slender members; gravity-dominated loading combined with dynamic actions such as wind and

seismic excitation; aggressive environments including corrosion, dust, and thermal gradients; and a high-availability operational requirement in which stoppages are costly and on-site interventions are logistically constrained. In this setting, structural performance cannot be reduced to stress compliance alone. Global and local stability, serviceability control, robustness under operational uncertainty, and material efficiency as a strategic variable affecting capital expenditure and embodied impacts are required to be integrated within the design rationale. For these reasons, structural optimization is positioned as a methodological component of mining asset design rather than as an optional refinement [1].

Finite Element Analysis (FEM) is widely established as the technical standard for assessing global stiffness, stress states, and structural response in systems characterized by complex geometry and nontrivial boundary conditions. However, in conventional engineering workflows, FEM is predominantly applied as a verification tool for a geometry selected a priori based on experience-driven typologies. While this approach is effective for code compliance, such separation introduces a fundamental methodological limitation: topology is determined outside the physical–mathematical model and is only subsequently evaluated numerically. Consequently, in large-scale structures, this dissociation between form selection and analysis has been associated with conservative material usage and sub-optimal load-path distributions under complex load combinations [2–4].

Topology optimization provides a qualitatively different alternative by determining the distribution of material that optimizes performance under explicit constraints, rather than assuming connectivity and adjusting only member sizes. In its modern form, topology optimization is treated as an optimization problem constrained by governing equations, typically posed as PDE- or FEM-constrained optimization. The SIMP approach parameterizes design through a continuous pseudo-density field and introduces penalization to promote near-binary solutions. Its widespread adoption has been driven by computational efficiency and reproducibility. However, the underlying continuous problem is, in general, ill-posed unless regularization is introduced through filtering, projection, or length-scale control, as otherwise mesh dependence and spurious patterns may be produced. Recent work has further refined the mathematical understanding and numerical analysis of SIMP-type models, reinforcing the requirement that the effective model that is actually solved must be reported explicitly [5–8].

Despite the maturity of the computational framework, systematic adoption in mining infrastructure remains limited and fragmented across disciplines. Constraints specific to mining practices, including constructability requirements such as minimum thicknesses, connectivity, radii, and availability of profiles, inspection and maintenance accessibility, and governing mechanisms for which static stiffness alone is insufficient, such as buckling, vibration, cyclic loading, and degradation, introduce additional complexity. These conditions motivate the need for multi-case and multi-constraint formulations and for parameterizations that facilitate crisp boundaries and downstream transfer to CAD and detailing. Level-set methods have been used as a natural bridge from synthesis to constructible geometries by incorporating geometric control and reducing ambiguity associated with diffuse densities, while recent developments have emphasized explicit constructability constraints in truss and frame contexts [9–17].

In this work, a systematic literature review is conducted to consolidate and assess the evidence on variationally grounded computational approaches for structural modeling and synthesis relevant to industrial applications, with emphasis on mining infrastructure. The review is intended to produce a structured map of the literature, identify recurrent methodological patterns, and expose gaps that currently limit the transfer to engineering practice. The review is positioned as the central contribution of the manuscript and is

used to derive recommendations focused on traceability between physical formulation, numerical discretization, and the effective design model reported in each study.

In addition to the review, an illustrative case study is presented as an initial, minimal demonstration intended to operationalize one recommendation derived from the review and discussion. The illustrative case is not introduced as a validation of a novel optimization method. Instead, it is used to demonstrate reproducibility and reporting discipline by making explicit the effective model components that materially affect outcomes, including penalization, filtering, projection, and associated parameters, and by clarifying how such choices influence interpretation and transferability.

A variational viewpoint is adopted to provide a coherent theoretical backbone connecting mechanics, FEM, and synthesis. In the quasi-static regime, structural mechanics are derived from the stationarity of the total potential. FEM is obtained as a Ritz or Galerkin approximation of that variational statement. Synthesis is then introduced as an outer optimality level acting over the design space. Under this perspective, an optimized topology is interpreted as a discrete and regularized approximation of the solution of an underlying continuous variational problem, enabling the separation of physical content, including equilibrium, constitutive behavior, and stability, from numerical content, including discretization, material interpolation, and regularization [18–20].

This variational backbone can be read as a direct inheritance of the classical principles of analytical mechanics. In its most general form, Hamilton’s principle states that admissible motions are stationary points of an action functional, $\delta S = 0$, and that the governing field equations arise as the associated Euler–Lagrange optimality conditions posed over an infinite-dimensional space [18,21,22]. In the quasi-static setting adopted here, the action-based statement reduces to stationarity of the total potential energy, $\delta \Pi(\mathbf{u}; \rho) = 0$, which is equivalent to the weak form of equilibrium [2,19,23]. Accordingly, what is commonly introduced in topology optimization as a “physics constraint” is precisely this variational stationarity condition, i.e., the definition of the physically admissible state for each design. A Ritz–Galerkin restriction of the admissible space then yields the discrete state operator used in FEM, $R(\mathbf{u}, \rho) = \mathbf{K}(\tilde{\rho})\mathbf{u} - \mathbf{f} = \mathbf{0}$ [2,23]. Synthesis is subsequently posed as an outer optimization over the design variables ρ , acting on this inner variational state problem, so that the optimized topology is obtained by minimizing a design objective subject to $\mathbf{u}$ remaining a stationary point of the underlying energy functional [1,6].

At the structural level, quasi-static equilibrium is expressed as a stationarity of the total potential,

$$\Pi(\mathbf{u}; \rho) = U(\mathbf{u}; \rho) - W(\mathbf{u}), \quad \text{find } \mathbf{u}(\rho) \in \mathcal{V}_{\text{ad}} \text{ such that } \delta \Pi(\mathbf{u}; \rho)[\delta \mathbf{v}] = 0 \quad \forall \delta \mathbf{v} \in \mathcal{V}_0, \quad (1)$$

where $\mathbf{u}$ is the displacement field, U is the internal energy dependent on the design ρ through an effective stiffness, and W is the potential of conservative external loads. Topology optimization introduces an outer level in which ρ is selected to optimize a performance functional, commonly compliance, subject to a volume constraint and the discrete state equation,

$$\min_{\rho \in \mathcal{U}_{\text{ad}}} C(\rho) = \mathbf{f}^T \mathbf{u}(\rho) \quad \text{subject to} \quad \mathbf{K}(\tilde{\rho}) \mathbf{u}(\rho) = \mathbf{f}, \quad \int_{\Omega} \rho \, d\Omega \leq V^*, \quad (2)$$

where $\tilde{\rho}$ denotes the regularized field after filtering and projection. Under linear elasticity, minimizing C is equivalent to maximizing global stiffness for a prescribed volume, which makes explicit the energetic logic of synthesis: material is redistributed toward regions that contribute most to strain energy while regions with marginal contribution are suppressed [1,6].

The aim of this work is to deliver a systematic synthesis of the literature and to provide a coherent variational framing for the interpretation and reporting of topology optimization workflows in mining infrastructure contexts. A conceptual synthesis is provided to clarify the transition from action and energy principles to quasi-static total potential, with emphasis on the distinction between minima and general critical points and on the role of stability in physical admissibility [18,24]. FEM is articulated as a direct consequence of the variational principle through Ritz and Galerkin approximations, emphasizing energetic consistency when the state is used as a design constraint [2,23]. Topology synthesis is formulated as constrained optimization, and regularization through filtering, projection, and length-scale control is treated as constitutive of the effective problem that is solved and therefore as necessary for reproducibility and for avoiding mesh dependence [7,8,25]. Transfer to mining practice is addressed through the articulation of industrial constraints, including buckling, frequency, and cyclic loading considerations, and through representations aimed at constructible boundaries and verifiable detail-level engineering [9,10,12].

The manuscript is organized to first establish the review scope and method and then to present a structured synthesis of findings, patterns, and gaps. The discussion is used to translate those findings into actionable recommendations for mining infrastructure design workflows, with particular emphasis on reporting of the effective model and on constraints required for physically meaningful and constructible solutions. The illustrative case study is then used to demonstrate how a recommendation can be applied in a reproducible manner, and final remarks are provided to summarize implications and directions for subsequent research and industrial transfer.

2. Methodology

This systematic literature review was conducted in accordance with the PRISMA 2020 guidelines (Preferred Reporting Items for Systematic Reviews and Meta-Analyses) to ensure transparency, completeness, and reproducibility of the identification, selection, and synthesis process. The primary aim was to identify, assess, and synthesize peer-reviewed scientific evidence on variational and Hamiltonian-based formulations applied to industrial and advanced engineering contexts, with particular relevance to structural modeling and synthesis workflows [26,27]. The review was designed to support a structured mapping of the literature and to derive methodological recommendations grounded in observed reporting practices and computational frameworks.

2.1. Search Strategy

A comprehensive and systematic search strategy was defined to identify relevant studies from high-impact scientific databases. Searches were conducted between 10 January and 15 January 2026, using Scopus, Web of Science, IEEE Xplore, and ScienceDirect. Search strings were formulated to capture contributions related to Hamiltonian and variational principles and their application in engineering and industrial domains, and were adapted to the syntax and indexing rules of each database. The primary search query employed was as follows:

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("Hamilton's Principle" OR "Hamiltonian Principle" OR "variational principle"
OR "Lagrangian mechanics") AND ("industrial application" OR "engineering"
OR "structural analysis" OR "robotics" OR "aerospace" OR "control systems"
OR "finite element method" OR "mechatronics")
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In addition to the automated database search, backward reference screening was performed on the most relevant articles to identify additional studies not captured by the initial queries. Selected conference proceedings and specialized venues were also examined when they were deemed influential within the scope of the review.

2.2. Eligibility Criteria

Eligibility criteria were defined a priori to ensure consistency with the objectives of the review. Studies were included if they consisted of original research articles, systematic reviews, or scholarly book chapters published in peer-reviewed journals ranked in the first or second quartile. An explicit application of Hamilton's principle or closely related variational formulations in an engineering or industrial context was required. Only studies with full-text availability in English or Spanish were considered.

Studies were excluded if they consisted of opinion pieces, editorials, letters, or conference abstracts without full-text availability. Contributions that were purely theoretical or mathematical, without a clear engineering application or computational implementation, were excluded. Articles whose primary focus was not aligned with the review question or that represented duplicate publications were also removed from consideration.

2.3. Study Selection Process

Study selection was performed in two sequential stages by two independent reviewers. In the first stage, titles and abstracts of all identified records were screened to assess potential relevance. In the second stage, full-text versions of the preselected studies were evaluated against the eligibility criteria to determine final inclusion. Disagreements at any stage were resolved through discussion and consensus, with arbitration by a third reviewer when necessary. The complete selection procedure is documented using a PRISMA 2020 flow diagram, which reports the number of records identified, screened, excluded, and included at each stage (Figure 1).

2.4. Data Extraction

A standardized data extraction form was developed to ensure consistent collection of relevant information across all included studies. Extracted data comprised author names, year of publication, journal outlet, type of contribution, industrial or engineering application domain, principal methodological approach, and reported implications for practice or implementation. Data extraction was performed by one reviewer and independently verified by a second reviewer to reduce transcription errors and ensure internal consistency [26,28].

2.5. Data Analysis and Synthesis

A descriptive statistical analysis was conducted to characterize the body of literature included in the review. Publication trends were analyzed by year of publication and by journal outlet to identify temporal evolution and concentration across venues (Figures 2 and 3). Evidence synthesis was performed using a narrative approach, with studies grouped according to their primary industrial application domain and methodological orientation. This structure enabled identification of recurring patterns, dominant modeling strategies, and underexplored areas. Key contributions and representative studies were summarized in tabular form to facilitate comparison and to support the identification of methodological gaps and reporting inconsistencies.

PRISMA 2020 Flow Diagram for Systematic Review

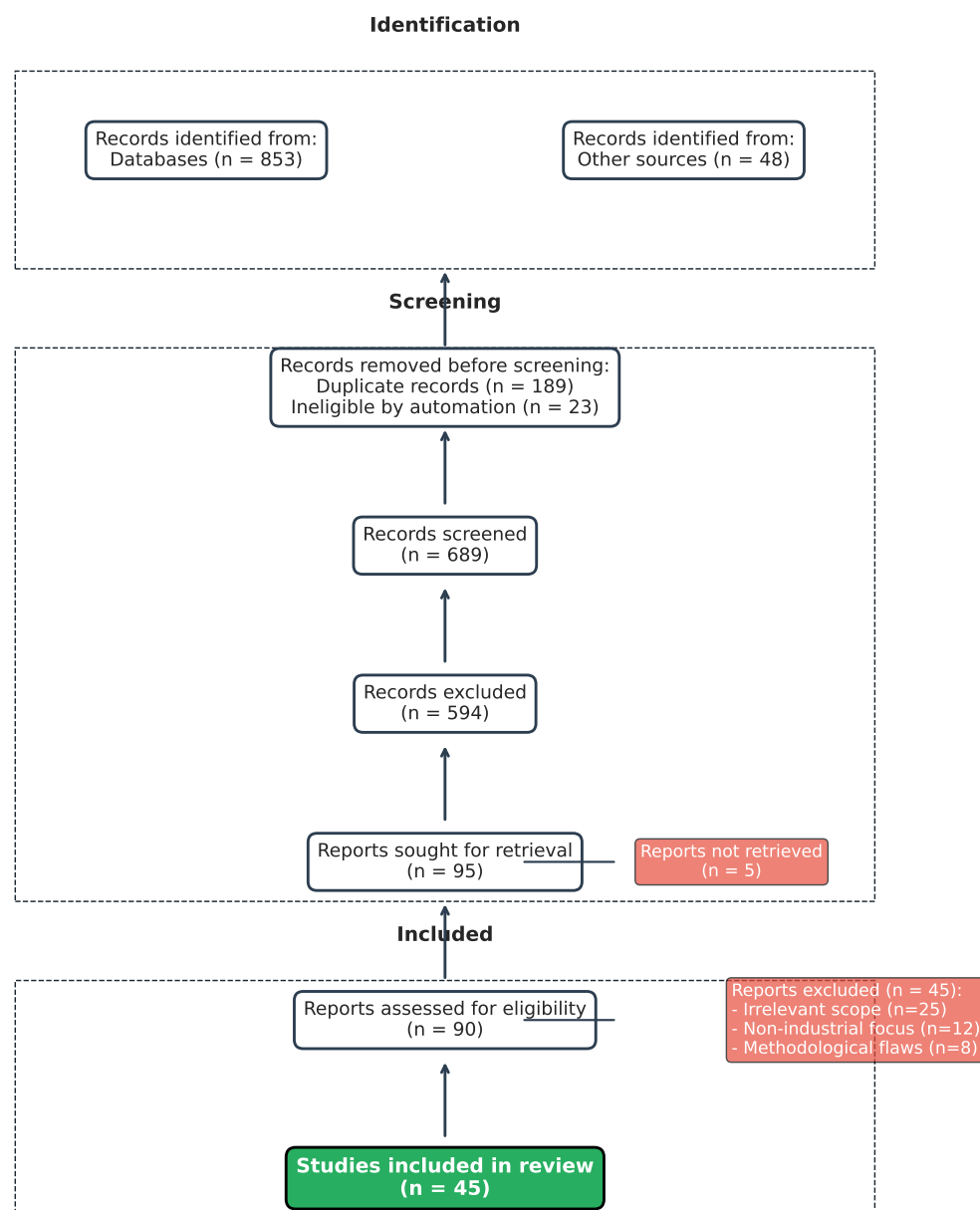


Figure 1. PRISMA 2020 flow diagram for study selection.

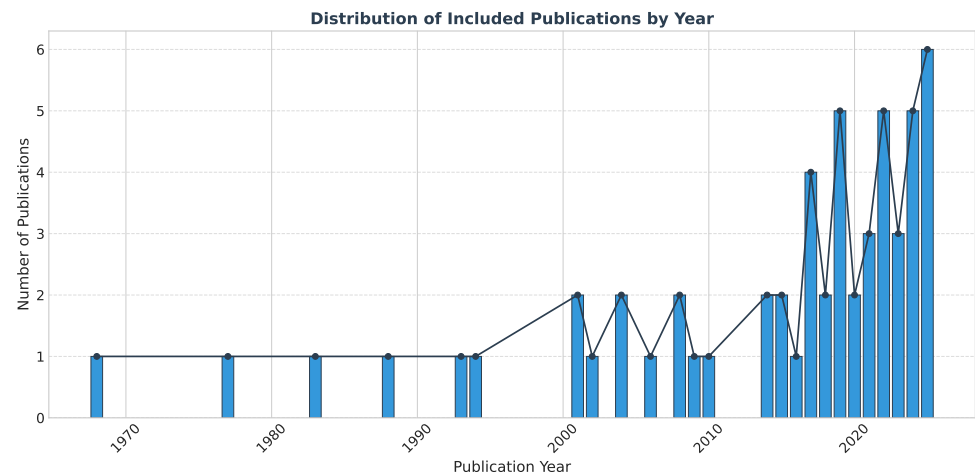


Figure 2. Distribution of included publications by year.

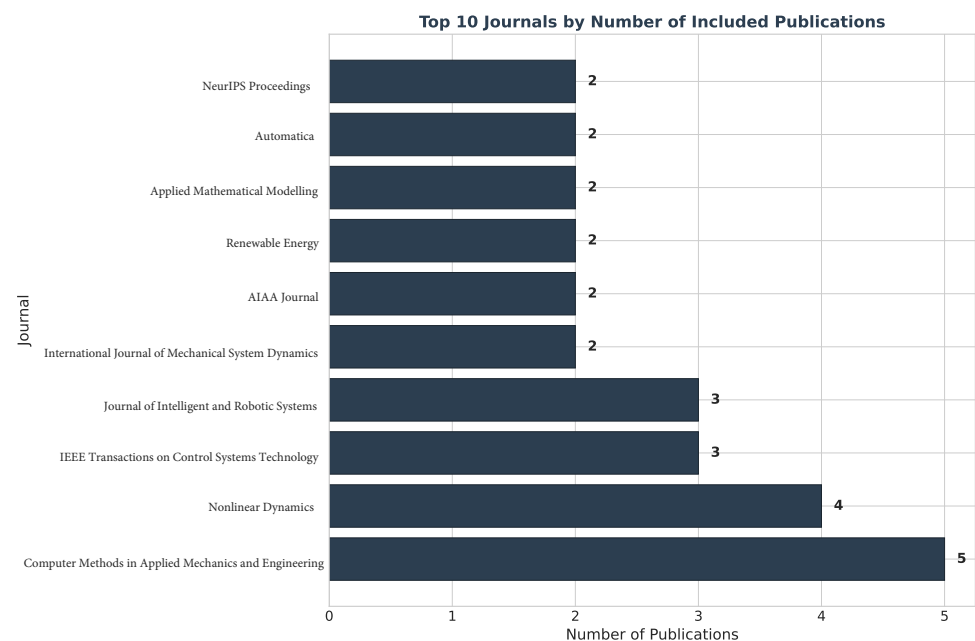


Figure 3. Top journals by number of included publications.

3. Historical and Philosophical Evolution of the Principle of Least Action

The Principle of Least Action is, in modern terms, a principle of stationary action: physical dynamics is characterized as a critical point of a functional defined on an infinite-dimensional space of admissible trajectories. For generalized coordinates $q_i(t)$ with fixed endpoints at t_1 and t_2 , the action is introduced as

$$S[q] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt, \quad \delta S = 0, \quad (3)$$

where $\delta q_i(t_1) = \delta q_i(t_2) = 0$ and L is the Lagrangian. From the viewpoint of the calculus of variations, $\delta S = 0$ indicates that the first variation vanishes for every admissible perturbation. The critical point may be a minimum, a maximum, or a saddle, and the qualifier “least” is historically inherited rather than mathematically required [18,20]. As a structural consequence, equations of motion are obtained as optimality conditions in function space through the Euler–Lagrange equations, which are developed in Section 4.

The historical relevance of the principle extends beyond mere chronology. It represents a methodological shift from mechanics formulated in terms of local forces toward a global

formulation based on functionals, invariances, and constraints. Ideally, a persistent philosophical tension has been associated with the principle, often expressed as local causality versus teleological readings. However, such tension is not inherent to the mathematical content of the principle but is rather introduced by metaphysical interpretations. In modern treatments, a strict distinction is maintained between (i) a local variational condition in trajectory space and (ii) finalistic narratives regarding a “choice” of paths [29,30]. A graphical synthesis of the variational chain $L = T - V \Rightarrow \delta S = 0 \Rightarrow$ Euler–Lagrange may be referenced, if included in the manuscript, in Section 3.10.

3.1. Precursors: From Heron’s Shortest Path to Fermat’s Least Time

Early extremal thinking appeared in geometric optics. In Heron of Alexandria, reflection in a plane mirror was interpreted as the shortest path under the imposed geometric constraint. The law of reflection was obtained by a symmetry argument that reduces the problem to a straight segment between a point and the mirror image of the other [20]. Although dynamics was not yet addressed, an essential feature was introduced: observable laws can be formulated as optimality conditions.

In Fermat’s principle, the criterion was shifted from length to a physically meaningful quantity, the travel time. The principle is expressed as stationarity of the functional

$$\delta \left(\int_A^B \frac{ds}{v(\mathbf{x})} \right) = 0, \quad (4)$$

or equivalently as $\delta \left(\int_A^B n(\mathbf{x}) ds \right) = 0$ with $n = c/v$ [18]. Refraction was thereby transformed from an empirical rule into a stationarity condition: a local law of Snell-type was obtained from a global variational statement. Teleological interpretations were historically influential, but are methodologically secondary. The central point is that the phenomenon is reduced to a well-posed variational problem [29].

3.2. The Brachistochrone Problem and the Emergence of the Calculus of Variations

The decisive bridge to mechanics was established by the brachistochrone problem (Johann Bernoulli, 1696), in which the curve that minimizes frictionless descent time under uniform gravity is sought [31]. By energy conservation, v is expressed as a function of height and the transit time is written as the functional

$$\mathcal{T}[y] = \int_{x_1}^{x_2} \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy(x)}} dx, \quad (5)$$

whose extremal is a cycloid. The significance of the result is not the curve itself, but the methodological shift: optimization is performed over a space of functions rather than over a finite set of parameters. Formalization of the first variation and of necessary conditions expressed as differential equations is thereby forced, marking the operational emergence of the calculus of variations [20].

A further structural feature was exploited through a deep analogy: the mechanical problem was interpreted optically as propagation in a medium with variable refractive index, so that a local Snell-type law replicates the condition of extremal time. This transversal connection (optics $\leftrightarrow$ mechanics) anticipates the later Hamilton–Jacobi unification and reinforces the structural role of variational reasoning [20].

3.3. The Definition of Action and Associated Metaphysical Controversies

In the eighteenth century, the idea of an “economy of nature” was made explicit by Maupertuis (1744), by whom the term action was introduced and one historical form was proposed as an integral proportional to $m v ds$ [32]. Beyond the particular historical form,

a durable contribution was produced by the change of object: dynamics was encapsulated in a scalar integral quantity susceptible to generalization, thereby opening a path to generating principles.

Associated controversies, including priority disputes, finalistic interpretations, and ambiguities of formulation, are historically revealing because the power of the principle is not dependent on teleology. What remains scientifically stable is the reduction in dynamics to a stationarity condition. What remains metaphysically disputable is the interpretation of stationarity as intrinsic purpose [20,30]. This distinction is required for the maturation of analytical mechanics.

3.4. Analytical Formalization: Euler, Lagrange, and Variational Power

Euler systematized the theory of extremals of functionals and established general differential conditions, thereby consolidating the technical apparatus of the calculus of variations (as illustrated in Figure 4). In *Mécanique Analytique* (1788), Lagrange produced the foundational turn by defining the Lagrangian $L = T - V$ and rewriting mechanics as a stationary-action problem [33,34]. The canonical form

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = 0 \quad (6)$$

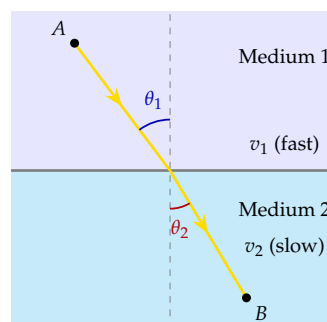
yields, systematically, the Euler–Lagrange equations (shown in Figure 5, if included in the text):

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0.$$

Methodologically, the focus is shifted from vector forces to an energetic description that is invariant under coordinate changes, which is particularly effective in the presence of constraints and many degrees of freedom [18]. In this way, the principle ceases to be heuristic and is established as a generating principle.

Fermat's Principle (Optics)

A path of stationary travel time is followed

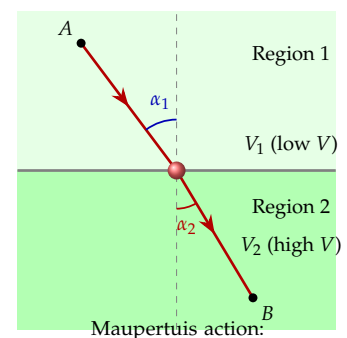


Snell's law:

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

Maupertuis' Principle (Mechanics)

A path of stationary action is followed



Maupertuis action:

$$S = \int m v ds$$

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{v_2}{v_1}$$

← ANALOGY →
A scalar functional is extremized in both cases

Unified variational principle

Both phenomena are governed by stationarity of a functional: $\delta S = 0$.
A local law of Snell-type emerges as a first-order necessary condition.

Figure 4. Variational analogy between optics (stationary time) and mechanics (stationary action): a discontinuity in medium properties induces a local transmission condition analogous to refraction.

Hamilton's Principle: Variational Formulation

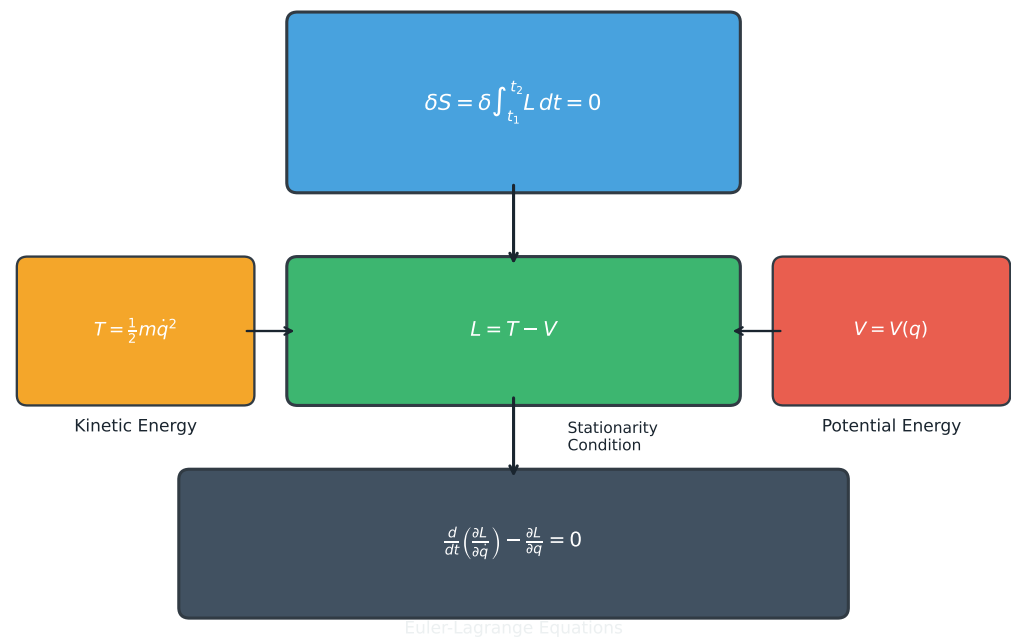


Figure 5. Conceptual diagram of Hamilton's principle showing the variational formulation, the Lagrangian as the difference between kinetic and potential energy, and the Euler-Lagrange equations are obtained from stationarity.

3.5. Derivation of the Stationary-Action Principle: Euler-Lagrange, Newton, and the Hamilton Bridge

In this section, the logical and mathematical chain is developed rigorously and step by step:

$$\delta S = 0 \implies \text{Euler-Lagrange} \iff \text{Newton's second law} \iff \text{Hamilton's canonical equations.}$$

The central point is that the variational formalism yields dynamics as a first-order optimality condition for a scalar functional, and that Newtonian, Lagrangian, and Hamiltonian representations are equivalent, under suitable regularity assumptions, for the same physical content [18,21,22].

3.6. Variational Setup: Trajectory Space and Admissible Variations

Let Q denote the configuration space, which in classical applications may be taken as an open subset of $\mathbb{R}^n$ or as a differentiable manifold. A trajectory is defined as a sufficiently regular curve

$$\mathbf{q} : [t_1, t_2] \rightarrow Q, \quad \mathbf{q}(t) = (q_1(t), \dots, q_n(t)).$$

The action functional is defined as

$$S[\mathbf{q}] = \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) dt, \quad (7)$$

where $L : TQ \times [t_1, t_2] \rightarrow \mathbb{R}$ is the Lagrangian and TQ denotes the tangent bundle (pairs $(\mathbf{q}, \dot{\mathbf{q}})$). The fixed-endpoint problem is posed as

$$\mathbf{q}(t_1) = \mathbf{q}_1, \quad \mathbf{q}(t_2) = \mathbf{q}_2. \quad (8)$$

A one-parameter family is introduced as a Gâteaux variation

$$\mathbf{q}_\varepsilon(t) = \mathbf{q}(t) + \varepsilon \boldsymbol{\eta}(t), \quad \boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = \mathbf{0}, \quad (9)$$

with $\boldsymbol{\eta}$ taken as an arbitrary test function (smooth, or more generally belonging to the appropriate function space).

The first variation of S in the direction $\boldsymbol{\eta}$ is defined by

$$\delta S[\mathbf{q}; \boldsymbol{\eta}] := \left. \frac{d}{d\varepsilon} S[\mathbf{q}_\varepsilon] \right|_{\varepsilon=0}. \quad (10)$$

Hamilton's principle of stationary action is stated as

$$\delta S[\mathbf{q}^*; \boldsymbol{\eta}] = 0 \quad \forall \boldsymbol{\eta} \text{ admissible.} \quad (11)$$

3.7. Step-by-Step Derivation of Euler–Lagrange from $\delta S = 0$

Step 1 is performed by differentiating under the integral sign. Using (7)–(10) and the chain rule,

$$\begin{aligned} \delta S[\mathbf{q}; \boldsymbol{\eta}] &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{t_1}^{t_2} L(\mathbf{q} + \varepsilon \boldsymbol{\eta}, \dot{\mathbf{q}} + \varepsilon \dot{\boldsymbol{\eta}}, t) dt \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} \cdot \boldsymbol{\eta} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\boldsymbol{\eta}} \right) dt, \end{aligned} \quad (12)$$

where $\partial L / \partial \mathbf{q}$ and $\partial L / \partial \dot{\mathbf{q}}$ are interpreted as row gradients (or vectors) with respect to their arguments.

Step 2 is performed by integration by parts on the second term in (12),

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\boldsymbol{\eta}} dt = \left[\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \boldsymbol{\eta} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \cdot \boldsymbol{\eta} dt. \quad (13)$$

Because $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = \mathbf{0}$, the boundary term vanishes. Substitution into (12) yields

$$\delta S[\mathbf{q}; \boldsymbol{\eta}] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \right) \cdot \boldsymbol{\eta} dt. \quad (14)$$

Step 3 uses the fundamental lemma of the calculus of variations. If (14) is zero for every admissible $\boldsymbol{\eta}$, then the integrand must vanish (almost everywhere), and the Euler–Lagrange equations are obtained:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n. \quad (15)$$

This result provides the methodological core: dynamics is produced as a first-order necessary condition for a scalar functional [18,21,22].

3.8. Equivalence with Newton's Second Law (Particle in Cartesian Coordinates)

A particle of mass m with position $\mathbf{r}(t) \in \mathbb{R}^3$ subjected to a potential $V(\mathbf{r})$ is considered. The classical Lagrangian is given by

$$L(\mathbf{r}, \dot{\mathbf{r}}) = T - V = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - V(\mathbf{r}). \quad (16)$$

The required derivatives are obtained as

$$\frac{\partial L}{\partial \dot{\mathbf{r}}} = m \dot{\mathbf{r}}, \quad \frac{\partial L}{\partial \mathbf{r}} = -\nabla V(\mathbf{r}). \quad (17)$$

Substitution into (15) in vector form yields

$$\frac{d}{dt}(m\dot{\mathbf{r}}) - (-\nabla V(\mathbf{r})) = \mathbf{0} \implies m\ddot{\mathbf{r}} = -\nabla V(\mathbf{r}). \quad (18)$$

By defining the conservative force as

$$\mathbf{F}(\mathbf{r}) := -\nabla V(\mathbf{r}), \quad (19)$$

(18) is written exactly as Newton's second law:

$$m\ddot{\mathbf{r}} = \mathbf{F}. \quad (20)$$

Therefore, in Cartesian coordinates with conservative forces, (15) is Newton's law expressed in variational form [18,29].

If $\mathbf{r} = \mathbf{r}(\mathbf{q}, t)$ with $\mathbf{q} \in \mathbb{R}^n$, the projection of Newton's law onto virtual directions $\delta\mathbf{r} = \sum_i (\partial\mathbf{r}/\partial q_i) \delta q_i$ leads to d'Alembert's principle and to Lagrange's equations in the generalized coordinates. In the conservative case, equivalence with (15) is obtained, with generalized forces $Q_i = -\partial V/\partial q_i$ [18].

3.9. From Euler–Lagrange to Hamilton: Legendre Transform and Canonical Equations

The Hamiltonian formulation does not introduce new physics. The conjugate structure of coordinates and momenta and the geometry of phase-space flow are made explicit. The canonical momentum is defined as

$$p_i := \frac{\partial L}{\partial \dot{q}_i}. \quad (21)$$

Regularity of the Lagrangian is assumed in the sense that

$$\det \left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right] \neq 0, \quad (22)$$

which allows $\mathbf{p} = \partial L/\partial \dot{\mathbf{q}}$ to be locally inverted and $\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)$ to be expressed.

The Hamiltonian is defined by the Legendre transform

$$H(\mathbf{q}, \mathbf{p}, t) := \sum_{i=1}^n p_i \dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t), \quad \dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t). \quad (23)$$

Differentiation of (23) and use of $p_i = \partial L/\partial \dot{q}_i$ yield

$$\begin{aligned} dH &= \sum_i \dot{q}_i dp_i + \sum_i p_i d\dot{q}_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt \\ &= \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt. \end{aligned} \quad (24)$$

By the Euler–Lagrange equations, $\dot{p}_i = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$, and (24) is rewritten as

$$dH = \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \frac{\partial H}{\partial t} dt. \quad (25)$$

Comparison with

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt, \quad (26)$$

identifies Hamilton's canonical equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad (27)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (28)$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (29)$$

Under the regularity assumption (22), the Euler–Lagrange Equation (15) are equivalent to the canonical Equations (27) and (28). This equivalence constitutes the essential content of the Lagrange–Hamilton bridge: a single dynamics may be represented as a variational problem in configuration space or as a phase-space flow generated by H [18,29,35].

3.10. Closure of Equivalences: Hamilton $\Rightarrow$ Newton (Particle Case)

For the particle Lagrangian (16), the momentum is $\mathbf{p} = m\dot{\mathbf{r}}$ and the Hamiltonian is

$$H(\mathbf{r}, \mathbf{p}) = \frac{1}{2m} \mathbf{p} \cdot \mathbf{p} + V(\mathbf{r}). \quad (30)$$

Hamilton's equations yield

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}} = -\nabla V(\mathbf{r}). \quad (31)$$

By differentiating the first relation and substituting the second,

$$m\ddot{\mathbf{r}} = \dot{\mathbf{p}} = -\nabla V(\mathbf{r}) = \mathbf{F}(\mathbf{r}), \quad (32)$$

Newton's law (20) is recovered. Consequently, the route

$$\delta S = 0 \Rightarrow \text{Euler–Lagrange} \Rightarrow \text{Hamilton} \Rightarrow \text{Newton}$$

is not a sequence of distinct methods, but a chain of equivalent representations of the same physical principle, where formalism is selected for geometric convenience, computational structure, or control-theoretic utility [36].

$$u(\rho) \in \mathcal{V}_{ad} : \quad \delta \Pi(u; \rho)[\delta v] = 0 \quad \forall \delta v \in \mathcal{V}_0 \iff R(u, \rho) = 0. \quad (33)$$

Here $R(u, \rho) = 0$ denotes the equivalent residual form (continuous or discretized); under FEM one typically has $R(u, \rho) = K(\tilde{\rho})u - f$. The outer level then acts on the reduced objective $\hat{J}(\rho) = J(\rho, u(\rho))$ and enforces design constraints (volume bounds, box constraints, manufacturability, etc.):

$$\min_{\rho \in \mathcal{U}_{ad}} \hat{J}(\rho) = J(\rho, u(\rho)) \quad \text{s.t.} \quad R(u(\rho), \rho) = 0, \quad g(\rho, u(\rho)) \leq 0. \quad (34)$$

The coupling mechanism becomes transparent by differentiating the reduced functional. Since the state depends *implicitly* on ρ through equilibrium, the chain rule gives

$$\frac{d\hat{J}}{d\rho} = \partial_\rho J(\rho, u) + \partial_u J(\rho, u) \frac{\partial u}{\partial \rho}. \quad (35)$$

Implicit differentiation of the equilibrium constraint $R(u(\rho), \rho) = 0$ yields

$$\partial_u R(u, \rho) \frac{\partial u}{\partial \rho} + \partial_\rho R(u, \rho) = 0, \quad (36)$$

and the sensitivity $\partial u / \partial \rho$ is eliminated via an adjoint variable λ solving

$$(\partial_u R(u, \rho))^T \lambda = -(\partial_u J(\rho, u))^T. \quad (37)$$

This leads to the closed-form gradient

$$\frac{d\hat{J}}{d\rho} = \partial_\rho J(\rho, u) + \lambda^T \partial_\rho R(u, \rho), \quad (38)$$

which is the sought *mathematical coupling* between the inner variational equilibrium and the outer design optimization: the design gradient depends on the state $u(\rho)$ and the adjoint $\lambda(\rho)$, both governed by the same physical operator (e.g., the tangent stiffness). In the prototypical linear-elastic compliance setting with $J = f^T u$ and $R = K(\tilde{\rho})u - f$, one recovers $\frac{d\hat{J}}{d\rho} = -u^T \frac{\partial K}{\partial \rho} u$, making explicit how design affects equilibrium through $K(\tilde{\rho})$.

4. The Mathematical Framework of Variational Mechanics

Variational mechanics formalizes dynamics as a stationarity problem for a functional defined on an infinite-dimensional space of admissible trajectories. Instead of postulating differential equations a priori, a global scalar quantity is defined, such as an action, total potential, or cost functional, whose first variation encodes admissible kinematics, energetic exchange, and constraints. Equations of motion are obtained as necessary optimality conditions in the sense of Gâteaux or Fréchet under perturbations compatible with boundary conditions [18,21,22]. This viewpoint provides a natural language for weak formulations, mixed boundary conditions, constraint incorporation, and the construction of discretizations that preserve structure (see Figure 6).

4.1. Variational Problem: Admissible Spaces, First Variation, and Stationarity Criterion

In a prototypical form, the fundamental problem of the calculus of variations consists of determining the y that renders stationary the functional

$$J[y] = \int_{x_1}^{x_2} F(x, y(x), y'(x)) dx, \quad (39)$$

over an admissible set $\mathcal{A}$ specified by boundary conditions and regularity requirements. For rigor and applicability in continuum mechanics, Sobolev spaces are naturally employed. Typically, $y \in H^1(x_1, x_2)$ is assumed, with variations $\eta \in H_0^1(x_1, x_2)$ vanishing at the endpoints in the trace sense, which enables the treatment of limited regularity solutions and formulation of stationarity as an integral identity [21,24].

A variational family is introduced as a Gâteaux directional perturbation,

$$\bar{y}(x; \alpha) = y(x) + \alpha \eta(x), \quad \eta \in \mathcal{V}, \quad (40)$$

and the first variation is defined by

$$\delta J[y; \eta] := \left. \frac{d}{d\alpha} J[\bar{y}(\cdot; \alpha)] \right|_{\alpha=0}. \quad (41)$$

The stationarity condition is written as

$$\delta J[y^*; \eta] = 0 \quad \forall \eta \in \mathcal{V}, \quad (42)$$

by which critical points of the functional are characterized in the chosen space [21,22]. The condition $\delta J = 0$ is first order and does not distinguish minima from maxima or saddle

points. Physical and engineering selection requires additional information such as the second variation, convexity or semiconvexity, coercivity, and weak lower semicontinuity, particularly in nonlinear systems with bifurcations and multiple stationary states [24].

Variational Integrators: Structure Preservation

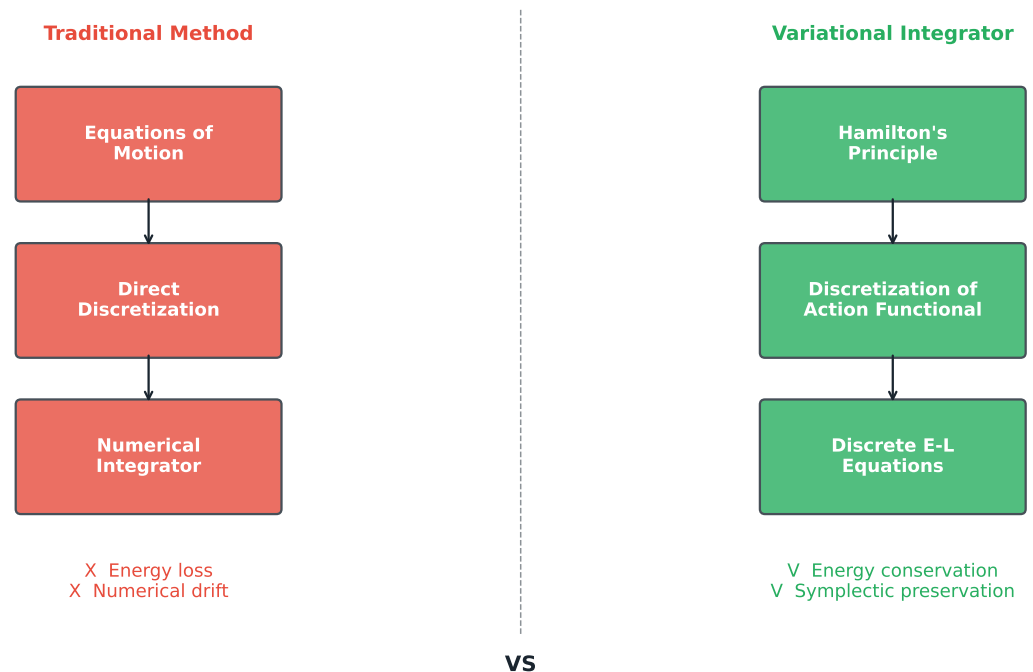


Figure 6. Comparison between direct equation discretization (**left**) and action discretization (**right**). Variational integrators preserve symplectic structure and momentum maps, which reduces long-term drift.

4.2. Euler–Lagrange Equations, Boundary Terms, and Natural Conditions

Under sufficient regularity, or by interpreting derivatives in the weak sense, differentiation under the integral yields

$$\delta J[y; \eta] = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right) dx. \quad (43)$$

Integration by parts, which remains valid in Sobolev settings through duality, gives

$$\delta J[y; \eta] = \left[\frac{\partial F}{\partial y'} \eta \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right) \eta dx. \quad (44)$$

If variations vanish at the endpoints, corresponding to Dirichlet-type conditions, the boundary term disappears and the fundamental lemma of the calculus of variations yields the strong Euler–Lagrange form

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0. \quad (45)$$

For $y \in \mathbb{R}^n$, the componentwise form is obtained as

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'_i} \right) = 0, \quad i = 1, \dots, n. \quad (46)$$

The strong Equation (45) is a consequence, whereas the primary object is the variational identity (44). When the boundary is free or mixed, variations are not required to vanish and

the boundary term imposes natural conditions of Neumann-, Robin-, or transversality-type. This provides the mathematical basis of the principle of virtual work, where boundary conditions are obtained from the functional rather than appended after the fact [21,24]. In structural mechanics, this logic explains the compatibility between energetic formulations and traction boundary conditions, and supports weak formulations in the presence of mixed boundaries and distributed loads.

If F does not depend explicitly on x , a first-integral identity of Beltrami-type holds,

$$F - y' \frac{\partial F}{\partial y'} = C, \quad (47)$$

which is interpreted as a conservation law associated with translation invariance of the independent variable [22]. This idea provides a variational precursor to more general conservation results of Noether-type in mechanics [18].

4.3. Hamilton's Principle, Weak Form, and Connection to Structural Mechanics and Discretization

In mechanics, the central functional is the action

$$S[\mathbf{q}] = \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) dt, \quad (48)$$

and Hamilton's principle enforces $\delta S = 0$ for variations consistent with fixed-endpoint states, yielding

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n. \quad (49)$$

A conceptual synthesis of the chain $\delta S = 0 \Rightarrow$ Euler–Lagrange may be referenced, if included, in Figure 5.

The condition $\delta S = 0$ defines an integral identity valid for a broad class of test functions, and this weak statement is preserved in Ritz–Galerkin approximations. In quasi-static structural analysis, the analogue is $\delta \Pi[\mathbf{u}] = 0$ for the total potential, by which the finite element method is obtained through restriction to a finite-dimensional subspace [2,3,37]. Interpretations based on differential complexes, such as de Rham complexes, clarify which structures must be preserved to avoid spurious modes, locking, and numerical stability failures in mixed and coupled formulations [38].

For holonomic constraints $g_a(\mathbf{q}, t) = 0$, variationality is preserved by introducing Lagrange multipliers through

$$\tilde{L} = L + \sum_a \lambda_a g_a,$$

and Euler–Lagrange equations are obtained for $(\mathbf{q}, \lambda)$ [18]. In the presence of nonpotential forces, a consistent extension is provided by the Lagrange–d'Alembert principle, in which $\delta S = 0$ is replaced by $\delta S + \int Q_i \delta q_i dt = 0$, thereby maintaining the virtual-work structure and avoiding energetic inconsistencies [18].

4.4. Hamiltonian Reformulation: Legendre Transform, Symplectic Structure, and Geometric Preservation

Generalized momenta are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad (50)$$

and regularity of the Lagrangian is assumed in the sense that the Hessian $\partial^2 L / \partial \dot{q}_i \partial \dot{q}_j$ is nonsingular. Under this condition, $\mathbf{p} = \partial L / \partial \dot{\mathbf{q}}$ is locally invertible and $\dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)$ is defined, and the Hamiltonian is introduced through the Legendre transform as

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^n p_i \dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t). \quad (51)$$

Hamilton's canonical equations are obtained as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad (52)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (53)$$

which describe a phase-space flow with symplectic structure [18]. This geometric structure supports conservation theorems associated with symmetries, qualitative stability and integrability analysis, and criteria for the construction of structure-preserving numerical integrators.

A central methodological consequence is that discretization of the action, rather than direct discretization of (49), yields variational integrators that inherit geometric properties by construction, including symplecticity and conservation of momenta associated with discrete symmetries. Long-term numerical drift is thereby controlled, and a bridge is formed toward contemporary formulations in dynamics, control, and robust simulation. This idea is summarized visually in Figure 6 and provides a pathway to energetic extensions with interconnection and dissipation, including port-Hamiltonian formulations (see Figure 7), in which power balance is represented as a structural invariant [18,37].

Port-Hamiltonian Systems: Energy-Based Control

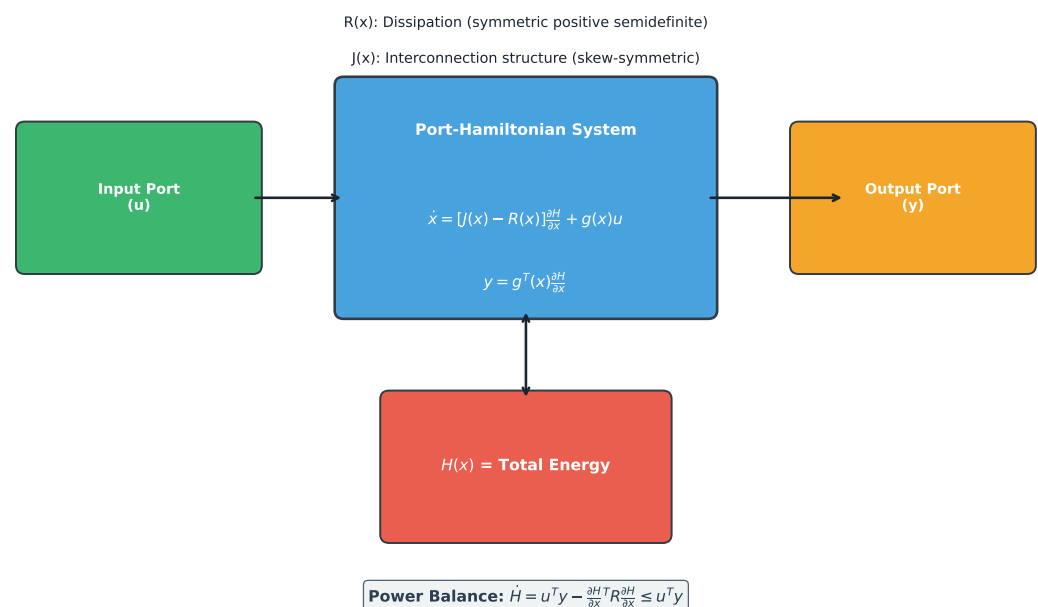


Figure 7. Structure of a port-Hamiltonian system: power ports, Hamiltonian as total energy, interconnection, and dissipation. Power balance guarantees passivity and supports robust energy-based control design.

5. Industrial Applications of Hamilton's Principle and Variational Optimization: Mining Infrastructure, Simulation, and Energy-Based Control

Hamilton's principle and, more broadly, variational methods provide a unified framework by which physically consistent models are derived, discretized without loss of energetic meaning, and used to pose design and control tasks as optimization problems constrained by physics [21,22,24]. In this section, two perspectives are integrated that are methodologically continuous: the structural perspective with emphasis on mining infrastructure, and the broader industrial landscape that includes robotics, aerospace, energy systems, control, and structure-informed machine learning. Dynamics and equilibrium are obtained from the stationarity of a functional (see Figure 5), while synthesis in design and control is posed as an optimization problem defined over that model [18,19]. In historical terms, continuity has been established from stationary time in optics and stationary action in mechanics, from Fermat and Maupertuis to modern analytical mechanics [18,31,32]. In classical form and in contemporary geometric interpretations, a rigorous connection is provided between Lagrangian and Hamiltonian descriptions, and local laws are obtained as first-order necessary conditions derived from $\delta S = 0$ [20,33,35].

5.1. From Action to Equilibrium and Discretized Dynamics: The Operational Bridge to FEM

For a deformable continuum within a mining industrial context, where dissipative mechanisms such as bolted joint friction and material damping are non-negligible, the classical conservative statement must be augmented. The extended Hamilton's principle enforces stationarity of the action accounting for nonconservative virtual work δW_{nc} :

$$\delta \int_{t_1}^{t_2} (T[\mathbf{u}, \dot{\mathbf{u}}] - U[\mathbf{u}]) dt + \int_{t_1}^{t_2} \delta W_{nc} dt = 0. \quad (54)$$

Here, δW_{nc} typically incorporates Rayleigh dissipation terms or nonlinear friction functionals proportional to velocity fields, $\delta W_{nc} = - \int_{\Omega} \mathbf{C} \dot{\mathbf{u}} \cdot \delta \mathbf{u} d\Omega$. This distinction is critical for mining infrastructure, where unmodeled dissipation can lead to catastrophic resonance predictions under harmonic excitation from vibrating screens or crushers [18,19].

In the quasi-static regime where $T \simeq 0$, the variational condition reduces to stationarity of the total potential

$$\Pi[\mathbf{u}] := U[\mathbf{u}] - W[\mathbf{u}, t], \quad \delta \Pi[\mathbf{u}; \delta \mathbf{u}] = 0, \quad (55)$$

which is equivalent to the weak form of equilibrium and shows that balance laws and natural boundary conditions emerge from the same variational statement [2,19].

Galerkin or Ritz discretization of the weak problem induces the finite element method as a direct consequence. When $\mathbf{u}$ is approximated by $\mathbf{u}_h = \mathbf{N} \mathbf{d}$, the linear static system is obtained as

$$\mathbf{K} \mathbf{d} = \mathbf{f}, \quad (56)$$

and the corresponding dynamic system, including damping such as Rayleigh damping or extensions consistent with d'Alembert-type virtual work, is obtained as

$$\mathbf{M} \ddot{\mathbf{d}} + \mathbf{C} \dot{\mathbf{d}} + \mathbf{K} \mathbf{d} = \mathbf{f}(t). \quad (57)$$

This variational origin ensures symmetry in conservative problems, provides interpretation in an energy norm, and supports error analysis, stability analysis, and advanced mixed formulations [2,3,39]. The functional status of the weak formulation, including Sobolev spaces, coercivity, and consistency, enables the rigorous control of convergence and avoids

purely algebraic readings of the discrete equilibrium equations [23,37,38]. In industrial applications, this variational bridge supports traceable transfer from constitutive laws and complex boundaries to robust computational models implemented in modern FEM software [40,41].

5.2. Mining Infrastructure as a High-Demand Case: Equilibrium, Stability, and Reliability Under Severe Loading

Mining infrastructure concentrates, in an extreme form, the main reasons for which a variational approach is methodologically advantageous. Large geometric scales and sustained operation are combined with severe load combinations. Here, *large geometric scales* refers to mining structural infrastructure with large characteristic dimensions (typically $\mathcal{O}(10^1\text{--}10^2)$ m spans/lengths and/or large elevations), frequently realized as slender steel systems with relevant unbraced lengths—e.g., conveyor galleries, transfer bridges/gantries, and equipment support frames—for which second-order effects, global stability margins, and dynamic sensitivity cannot be treated as marginal. *Sustained operation* denotes high duty-cycle service (continuous or near-continuous operation with constrained maintenance windows), where process loads are not only severe but also persistent and time-varying; consequently, the performance rationale is naturally coupled to reliability/maintainability requirements over a service horizon T . In probabilistic terms, this motivates a time-dependent reliability reading, e.g.,

$$P_f(T) = \mathbb{P}\left(\min_{t \in [0, T]} g(X, t) \leq 0\right) \leq P_f^*, \quad (58)$$

where $g(X, t)$ denotes a limit-state function (strength, serviceability, or stability margin) and X collects uncertain loads, resistances, and degradation-related parameters. Refs. [42,43] including self-weight, process loads, wind, seismic actions, and thermal effects, and with simultaneous requirements of reliability, maintainability, and material efficiency. Under these conditions, the energetic formulation (55) and its dynamic extension (57) enable a coherent workflow in which a traceable physical model is obtained through the explicit definition of energies, external work, and constraints, while ambiguities associated with constraints, contact, and mixed boundaries are reduced [2,19].

In this setting, structural stability is rigorously interpreted variationally. While equilibrium is obtained as a stationary point ($\delta\Pi = 0$), physical stability requires the positivity of the second variation of the total potential energy:

$$\delta^2\Pi(\mathbf{u})[\delta\mathbf{u}, \delta\mathbf{u}] = \int_{\Omega} \nabla\delta\mathbf{u} : \mathbb{C} : \nabla\delta\mathbf{u} \, d\Omega + \int_{\Omega} \nabla\delta\mathbf{u} : \boldsymbol{\tau} \cdot \nabla\delta\mathbf{u} \, d\Omega > 0, \quad (59)$$

where the second term involves the initial stress tensor $\boldsymbol{\tau}$, responsible for geometric stiffness effects. In computational practice, this implies that the tangent stiffness matrix $\mathbf{K}_T = \mathbf{K}_0 + \mathbf{K}_\sigma$ must remain positive and definite. This check is mandatory for slender mining assets—such as conveyor galleries, transfer towers, and gantries—where limit-point buckling or bifurcation often precedes material yield [18,19]. Furthermore, when large deformations, damage, or degradation are present, an incremental energetic interpretation based on free energy and dissipation is naturally induced, ensuring consistent tangent operators for stability analysis and solver convergence [44–46]. This framework ultimately supports a verifiable design loop, as the FEM state operator enables systematic multi-case evaluation and traceability between assumptions, response, and engineering decisions [2,3].

The advantage of a variational approach in mining is not limited to computation. A unified language is provided for equilibrium, stability, and dynamic response, consistent

with energy and constraints, and thereby suited to design decisions under operational uncertainty and progressive degradation [47].

5.3. Physics-Constrained Structural Optimization: From Sizing to Topology Under Performance Constraints

Once equilibrium or dynamics is enforced as an implicit constraint through (56) or (57), structural design can be formulated as an optimization problem constrained by the governing equations, commonly referred to as PDE- or FE-constrained optimization. A standard abstract formulation, used throughout this work, is given in (60).

In PDE/FE-constrained optimization, the discrete state equation $\mathbf{R}(\mathbf{d}, \theta) = \mathbf{0}$ can be interpreted most naturally as the finite-dimensional expression of an underlying variational stationarity condition. For conservative mechanics, the state is characterized by $\delta\Pi(\mathbf{u}; \theta) = 0$, whereas in dynamics it is characterized by $\delta S(\mathbf{q}; \theta) = 0$ over admissible trajectories; in both cases, these are first-order optimality conditions in the calculus of variations [18,19,21,22]. A Ritz–Galerkin restriction of the admissible space yields the FEM residual, e.g., $\mathbf{R}(\mathbf{d}, \theta) = \mathbf{K}(\theta)\mathbf{d} - \mathbf{f}$, as the discrete counterpart of that stationarity statement [2,23]. Under this perspective, the optimization becomes hierarchically variational: an outer optimality problem is posed over the design variables, while the inner constraint enforces that the state remains a stationary point of the associated energy/action functional for each design [1,6]:

$$\min_{\theta \in \Theta} J(\theta, \mathbf{d}(\theta)) \quad \text{subject to} \quad \mathbf{R}(\mathbf{d}(\theta), \theta) = \mathbf{0}, \quad \mathbf{g}(\theta, \mathbf{d}) \leq \mathbf{0}. \quad (60)$$

where θ parameterizes geometry, cross-sections, materials, or densities, $\mathbf{R} = \mathbf{0}$ represents equilibrium, and $\mathbf{g}$ collects service, strength, and stability constraints such as displacements, stresses, frequencies, or buckling measures.

This same variational viewpoint also makes the Lagrangian (KKT) structure of physics-constrained optimization essentially inevitable. With the state defined implicitly by stationarity and enforced in practice through the residual constraint, the problem admits an augmented Lagrangian in which multipliers enforce the state equation, yielding coupled first-order conditions for (i) the state, (ii) the adjoint (multiplier), and (iii) the design variables. In this way, adjoint-based sensitivities and KKT conditions emerge as the finite-dimensional analogue of constrained variational problems, and the computational “physics constraint” is seen as inherited from the generating variational principle rather than appended post hoc [1,2,18,19].

To explain the coupling mechanism between the inner state equilibrium and the outer design optimization, consider the reduced objective functional $\hat{J}(\theta) := J(\theta, \mathbf{d}(\theta))$. Since the state $\mathbf{d}$ depends implicitly on θ via the residual $\mathbf{R}(\mathbf{d}, \theta) = \mathbf{0}$, the total design gradient is derived utilizing the chain rule and the adjoint vector λ , which enforces the state constraint variationally:

$$\frac{d\hat{J}}{d\theta} = \frac{\partial J}{\partial \theta} + \lambda^T \frac{\partial \mathbf{R}}{\partial \theta}. \quad (61)$$

Here, λ is the solution to the adjoint equation $\mathbf{K}_T^T \lambda = -(\partial J / \partial \mathbf{d})^T$, where $\mathbf{K}_T = \partial \mathbf{R} / \partial \mathbf{d}$ denotes the tangent stiffness matrix characterizing the physical equilibrium. This system demonstrates that the inner variational layer acts as a physics-based filter for sensitivity information, back-propagating the objective gradient through the structural stiffness operator.

In large design domains, scalable optimization requires efficient gradients, and adjoint methods follow directly from the above Lagrangian structure [1]. At the methodological level, shape and topology optimization frameworks and their regularization techniques

define the effective solvable model through length-scale control, suppression of spurious micro-structures, and numerical robustness [7,25,48].

In linear elasticity topology optimization, a widely adopted prototype consists of compliance minimization under a volume constraint:

$$\min_{\rho(\mathbf{x}) \in [0,1]} C(\rho) = \mathbf{f}^T \mathbf{d}(\rho) \quad \text{subject to a} \quad V(\rho) \leq V^*, \quad \mathbf{K}(\rho) \mathbf{d}(\rho) = \mathbf{f}, \quad (62)$$

where ρ is a pseudo-density field. In linear equilibrium, the minimization of C is equivalent to the maximization of global stiffness under a material budget, and material is redistributed toward regions of high strain-energy density [5,6,14]. In SIMP parameterization with penalization $p > 1$, a typical sensitivity expression is given by

$$\frac{\partial C}{\partial \rho_e} = -p \rho_e^{p-1} \mathbf{d}_e^T \mathbf{k}_0 \mathbf{d}_e, \quad (63)$$

which formalizes a local energy-efficiency criterion because $\mathbf{d}_e^T \mathbf{k}_0 \mathbf{d}_e$ is proportional to the element strain energy, and the optimal update privilege regions where material addition yields the largest compliance reduction [6,14]. In practice, convergence toward near-binary solutions and suppression of numerical instabilities depends on penalization, filtering, and projection, which are treated as constitutive components of the solved problem and must be reported as such [8,49].

However, the continuous topology optimization problem is generally ill-posed, leading to mesh-dependent solutions such as checkerboards. To ensure the existence of solutions and to introduce an explicit length scale associated with manufacturability, the formulation is regularized through a Helmholtz-type partial differential equation that defines a filtered density field $\tilde{\rho}$.

The density filter employed in this work is defined by a Helmholtz-type boundary value problem,

$$-r_{\min}^2 \nabla^2 \tilde{\rho} + \tilde{\rho} = \rho \quad \text{in } \Omega, \quad \frac{\partial \tilde{\rho}}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad (64)$$

which admits the following weak formulation: find $\tilde{\rho} \in H^1(\Omega)$ such that, for all $v \in H^1(\Omega)$,

$$\int_{\Omega} (\tilde{\rho} v + r_{\min}^2 \nabla \tilde{\rho} \cdot \nabla v) d\Omega = \int_{\Omega} \rho v d\Omega. \quad (65)$$

The associated bilinear form $a(\phi, v) = \int_{\Omega} (\phi v + r_{\min}^2 \nabla \phi \cdot \nabla v) d\Omega$ is continuous and coercive on $H^1(\Omega)$; therefore, by standard elliptic theory, the filtered field $\tilde{\rho}$ exists and is unique for any $\rho \in L^2(\Omega)$, and the filter defines a bounded smoothing operator $F = (I - r_{\min}^2 \Delta)^{-1} : L^2(\Omega) \rightarrow H^1(\Omega)$. Equivalently, $\tilde{\rho}$ can be characterized as the unique minimizer of the Tikhonov-type functional

$$\tilde{\rho} = \arg \min_{\eta \in H^1(\Omega)} \frac{1}{2} \int_{\Omega} \left((\eta - \rho)^2 + r_{\min}^2 |\nabla \eta|^2 \right) d\Omega, \quad (66)$$

which makes explicit that filtering penalizes high spatial frequencies and enforces a minimum feature size of order $r_{\min}$. Consequently, the SIMP formulation must be interpreted through the effective design map $\rho \mapsto \tilde{\rho} = F(\rho)$ and, when employed, the subsequent smooth Heaviside projection $\bar{\rho} = P(\tilde{\rho})$, so that the equilibrium constraint is solved using $K(\bar{\rho})$ rather than $K(\rho)$. This explicit regularization is precisely what removes checkerboard instabilities and mitigates mesh dependence: the boundedness of $\tilde{\rho}$ in $H^1(\Omega)$ implies compactness in $L^2(\Omega)$, which in turn supports subsequential convergence under mesh refinement, existence of minimizers for the regularized topology optimization problem, and mesh-independent solutions in the limit of mesh refinement [6,7,50].

In mining infrastructure, the methodological role of (62) is not to replace constructible final design, but to synthesize load-bearing architecture. Load paths are identified, members are prioritized, and transition to fabricable typologies is guided through commercial profiles, joints, tolerances, accessibility, and maintainability, while consistency with equilibrium physics and performance constraints is retained [1]. Transfer is strengthened when geometric constraints and crisp-boundary representations are incorporated, such as level-set approaches, by which ambiguity associated with diffuse densities is reduced and traceability to CAD and detailing is facilitated [12,13,16].

5.4. Long-Term Simulation: Variational Integrators and Qualitative Preservation of Invariants

In high-fidelity industrial simulation, including multi-body dynamics, flexible structures, aeroelasticity, and rotating machinery, direct discretization of ODE or PDE models may induce energy drift and qualitative degradation over long horizons. A structure-preserving alternative is provided by discretization of the action functional, by which variational and symplectic integrators are obtained that preserve system geometry by construction (see Figure 6) [51,52]. Recent applications have emphasized schemes that conserve invariants or enforce controlled dissipation when long-horizon qualitative fidelity is required for iterative simulation, optimization, and control [53–55].

Qualitative stability and long-term drift control are obtained in practice, particularly when conservation of energy or momentum, or controlled dissipation, is an operational requirement rather than a numerical detail [56–58]. In industrial contexts in which models are used iteratively, such as in optimization, control, and digital twins, systematic violation of invariants may feed back into decision loops and degrade performance. For this reason, structure-preserving discretization is treated as a natural extension of variational modeling: model quality is not sufficient unless discretization is compatible with model structure [51,52]. Phase-space comparisons and energy evolution illustrate the qualitative contrast between non-structure-preserving methods and symplectic integrators (Figure 8) [59,60].

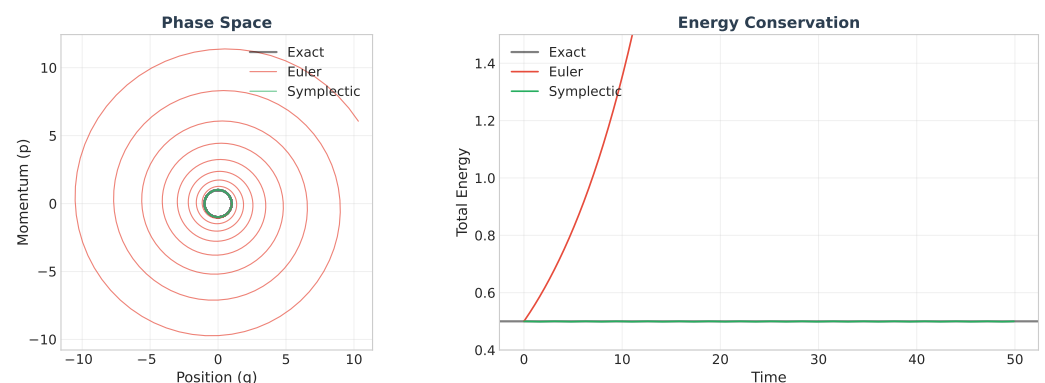


Figure 8. Numerical demonstration of energy behavior in a simple harmonic oscillator. The Euler method exhibits energy drift, whereas a symplectic integrator keeps energy bounded around the exact value and preserves qualitative dynamics.

5.5. Energy-Based Modeling and Control: Port-Hamiltonian Systems and Passivity

A modern extension of the classical formalism that is particularly effective for multi-domain industrial engineering, including mechanical, electrical, hydraulic, and thermal systems, is provided by port-Hamiltonian system theory. The core representation is based on storage through a Hamiltonian H , interconnection through a skew-symmetric operator

J , and dissipation through a positive semidefinite operator R , together with power ports (u, y) (see Figure 7). A prototypical form is given by

$$\dot{x} = (J(x) - R(x))\nabla H(x) + g(x)u, \quad (67)$$

$$y = g(x)^T \nabla H(x), \quad (68)$$

with a power balance and passivity inequality expressed as

$$\dot{H} = u^T y - \nabla H(x)^T R(x) \nabla H(x) \leq u^T y. \quad (69)$$

This framework unifies modeling and control because energy shaping and interconnection assignment are used to construct controllers that ensure robustness through passivity, as in the interconnection and damping assignment passivity-based control [61–63]. Industrial relevance is immediate in process control, mechatronics, and coupled systems in which energy conservation and transfer dominate system behavior [61]. Recent extensions incorporate thermodynamic criteria and irreversible processes with explicit dissipation, thereby broadening applicability to complex industrial processes [64,65].

5.6. Industrial Panorama and Methodological Convergence: From Structural Modeling to Structure-Informed Machine Learning

Applied evidence indicates sustained expansion of variational and Hamiltonian principles in modern engineering, with recent acceleration in renewable energy and structure-informed machine learning (see Figure 9). A domain map is summarized in Figure 10. Methodologically, applications are grouped by the type of gain provided by variational formulation.

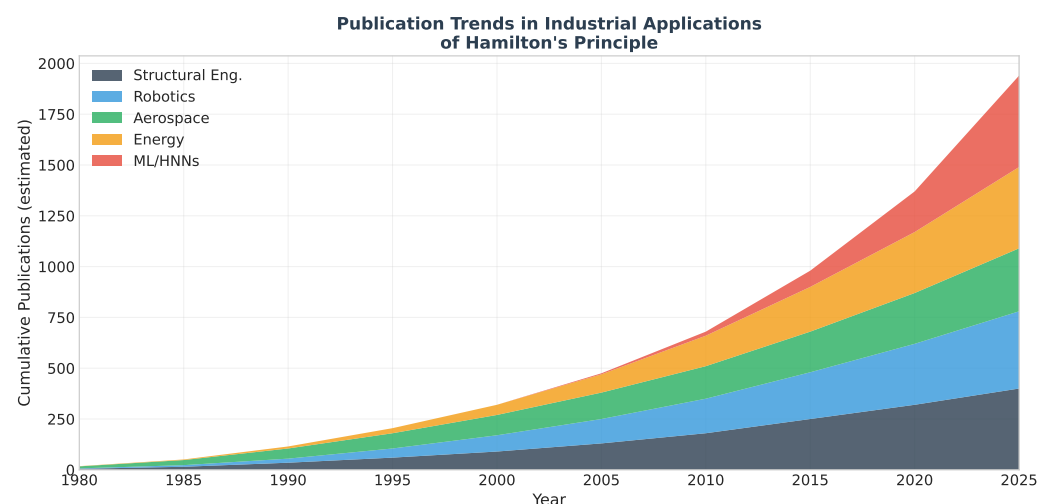


Figure 9. Temporal evolution of publications on industrial applications of Hamilton's principle by area. Accelerated growth has been reported in renewable energy and structure-informed machine learning.

- Systematic derivation of equations of motion in robotics and mechatronics is enabled by Lagrangian and Hamiltonian modeling, by which multi-degree-of-freedom equations are constructed in a traceable form suitable for nonlinear control and planning, including unmanned aerial vehicles and cooperative systems [66–68]. Energy-based control uses Hamiltonian structure to design laws that exploit passivity and stored-energy functions for robustness and performance [36,69,70].
- Structure–fluid coupling in aerospace and wind turbines is supported by variational formulation through derivation of coupled equations and coherent extensions using

nonconservative virtual work for aerodynamic loading, which is required in flutter, divergence, and aeroelastic stability analyses [71–76]. In wind turbines, energetic decomposition and modal and control coupling are typically represented as in Figure 11.

- Robust simulation is supported through variational integrators that preserve symplectic structure and momenta for qualitative fidelity over long horizons [51,52]. Recent industrial applications emphasize structure preservation and drift control under iteration with design and control loops [53,54,59].
- Energy-based control is supported through port-Hamiltonian systems, where passivity provides a structural guarantee for robustness and modularity in multi-domain systems [61,62]. In grid-connected energy domains, port-Hamiltonian modeling supports control designs that respect power balance and stability by construction [63].



Figure 10. Conceptual map of industrial applications of Hamilton's principle, highlighting its role as a generator of energetically consistent models and as a basis for simulation, control, and structure-informed learning.

A high-impact industrial frontier is provided by structure-informed machine learning. Hamiltonian neural networks learn a Hamiltonian H_θ from data and enforce the learned dynamics to satisfy the canonical form

$$\dot{z} = J \nabla H_\theta(z), \quad (70)$$

by which conservation properties, or controlled dissipation in extended models, are induced and physical generalization is improved relative to purely phenomenological models [77,78]. This approach is relevant when first-principles modeling is partial or costly, but invariants, stability, and long-term coherence are required, including the identification and control of complex industrial assets [79]. Recent lines have explored extensions to constrained and dissipative systems and integration within identification and control workflows [80,81] (see Figure 12).

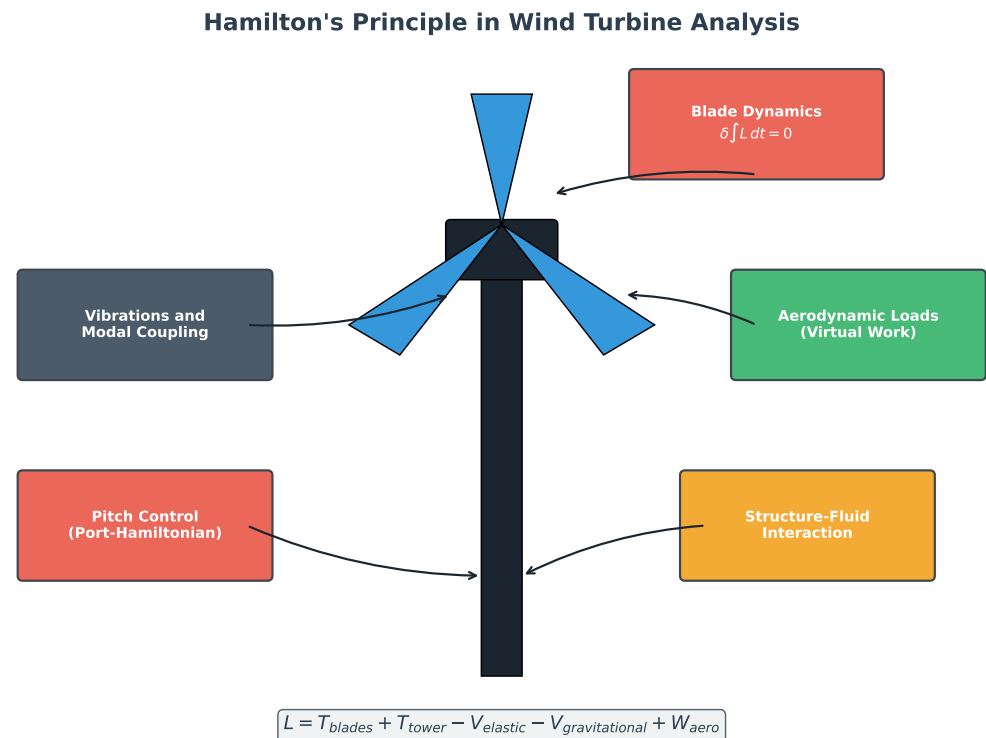


Figure 11. Application of Hamilton's principle in dynamic and aeroelastic modeling of wind turbines: blade structural dynamics, aerodynamic loads represented through virtual work, modal coupling, and integration with control such as pitch regulation.

Hamiltonian Neural Networks (HNNs)

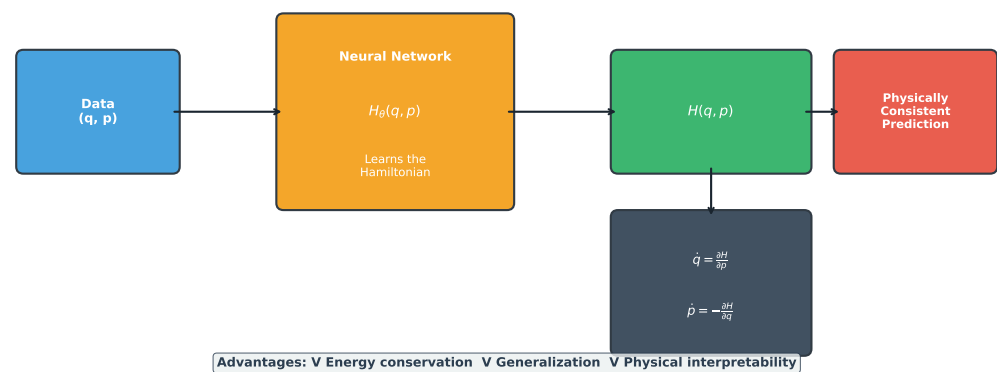


Figure 12. Conceptual architecture of a Hamiltonian neural network: H_θ is learned from data and dynamics is generated through canonical equations, which promotes invariant preservation and physical consistency.

5.6.1. Methodological Note on Mining 4.0 and Digital Twins

In instrumented mining assets, the same variational structure, with equilibrium or dynamics imposed as a constraint, may be used as an interpretable backbone of a digital twin. Parameter identification and updating may be posed as physics-constrained optimization rather than as purely statistical fitting, by which energetic consistency and stability are preserved in the data–model–decision cycle [15,82,83]. This interpretation is consistent with industrial requirements of traceability with respect to what is preserved, such as equilibrium, energy, and structure, and what is approximated, such as parameterization, inference, and regularization.

5.6.2. Industrial Extensions: Dissipation and Multiphysics

In scenarios with irreversibility, including plasticity, damage, friction, and corrosion, or with thermomechanical and structure–fluid couplings, methodological coherence requires extension of the functional by a free-energy and dissipation structure and use of discretizations and solvers consistent with the resulting model structure [45–47]. Hamiltonian formulations in fluids and geophysics have also been used to analyze structure and invariants and to guide compatible discretizations in computational fluid dynamics [84–86]. In industrial applications involving vibration and noise and structural–acoustic coupling, variational methods provide a natural language for field coupling and response-control formulations [87,88].

5.6.3. Synthesis

A single methodological chain is represented. Hamilton’s principle provides the generating mechanism of models, weak formulation and FEM provide a computational state operator for performance evaluation, variational optimization including topology transforms energy and constraints into explicit synthesis rules, variational integrators preserve structure over time, and port-Hamiltonian systems extend the approach to interconnection, dissipation, and control. This chain explains the transversal relevance of variational methods from mining infrastructure to advanced industrial applications [1,18,51,61].

6. Evidence-Grounded Comparative Case and Variational Synthesis

6.1. Comparative Case Study: Solid Mechanics, PINNs, and Hamiltonian Formulation

6.1.1. Scope, Assumptions, and Reproducibility Statement

A controlled, fully specified two-dimensional pin-jointed truss (axial bars, small strains, linear elasticity) is adopted to anchor the strain-energy density (SED) interpretation in Figure 13. The primary objective is to compare three representations of the same physical content: (i) the discrete solid equations, (ii) a physics-informed approximation (PINN in residual form), and (iii) a variational/action statement (Hamilton). It is important to clarify that the goal is methodological traceability—that is, to demonstrate how equilibrium, constitutive behavior, compatibility, and energy localizations are encoded in each route—rather than to generalize design conclusions from a single geometry or loading regime.

All modeling assumptions are explicit: (a) plane truss kinematics with two translational DOFs per node, (b) bars carry axial force only, (c) linear elastic law with constant E and constant cross-section A in all members, and (d) static regime unless otherwise stated. Under these assumptions, the solid-equation solution and the static limit of Hamilton’s principle are mathematically equivalent (same admissible space, same functional, same stationary point), and any discrepancy arises only from approximation (PINN capacity) or numerical conditioning.

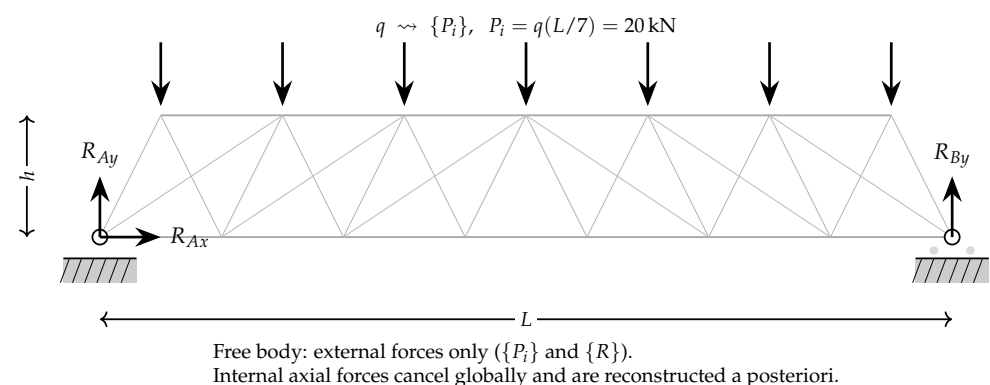


Figure 13. Truss free-body diagram (external actions) superposed on the computational topology (overbraced Warren truss). A common starting point is provided for: (i) matrix equilibrium, (ii) physics residuals in PINNs, and (iii) action-based (Hamilton) formulations.

6.1.2. Case Definition: Geometry, Topology, DOF Ordering, and Load Discretization

A span $L = 14$ m and height $h = 2$ m were prescribed, together with linear elastic material behavior with $E = 200$ GPa and uniform area $A = 2 \times 10^{-3}$ m² (see Table 1). The structural idealization is an overbraced Warren-type truss: a Warren topology (adjacent diagonals) augmented by six redundant diagonals. This redundancy is representative of conveyor-gallery bracing (stiffness and redundancy demands) and renders the truss statically indeterminate of degree

$$s = m + r - 2j = 33 + 3 - 2(15) = 6, \quad (71)$$

where m is the number of bars, r the number of external reaction components, and j the number of joints.

Table 1. Base-case parameters for Figure 13 (2D axial-bar truss).

Parameter	Value
Span/Height	$L = 14$ m, $h = 2$ m
Elastic modulus/Area	$E = 200$ GPa, $A = 2 \times 10^{-3}$ m ² (uniform in all members)
Nodes/DOFs	$j = 15$ nodes (30 translational DOFs)
Members/Indeterminacy	$m = 33$ bars; degree of indeterminacy $s = 6$
Distributed load idealization	$q = 10$ kN/m along the span (belt + ore)
Nodal load rule (discretization)	Midpoint lumping over $n = 7$ equal panels: $\Delta x = L/n = 2$ m, $P_i = q \Delta x = 20$ kN
Equivalence checks	$\sum_{i=1}^7 P_i = qL = 140$ kN and $\sum P_i x_i = qL(L/2)$ (centroid preserved)
Supports	Left: pin ($u_x = u_y = 0$ at N1), Right: roller ($u_y = 0$ at N8)

Node Set and Coordinates

To remove any ambiguity in member labels and connectivity, the joint set is explicitly reported in Table 2. Bottom nodes are denoted $N1, \dots, N8$ and top nodes $T1, \dots, T7$.

Table 2. Node coordinates (m). Bottom chord $N1$ – $N8$ and top chord $T1$ – $T7$.

Node	x	y	Node	x	y
$N1$	0	0	$T1$	1	2
$N2$	2	0	$T2$	3	2
$N3$	4	0	$T3$	5	2
$N4$	6	0	$T4$	7	2
$N5$	8	0	$T5$	9	2
$N6$	10	0	$T6$	11	2
$N7$	12	0	$T7$	13	2
$N8$	14	0	—	—	—

Member Set (Topology)

The full bar set $\mathcal{E}$ is defined as the union of four subsets:

$$\mathcal{E} = \mathcal{E}_{\text{bot}} \cup \mathcal{E}_{\text{top}} \cup \mathcal{E}_W \cup \mathcal{E}_R, \quad (72)$$

$$\mathcal{E}_{\text{bot}} := \{(N_i, N_{i+1})\}_{i=1}^7, \quad L_e = 2 \text{ m}, \quad (73)$$

$$\mathcal{E}_{\text{top}} := \{(T_i, T_{i+1})\}_{i=1}^6, \quad L_e = 2 \text{ m}, \quad (74)$$

$$\mathcal{E}_W := \{(T_i, N_i), (T_i, N_{i+1})\}_{i=1}^7, \quad L_e = \sqrt{5} \text{ m}, \quad (75)$$

$$\mathcal{E}_R := \{(N_1, T_2), (N_2, T_3), (N_3, T_4), (N_6, T_4), (N_7, T_5), (N_8, T_6)\},$$

$$L_e = \sqrt{13} \text{ m}. \quad (76)$$

This specification uniquely defines the stiffness matrix $\mathbf{K}$ by standard assembly and makes member labels (e.g., $N4$ – $N5$, $T4$ – $T5$) unambiguous.

DOF Ordering

The global displacement vector is ordered as

$$\mathbf{u} = (u_{x,N1}, u_{y,N1}, \dots, u_{x,N8}, u_{y,N8}, u_{x,T1}, u_{y,T1}, \dots, u_{x,T7}, u_{y,T7})^T \in \mathbb{R}^{30}. \quad (77)$$

6.1.3. Free-Body Statement and Boundary Conditions

The free-body diagram in Figure 13 treats the *entire truss* as the body. Only external actions are admissible in the free body: the lumped nodal loads $\{P_i\}$ and the support reactions $\{R\}$. All internal bar forces are *excluded* from the global free body, since they appear in equal-and-opposite pairs at joints and cancel in the global balance.

With purely vertical loading and the symmetric discretization $\{P_i\}_{i=1}^7$ (centroid preserved), global equilibrium yields the reactions

$$R_{Ax} \approx 0, \quad R_{Ay} = R_{By} = \frac{qL}{2} = 70 \text{ kN}, \quad (78)$$

which are also recovered numerically from the constrained DOFs after solving $\mathbf{Ku} = \mathbf{f}$.

6.1.4. From Continuum Balance Laws to the Discrete Truss Equations

The governing equations of a linear elastic continuum are stated to fix the physical content subsequently discretized (Ritz/Galerkin) or imposed as a residual (PINN). Under small strains, the displacement field $\mathbf{u}(\mathbf{x}, t)$ satisfies

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{b} = \rho \ddot{\mathbf{u}} \quad \text{in } \Omega, \quad (79)$$

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \nabla^s \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad (80)$$

with the essential boundary condition $\mathbf{u} = \bar{\mathbf{u}}$ on Γ_u and the natural boundary condition $\boldsymbol{\sigma} \mathbf{n} = \bar{\mathbf{t}}$ on Γ_t . In the static regime, $\rho \ddot{\mathbf{u}} = \mathbf{0}$ and static equilibrium is obtained. [2,3,23]

The associated weak form (principle of virtual work) is as follows: find $\mathbf{u} \in \mathcal{V}$ such that for all admissible virtual fields $\delta \mathbf{u} \in \mathcal{V}_0$,

$$\int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\delta \mathbf{u}) d\Omega = \int_{\Omega} \mathbf{b} \cdot \delta \mathbf{u} d\Omega + \int_{\Gamma_t} \bar{\mathbf{t}} \cdot \delta \mathbf{u} d\Gamma. \quad (81)$$

In linear elastostatics, this weak form is equivalent to stationarity of the total potential energy $\Pi(\mathbf{u}) = U(\mathbf{u}) - W(\mathbf{u})$ on $\mathcal{V}$ [3,19,89].

Reduction to an Axial-Bar Truss

Each bar $e = (i, j) \in \mathcal{E}$ is modeled as a one-dimensional axial member. Let $\mathbf{n}_e$ be the unit vector along the member and L_e its length. The axial strain, axial force, SED, and strain energy are

$$\epsilon_e(\mathbf{u}) = \frac{(\mathbf{u}_j - \mathbf{u}_i) \cdot \mathbf{n}_e}{L_e}, \quad N_e(\mathbf{u}) = EA \epsilon_e(\mathbf{u}), \quad (82)$$

$$\psi_e(\mathbf{u}) = \frac{1}{2} \sigma_e \epsilon_e = \frac{N_e(\mathbf{u})^2}{2A^2E}, \quad U_e(\mathbf{u}) = \int_{V_e} \psi dV = \psi_e(\mathbf{u}) (AL_e) = \frac{N_e(\mathbf{u})^2 L_e}{2AE}. \quad (83)$$

Nodal equilibrium at free DOFs is expressed in residual form as

$$\mathbf{r}(\mathbf{u}) = \mathbf{Ku} - \mathbf{f} = \mathbf{0} \quad \text{on free DOFs}, \quad (84)$$

and the assembled global system is written as

$$\mathbf{Ku} = \mathbf{f}. \quad (85)$$

For a bar e with direction cosines (c_e, s_e) , the element stiffness in global coordinates is

$$\mathbf{k}_e = \frac{EA}{L_e} \begin{bmatrix} c_e^2 & c_e s_e & -c_e^2 & -c_e s_e \\ c_e s_e & s_e^2 & -c_e s_e & -s_e^2 \\ -c_e^2 & -c_e s_e & c_e^2 & c_e s_e \\ -c_e s_e & -s_e^2 & c_e s_e & s_e^2 \end{bmatrix}, \quad (86)$$

and $\mathbf{K}$ is obtained by standard assembly over all $e \in \mathcal{E}$ using the DOF ordering stated above. In this setting, the energy relation used throughout the manuscript is recovered:

$$\Pi(\mathbf{u}) = U(\mathbf{u}) - W(\mathbf{u}), \quad U(\mathbf{u}) = \sum_{e \in \mathcal{E}} U_e(\mathbf{u}) = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u}, \quad W(\mathbf{u}) = \mathbf{u}^T \mathbf{f}, \quad (87)$$

and

$$\delta \Pi(\mathbf{u}) = 0 \iff \mathbf{K} \mathbf{u} = \mathbf{f}, \quad (88)$$

so the SED provides a local indicator of the contribution to U [3,19,89].

6.1.5. PINN Route on the Same Discrete Geometry (Residual Minimization in a Parameterized Space)

In contrast to directly solving (85), a differentiable displacement approximation $\mathbf{u}_\theta$ is introduced. Its parameters θ are fitted by minimizing a loss function that penalizes (i) equilibrium residuals and (ii) violations of essential boundary conditions. In the context of this case study, the reported PINN results correspond to a *linear-in-parameters* PINN (Ritz-type parameterization) defined by a total-degree-5 polynomial basis in (x, y) :

$$u_{x,\theta}(x, y) = \sum_{k+\ell \leq 5} a_{k\ell} x^k y^\ell, \quad u_{y,\theta}(x, y) = \sum_{k+\ell \leq 5} b_{k\ell} x^k y^\ell, \quad \theta = \{a_{k\ell}, b_{k\ell}\} \in \mathbb{R}^{42}, \quad (89)$$

where the number of monomials is $n_\phi = \binom{5+2}{2} = 21$ per component.

Evaluating $\mathbf{u}_\theta$ at the nodes defines a linear map $\mathbf{u}_\theta = \mathbf{P}\theta$ with $\mathbf{P} \in \mathbb{R}^{30 \times 42}$. The discrete equilibrium residual is then

$$\mathbf{r}_f(\theta) = (\mathbf{K} \mathbf{u}_\theta - \mathbf{f})|_{\text{free DOFs}} = (\mathbf{K} \mathbf{P} \theta - \mathbf{f})|_{\text{free DOFs}}, \quad (90)$$

and a canonical quadratic loss is adopted:

$$\mathcal{L}(\theta) = \underbrace{\|\mathbf{r}_f(\theta)\|_2^2}_{\text{physics (equilibrium)}} + \beta \underbrace{\|\mathbf{u}_\theta|_{\Gamma_u} - \bar{\mathbf{u}}\|_2^2}_{\text{essential conditions}} + \lambda \Pi(\mathbf{u}_\theta). \quad (91)$$

In the reported results, $\lambda = 0$ and a strong penalty $\beta = 10^4$ is used to enforce supports (in practice, any $\beta \gtrsim 10^4$ leads to the same minimizer for this case). Because $\mathcal{L}$ is quadratic in θ for this linear-in-parameters PINN, the minimizer is obtained deterministically by linear least squares (global optimum; no dependence on stochastic training or initialization). An energy term $\lambda \Pi(\mathbf{u}_\theta)$ can be activated as a variational regularizer to improve physical selectivity for under-determination or noisy data [4,36].

Once θ^* is obtained, the axial forces $N_e(\theta^*)$ and $\psi_e(\theta^*)$ are reconstructed member-by-member through the same constitutive law, so the SED is not treated as a learned output but as a thermodynamically consistent consequence of enforcing equilibrium and constitutive relations in (91).

6.1.6. Hamilton's Principle: Discrete Action and Static Limit

Hamilton's principle is stated in a discrete form for the same truss model by taking $\mathbf{q}(t) \equiv \mathbf{u}(t)$ as the vector of generalized nodal displacements. Stationarity of the action is imposed as

$$\delta \int_{t_1}^{t_2} (T(\dot{\mathbf{q}}) - U(\mathbf{q}) + W(\mathbf{q}, t)) dt = 0, \quad (92)$$

where, for a linear model,

$$T = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}, \quad U = \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q} = \sum_e U_e, \quad W = \mathbf{q}^T \mathbf{f}(t). \quad (93)$$

By applying variations with $\delta \mathbf{q}(t_1) = \delta \mathbf{q}(t_2) = \mathbf{0}$, the equation of motion is obtained:

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{K} \mathbf{q} = \mathbf{f}(t). \quad (94)$$

In the static regime ($\ddot{\mathbf{q}} = \mathbf{0}$ and $\mathbf{f}(t) = \mathbf{f}$), (94) reduces to (85), and Hamilton's principle reduces to stationarity of the total potential $\Pi = U - W$, consistent with the variational framework used across the manuscript [18,20,35].

6.1.7. Metrics for Comparison (Statics)

To compare the three techniques on the same geometry, the following metrics are reported:

$$u_{y,\min} := \min_i u_{y,i}, \quad (95)$$

$$C := \mathbf{f}^T \mathbf{u} \quad (\text{compliance, } [C] = \text{J}), \quad (96)$$

$$U := \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} = \frac{1}{2} C \quad (\text{strain energy}), \quad (97)$$

$$\psi_{\max} := \max_e \psi_e, \quad \psi_e = \frac{N_e^2}{2A^2E}, \quad (98)$$

$$\eta_r := \frac{\|\mathbf{K} \mathbf{u} - \mathbf{f}\|_2}{\|\mathbf{f}\|_2} \quad (\text{equilibrium residual on free DOFs, dimensionless}). \quad (99)$$

As a reference, the solid-equation solution of (85) is adopted, which coincides with the static limit of (94). For the PINN alternative, the order-5 polynomial parameterization described in Table 3 is used, with $\theta^* = \arg \min \mathcal{L}(\theta)$ in (91). The purpose is to expose the classical trade-off between parameterization capacity and satisfaction of equilibrium.

Table 3. PINN specification used for the reported “order 5” result (fully reproducible).

Item	Specification
Parameterization (“architecture”)	Total-degree-5 polynomial basis in (x, y) for each component (u_x, u_y)
Basis size	$n_\phi = \binom{7}{2} = 21$ per component; $n_\theta = 42$ parameters
Discrete evaluation	$\mathbf{u}_\theta = \mathbf{P}\theta$ at the 15 nodes (Table 2)
Physics residual	$\mathbf{r}_f(\theta) = (\mathbf{K} \mathbf{u}_\theta - \mathbf{f}) _{\text{free DOFs}}$
BC penalty	$\beta = 10^4$ (supports: $u_x = u_y = 0$ at N1; $u_y = 0$ at N8)
Energy regularizer	$\lambda = 0$ (optional $\lambda \Pi(\mathbf{u}_\theta)$)
Optimizer/stopping	Exact linear least squares (unique minimizer; no seed/early stopping required)

The quantitative results are linked to Figure 13 and summarized in Table 4. The maximum value of ψ is observed in the central lower chord member, where axial demand is highest and the local contribution to U is maximal. This localization supports the standard stiffness-driven design interpretation in which compliance minimization tends to retain or thicken material in regions that concentrate strain energy [5,49].

Table 4. Quantitative comparison (base case). In statics, the solid equations and the Hamilton formulation coincide (same physics and same functional).

Method	$u_{y,\min}$ [mm]	C [J]	U [J]	$\psi_{\max}$ [kJ/m ³]	η_r [%]
Solid equations ($\mathbf{Ku} = \mathbf{f}$)	−7.247	664.997	332.499	9.766	≈0
PINN (order 5, Table 3)	−7.040	645.443	313.320	9.361	14.153
Hamilton (static limit)	−7.247	664.997	332.499	9.766	≈0

Taken together, the tabulated member-level indicators (Table 5) and the normalized metric comparisons (Figures 14 and 15) provide a consistent picture at the level of global response and strain-energy distribution. However, when approximate equilibrium states are introduced through residual-based surrogates, such as physics-informed neural networks (PINNs), these global metrics alone are not sufficient to assess safety relevance, because equilibrium violations may bias internal forces and limit-state indicators even when energy-based quantities appear close to the reference solution.

Table 5. Most demanded members according to ψ_e (reference). The sign of N_e indicates tension (+) or compression (−).

Member ($i-j$)	L_e [m]	N_e [kN]	ψ_e [kJ/m ³]
N4–N5	2.0	+125.000	9.766
N5–N6	2.0	+119.932	8.990
N3–N4	2.0	+119.932	8.990
T4–T5	2.0	−115.068	8.275
T3–T4	2.0	−115.068	8.275
N6–N7	2.0	+96.051	5.766

In the static truss example, the residual PINN yields an approximate state $\tilde{u}$ by minimizing equilibrium violations on the free degrees of freedom. Let u^* denote the exact equilibrium solution satisfying $Ku^* = f$ on the free DOFs, and define the residual $r := K\tilde{u} - f$ and the error $e := \tilde{u} - u^*$. Then,

$$Ke = r \implies \|e\|_K^2 = e^\top Ke = r^\top K^{-1}r, \quad \frac{\|e\|_2}{\|u^*\|_2} \leq \kappa(K) \eta_r, \quad \eta_r := \frac{\|r\|_2}{\|f\|_2}. \quad (100)$$

A non-negligible η_r therefore constitutes a direct certificate of equilibrium violation and may lead to biased internal forces and limit-state indicators, even if global energy metrics remain close to the reference. Accordingly, in safety-critical mining assessments, residual-based PINN surrogates are restricted to conceptual load-path screening and sensitivity studies, or to monitoring and decision support as a trigger for a subsequent verified equilibrium solve; they are not used as stand-alone evidence for code compliance or for limit states governed by instability, fatigue, or degradation without an explicit verification-and-translation stage [4,90].

6.1.8. Engineering Synthesis and Methodological Comparison

A methodological comparison is established at an engineering level and summarized in Table 6. The three routes share the same physical basis (compatibility, constitutive behavior, and equilibrium), but differ in how these laws are enforced: direct solution of discrete equations, minimization of a physics residual in a parameterized function space (PINN), and stationarity of an action functional (Hamilton) that yields the equations of motion and, in statics, the minimum total potential principle. The SED links the global energetic criterion (U or Π) to a local interpretation of where the structure stores strain energy and therefore where stiffness-driven redesign would preferentially allocate material.

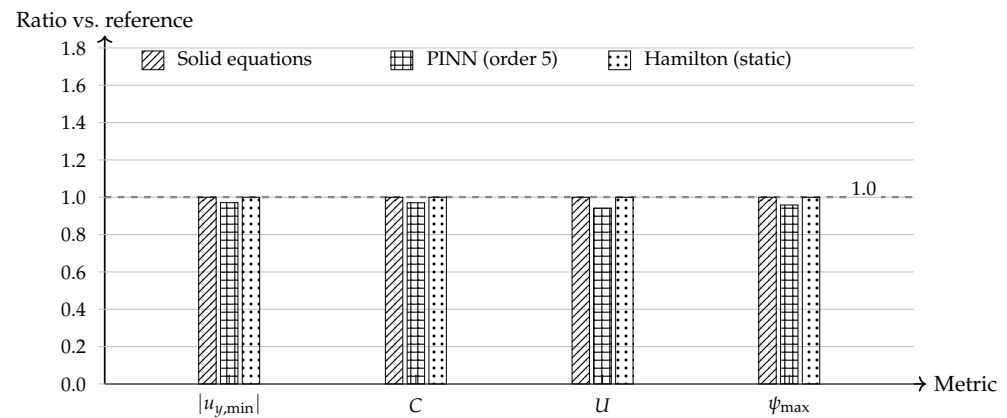


Figure 14. Normalized comparison of static metrics relative to the reference solution (solid equations). The Hamilton formulation coincides with the reference; the PINN approximation depends on capacity and the adopted residual minimization.

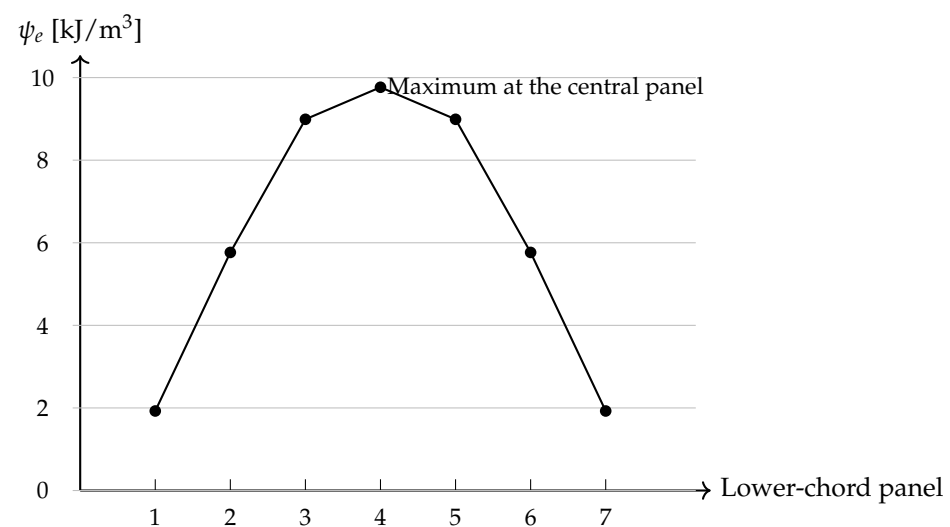


Figure 15. SED profile ψ_e along the lower chord (reference). The SED peaks at the center, supporting material concentration under stiffness-driven criteria (compliance minimization).

Table 6. Methodological comparison between (1) solid equations, (2) PINNs, and (3) Hamilton.

Criterion	Solid Equations	PINNs (Residual)	Hamilton (Action/Energy)
Primary object	$\mathbf{Ku} = \mathbf{f}$ (static)/continuum PDE	$\min_{\theta} \mathcal{L}(\theta)$ with physics residual + BC	$\delta \int (T - U + W) dt = 0$
Equilibrium guarantee	Exact (up to numerical error; on free DOFs)	Approximate (capacity-dependent)	Exact (equations are generated; in statics $\delta \Pi = 0$ is recovered)
Error nature	Discretization/conditioning	Approximation bias (subspace) + optimization (if nonlinear PINN)	Discretization/conditioning
Computational scaling	Robust sparse linear solve	Iterative training or deterministic LS (if linear-in-parameters)	Identical to solid equations after discretization
SED and energy	Direct: ψ_e from $\mathbf{u}$	Direct: ψ_e from $\mathbf{u}_{\theta}$	Direct: $U = \sum U_e$, $\psi_e = U_e / (AL_e)$

6.1.9. Micro-Sensitivity: Uniform Area Scaling ($A \rightarrow \alpha A$) and Stability of the Qualitative Conclusions

To avoid over-inference from a single parameter set while keeping the case lightweight, a simple sensitivity is reported. Consider a uniform scaling of all member areas $A \rightarrow \alpha A$ with E and $\mathbf{f}$ fixed. Since $\mathbf{K} \propto EA$, the exact solid-equation solution scales as

$$\mathbf{K} \rightarrow \alpha \mathbf{K} \implies \mathbf{u} \rightarrow \frac{1}{\alpha} \mathbf{u}, \quad C = \mathbf{f}^T \mathbf{u} \rightarrow \frac{1}{\alpha} C, \quad U \rightarrow \frac{1}{\alpha} U. \quad (101)$$

Moreover, the axial forces satisfy $N_e = EA \epsilon_e$ with $\epsilon_e \propto \mathbf{u}$, so N_e is invariant under uniform scaling ($N_e \rightarrow N_e$), whereas stresses and SED scale as

$$\sigma_e = \frac{N_e}{A} \rightarrow \frac{1}{\alpha} \sigma_e, \quad \psi_e = \frac{N_e^2}{2A^2E} \rightarrow \frac{1}{\alpha^2} \psi_e. \quad (102)$$

Therefore, the *location* of $\psi_{\max}$ is unchanged (same critical member), and only magnitudes scale. For completeness, Table 7 reports the resulting values for $\alpha = 0.8$ and $\alpha = 1.2$.

Table 7. Sensitivity to uniform area scaling $A \rightarrow \alpha A$ (solid equations/Hamilton static). Axial forces are invariant; displacements and energies scale with $1/\alpha$; SED scales with $1/\alpha^2$.

α	$u_{y,\min}$ [mm]	C [J]	U [J]	$\psi_{\max}$ [kJ/m ³]	Critical Member
0.8	−9.059	831.247	415.623	15.259	N4–N5
1.0	−7.247	664.997	332.499	9.766	N4–N5
1.2	−6.039	554.165	277.082	6.782	N4–N5

6.1.10. Explicit Output: Minimal Reproducibility Checklist

To make the section a reusable methodological artifact (beyond a standalone demonstration), the following checklist defines the *minimal* reporting items required to reproduce the case study:

- **Geometry and node set:** coordinates (x_i, y_i) for all nodes (Table 2), including the node labeling convention (N_i, T_i) .
- **Topology/connectivity:** complete member set $\mathcal{E}$, including chords, Warren diagonals, and redundant members $\mathcal{E}_R$ (explicitly listed above).
- **Material and section data:** E , A (uniform or per member), and statement of linear elasticity and axial-bar idealization.
- **DOF ordering and constraints:** definition of $\mathbf{u}$ ordering and essential boundary conditions (pin at N1, roller at N8).
- **Loads and discretization rule:** q and the rule $q \rightsquigarrow \{P_i\}$ (midpoint lumping, $P_i = qL/7$) plus verification $\sum P_i = qL$.
- **Assembly:** element stiffness $\mathbf{k}_e$ and global assembly rule for $\mathbf{K}$ (or the full $\mathbf{K}$ provided as supplemental data).
- **Derived quantities:** definitions of C , U , ψ_e , $\psi_{\max}$, and the residual metric η_r (on free DOFs).
- **PINN specification (if used):** basis/architecture, parameter count, β and λ in (91), and solver procedure (Table 3).

7. Discussion

A stronger methodological claim is supported than the statement that topology optimization is computationally effective for mining infrastructure. Structural synthesis can be formulated as a hierarchical variational problem in which the state is obtained as a stationarity condition of an energetic functional, while the design is obtained as the solution of an

optimization problem constrained by that state. This framing places structural engineering within the same conceptual continuum that links extremal principles in optics and analytic mechanics and culminates in Hamilton's principle as a generator of the equations of motion (Figures 4 and 5) [18,19,33,35].

The SIMP+FEM topologies are interpreted as discrete, regularized extrema of an effective problem rather than as shapes arbitrarily imposed by an algorithm. Therefore, a rigorous separation is required between the physical content (constitutive assumptions, equilibrium, admissibility, stability) and numerical or effective-model content (mesh, material interpolation, penalization, filtering, projection, and solver tolerances). Unless this separation is made explicit, discretization artifacts may be misinterpreted as physical causality. Furthermore, without such clarity, the criteria for transferring an optimum to detailed engineering remain underspecified [1–3,6–8,25].

An integrative contribution is obtained by making three connections explicit. First, continuity is established from action to total potential to weak form to FEM, with equilibrium treated as stationarity. Second, structural synthesis is posed as an optimal control problem constrained by equilibrium in the PDE/FEM sense. Third, implications of the Hamiltonian structure are carried into simulation and optimization through structure-preserving discretization and through energy-based interconnection and dissipation models, including variational integrators and port-Hamiltonian representations (Figures 6 and 7) [18,51,52,61,62]. The comparative example in Section 6.1 is accordingly interpreted as a taxonomy of what is preserved (physics, energy, structure) and what is approximated (parameterization, regularization, optimization) rather than as a competition between methods [4,36].

For mining structures, the methodological framing above becomes design-relevant only when the variational and optimization results are translated into the limit states that govern real assets. Mining infrastructure is typically code-governed and highly utilized, and its acceptability is determined not only by equilibrium and volume constraints but also by stability and serviceability requirements that constrain admissibility in practice. Consequently, the admissible set is effectively filtered by code-type checks on member slenderness, global and local buckling resistance, connection detailing, and serviceability (deflections and vibration-related criteria), which determine whether a numerically "stiff" layout is buildable and certifiable in steelwork [91,92].

This observation also provides a critical lens for interpreting the reviewed topology optimization literature: improvements are often demonstrated under simplified loading, idealized boundary conditions, and homogenized constitutive assumptions, so the most transferable contribution to mining practice is frequently the identification of dominant load paths and redundancy patterns rather than the direct adoption of the produced geometry. A rigorous transfer therefore requires an explicit verification-and-translation stage, in which synthesized layouts are mapped to fabricable typologies (profiles, plates, stiffeners, bracing systems) and then rechecked under code-consistent combinations and detailing rules, so that the performance claim is anchored to the final buildable artifact rather than to an intermediate density field [1,6,12–14].

In addition, mining assets are frequently governed by mechanisms that are not captured by stiffness-only objectives: slender conveyor galleries and transfer towers can be controlled by stability, vibrating equipment frames by resonance and fatigue, and aggressive environments by degradation, corrosion, and connection deterioration. The discussion is therefore sharpened by asking, for each variational/optimization paradigm, which governing mechanisms it can represent (or approximate) within the objective–constraint structure and which require explicit augmentation through stability-, dynamic-, and degradation-

aware constraints to avoid a mismatch between the reported optimum and the governing mining design challenge [2,19,45–47].

7.1. Bi-Level Variational Synthesis and the Meaning of the Discrete Optimum

A bi-level variational structure is obtained. For a given design, such as a density field $\rho(\mathbf{x})$ that induces an effective stiffness, the state is not primarily treated as the solution of equations but as a critical point of the total potential,

$$\text{Find } \mathbf{u}(\rho) \in \mathcal{V}_{\text{ad}} \text{ s.t. } \delta\Pi(\mathbf{u};\rho)[\delta\mathbf{v}] = 0 \quad \forall \delta\mathbf{v} \in \mathcal{V}_0, \quad \Pi(\mathbf{v};\rho) = U(\mathbf{v};\rho) - W(\mathbf{v}). \quad (103)$$

In conservative systems, the physical selection among critical points is governed by stability. A local minimum is ensured when the second variation is positive and the functional is coercive in the relevant space, which provides the precise meaning of minimum energy in the admissible setting and separates admissible equilibria from unstable critical points under buckling and bifurcation mechanisms that are relevant in slender mining structures [2,18,19,24].

The outer level is obtained when design is treated as a decision variable, where $\mathbf{u}(\rho)$ is defined by (103).

$$\min_{\rho \in \mathcal{U}_{\text{ad}}} J(\rho, \mathbf{u}(\rho)) \quad \text{s.t.} \quad \mathbf{u}(\rho) \text{ satisfies } \int_{\Omega} \rho \, d\Omega \leq V^*, \quad 0 \leq \rho \leq 1. \quad (104)$$

The FEM approximation is introduced through a Ritz/Galerkin restriction of $\mathcal{V}_{\text{ad}}$ to a finite-dimensional subspace $\mathcal{V}_h$, so that a discretized effective problem is solved. The computed optima are therefore interpreted as extrema of the discretized and regularized functional actually solved rather than of an idealized continuous problem. As a consequence, regularization that fixes a characteristic length and mitigates mesh dependence is reported explicitly as part of the model definition, together with projection and penalization choices and with convergence tolerances that define the effective solution obtained [1–3,6–8,25,48].

Homogenization and shape-optimization viewpoints provide an interpretation of sequences of discrete designs and clarify the role of regularization as an effective micro-structure selection mechanism. This interpretation supports the suppression of spurious patterns and improves transferability to manufacturable geometries [1,14,25,48].

The FEM is obtained as a Ritz/Galerkin consequence of the variational state problem, so the discrete solution approximates an extremum of a discretized functional rather than only an algebraic solution. This origin explains symmetry and energy-norm interpretation under conservative assumptions and provides the basis for reliability, error control, and stable sensitivity evaluation when analysis is embedded in the design loop [2,3,90,93,94].

For mining assets, the bi-level reading in (104) becomes operational only if the admissible design set encodes governing limit states beyond equilibrium, because equilibrium stationarity is necessary but not sufficient for a certifiable steel structure. This supports interpreting mining-relevant bi-level formulations as physics-constrained optimization under a code-filtered admissible set, where constraints represent stability and serviceability margins checked in practice, rather than as purely mathematical add-ons [91,92].

From the variational standpoint, this extension is coherent with the separation already emphasized: equilibrium remains the first-order condition $\delta\Pi = 0$, while physical acceptability is linked to stability (second variation) and to constraint satisfaction under prescribed scenarios. In slender mining structures, where bifurcations and slenderness-driven failures are credible, the “meaning” of an optimum must therefore include a stability statement: a design that reduces compliance but produces low buckling margins is not mining-feasible in the admissible sense implicit in minimum-energy interpretations [18,19,24,44].

A practical representation of this requirement within topology optimization is to treat stability margins as explicit inequality constraints coupled to the state problem, for example by enforcing lower bounds on critical buckling load factors or on eigenvalue-based stability indicators. This makes the discussion more incisive because it clarifies why “stiffness-optimal” and “mining-feasible” are not equivalent labels, and it aligns the variational optimum with the governing mechanisms that steel design standards explicitly check [91,92,95].

7.2. Compliance, Energy, and Load Paths: Physical Interpretation and Numerical Cautions

In linear elasticity with conservative loading, compliance minimization $C = \mathbf{f}^T \mathbf{u}$ is equivalent to maximizing global stiffness and is linked to stored strain energy by

$$C = \mathbf{f}^T \mathbf{u} = 2U. \quad (105)$$

Material is redistributed to reduce global energy storage at a fixed volume, which yields an interpretable physical reading in terms of dominant load paths and an emergent structural hierarchy. This reading is consistent with the use of strain-energy density as a local indicator that connects global objectives with local demand, as discussed in Section 6.1 [1,5,6,96,97].

This mechanism operates on a regularized effective problem. Penalization in SIMP, filtering, and projection determine the set of accessible solutions and are not treated as implementation details. Without them, mesh dependence, non-uniqueness, and spurious patterns such as checkerboarding are obtained, and physical inferences become unreliable unless the minimum resolvable length scale and the filtering operator are specified [1,6–8,25,49].

Mining infrastructure is rarely governed only by static stiffness. When compliance alone is optimized, slender members and geometric concentrations can be produced, and performance can be degraded by buckling, operational vibration, or cyclic loading. The template (104) is therefore treated as a prototype that is extended by additional functional constraints, including serviceability, frequency targets, stability constraints, and fatigue-related measures, while preserving the same variational language [2,3,9–11,19,44].

Mining-Relevant Limit States Beyond Compliance: Multi-Scenario Loading, Stability, Vibration, and Fatigue

In mining design, the physical interpretation of compliance as a proxy for “good design” is conditional: $C = 2U$ in (105) holds under linear elasticity with conservative loading, but many mining actions include operational variability, moving loads, and dynamic amplification. The discussion is therefore strengthened by stating explicitly when compliance is used as a stiffness surrogate for preliminary load-path synthesis under representative quasi-static scenarios, and when it must be replaced or complemented by constraints tied to the governing mechanism (buckling, resonance, fatigue, or degradation) [2,3,19].

A direct mining-driven extension is a multi-scenario design, because conveyor galleries and transfer structures are assessed under multiple load combinations rather than a single load case. In a variational-optimization reading, this is expressed by coupling multiple equilibrium states $\mathbf{u}_k(\rho)$ to the same design, so that the optimum is scenario-robust in the sense of satisfying constraints across the relevant operating envelope; without such coupling, single-case optima should be interpreted primarily as informative load-path prototypes rather than as transferable designs [1,6].

Stability is a first-order mining concern because stiffness-driven topology optimization can generate slender compression members and long unbraced segments that are optimal in an energy norm but poor in buckling resistance. This is not a marginal detail: global stability and local member buckling are explicit checks in steel design standards, so mining-oriented

interpretation must evaluate whether the reviewed optimization evidence accounts for stability or whether it implicitly assumes it away [91,92,95].

Fatigue is another mining-dominant mechanism that changes how “good topology” should be interpreted, because compliance-driven redistribution can increase stress gradients and introduce geometric discontinuities that are benign in static stiffness but harmful under cyclic excitation. Embedding finite-life fatigue constraints into topology optimization is possible, but it requires explicit selection of a fatigue damage model and its coupling to stress/strain outputs, which reorients the optimization from pure stiffness toward life-cycle performance and detail sensitivity [98].

This fatigue sensitivity also highlights a transfer gap that is methodological rather than cosmetic: code-level fatigue verification is governed by detail categories and connection-specific provisions that depend on geometry and joints rather than on continuum densities. Thus, fatigue-constrained topology optimization ultimately must be reconciled with fatigue checks on the translated, fabricated details, because governing damage often localizes at weld toes, bolted joints, and attachments not represented in the density field [98,99].

Accordingly, the case study in Section 6.1 remains valuable for mining practice because it delineates what compliance and SED can and cannot claim: SED localization is an interpretable indicator of where stiffness is “paid for” in strain-energy terms, but it is not, by itself, a proof of stability margin, vibration safety, or fatigue life. This boundary of interpretability is central to addressing mining design challenges without overclaiming [5,6,9,10].

A reporting structure is suggested to limit over-interpretation and to increase transferability. Physical assumptions are reported at the level of constitutive laws, loading, boundary conditions, and stability criteria. Numerical definitions are reported at the level of mesh, SIMP exponent p , filter and projection radii, tolerances, and convergence criteria. Constructability translation rules and verification on a detailed model are reported to support the transfer to engineering practice [1,6,12–14].

7.3. Mining 4.0: Variational Digital Twins and Adaptive Design or Rehabilitation

In instrumented mining assets, the variational structure enables a rigorous framework for data assimilation (4D-Var), where the discrepancy between the model response and sensor data $\mathbf{u}_{\text{obs}}$ is minimized subject to physical laws. In this setting, the optimization problem shifts from pure design to parameter identification,

$$\min_{\boldsymbol{\theta}} \mathcal{J}(\boldsymbol{\theta}) = \frac{1}{2} \|\mathbf{u}(\boldsymbol{\theta}) - \mathbf{u}_{\text{obs}}\|_{\mathbf{W}}^2 + \gamma \mathcal{R}(\boldsymbol{\theta}) \quad \text{subject to} \quad \delta \Pi(\mathbf{u}; \boldsymbol{\theta}) = 0, \quad (106)$$

where the admissible state is defined by the equilibrium at each update. When degradation mechanisms and damage accumulation become relevant over the service life of the asset, equilibrium alone is no longer sufficient to characterize physically admissible states, and the variational formulation must be extended to include an explicit energy-dissipation structure.

Let $\mathbf{z}$ denote internal variables associated with degradation processes (such as damage-driven stiffness reduction, connection deterioration, or corrosion-induced loss parameters), and let $\Psi(u, \mathbf{z}; \boldsymbol{\theta})$ denote the stored or free energy. A standard incremental variational construction then computes, at each update step $t_k \rightarrow t_{k+1}$,

$$(\mathbf{u}_{k+1}, \mathbf{z}_{k+1}) \in \arg \min_{\mathbf{u} \in \mathcal{V}_{\text{ad}}, \mathbf{z} \in \mathcal{Z}_{\text{ad}}} \left[\Pi(\mathbf{u}, \mathbf{z}; \boldsymbol{\theta}, t_{k+1}) + \mathcal{D}(\mathbf{z}, \mathbf{z}_k) \right], \quad (107)$$

with irreversibility constraints (e.g., $z_{k+1} \succeq z_k$) imposed when applicable, where $\Pi(u, z; \theta, t) = \Psi(u, z; \theta) - W(u, t)$ and $\mathcal{D}(z, z_k) \geq 0$ is an incremental dissipation functional. This construction induces a discrete energy inequality of the form

$$\Psi(u_{k+1}, z_{k+1}; \theta) - \Psi(u_k, z_k; \theta) + \mathcal{D}(z_{k+1}, z_k) \leq W(u_{k+1}, t_{k+1}) - W(u_k, t_k), \quad (108)$$

where $\mathbf{W}$ is the inverse sensor noise covariance and $\delta\Pi = 0$ ensures that the updated Digital Twin state remains a valid equilibrium solution. Unlike purely statistical fitting, this physics-constrained approach preserves energetic consistency and stability in the data–model–decision cycle, which is vital when extrapolating maintenance decisions [15,82,83].

This inequality prevents non-physical energy creation and provides a stability-relevant admissibility check in long-horizon iterative loops. In the context of Mining 4.0 digital twins, parameter updating is therefore posed as physics-constrained optimization in which the data-misfit objective is minimized subject to energetic admissibility and, when required, additional stability conditions (such as $\delta^2\Pi > 0$ or positive-definiteness of the consistent tangent in the admissible regime), rather than as unconstrained statistical fitting. Finally, when repeated long-horizon propagation is required, discretization of the underlying variational principle provides structure-preserving time stepping that mitigates numerical drift and supports stable decision-making in iterative data–model–decision cycles [46,47,51].

Repeated state evaluation in optimization, transient analyses, and digital-twin loops makes structure preservation in time integration operationally relevant. Discretization of the action yields variational and symplectic integrators that preserve structure by construction and improve long-horizon qualitative fidelity, which supports stable decision-making in industrial contexts [51,52,54].

The FEM/PINN/Hamilton comparison can be used as a practical guideline for industrial digital twins. Conservation and consistency of equilibrium, energy, and stability are prioritized over purely predictive fitting, especially under extrapolation and over long horizons [4,36,77–79].

Energy-based interconnection and dissipation models provide a compatible extension for multi-domain systems. Port-Hamiltonian formulations provide explicit power-balance structure for robust control and modular interconnection, and structure-informed hybrid models are used when validated by equilibrium, energy, and structural consistency metrics [17,61,62,65].

For mining infrastructure, the principal contribution of a variational digital twin is not merely “better prediction” but a controlled decision loop in which updates remain physically admissible and interpretable in terms of engineering quantities of interest, such as utilization ratios, stability margins, and damage indicators. This makes the choice of state/parameter vector θ and regularization $\mathcal{R}(\theta)$ in (106) design-critical: updating parameters that compensate model error without preserving stability can fit the data yet mislead maintenance decisions under extrapolation beyond the calibration window [15,82,83].

A mining-feasible twin must also confront model discrepancy explicitly, because operational environments include evolving boundary conditions (e.g., support settlement, joint relaxation), changing loads, and degradation. Within the variational framework, these effects should be represented either as explicit state variables or as structured uncertainty, so that “data fidelity” does not silently absorb physics omissions [45,47,82].

This closes the loop with design: the more the design model anticipates mining-relevant limit states (stability, fatigue, degradation), the more meaningful twin calibration becomes, because inferred parameters map to physically interpretable mechanisms rather than compensating artifacts. The shared admissibility logic (equilibrium + stability +

controlled dissipation) thus supports a coherent workflow linking variational synthesis and variational twinning [19,46,47].

7.4. Variational Structural Design vs. Variational Digital Twins: Shared Backbone and Operational Differences

A direct methodological continuity exists between variationally grounded a structural design (including topology optimization) and physics-based digital twins, but their roles, objectives, and uncertainty interfaces differ in operationally relevant ways. Both paradigms can be formulated as optimization problems constrained by a variationally defined state, i.e., a stationarity condition of an energy/action functional. In structural design, the inner problem typically enforces quasi-static equilibrium as $\delta\Pi(\mathbf{u};\rho) = 0$ (or dynamics as $\delta S(\mathbf{q};\theta) = 0$), while the outer level selects design variables to optimize performance metrics (e.g., compliance, weight, stability margins). In digital twins, the same variational state acts as the physical admissibility constraint, but the outer level is reinterpreted as inference: parameters, states, or loads are updated to minimize data misfit under model consistency.

The common backbone is a physics-constrained optimization in which the residual equation $\mathbf{R}(\mathbf{u},\theta) = \mathbf{0}$ is the discrete expression of a stationarity statement. Consequently, both admit a KKT/Lagrangian structure, enabling adjoint sensitivities and gradient-based updates. In this sense, the “physics constraint” used in topology optimization and in twin calibration is not appended but inherited from the generating variational principle: the state remains an extremum of Π (statics) or S (dynamics) for any admissible design or parameter vector.

Key differences arise from the semantics of the outer problem. Structural design modifies geometry and/or material distribution (e.g., ρ , shape, topology), whereas a digital twin typically updates states/parameters/inputs (e.g., θ , boundary conditions, degradation variables) while preserving the physical configuration. Design objectives encode performance and resource trade-offs, while twin objectives encode data fidelity and uncertainty-weighted consistency (e.g., $\frac{1}{2}\|\mathbf{u}(\theta) - \mathbf{u}_{obs}\|_{\mathbf{W}}^2$) together with regularization reflecting prior knowledge. Design is commonly an offline synthesis task, while twins operate in an online or recurrent loop with sequential assimilation; thus, numerical structure preservation (e.g., variational/symplectic integration, controlled dissipation) becomes more critical for twins due to long-horizon iteration and feedback into decisions.

Ill-posedness and regularization also manifest differently but play an analogous mathematical role. Topology optimization is ill-posed without length-scale regularization (filtering/projection), whereas twin updating is ill-posed without statistical/variational regularization (priors, sensor covariance, parameter bounds). In both cases, the chosen regularization defines the effective solvable problem and should be reported as part of the model. This comparison clarifies why variational mechanics provides a unified backbone for both synthesis and operational digitalization in mining: the same stationarity principle defines admissible physics, while the outer level distinguishes design (improving performance by changing the artifact) from twinning (improving state-of-knowledge by reconciling the model with measurements), and highlights mining-specific extensions such as stability constraints, degradation/dissipation, and multiphysics coupling within closed-loop SHM and maintenance workflows.

7.5. Beyond Linear Elasticity: Dissipation, Multiphysics, and Structure Preservation

Industrial validity requires extensions when irreversibilities (plasticity, damage, friction, corrosion) or large deformations become active. A purely conservative principle is then insufficient and is coupled with irreversible thermodynamics through incremental formulations based on free-energy and dissipation potentials. Energetic consistency and coherent tangent operators are obtained, which are required for stability and convergence

in nonlinear solvers [45–47]. The meaning of optimum is also modified: compliance minimization can be insufficient when damage initiation or energy concentration governs performance, and local objectives or constraints that control where energy concentrates become natural [44,47,100].

Mining infrastructure often operates in regimes where irreversible mechanisms are drivers of maintenance and life-cycle cost, so dissipation and degradation are not merely secondary modeling refinements. Variational formulations based on free-energy and dissipation potentials provide a disciplined extension that introduces plasticity, damage, frictional effects, and corrosion-like degradation while retaining consistent tangents and stability checks, which is essential if such models are embedded in optimization loops or in twin updating [45–47].

From a mining design standpoint, this implies that stiffness-based objectives can be misleading when failure is governed by localized dissipation or progressive loss of stiffness in sections and connections. Thus, results derived under purely elastic, conservative assumptions are most transferable as early-stage structural layouts, whereas claims about operational reliability require explicit modeling (or bounding) of dissipation, degradation, and long-horizon stability under the relevant excitation envelope [44,45,47].

In multiphysics settings such as fluid–structure interaction and thermal coupling, functionals and constraints provide a traceable formulation for coupled synthesis, provided that discretization and regularization control spurious coupling modes. Structure preservation in time integration and energy-based interconnection models provide a consistent basis for robust simulation and control in multi-domain systems [8,14,51,52,61,62,65,101].

7.6. Methodological Aspects for Mining Transfer: Critical Validity, Verification, and Traceability

A mining-relevant discussion must make explicit three validity layers: (i) mathematical validity of the variational statement and discretization (consistency, stability), (ii) numerical validity of the effective optimization problem actually solved (regularization, mesh dependence, solver tolerances), and (iii) engineering validity of the translated, buildable structure under code-governed limit states. This layered view prevents over-claiming from visually compelling density plots and clarifies which results are directly actionable and which require nontrivial verification and detailing [1,2,6,8,23].

Verification in mining should be understood as a multi-fidelity pipeline: the topology-optimized field defines a conceptual layout, but the final design must be re-analyzed as a detailed steel model (profiles, plates, joints, eccentricities), because stiffness, stability, and fatigue are strongly influenced by detailing and connection flexibility. This requirement does not weaken topology optimization; it defines its correct role as the synthesis whose outputs must be translated into a certifiable artifact and checked under recognized standards [91,92].

Traceability of the effective problem is critical in mining because designs are audited and revised over long lifetimes, and small modeling choices can change the reported optimum. For topology optimization, SIMP exponent, filter type/radius, projection parameters, continuation, and stopping criteria are part of the model definition, because they define the attainable family of layouts and the minimum feature size that can be fabricated and inspected [7,8,25,49].

Operational variability (loads, boundary conditions, degradation rates) is a primary reason for conservative mining design. When uncertainty is not modeled, deterministic optima should not be interpreted as reliability statements; when uncertainty is modeled, the discussion should specify whether robustness is enforced by scenario sets or other formulations, and how that choice relates to mining decision criteria (availability risk, maintenance windows, safety margins) [1,6].

7.7. Manufacturability, External Validity, and Reproducible Reporting

To bridge synthesis and construction, manufacturability constraints are included in the admissible design set. Density-based topology optimization produces diffuse density fields that typically require thresholding and post-processing, which may degrade performance when geometric control is not enforced. In contrast, sharp-boundary approaches such as level-set methods and structural parameterizations improve transfer to CAD and allow minimum thickness, curvature radii, connectivity, and tolerance constraints to be imposed more directly [6,12–14].

In density-based topology optimization, the reported geometry is a post-processed representation—often obtained by thresholding—of a design field that was computed by solving a regularized optimization problem. Geometry alone is therefore not a sufficient scientific or engineering specification, because it does not encode the regularization and discretization choices that define the admissible design family, the effective minimum feature size, or the numerical criteria that determine which stationary point is attained. A concise way to state the corresponding effective model is through the composed design map

$$\rho \xrightarrow{F_{\min}} \tilde{\rho} \xrightarrow{P_{\beta,\eta}} \bar{\rho}, \quad K = K(\bar{\rho}), \quad R(u, \bar{\rho}) = 0, \quad (109)$$

together with the parameter tuple $\mathcal{M}_{\text{eff}} := (\mathcal{T}_h, p, r_{\min}, \beta, \eta, \text{continuation}, \text{tolerances/stop})$. Because the underlying continuous problem is ill-posed without explicit regularization, $\mathcal{M}_{\text{eff}}$ is constitutive of the solved problem; reporting only the final geometry is therefore insufficient for reproducibility, comparability, and auditable transfer to mining workflows [6–8].

To support reproducibility and deployment in industrial settings, the effective design specification must include, at a minimum, the objective and constraints with associated load scenarios; the finite element discretization (mesh, element type, degrees of freedom, and boundary conditions); the material interpolation and SIMP exponent (including any continuation strategy); the filter operator or PDE filter and radius together with projection parameters; solver settings, tolerances, and stopping criteria; and the post-processing or translation rules (thresholding and CAD reconstruction) together with the verification step performed on the translated steel model.

In density-based topology optimization (SIMP), the design is parameterized by a relaxed pseudo-density field $\rho(x) \in [0, 1]$ coupled to the state through a material interpolation (e.g., $E(\rho) \sim \rho^p$) and an explicit regularization pipeline (filtering and projection) that defines the effective solvable problem and its minimum feature size. The downstream CAD step is commonly performed by thresholding, i.e., constructing a binary indicator $\chi(x) = \mathbb{I}[\rho(x) \geq \rho_{\text{th}}]$. This binarization constitutes a non-smooth projection applied a posteriori to a continuous optimum, and it may alter stiffness or volume locally; from an engineering workflow standpoint, the thresholded geometry should therefore be treated as a translated artifact that requires re-analysis and, if necessary, local geometric repair before detailing.

By contrast, level-set methods represent the material domain through an implicit boundary $\partial\Omega_m = \{x : \phi(x) = 0\}$, with $\Omega_m = \{x : \phi(x) > 0\}$, so that crisp boundaries are intrinsic to the representation and can be transferred to CAD with reduced ambiguity. Manufacturing constraints that are central in mining steelwork—minimum thickness, minimum curvature radii, and connectivity of load-carrying features—can then be imposed more directly at the geometric level, even though the underlying PDE-constrained state problem remains unchanged. A compact comparison oriented to manufacturability interpretation in mining is summarized in Table 8 [6,12,25].

Table 8. Compact comparison for manufacturability-oriented interpretation in mining.

Aspect	SIMP (Density-Based)	Level-Set (Boundary-Based)
Design variable	$\rho(x) \in [0, 1]$ (relaxed density)	$\phi(x)$ with $\partial\Omega_m = \{\phi = 0\}$
Geometry output	Diffuse field; requires thresholding/projection to obtain $\chi(x)$	Crisp boundary by construction
Length-scale control	Filter/projection parameters define the effective feature size	Geometric regularization and boundary evolution control feature size
Manufacturing constraints	Imposed indirectly (filters/projection; post-processing often required)	Imposed more directly at the geometric level (thickness, radius, connectivity)
CAD transfer	Requires binarization and typically repair and re-check	Direct boundary export with reduced ambiguity

In mining, manufacturability is not only a geometric constraint but also an inspection-and-maintenance constraint, because access for bolting, welding, NDT inspection, and component replacement is integral to operational availability. Sharp-boundary representations are therefore practically relevant insofar as they allow fabrication, assembly, and access requirements to be encoded early enough that optimized layouts survive translation into workshop drawings [12–14].

External validity is limited by the assumptions under which energetic identities hold. The identity (105) is restricted to linear elasticity under conservative loading, and non-conservative loading and nonlinearities require explicit redefinition of functionals and stationarity statements. Compliance prioritizes global stiffness and does not guarantee performance under buckling, fatigue, or corrosion, which are frequently governing the mechanisms in mining assets [2,19,45].

External validity should therefore be tied to the limit states for which the model is truthful: linear-elastic compliance optima are externally valid for stiffness-dominated behavior under conservative assumptions, but do not automatically transfer to buckling-, vibration-, or fatigue-dominated mining assets. This supports treating code checks and mechanism-specific constraints as the boundary between variational synthesis and engineering design, and encourages reporting that makes that boundary explicit [91,92,99].

Reproducibility requires that the effective problem is fully specified. Geometry, load cases, boundary conditions, properties, and stability criteria are reported. Mesh, interpolation, SIMP exponent p , filter and projection parameters, tolerances, and convergence metrics are reported. Post-processing and verification procedures on a detailed model are reported [1,6,7,25].

Reproducible reporting is especially consequential in mining because designs are revisited during expansions, retrofits, and life-extension projects, and the traceability of assumptions becomes operationally safety-relevant. This motivates a discipline in which (i) the effective optimization problem is fully specified, (ii) verification on the translated steel model is documented, and (iii) the link between objective/constraints and the mining-governing limit states is stated explicitly, enabling later re-evaluation under updated loads, degradation states, or requirements [1,6–8,25].

A unified interpretation is supported. Mining structural optimization is positioned as computational variational mechanics in which the optimal form is obtained from energetic principles under explicit discretization, explicit regularization, and physically informed constraints. The methodological implication is that stability, dissipation, multiphysics, manufacturability, and long-horizon structure preservation determine whether an obtained optimum remains valid, interpretable, and transferable under operational conditions [1,14,18,51,61].

7.8. Opportunities and Limitations for Deployment in Real Mining Projects: Where Variational Synthesis Delivers Value and Where It Can Mislead

The transfer of variationally grounded optimization (SIMP/FEM, level-set, and related PDE-constrained formulations) to real mining projects is not limited by the availability of algorithms, but by the alignment between what the *effective problem* optimizes and what the project must ultimately certify, fabricate, inspect, and maintain. In mining, this alignment is particularly demanding because the governing criteria are typically expressed through code-governed limit states (stability, serviceability, fatigue) and through constructability and availability constraints (access for maintenance, robustness to operational variability). Consequently, the sharpest discussion of “opportunities and limitations” is obtained by framing deployment as a multi-stage engineering workflow: topology optimization is most valuable as a *synthesis and decision-support layer* that produces load-path evidence and structural hierarchies, while certification requires a subsequent translation and verification stage on a buildable steel model under code-consistent checks [1–3,6,91,92].

7.8.1. Opportunities: Mining-Relevant Use Cases Where These Techniques Provide Decisive Leverage

This section does not revisit the mechanical interpretation of compliance and related limit states, which has been addressed above; instead, it frames where variational synthesis and topology optimization provide practical leverage in mining workflows when interpreted as *decision-support* for structural architecture rather than as directly buildable deliverables [1,5,6].

A first opportunity is *early-stage structural synthesis* under competing load-path hypotheses. For conveyor galleries, transfer towers, and equipment support frames, several bracing and framing concepts can satisfy basic serviceability targets, yet their relative efficiency and redundancy are difficult to compare without extensive manual iteration. Compliance- or energy-driven synthesis is valuable here because it makes the implicit design choice explicit: it produces a ranked structural *architecture* (dominant load paths, hierarchy, and redundancy patterns) under stated constraints, providing evidence for concept selection that would otherwise be based on intuition and only checked *a posteriori*. The appropriate interpretation is therefore *conceptual layout*, not final member sizing or detailing [1,5,6].

A second opportunity is *scenario-robust conceptual design* aligned with the mining design envelope. Mining structures are assessed through multiple combinations and operational scenarios; variational formulations can couple multiple equilibrium states $\mathbf{u}_k(\rho)$ to a single design, enabling synthesis toward layouts that remain effective across the scenario set. This directly matches mining practice and also clarifies how to read the literature: when coupling is absent, single-case optima should be treated as informative prototypes that reveal load-path tendencies rather than as evidence of project-level robustness [1,6].

A third opportunity is *translation-ready geometry control* for steelwork through sharp-boundary representations. In mining infrastructure, value is realized only if synthesized layouts can be mapped to plates, profiles, stiffeners, and bracing systems compatible with fabrication and detailing. Level-set and related crisp-boundary approaches reduce the ambiguity of diffuse density fields and make geometric controls (minimum thickness, curvature, and member-like features) more directly enforceable, improving continuity from synthesis to CAD-oriented interpretation and subsequent detailing [12–14].

A fourth opportunity is *rehabilitation and life-extension planning* under constrained interventions. When the objective is not a greenfield design but targeted strengthening with limits on intervention locations, downtime, and added mass, variational indicators can be used to prioritize *where* reinforcement is structurally most effective and to compare alternative strengthening topologies consistently. The downstream deliverable remains a

conventional detailed steel model, but synthesis reduces the search space and supports transparent justification of the intervention strategy [1–3,6].

A fifth opportunity is the Mining 4.0 interface between synthesis and operations: *variational digital twins* can update state/parameter estimates under explicit physical admissibility, so that design intent (load paths and stiffness distribution) can be tracked against evolving conditions such as support settlements, operational boundary changes, or degradation. The value is conditional: updated rules must preserve equilibrium and interpretable quantities of interest, rather than absorbing model error through unphysical parameter drift [15,82,83].

7.8.2. Limitations: What Prevents Direct Adoption as “Buildable Designs” and What Must Be Made Explicit in a Mining Workflow

A first limitation is the *model-to-artifact gap*. Many results are produced on continuum density fields with idealized supports, simplified connection behavior, and homogenized materials. Mining steelwork, by contrast, is often governed by joint flexibility, eccentricities, local plate behavior, and constructability constraints. Consequently, a density-field optimum is not a certifiable artifact: it is an extremum of the *effective numerical model solved* and must be translated into a buildable steel typology (members, plates, joints) and then re-verified under code-consistent combinations and detailing rules [1,6,12–14,91,92].

A second limitation is the *objective-to-acceptance mismatch*. Mining acceptance is ultimately determined by code-governed verification and documented engineering judgment, whereas many formulations optimize stiffness surrogates. Even when additional constraints are embedded, optimization outputs are best positioned as evidence for structural architecture (layout, redundancy, and feasible load paths), while certification remains anchored to the translated steel model and its checks for stability, serviceability, vibration, and fatigue as applicable. Therefore, mining deployment requires an explicit statement of scope: what mechanisms are represented within synthesis and what are deferred to downstream verification [1,6,91,92,99].

A third limitation is *regularization-dependence and traceability*. In SIMP-type workflows, filtering, projection, penalization, continuation, and stopping criteria define the attainable design family and minimum feature size. In long-life mining assets subject to audit and revision, this dependence is not a numerical footnote: it affects what layout is obtained and what can be fabricated and inspected. Project-relevant use therefore requires reporting the effective numerical definition (mesh, filter type/radius, projection parameters, SIMP exponent, tolerances, convergence/termination criteria) as part of the design specification, otherwise results are not reproducible nor transferable [6–8,25,49].

A fourth limitation is *fatigue-critical detail semantics*. In fatigue-dominated assets, governing damage localizes at weld toes, bolts, attachments, and connection geometries that are not represented in continuum densities. Thus, even when synthesis informs member layout or reinforcement placement, acceptance requires explicit reconciliation with code-level fatigue verification of the fabricated details, including the adopted detail categories and joint assumptions. This constraint is methodological rather than cosmetic: without a defined translation-and-check step, fatigue claims cannot be transferred from the continuum optimum to the engineered artifact [98,99].

7.8.3. A Deployment Pathway That Turns “Optimized Topologies” into Mining-Actionable Engineering Outputs

To operationalize these techniques in mining projects without over-claiming, a disciplined deployment pathway is recommended: (i) define the design envelope as a scenario set with explicit quantities of interest (e.g., drift/stiffness targets, frequency targets, stability margins, fatigue-relevant indicators), (ii) choose a synthesis representation con-

sistent with downstream translation (density-based with explicit regularization, or sharp-boundary/level-set when geometry control is required), (iii) embed only those constraints that are intended to be represented at synthesis stage and state explicitly which checks are deferred, (iv) translate the synthesized layout into a fabricable steel typology (profiles, plates, bracing, joints) with explicit detailing assumptions, and (v) perform verification—and iterate if needed—on the detailed model under code-consistent checks. Within this workflow, topology optimization is positioned correctly as a variational synthesis engine whose outputs become engineering decisions only after translation and verification [1–3,6,12–14,91,92].

8. Conclusions

This systematic review supports a clear technical thesis: a variational reading provides a single, traceable backbone that connects (i) structural analysis (equilibrium as stationarity), (ii) structural synthesis (PDE/FEM-constrained optimization), and (iii) operational updating (physics-constrained digital twins). This unification is not merely conceptual: it clarifies what is physically enforced (admissible states defined by a variational statement) versus what is numerically defined by discretization and regularization, which ultimately determines what an “optimized” solution means in practice [1–3,6,18].

The review also yields specific insights that sharpen how the literature should be interpreted for mining infrastructure. Across the reviewed evidence, the dominant demonstration pattern remains the quasi-static, linear-elastic synthesis driven by compliance (or equivalent energy objectives), often under a limited number of load cases and idealized boundary and connection behavior. In mining terms, this body of work is most defensible when interpreted as conceptual load-path and bracing architecture synthesis, not as a directly certifiable steel design deliverable [1,5,6].

Three recurrent transfer gaps emerge consistently from the reviewed studies and explain why conclusions based on stiffness-only optima cannot be generalized to mine-scale assets without additional steps:

- **Effective-problem traceability gap.** Many studies do not fully report the effective numerical model (mesh dependence controls, filter and projection settings, SIMP penalization and continuation strategies, termination criteria). Because these choices define the attainable design family and length scale, incomplete reporting directly limits the reproducibility and transfer to audited mining projects [7,8,25,49].
- **Limit-state mismatch gap.** Mining structures are frequently governed by mechanisms beyond static stiffness, including global and local buckling, vibration and resonance, fatigue, and degradation. The review shows that incorporating such mechanisms is feasible but changes the character and computational cost of optimization; therefore, compliance-based results must either include explicit stability, dynamic, or fatigue constraints or state clearly that these mechanisms are deferred to downstream verification [91,92,95,98,99].
- **Model-to-artifact translation gap.** A density field or continuum topology is not a fabricated steel system. Transferable contributions are those that document a disciplined translation to profiles, plates, and joints and then demonstrate verification on the translated model under code-consistent combinations. Fatigue-critical acceptance is particularly sensitive because governing damage localizes at details not represented in continuum densities [12–14,91,99].

From these findings, a mining-feasible conclusion is not that topology optimization produces better designs, but that it reduces conceptual uncertainty early in the workflow by making the load-path competition explicit, provided that deployment is framed as a synthesis–translation–verification pipeline. Practically, this implies defining a scenario

envelope and quantities of interest, selecting a representation consistent with downstream constructability, embedding governing constraints where they are known to dominate (or bounding the claim when they are not modeled), and translating and re-verifying on a detailed steel model under code checks before the design decisions are finalized [1,6,14,91].

Finally, the review highlights a high-impact frontier for Mining 4.0. Variational digital twins and assimilation framed as physics-constrained optimization can maintain physical admissibility during model updating, enabling condition tracking in terms that remain interpretable for engineering decisions. However, the same evidence also implies a strict requirement: update mechanisms must preserve equilibrium, stability, and meaningful parameters, rather than compensating model error through unphysical drift [15,82,83]. The illustrative case in this manuscript is positioned accordingly: it demonstrates reporting discipline and interpretable energy localization, while explicitly delimiting what stiffness and SED evidence can and cannot certify about stability, vibration, or fatigue performance.

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Abbreviations

The following abbreviations are used in this manuscript:

API	Application Programming Interface (interfaz de programación de aplicaciones)
BC	Boundary condition(s) (condición(es) de borde)
BVP	Boundary value problem (problema de valores de borde)
CAD	Computer-Aided Design (diseño asistido por computador)
CAPEX	Capital expenditure (inversión de capital)
CFD	Computational Fluid Dynamics (dinámica de fluidos computacional)
CFL	Courant–Friedrichs–Lewy (condición de estabilidad numérica)
CO ₂	Carbon dioxide (dióxido de carbono)
CPU	Central Processing Unit (unidad central de procesamiento)
DOF	Degree(s) of freedom (grado(s) de libertad)
DP	Dissipation potential (potencial disipativo)

EDP	Ecuación(es) en derivadas parciales
FE	Finite element(s) (elemento(s) finito(s))
FEA	Finite Element Analysis (análisis por elementos finitos)
FEM	Finite Element Method (método de elementos finitos)
FO	First-order optimality (condiciones de optimalidad de primer orden)
FOSD	First-order sensitivity derivative(s) (derivada(s) de sensibilidad de primer orden)
FVM	Finite Volume Method (método de volúmenes finitos)
GPU	Graphics Processing Unit (unidad de procesamiento gráfico)
HNN	Hamiltonian Neural Network(s) (red(es) neuronal(es) hamiltoniana(s))
IDA-PBC	Interconnection and Damping Assignment Passivity-Based Control
KKT	Karush–Kuhn–Tucker (condiciones necesarias de optimalidad con restricciones)
LHS	Left-hand side (lado izquierdo de una ecuación)
ML	Machine learning (aprendizaje automático)
MMA	Method of Moving Asymptotes (método de asíntotas móviles)
ODE	Ordinary Differential Equation(s) (ecuación(es) diferenciales ordinarias)
PDE	Partial Differential Equation(s) (ecuación(es) en derivadas parciales)
PHS	Port-Hamiltonian system(s) (sistema(s) port-Hamiltoniano(s))
PMA	Principle of (stationary) action (Principio de Mínima/Acción Estacionaria)
QoI	Quantity(ies) of interest (cantidad(es) de interés)
RHS	Right-hand side (lado derecho de una ecuación)
RVE	Representative volume element (elemento de volumen representativo)
SED	Strain-energy density (densidad de energía de deformación)
SHM	Structural health monitoring (monitoreo de salud estructural)
SIMP	Solid Isotropic Material with Penalization
TO	Topology Optimization (optimización topológica)
UAV	Unmanned Aerial Vehicle(s) (vehículo(s) aéreo(s) no tripulado(s))
VC	Virtual work (trabajo virtual)
VI	Variational integrator(s) (integrador(es) variacional(es))

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