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Stability and Bifurcation Analysis of a Discrete Tumor-Immune System with Allee Effects

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Abstract

Differential equations are usually employed to accurately represent the ongoing relationships between tumor cells and immune effector populations, enabling scientists to discover how variation in growth and response rates affects tumor development or elimination. The essential objective of this work is to analyze the dynamical development of a discrete tumor-immune interaction model, with a particular focus on finding out how the combined effects of tumor growth and immune response influence tumor progression. The forward Euler approach is effectively used to discretize the governed system. The bifurcation theory is used to establish the fixed points of the considered system, the stability about the fixed points, and Neimark–Sacker and period-doubling bifurcations. We identify parameter domains that result in tumor existence, restricted oscillations, or full-tumor elimination utilizing stability evaluation, bifurcation examination, and computational simulations. In addition, the 0–1 test is presented. Chaos control is also developed. This article successfully discusses some numerical simulations to verify the results obtained. In general, the research gives an overall insight into this interaction and highlights the circumstances under which the immune system is capable of suppressing or removing tumor cells.

Keywords: tumor-immune model; stability analysis; bifurcations; 0–1 test; chaos control

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1. Introduction

Discrete models are often developed from continuous differential equations employing numerical discretization approaches such as the Euler or Runge–Kutta strategies. Differential equations are essential for simulating tumor-immune system interactions, specifically when the Allee effect is taken into consideration, which describes a threshold population density below which an organism (in this case, tumor or immune cells) cannot sustain itself. Systems of difference equations are commonly employed in discrete tumor-immune simulations to analyze fluctuations in the population over discrete time steps, which enables computational simulations and determination of stability under physiologically appropriate limitations [1]. The immune system plays a vital role in protecting the body against cancer by detecting and eliminating abnormal cells before they develop into tumors, a

process known as immune surveillance. However, tumors are not passive targets; they can evolve strategies to evade or suppress immune responses, giving rise to a dynamic interplay referred to as tumor-immune interactions. This relationship encompasses both anti-tumor mechanisms, such as cytotoxic T lymphocyte and natural killer (NK)-cell-mediated killing, and pro-tumor effects, including immune suppression and inflammation-driven tumor progression. A deeper understanding of these interactions is crucial for the design of effective cancer immunotherapies and the improvement of patient outcomes.

The study of virus transmission between populations using dynamical systems plays a crucial role in the biological, medical, and social sciences. Many researchers have focused on modeling the spread of diseases between individuals. For instance, Danane et al. [2] analyzed a model consisting of fractional differential equations that describe the interactions between susceptible, exposed, infectious, and removed individuals. In [3], the authors developed a system to examine the evolution of the second wave of the global outbreak in Nigeria. Their model was based on data from the World Health Organization (WHO), Nigeria Center for Disease Control (NCDC), and Wolfram Data Repository, employing the Quasi-Newton algorithm for parameter estimation. Rezapour and Mohammadi [4] investigated the spread of A/H1N1 influenza using an SEIR epidemic model with fractional derivatives defined via the Caputo–Fabrizio fractional-order derivative. Solutions were obtained using the fractional Euler method and the stability of equilibrium points was thoroughly analyzed. Furthermore, Li and Yousef [5] employed the susceptible–infected–recovered (SIR) epidemic model with a saturation function to study virus propagation, comparing patient numbers with treatment capacity.

Several researchers have extensively studied the long-term behavior of models that describe virus transmission. For example, Wang et al. [6] introduced a mathematical model that incorporates seasonality effects to analyze the spread of COVID-19, focusing on bifurcation and stability properties. Yu et al. [7] examined a discrete-time SIR epidemic system, identifying transcritical, period-doubling, and Neimark–Sacker bifurcations, as well as a codimension-two bifurcation, to investigate virus transmission dynamics. The phase portraits illustrating the long-term behavior of their proposed model were also presented therein. Furthermore, Cui et al. [8] proposed SIR epidemic models to explore the impact of healthcare resource availability, specifically hospital beds, revealing the occurrence of backward and saddle-node bifurcations under bed shortages. Ahmed et al. [9] conducted a detailed study on the spread of COVID-19, including bilinear phenomena and recovery rates, and derived bifurcations, control strategies, and insights on vaccination-related social awareness. For further information on virus transmission modeling using dynamical systems, see references [10–14].

Biological and mathematical concerns motivate the investigation of tumor-immune interaction models without Allee effects. This investigation of tumor-immune relationship models without Allee effects is biologically motivated by the need to improve understanding of cancer-immune interactions under the assumption that tumor development is not restricted by a vital population threshold. In certain biological applications, especially those involving immediately reproducing tumor cell populations, the Allee effect may be insignificant, causing cancer cells to develop from extremely tiny starting sizes. In addition, discrete-time examination is particularly beneficial for mimicking clinical interventions such as chemotherapy or immunotherapy provided at certain times, when the timing and strength of immune responses are vital for the success of treatment. From the point of view of mathematics, avoiding the Allee factor facilitates the growth function, decreasing non-linearity and making the model simpler for analytical stability study, bifurcation examinations, and parameter sensitivity evaluations.

The principal objective of this research is to explore the dynamical behavior of the following tumor-immune interaction model [1], with a specific emphasis on figuring out how a combination of tumor growth and immune response affects tumor growth or elimination:

$$\begin{cases} \frac{dT}{dt} = RT(1 - BT)(T - m) - \left(\frac{AT}{K + T}\right)E, \\ \frac{dE}{dt} = E(CT - D). \end{cases} \quad (1)$$

Here, T represents the density of the tumor cell population, and E denotes the density of the effector cell population at time t . The function $RT(1 - BT)(T - m)$ describes the growth rate of tumor cells, $\frac{AT}{K + T}$ is the killing rate function, and CTE denotes the activation function of effector cells. Furthermore, the parameter R denotes the intrinsic tumor growth rate, $\frac{1}{B}$ is the tumor carrying capacity, m is the strong Allee threshold that affects the tumor-immune dynamics, A is the tumor clearance rate by immune cells, K is the half-saturation constant for the immune response, C is the tumor antigenicity parameter responsible for activating effector cells and D is the natural death rate of immune effectors. Hernandez-Lopez and Wang [1] discussed the dynamical behavior of model (1). However, we will discuss the dynamical behavior of the discrete version of model (1). More specifically, our objective is to determine the existence of equilibrium points, stability investigation, bifurcation analysis, 0–1 test, chaos, control chaos, and how discrete-time dynamics influence these outcomes via some numerical examples.

The structure of this work is given as follows. The discretization of the model is presented in Section 2. The dynamics of system (3) with weak Allee effects are discussed in Section 3. In particular, Section 3.1 analyses the existence and stability of the fixed points. The Neimark–Sacker and period-doubling bifurcations are investigated in Section 3.2. The behavior of the model with the Allee effect is nicely discussed in Section 4. Moreover, Section 5 shows the numerical investigation in this paper while Section 6 illustrates the 0–1 test method. In Section 7, we control the chaos while Section 8 concludes this work.

2. Discretization of Model (1)

In this section, we construct a discrete-time analogue of model (1) by employing the forward Euler discretization method, which is commonly used for approximating continuous-time systems. Under this scheme, the time derivatives of the state variables are approximated as

$$\frac{dT(t_n)}{dt} \approx \frac{T_{n+1} - T_n}{h} \quad \text{and} \quad \frac{dE(t_n)}{dt} \approx \frac{E_{n+1} - E_n}{h},$$

where $h > 0$ denotes the time step size and $T_n := T(t_n)$, $E_n := E(t_n)$, $t_n = nh$, $n = 0, 1, 2, \dots$, represents the discrete time instants [15]. Applying this numerical approximation to model (1) yields

$$\begin{cases} \frac{T_{n+1} - T_n}{h} = RT_n(1 - BT_n)(T_n - m) - \frac{AT_n}{K + T_n}E_n, \\ \frac{E_{n+1} - E_n}{h} = E_n(CT_n - D). \end{cases} \quad (2)$$

After rearranging the above expressions, the discrete dynamical system can be written as

$$\begin{cases} T_{n+1} = T_n + h \left(RT_n(1 - BT_n)(T_n - m) - \frac{AT_n}{K + T_n} E_n \right), \\ E_{n+1} = E_n + h E_n (CT_n - D). \end{cases} \quad (3)$$

Here, n denotes the discrete time index, $h > 0$ is the discretization step size and with nonnegative initial conditions

$$T_0 = T(0) > 0, \quad E_0 = E(0) > 0.$$

The logistic term $T(1 - bT)$ indicates that in the absence of effector cell interactions, the tumor population T would saturate at the carrying capacity $\frac{1}{B}$. The presence of the effector cell term $-\frac{AE}{K + T}$ lowers this effective maximum, reflecting the inhibitory influence of E . Thus, when effector cells are present, the maximum achievable value of T is reduced and remains below $\frac{1}{B}$.

For the effector cell population E , the dynamics governed by the equation for $\frac{dE}{dt}$ indicate that E exhibits exponential growth whenever $T > \frac{D}{C}$. As long as T remains above this threshold, the effector cell population increases exponentially without an intrinsic upper bound. Conversely, if T falls below $\frac{D}{C}$, E decreases exponentially toward zero. Therefore, in model (1), the growth of E is entirely driven by the dynamics of the tumor cell population T and does not possess an intrinsic carrying capacity. In discrete time (system (3)) it is sufficient for the time step h to satisfy $h < \frac{1}{D}$.

3. Dynamics with Weak Allee Effects

We begin our analysis of tumor-immune interaction dynamics by examining the simplified case with weak Allee effects, implemented through $m = 0$. This leads to the following model:

$$\begin{cases} T_{n+1} = T_n + h T_n \left(RT_n(1 - BT_n) - \frac{AE_n}{K + T_n} \right), \\ E_{n+1} = E_n + h E_n (CT_n - D). \end{cases} \quad (4)$$

The first equation in model (4) describes the tumor growth through a logistic term $T_n(1 - BT_n)$ combined with a clearance term $\frac{AE_n}{K + T_n}$, representing elimination by effector cells. The second equation represents the activation of effector cells in the presence of tumor cells, regulated by the antigenicity parameter C . A higher value of C indicates a stronger immune capacity to detect and respond to tumor cells, whereas a lower value of C suggests the possible need for therapeutic interventions that enhance effector activation, as noted in [16,17]. The term DE_n accounts for the natural death of immune cells. Next, we find the fixed points of this system. Basic calculations show that model (4) has three fixed points given as follows:

- $M_0 = (0, 0)$ and $M_1 = \left(\frac{1}{B}, 0\right)$. These fixed points always exist.
- $M_2 = \left(\frac{D}{C}, \frac{RD(KC + D)(C - BD)}{AC^3}\right)$. This fixed point exists if and only if $C > BD$.

3.1. Existence and Stability of Fixed Points of System (4)

In this section, we analyze the stability of the fixed points of model (4). Using Lemmas A1 and A2 (see Appendix A), we examine the local stability of the fixed points obtained. To achieve this, we evaluate the Jacobian matrix of model (4) at the fixed point (T, E) :

$$\mathcal{J}^*(T, E) = \begin{pmatrix} 1 + h \left(R(2T - 3BT^2) - \frac{AKE}{(K+T)^2} \right) & -\frac{hAT}{K+T} \\ hCE & 1 + h(CT - D) \end{pmatrix}. \quad (5)$$

The constraints of the stability of all fixed points are successfully summarized in the following theorems.

Theorem 1. *The origin $M_0 = (0, 0)$, which represents a degenerate fixed point of the system, is non-hyperbolic.*

Proof. To examine the local stability of the fixed point $M_0 = (0, 0)$, we compute the Jacobian matrix at M_0 :

$$\mathcal{J}(M_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - hD \end{pmatrix}.$$

From Lemma A1, the result is proved. $\square$

Theorem 2. *For all positive parameters, the fixed point $M_1 = \left(\frac{1}{B}, 0\right)$ exhibits the following stability characteristics:*

- When $C < DB$, we have
 - M_1 is locally asymptotically stable (sink) if $0 < h < \min\left\{\frac{2B}{R}, \left(\frac{2B}{DB-C}\right)\right\}$.
 - M_1 is a source if $h > \max\left\{\frac{2B}{R}, \left(\frac{2B}{DB-C}\right)\right\}$.
 - M_1 is a saddle if $\frac{2B}{R} < h < \left(\frac{2B}{DB-C}\right)$ with eigenvalues $|\zeta_1| > 1$ and $|\zeta_2| < 1$, or $\left(\frac{2B}{DB-C}\right) < h < \frac{2B}{R}$ with eigenvalues $|\zeta_1| < 1$ and $|\zeta_2| > 1$.
 - M_1 is non-hyperbolic if $h = \left\{\frac{2B}{R}, \left(\frac{2B}{DB-C}\right)\right\}$.
- When $C = DB$, we have that M_1 is non-hyperbolic.
- When $C > DB$, we have
 - M_1 is a saddle if $0 < h < \frac{2B}{R}$ with eigenvalues $|\zeta_1| < 1$ and $|\zeta_2| > 1$.
 - M_1 is a source if $h > \frac{2B}{R} > 0$.
 - M_1 is non-hyperbolic if $h = \frac{2B}{R}$.

Proof. The Jacobian matrix at the fixed point $M_1 = \left(\frac{1}{B}, 0\right)$ can be written as

$$\mathcal{J}(M_1) = \begin{pmatrix} 1 - \frac{hR}{B} & -h\frac{\frac{A}{B}}{K + \frac{1}{B}} \\ 0 & 1 + h\left(\frac{C}{B} - D\right) \end{pmatrix}.$$

Hence, the eigenvalues are

$$\zeta_1 = 1 - \frac{hR}{B}, \quad \zeta_2 = 1 + h\left(\frac{C}{B} - D\right).$$

From Lemma A1, the result is proved. $\square$

Theorem 3. For the unique positive fixed point M_2 of model (4), assuming that $C > BD$ and defining

$$\Delta = R^2 D^2 (3BD^2 - KC^2 - 2CD + 2BDKC)^2 - 4RD^2 C^2 (C - BD)(KC + D)^2,$$

the following statements hold.

i- If any one of the following conditions holds, then M_2 is locally asymptotically stable (i.e., a sink):

1. $\Delta < 0$ and $0 < h \leq h_0 = \left(\frac{RD(3BD^2 - KC^2 - 2CD + 2BDKC)}{RD^2(C - BD)(KC + D)} \right),$
2. $\Delta \geq 0$ and $0 < h \leq h_1 = \left(\frac{RD(3BD^2 - KC^2 - 2CD + 2BDKC) - \sqrt{\Delta}}{RD^2(C - BD)(KC + D)} \right).$

ii- If any one of the following conditions holds, then M_2 is unstable (i.e., a source):

1. $\Delta < 0$ and $h > h_0 = \left(\frac{RD(3BD^2 - KC^2 - 2CD + 2BDKC)}{RD^2(C - BD)(KC + D)} \right),$
2. $\Delta \geq 0$ and $h > h_2 = \left(\frac{RD(3BD^2 - KC^2 - 2CD + 2BDKC) + \sqrt{\Delta}}{RD^2(C - BD)(KC + D)} \right).$

iii- The fixed point M_2 is unstable (i.e., a saddle point) if

$$\Delta \geq 0 \quad \text{and} \quad h_1 < h \leq h_2.$$

iv- The fixed point M_2 is non-hyperbolic if one of the following conditions holds:

1. $\Delta < 0$ and $h = h_0 = \left(\frac{RD(3BD^2 - KC^2 - 2CD + 2BDKC)}{RD^2(C - BD)(KC + D)} \right),$
2. $\Delta \geq 0$ and $h = h_{1,2} = \left(\frac{RD(3BD^2 - KC^2 - 2CD + 2BDKC) \mp \sqrt{\Delta}}{RD^2(C - BD)(KC + D)} \right)$ and $h \neq \left(\frac{2C^2(KC + D)}{RD(3BD^2 - KC^2 - 2CD + 2BDKC)} \right).$

Proof. The Jacobian matrix of model (4) at the fixed point M_2 is given by

$$\mathcal{J}(M_2) = \begin{pmatrix} 1 + \frac{hRD(KC^2 + 2CD - 2BKCD - 3BD^2)}{C^2(KC + D)} & -\frac{hAD}{KC + D} \\ \frac{hRD(KC + D)(C - BD)}{AC^2} & 1 \end{pmatrix}.$$

The characteristic polynomial associated with $\mathcal{J}(M_2)$ is

$$Z_{\mathcal{J}}(\zeta) = \zeta^2 - \text{Tr}(\mathcal{J})\zeta + \text{Det}(\mathcal{J}),$$

with

$$\begin{aligned}\text{Tr}(\mathcal{J}) &= 2 - \frac{hRD(3BD^2 - KC^2 - 2CD + 2BDKC)}{C^2(KC + D)}, \\ \text{Det}(\mathcal{J}) &= 1 - \frac{hRD(3BD^2 - KC^2 - 2CD + 2BDKC)}{C^2(KC + D)} + \frac{h^2RD^2(C - BD)}{C^2}.\end{aligned}$$

Using the characteristic polynomial, the following conditions are calculated:

$$\begin{aligned}Z_{\mathcal{J}}(1) &= \frac{h^2RD^2(C - BD)}{C^2} > 0 \quad (\text{because } C > BD), \quad Z_{\mathcal{J}}(0) = \text{Det}(\mathcal{J}), \\ Z_{\mathcal{J}}(-1) &= 4 - \frac{2hRD(3BD^2 - KC^2 - 2CD + 2BDKC)}{C^2(KC + D)} + \frac{h^2RD^2(C - BD)}{C^2},\end{aligned}$$

and the discriminant of equation $Z_{\mathcal{J}}(\zeta) = 0$ is given by

$$\Delta = R^2D^2(3BD^2 - KC^2 - 2CD + 2BDKC)^2 - 4RD^2C^2(C - BD)(KC + D)^2.$$

Utilizing Lemma A2, we note that the conditions of Theorem 3 are satisfied. $\square$

3.2. Bifurcations of System (4)

Here, we provide an extensive analysis of the Neimark–Sacker and period-doubling bifurcations that occur at the fixed point M_2 . It should be noted that we focus on the bifurcation parameter h to explore the progression of these bifurcations.

3.2.1. Neimark–Sacker Bifurcation Analysis

In order to properly investigate the Neimark–Sacker bifurcation, we consider h as the main bifurcation parameter. In particular, we examine the parameter region defined by

$$\Delta_{NS} = \left\{ (R, A, B, C, D, h) \in \mathbb{R}^6 \mid \Delta < 0, h = h_0 = \left(\frac{RD(3BD^2 - KC^2 - 2CD + 2BDKC)}{RD^2(C - BD)(KC + D)} \right) \right\}.$$

This expression specifies the criteria for the fixed point where it undergoes a transition to quasi-periodic dynamics. The critical bifurcation value h_0 is derived from the ratio of characteristic determinants of the system, while the stability condition $\Delta < 0$ guarantees that the eigenvalues are complex conjugates. This confirms the existence of a Neimark–Sacker bifurcation. To further analyze the bifurcation dynamics, we introduce a small perturbation h^* to the bifurcation parameter h . This perturbation ends up with a modified system based on model (4), which can be expressed as

$$\begin{cases} T_{n+1} = T_n + (h + h^*) \left(RT_n^2(1 - BT_n) - \frac{AT_nE_n}{K + T_n} \right) = \mathcal{H}_1(T_n, E_n, h^*), \\ E_{n+1} = E_n + E_n(h + h^*)(CT_n - D) = \mathcal{H}_2(T_n, E_n, h^*), \end{cases} \quad (6)$$

where h^* represents a small perturbation of the bifurcation parameter.

For analytical convenience, we define the variables $\mathcal{Q}_n = T_n - T^*$ and $\mathcal{P}_n = E_n - E^*$, where (E^*, T^*) denotes the fixed point $\tilde{F}_2$ of the system. This translation shifts the fixed point to the origin, giving $(\mathcal{Q}_n, \mathcal{P}_n) = (0, 0)$. By performing a third-order Taylor expansion of $\mathcal{H}_1(T_n, E_n, h^*)$ and $\mathcal{H}_2(T_n, E_n, h^*)$ about (T^*, E^*) , the system in model (6) takes the following transformed form.

$$\begin{cases} \mathcal{Q}_{n+1} = \alpha_{11} \mathcal{Q}_n + \alpha_{12} \mathcal{P}_n + \alpha_{13} \mathcal{Q}_n^2 + \alpha_{14} \mathcal{Q}_n \mathcal{P}_n + \alpha_{15} \mathcal{P}_n^2 + \alpha_{16} \mathcal{Q}_n^3 + \alpha_{17} \mathcal{Q}_n^2 \mathcal{P}_n \\ \quad + \alpha_{18} \mathcal{Q}_n \mathcal{P}_n^2 + \alpha_{19} \mathcal{P}_n^3 + \mathcal{R}(\|(\mathcal{Q}_n, \mathcal{P}_n)\|^4), \\ \mathcal{P}_{n+1} = \alpha_{21} \mathcal{Q}_n + \alpha_{22} \mathcal{P}_n + \alpha_{23} \mathcal{Q}_n^2 + \alpha_{24} \mathcal{Q}_n \mathcal{P}_n + \alpha_{25} \mathcal{P}_n^2 + \alpha_{26} \mathcal{Q}_n^3 + \alpha_{27} \mathcal{Q}_n^2 \mathcal{P}_n \\ \quad + \alpha_{28} \mathcal{Q}_n \mathcal{P}_n^2 + \alpha_{29} \mathcal{P}_n^3 + \mathcal{R}(\|(\mathcal{Q}_n, \mathcal{P}_n)\|^4), \end{cases} \quad (7)$$

where the coefficients are given by

$$\begin{aligned} \alpha_{11} &= 1 + \frac{hRD(KC^2 + 2CD - 2BKCD - 3BD^2)}{C^2(KC + D)}, \quad \alpha_{12} = -\frac{hAD}{KC + D}, \quad \alpha_{21} = \frac{hRD(KC + D)(C - BD)}{AC^2}, \\ \alpha_{22} &= 1, \quad \alpha_{13} = -\frac{2R(h + h^*)(3BC^2DK^2 + 7BCD^2K + 3BD^3 - C^3K^2 - 3C^2DK - CD^2)}{C(CK + D)^2}, \\ \alpha_{14} &= -\frac{AC^2K(h + h^*)}{(CK + D)^2}, \quad \alpha_{15} = \alpha_{19} = \alpha_{23} = \alpha_{25} = \alpha_{26} = \alpha_{27} = \alpha_{28} = \alpha_{29} = 0, \quad \alpha_{17} = \frac{2AC^3K(h + h^*)}{(CK + D)^3}, \\ \alpha_{16} &= -\frac{6R(h + h^*)(BC^3K^3 + 3BC^2DK^2 + 2BCD^2K + BD^3 + C^2DK)}{(CK + D)^3}, \quad \alpha_{18} = \frac{2AC^3Kh}{(CK + D)^3}, \quad \alpha_{24} = C(h + h^*). \end{aligned}$$

For model (7), the characteristic equation is given by

$$Z_{\mathcal{J}}(\zeta) = \zeta^2 - \text{Tr}(\mathcal{J})_{h^*} \zeta + \text{Det}(\mathcal{J})_{h^*},$$

with

$$\begin{aligned} \text{Tr}(\mathcal{J})_{h^*} &= 2 - \frac{(h + h^*)RD(3BD^2 - KC^2 - 2CD + 2BDKC)}{C^2(KC + D)}, \\ \text{Det}(\mathcal{J})_{h^*} &= 1 - \frac{(h + h^*)RD(3BD^2 - KC^2 - 2CD + 2BDKC)}{C^2(KC + D)} + \frac{(h + h^*)^2RD^2(C - BD)}{C^2}. \end{aligned}$$

The corresponding conjugate complex eigenvalues are

$$\zeta_{1,2}(h^*) = \frac{\text{Tr}(\mathcal{J})_{h^*} \mp i\sqrt{4\text{Det}(\mathcal{J})_{h^*} - (\text{Tr}(\mathcal{J})_{h^*})^2}}{2}.$$

Enforcing the unit-modulus condition $|\zeta_{1,2}(h^*)| = 1$ at $h^* = 0$ yields the relation $|\zeta_{1,2}(0)| = \sqrt{\text{Det}(\mathcal{J})_0}$. Moreover, the rate of change of the modulus with respect to variations in the parameter is described by

$$\left. \frac{d|\zeta_{1,2}(h^*)|}{dh^*} \right|_{h^*=0} = \frac{RD(3BD^2 - KC^2 - 2CD + 2BDKC)}{C^2(KC + D)} \neq 0.$$

To ensure the absence of unit roots at $h^* = 0$, it is required that $\text{Tr}(\mathcal{J})_0$ avoid the critical values $\pm 2, 0, 1$. This condition prevents degeneracies that could interfere with the stability assessment.

To obtain the normal form, let us denote $\wp = \text{Re}(\zeta_{1,2})$ and $\rho = \text{Im}(\zeta_{1,2})$. We then apply the transformation matrix $\begin{pmatrix} 0 & 1 \\ \rho & \wp \end{pmatrix}$, we introduce a new coordinate system through the variable change

$$\begin{pmatrix} \mathcal{Q}_n \\ \mathcal{P}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \rho & \wp \end{pmatrix} \begin{pmatrix} \tilde{T}_n \\ \tilde{E}_n \end{pmatrix}.$$

Rewriting the system in the transformed coordinates, the model takes the form

$$\begin{cases} \tilde{T}_{n+1} = \wp \tilde{T}_n - \rho \tilde{E}_n + \mathcal{H}_1(\tilde{E}_n, \tilde{E}_n), \\ \tilde{E}_{n+1} = \rho \tilde{T}_n + \wp \tilde{E}_n + \mathcal{H}_2(\tilde{T}_n, \tilde{E}_n). \end{cases}$$

Here, the nonlinear terms are expressed as

$$\begin{aligned} \tilde{\mathcal{H}}_1(\tilde{T}_n, \tilde{E}_n) &= \frac{\alpha_{13}}{\alpha_{12}} \mathcal{Q}^2 + \frac{\alpha_{14}}{\alpha_{12}} \mathcal{Q}\mathcal{P} + \frac{\alpha_{15}}{\alpha_{12}} \mathcal{P}^2 + \frac{\alpha_{16}}{\alpha_{12}} \mathcal{Q}^3 + \frac{\alpha_{17}}{\alpha_{12}} \mathcal{Q}^2\mathcal{P} + \frac{\alpha_{18}}{\alpha_{12}} \mathcal{Q}\mathcal{P}^2 + \frac{\alpha_{19}}{\alpha_{12}} \mathcal{P}^3 + \tilde{\mathcal{R}}(\|(\mathcal{Q}_n, \mathcal{P}_n)\|^4), \\ \tilde{\mathcal{H}}_2(\tilde{T}_n, \tilde{E}_n) &= \left(\frac{\alpha_{13}(\wp - \alpha_{11})}{\rho\alpha_{12}} - \frac{\alpha_{23}}{\rho} \right) \mathcal{Q}^2 + \left(\frac{\alpha_{14}(\wp - \alpha_{11})}{\rho\alpha_{12}} - \frac{\alpha_{24}}{\rho} \right) \mathcal{Q}\mathcal{P} + \left(\frac{\alpha_{15}(\wp - \alpha_{11})}{\rho\alpha_{12}} - \frac{\alpha_{25}}{\rho} \right) \mathcal{P}^2 \\ &\quad + \left(\frac{\alpha_{16}(\wp - \alpha_{11})}{\rho\alpha_{12}} - \frac{\alpha_{26}}{\rho} \right) \mathcal{Q}^3 + \left(\frac{\alpha_{17}(\wp - \alpha_{11})}{\rho\alpha_{12}} - \frac{\alpha_{27}}{\rho} \right) \mathcal{Q}^2\mathcal{P} + \left(\frac{\alpha_{18}(\wp - \alpha_{11})}{\rho\alpha_{12}} - \frac{\alpha_{28}}{\rho} \right) \mathcal{Q}\mathcal{P}^2 \\ &\quad + \left(\frac{\alpha_{19}(\wp - \alpha_{11})}{\rho\alpha_{12}} - \frac{\alpha_{29}}{\rho} \right) \mathcal{P}^3 + \tilde{\mathcal{R}}(\|(\mathcal{Q}_n, \mathcal{P}_n)\|^4). \end{aligned}$$

It is important to recall that the variables $\mathcal{Q}_n$ and $\mathcal{P}_n$ are specified as

$$\mathcal{Q} = \alpha_{12} \tilde{T}_n, \quad \mathcal{P} = (\wp - \alpha_{11}) \tilde{T}_n - \rho \tilde{E}_n.$$

The occurrence of a Neimark–Sacker bifurcation requires the critical parameter $\wedge$ to satisfy the non-degeneracy condition $\wedge \neq 0$.

$$\wedge = \left(\text{Real}(\kappa_2 e_{21}) - \text{Re} \left(\frac{(1 - 2\zeta_1)(\zeta_2)^2}{1 - \zeta_1} e_{20} e_{11} \right) - \frac{1}{2} |e_{11}|^2 - |e_{02}|^2 \right)_{h^*=0}. \quad (8)$$

where

$$\begin{aligned} e_{20} &= \frac{1}{8} \left[\frac{\partial^2 \tilde{\mathcal{H}}_1}{\partial \tilde{T}^2} - \frac{\partial^2 \tilde{\mathcal{H}}_1}{\partial \tilde{E}^2} + 2 \frac{\partial^2 \tilde{\mathcal{H}}_2}{\partial \tilde{T} \partial \tilde{E}} + i \left(\frac{\partial^2 \tilde{\mathcal{H}}_2}{\partial \tilde{T}^2} - \frac{\partial^2 \tilde{\mathcal{H}}_2}{\partial \tilde{E}^2} - 2 \frac{\partial^2 \tilde{\mathcal{H}}_1}{\partial \tilde{T} \partial \tilde{E}} \right) \right] \Big|_{h^*=0}, \\ e_{11} &= \frac{1}{4} \left[\frac{\partial^2 \tilde{\mathcal{H}}_1}{\partial \tilde{T}^2} + \frac{\partial^2 \tilde{\mathcal{H}}_1}{\partial \tilde{E}^2} + i \left(\frac{\partial^2 \tilde{\mathcal{H}}_2}{\partial \tilde{T}^2} + \frac{\partial^2 \tilde{\mathcal{H}}_2}{\partial \tilde{E}^2} \right) \right] \Big|_{h^*=0}, \\ e_{02} &= \frac{1}{8} \left[\frac{\partial^2 \tilde{\mathcal{H}}_1}{\partial \tilde{T}^2} - \frac{\partial^2 \tilde{\mathcal{H}}_1}{\partial \tilde{E}^2} - 2 \frac{\partial^2 \tilde{\mathcal{H}}_2}{\partial \tilde{T} \partial \tilde{E}} + i \left(\frac{\partial^2 \tilde{\mathcal{H}}_2}{\partial \tilde{T}^2} - \frac{\partial^2 \tilde{\mathcal{H}}_2}{\partial \tilde{E}^2} + 2 \frac{\partial^2 \tilde{\mathcal{H}}_1}{\partial \tilde{T} \partial \tilde{E}} \right) \right] \Big|_{h^*=0}, \\ e_{21} &= \frac{1}{16} \left[\frac{\partial^3 \tilde{\mathcal{H}}_1}{\partial \tilde{T}^3} + \frac{\partial^3 \tilde{\mathcal{H}}_1}{\partial \tilde{T} \partial \tilde{E}^2} + \frac{\partial^3 \tilde{\mathcal{H}}_2}{\partial \tilde{T}^2 \partial \tilde{E}} + \frac{\partial^3 \tilde{\mathcal{H}}_2}{\partial \tilde{E}^3} + i \left(\frac{\partial^3 \tilde{\mathcal{H}}_2}{\partial \tilde{T}^3} + \frac{\partial^3 \tilde{\mathcal{H}}_2}{\partial \tilde{T} \partial \tilde{E}^2} - \frac{\partial^3 \tilde{\mathcal{H}}_1}{\partial \tilde{T}^2 \partial \tilde{E}} - \frac{\partial^3 \tilde{\mathcal{H}}_1}{\partial \tilde{E}^3} \right) \right] \Big|_{h^*=0}. \end{aligned}$$

From the above analysis, the bifurcation characteristics are stated in the following theorem.

Theorem 4. Let $h = h_0$ and suppose that $\wedge \neq 0$. Then, system (4) undergoes a Neimark–Sacker bifurcation at the fixed point M_2 . Moreover,

- If $\wedge < 0$, a stable invariant closed curve bifurcates from M_2 for $h > h^*$.
- If $\wedge > 0$, an unstable invariant closed curve bifurcates from M_2 for $h < h^*$.

3.2.2. Period-Doubling Bifurcation Analysis

We here investigate the existence of the period-doubling bifurcation. This type of bifurcation describes a process by which a dynamical system transitions from stable periodic motion to chaos as a control parameter varies. It takes place when a stable periodic orbit loses stability. This unstable orbit doubles itself in the form of periodic orbits. Changing the control parameter produces successive period doublings at smaller intervals. This

phenomenon results in a cascade and chaotic behavior. For a two-dimensional discrete-time system,

$$X_{n+1} = \mathcal{H}(X_n, \epsilon), \quad X_n \in \mathbb{R}^2,$$

a period-doubling (flip) bifurcation arises when one eigenvalue of the Jacobian matrix $\mathcal{J}_{\mathcal{H}}(X^*, \epsilon)$ at the fixed point X^* satisfies

$$\zeta_1 = -1, \quad |\zeta_2| < 1,$$

along with a non-degeneracy condition ensuring that the fixed point loses stability as the parameter ϵ passes its critical value. As a consequence, a stable orbit with period 2 develops. Then, successive bifurcations generate a period-doubling cascade, marking the classical route to chaos. According to Theorem 3, a bifurcation of M_2 can occur in the parameter space.

$$(R, A, B, C, D, h) \in \Delta_{PD} = \left\{ \begin{array}{l} (R, A, B, C, D, h) \in \mathbb{R}^6 \mid h = \left(\frac{RD(3BD^2 - KC^2 - 2CD + 2BDKC) \mp \sqrt{\Delta}}{RD^2(C - BD)(KC + D)} \right), \\ \Delta = R^2D^2(3BD^2 - KC^2 - 2CD + 2BDKC)^2 - 4RD^2C^2(C - BD)(KC + D)^2 > 0. \end{array} \right\}.$$

Let $(R, A, B, C, D, h) \in \Delta_{PD}$. Then, model (4) can be easily written as

$$\begin{cases} T_{n+1} = T_n + (h + \bar{h}) \left(RT_n^2(1 - BT_n) - \frac{AT_nE_n}{K + T_n} \right) = \mathcal{H}_1(T_n, E_n, h), \\ E_{n+1} = E_n + (h + \bar{h})E_n(CT_n - D) = \mathcal{H}_2(T_n, E_n, h). \end{cases} \quad (9)$$

Here, we use $\bar{h}$ to be a small perturbation of r_1 , where $\bar{h} \ll 1$. Next, we use the new variables $\mathcal{Q}'_n = T_n - T^*$ and $\mathcal{P}'_n = E_n - E^*$, to shift the fixed point M_2 to the origin. We then expand the functions $\mathcal{H}_1$ and $\mathcal{H}_2$ by the Taylor series around the origin. Hence, system (9) becomes

$$\begin{cases} \mathcal{Q}'_{n+1} = \alpha_{11}\mathcal{Q}'_n + \alpha_{12}\mathcal{P}'_n + \alpha_{13}\mathcal{Q}'_n{}^2 + \alpha_{14}\mathcal{Q}'_n\mathcal{P}'_n + \alpha_{15}\mathcal{P}'_n{}^2 + \beta_{11}\mathcal{Q}'_n\bar{h} + \beta_{12}\mathcal{P}'_n\bar{h} + \alpha_{16}\mathcal{Q}'_n{}^3 \\ \quad + \beta_{13}\bar{h}^2 + \beta_{14}\mathcal{Q}'_n{}^2\bar{h} + \beta_{15}\mathcal{P}'_n{}^2\bar{h} + \alpha_{17}\mathcal{Q}'_n{}^2\mathcal{P}'_n + \alpha_{18}\mathcal{Q}'_n\mathcal{P}'_n{}^2 + \beta_{16}\mathcal{Q}'_n\mathcal{P}'_n\bar{h} + \beta_{17}\bar{h}^3 \\ \quad + \beta_{18}\mathcal{Q}'_n\bar{h}^2 + \beta_{19}\mathcal{P}'_n\bar{h}^2 + \alpha_{19}\mathcal{P}'_n{}^3 + \mathcal{R}'_1(|\mathcal{Q}'_n|, |\mathcal{P}'_n|, |\bar{h}|)^4, \\ \mathcal{P}'_{n+1} = \alpha_{21}\mathcal{Q}'_n + \alpha_{22}\mathcal{P}'_n + \alpha_{23}\mathcal{Q}'_n{}^2 + \alpha_{24}\mathcal{Q}'_n\mathcal{P}'_n + \alpha_{25}\mathcal{P}'_n{}^2 + \beta_{21}\mathcal{Q}'_n\bar{h} + \beta_{22}\mathcal{P}'_n\bar{h} + \alpha_{26}\mathcal{Q}'_n{}^3 \\ \quad + \beta_{23}\bar{h}^2 + \beta_{24}\mathcal{Q}'_n{}^2\bar{h} + \beta_{25}\mathcal{P}'_n{}^2\bar{h} + \alpha_{27}\mathcal{Q}'_n{}^2\mathcal{P}'_n + \alpha_{28}\mathcal{Q}'_n\mathcal{P}'_n{}^2 + \beta_{26}\mathcal{Q}'_n\mathcal{P}'_n\bar{h} + \beta_{27}\bar{h}^3 \\ \quad + \beta_{28}\mathcal{Q}'_n\bar{h}^2 + \beta_{29}\mathcal{P}'_n\bar{h}^2 + \alpha_{29}\mathcal{P}'_n{}^3 + \mathcal{R}'_2(|\mathcal{Q}'_n|, |\mathcal{P}'_n|, |\bar{h}|)^4. \end{cases} \quad (10)$$

Note that

$$\beta_{11} = \frac{RD(2C - 3BD)}{C^2} - \frac{RDK(C - BD)}{C(KC + D)}, \quad \beta_{12} = -\frac{AD}{KC + D}, \quad \beta_{21} = \frac{RD(KC + D)(C - BD)}{AC^2},$$

$$\beta_{13} = \beta_{22} = \beta_{23} = 0.$$

Utilizing a transformation, system (10) becomes

$$\begin{pmatrix} \mathcal{Q}'_n \\ \mathcal{P}'_n \end{pmatrix} = \begin{pmatrix} \alpha_{12} & \alpha_{12} \\ -1 - \alpha_{11} & \zeta_2 - \alpha_{11} \end{pmatrix} \begin{pmatrix} \tilde{T}_n \\ \tilde{E}_n \end{pmatrix}.$$

This shows that

$$\begin{cases} \tilde{T}_{n+1} = -\tilde{T}_n + \tilde{\mathcal{H}}_1(\tilde{T}, \tilde{E}, \tilde{h}), \\ \tilde{E}_{n+1} = \zeta_2 \tilde{E}_n + \tilde{\mathcal{H}}_2(\tilde{T}, \tilde{E}, \tilde{h}), \end{cases} \quad (11)$$

with

$$\mathcal{Q}'_n = \alpha_{12}(\tilde{T} + \tilde{E}), \quad \mathcal{P}'_n = -(1 + \alpha_{11})\tilde{T} + (\zeta_2 - \alpha_{11})\tilde{E},$$

It should be noted that $\tilde{\mathcal{H}}_1(\tilde{T}, \tilde{E}, \tilde{h})$ and $\tilde{\mathcal{H}}_2(\tilde{T}, \tilde{E}, \tilde{h})$ are defined in Appendix B.

In accordance with the center manifold theory (see [18]), there exists a center manifold which can be presented as follows:

$$\zeta^c(0, 0, 0) = \left\{ (\tilde{T}, \tilde{E}, \tilde{h}) \in \mathbb{R}^3 : \tilde{E} = \mathcal{K}(\tilde{T}, \tilde{h}) = \gamma_1 \tilde{T}^2 + \gamma_2 \tilde{T} \tilde{h} + \gamma_3 \tilde{h}^2 + \mathcal{R}_3(\tilde{T}, \tilde{h}) \right\},$$

where $\mathcal{R}_3(\tilde{T}, \tilde{h})$ denotes higher-order terms. This manifold satisfies the following invariance condition:

$$\mathcal{K}(-\tilde{T} + \tilde{\mathcal{H}}'_1(\tilde{T}, \mathcal{K}(\tilde{T}, \tilde{h}), \tilde{h}), \tilde{h}) - \zeta_2 \mathcal{K}(\tilde{T}, \tilde{h}) = \tilde{\mathcal{H}}'_2(\tilde{T}, \mathcal{K}(\tilde{T}, \tilde{h}), \tilde{h}).$$

Therefore, we obtain

$$\gamma_1 = \left(\frac{(\alpha_{13}\alpha_{12} - \alpha_{12}\alpha_{24})(1 + \alpha_{11}) - \alpha_{12}^2\alpha_{23} - (\alpha_{14} - \alpha_{26})(1 + \alpha_{11})^2}{1 - \zeta_2^2} + \frac{\alpha_{16}(1 + \alpha_{11})^3}{\alpha_{12}(1 - \zeta_2^2)} \right),$$

$$\gamma_2 = \left(\frac{(\beta_{11} - \beta_{22})(1 + \alpha_{11}) - \alpha_{12}\beta_{12}}{1 - \zeta_2^2} + \frac{\beta_{12}(1 + \alpha_{11})^2}{\alpha_{12}(1 - \zeta_2^2)} \right), \text{ and } \gamma_3 = \frac{\beta_{13}(1 - \alpha_{11}) - \alpha_{12}\beta_{23}}{\alpha_{12}(1 - \zeta_2^2)}.$$

Thus, system (11) is constrained to the center manifold $\asymp^c(0, 0, 0)$. That is

$$\Pi : \tilde{T} \mapsto -\tilde{T} + r_1 \tilde{T}^2 + r_2 \tilde{T} \tilde{h} + r_3 \tilde{T}^2 \tilde{h} + r_4 \tilde{T} \tilde{h}^2 + r_5 \tilde{T}^3 + R_4(\tilde{T}, \tilde{h}),$$

where the coefficients are given by

$$r_1 = \frac{(\zeta_2 - \alpha_{11})(\alpha_{13}\alpha_{12} - \alpha_{14}(1 + \alpha_{11})) + \alpha_{12}(\alpha_{24}(1 + \alpha_{11}) - \alpha_{23}\alpha_{12})}{1 + \zeta_2} + \frac{(1 + \alpha_{11})^2(\alpha_{15}(\zeta_2 - \alpha_{11}) - \alpha_{24}\alpha_{12})}{\alpha_{12}(1 + \zeta_2)},$$

$$r_2 = \frac{(\zeta_2 + \alpha_{11})(\alpha_{12}\beta_{11} - \beta_{12}(1 + \alpha_{11})) + \alpha_{12}(\beta_{22}(1 + \alpha_{11}) - \beta_{21}\alpha_{12})}{\alpha_{12}(1 + \zeta_2)}$$

$$r_3 = \frac{(\zeta_2 + \alpha_{11})(\alpha_{12}\beta_{14} - \beta_{16}(1 + \alpha_{11})) - \alpha_{12}(\alpha_{12}\beta_{24} + (1 + \alpha_{11})\beta_{26}) - \beta_{25}(1 + \alpha_{11})^2}{1 + \zeta_2} + \frac{\beta_{15}(\zeta_2 + \alpha_{11})(1 + \alpha_{11})^2}{\alpha_{12}(1 + \zeta_2)},$$

$$r_4 = \frac{\beta_{18}(\zeta_2 + \alpha_{11}) - \beta_{28}\alpha_{12} + \beta_{29}(1 + \alpha_{11})}{1 + \zeta_2} - \frac{\beta_{19}(\zeta_2 + \alpha_{11})^2}{\alpha_{12}(1 + \zeta_2)},$$

$$r_5 = \frac{(\zeta_2 - \alpha_{11})(\alpha_{14}\alpha_{12}^2 + \alpha_{16}(1 + \alpha_{11})^2 - \alpha_{15}\alpha_{12}(1 + \alpha_{11})) - \alpha_{24}\alpha_{12}^3 + \alpha_{25}\alpha_{12}^2(1 + \alpha_{11})}{1 + \zeta_2}$$

$$+ \frac{\alpha_{27}\alpha_{12}(1 + \alpha_{11})^3 - \alpha_{26}\alpha_{12}^2(1 + \alpha_{11})^2 - \alpha_{17}(\zeta_2 - \alpha_{11})(1 + \alpha_{11})^3}{\alpha_{12}(1 + \zeta_2)}.$$

Let

$$v_1 = \left(\frac{\partial^2 \Pi}{\partial \tilde{T} \partial \tilde{h}} + \frac{1}{2} \frac{\partial \Pi}{\partial \tilde{h}} \frac{\partial^2 \Pi}{\partial \tilde{T}^2} \right) \Big|_{(0,0)} = r_2, \quad v_2 = \left(\frac{1}{6} \frac{\partial^3 \Pi}{\partial \tilde{T}^3} + \left(\frac{1}{2} \frac{\partial^2 \Pi}{\partial \tilde{T}^2} \right)^2 \right) \Big|_{(0,0)} = r_5 + r_1^2.$$

The analysis shown previously leads us to the following theorem.

Theorem 5. If $v_1 \neq 0$ and $v_2 \neq 0$, then system (4) undergoes a period-doubling (flip) bifurcation at the positive coexistence fixed point M_2 as the parameter h varies in a small neighborhood of Δ_{PD} . Moreover,

- If $v_2 > 0$, system (4) bifurcates from the coexistence fixed point M_2 to a stable period-2 orbit.
- If $v_2 < 0$, system (4) bifurcates from the coexistence fixed point M_2 to an unstable period-2 orbit.

4. Dynamics with Allee Effects

This section is added to find the fixed points of the model including the Allee effect in the tumor cell population, where $0 < m < 1/B$. The fixed points of system (3) can be evaluated by solving the following nonlinear system:

$$\begin{cases} T = T + h \left(R T (1 - B T) (T - m) - \frac{A T E}{K + T} \right), \\ E = E + h E (C T - D). \end{cases} \quad (12)$$

Simplifying the previous system leads to

- $M'_0 = (0, 0)$, $M'_1 = \left(\frac{1}{B}, 0\right)$, and $M'_3 = (m, 0)$. These fixed points unconditionally exist for all accepted parameter values.
- $M'_2 = \left(\frac{D}{C}, \frac{R(KC + D)(C - BD)(D - mC)}{AC^3}\right)$. This fixed point exists if and only if the assumptions $BD < C < D/m$ and $m < 1/B$ are satisfied.

In addition, it is observed that if $C = BD$, the fixed point M'_2 coincides with the point M'_1 . Next, we introduce an extensive analysis for the stability of these fixed points in the following parts.

4.1. Existence and Stability of Fixed Points of Model (3)

This section is mainly devoted to presenting the stability of the fixed points of model (3). The generalized Jacobian matrix $\mathcal{J}'(T, E)$ of system (3) at a fixed point (T, E) is simply shown as

$$\mathcal{J}'(T, E) = \begin{pmatrix} 1 + h \left(R(-3BT^2 + 2(1 + Bm)T - m) - \frac{AEK}{(K + T)^2} \right) & -\frac{hAT}{K + T} \\ hCE & 1 + h(CT - D) \end{pmatrix}. \quad (13)$$

The stability properties of all fixed points are presented in the following theorems.

Theorem 6. The fixed point $M'_0 = (0, 0)$ is stable if $0 < h < \left\{ \frac{2}{D}, \frac{2}{Rm} \right\}$, non-hyperbolic if $h = \frac{2}{D}$ or $h = \frac{2}{Rm}$, and unstable otherwise.

Proof. We firstly begin with the Jacobian matrix of system (3) at the fixed point $M'_0 = (0, 0)$, which is shown as follows:

$$\mathcal{J}'(M'_0) = \begin{pmatrix} 1 - hRm & 0 \\ 0 & 1 - hD \end{pmatrix},$$

with

$$\zeta_1 = 1 - hRm, \quad \zeta_2 = 1 - hD.$$

From Lemma A1, one can observe that the fixed point M'_0 is stable if $0 < h < \left\{ \frac{2}{D}, \frac{2}{Rm} \right\}$, non-hyperbolic if $h = \frac{2}{D}$ or $h = \frac{2}{Rm}$, and unstable otherwise. $\square$

Theorem 7. *Considering the fixed point M'_1 , we establish the following properties:*

- When $C > DB$, we have
 - M'_1 is a saddle if $0 < h < \left(\frac{2B}{R(1-Bm)} \right)$ with eigenvalues $|\zeta_1| < 1$ and $|\zeta_2| > 1$.
 - M'_1 is a source if $h > \left(\frac{2B}{R(1-Bm)} \right) > 0$.
 - M'_1 is non-hyperbolic if $h = \frac{2B}{R(1-Bm)}$.
- When $C = DB$, we have that M'_1 is non-hyperbolic.
- When $C < DB$, we have
 - M'_1 is locally asymptotically stable if $0 < h < \min \left\{ \left(\frac{2B}{R(1-Bm)} \right), \left(\frac{2B}{DB-C} \right) \right\}$.
 - M'_1 is a source if $h > \max \left\{ \left(\frac{2B}{R(1-Bm)} \right), \left(\frac{2B}{DB-C} \right) \right\}$.
 - M'_1 is a saddle if $\left(\frac{2B}{R(1-Bm)} \right) < h < \left(\frac{2B}{DB-C} \right)$ with eigenvalues $|\zeta_1| > 1$ and $|\zeta_2| < 1$, or $\left(\frac{2B}{DB-C} \right) < h < \left(\frac{2B}{R(1-Bm)} \right)$ with eigenvalues $|\zeta_1| < 1$ and $|\zeta_2| > 1$.
 - M'_1 is non-hyperbolic if $h = \left\{ \left(\frac{2B}{R(1-Bm)} \right), \left(\frac{2B}{DB-C} \right) \right\}$.

Proof. First, we calculate the Jacobian matrix of system (3) at the fixed point $M'_1 = \left(\frac{1}{B}, 0 \right)$ as follows:

$$\mathcal{J}'(M'_1) = \begin{pmatrix} 1 - \frac{Rh(1-Bm)}{B} & -\frac{hA}{BK+1} \\ 0 & 1 + h\left(\frac{C}{B} - D\right) \end{pmatrix}.$$

We next find the corresponding eigenvalues which are presented as follows:

$$\zeta_1 = 1 - \frac{Rh(1-Bm)}{B}, \quad \zeta_2 = 1 + h\left(\frac{C}{B} - D\right).$$

It can be clearly seen from Lemma A1 that the outcomes of Theorem 7 are true. $\square$

Theorem 8. *The fixed point $M'_3 = (m, 0)$ of system (3) is characterized as follows:*

$$M'_3 \text{ is } \begin{cases} \text{a saddle point,} & \text{if } h < \left(\frac{2}{D-Cm} \right), \\ \text{a source,} & \text{if } h > \left(\frac{2}{D-Cm} \right), \\ \text{non-hyperbolic,} & \text{if } h = \left(\frac{2}{D-Cm} \right). \end{cases}$$

Proof. Note that the Jacobian matrix $\mathcal{J}'(M'_3)$ of system (3) takes the form

$$\mathcal{J}'(M'_3) = \begin{pmatrix} 1 + h R m (1 - B m) & -\frac{h A m}{K + m} \\ 0 & 1 + h (C m - D) \end{pmatrix}.$$

From the Jacobian matrix, we can find the corresponding eigenvalues as follows:

$$\zeta_1 = 1 + h R m (1 - B m) > 1 \quad (\text{since } 0 < m < 1/B) \quad \text{and} \quad \zeta_2 = 1 + h (C m - D).$$

The results of Theorem 8 follow directly from Lemma A1. $\square$

Theorem 9. Assume that $BD < C < D/m$ and $m < 1/B$. Then, M'_2 is a positive fixed point of system (3), and it satisfies $\Lambda_{\mathcal{J}'}(1) > 0$. Moreover,

i'- If any one of the following conditions holds, then M'_2 is locally asymptotically stable (i.e., a sink):

1. $\Delta' < 0$ and $0 < h' \leq h'_0$,
2. $\Delta' \geq 0$ and $0 < h' \leq h'_1$.

ii'- If any one of the following conditions holds, then M'_2 is unstable (i.e., a source):

1. $\Delta' < 0$ and $h' > h'_0$,
2. $\Delta' \geq 0$ and $h' > h'_2$.

iii'- The fixed point M'_2 is unstable (i.e., a saddle point) if

$$\Delta' \geq 0 \quad \text{and} \quad h'_1 < h' \leq h'_2.$$

iv'- The fixed point M'_2 is non-hyperbolic if one of the following conditions holds:

1. $\Delta' < 0$ and $h' = h'_0$,
2. $\Delta' \geq 0$ and $h' = h'_{1,2}$ and

$$h' \neq \left(\frac{2C^2(KC + D)}{R(D(3BD - 2C - 2BCm)(KC + D) + KC(C - BD)(D - mC) + C^2(KC + D)m)} \right).$$

where

$$\begin{aligned} h'_0 &= \frac{(D(2C - 3BD + 2BCm)(KC + D) - KC(C - BD)(D - mC) - C^2(KC + D)m)}{D(C - BD)(D - mC)(KC + D)}, \\ h'_1 &= \frac{R(D(2C - 3BD + 2BCm)(KC + D) - KC(C - BD)(D - mC) - C^2(KC + D)m) - \sqrt{\Delta'}}{RD(C - BD)(D - mC)(KC + D)}, \\ h'_2 &= \frac{R(D(2C - 3BD + 2BCm)(KC + D) - KC(C - BD)(D - mC) - C^2(KC + D)m) + \sqrt{\Delta'}}{RD(C - BD)(D - mC)(KC + D)}. \end{aligned}$$

Proof. It is worth noting that the Jacobian matrix (13) at the fixed point M'_2 is

$$\mathcal{J}'(M'_2) = \begin{pmatrix} 1 + hR \left(-\frac{3BD^2}{C^2} + \frac{2(1 + Bm)D}{C} - \frac{K(C - BD)(D - mC)}{C(KC + D)} - m \right) & -\frac{hAD}{KC + D} \\ \frac{hR(KC + D)(C - BD)(D - mC)}{AC^2} & 1 \end{pmatrix},$$

with the following characteristic equation:

$$\Lambda_{\mathcal{J}'}(\zeta) = \zeta^2 - \text{Tr}(\mathcal{J}')\zeta + \text{Det}(\mathcal{J}'). \quad (14)$$

Here, we have

$$\begin{aligned}\text{Tr}(\mathcal{J}') &= 2 - hR \left(\frac{D(3BD - 2C - 2BCm)}{C^2} + \frac{K(C - BD)(D - mC)}{C(KC + D)} + m \right), \\ \text{Det}(\mathcal{J}') &= 1 - hR \left(\frac{D(3BD - 2C - 2BCm)}{C^2} + \frac{K(C - BD)(D - mC)}{C(KC + D)} + m \right) + \left(\frac{h^2 RD(C - BD)(D - mC)}{C^2} \right).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\Lambda_{\mathcal{J}'}(-1) &= 4 + 2hR \left(\frac{D(3BD - 2C - 2BCm)}{C^2} + \frac{K(C - BD)(D - mC)}{C(KC + D)} + m \right) + \left(\frac{h^2 RD(C - BD)(D - mC)}{C^2} \right), \\ \Lambda_{\mathcal{J}'}(0) &= 1 - hR \left(\frac{D(3BD - 2C - 2BCm)}{C^2} + \frac{K(C - BD)(D - mC)}{C(KC + D)} + m \right) + \left(\frac{h^2 RD(C - BD)(D - mC)}{C^2} \right), \\ \text{and } \Lambda_{\mathcal{J}'}(1) &= \left(\frac{h^2 RD(C - BD)(D - mC)}{C^2} \right) > 0, \quad (BD < C < D/m \text{ and } m < 1/B).\end{aligned}$$

The discriminant of the equation $\Lambda_{\mathcal{J}'}(\zeta) = 0$ is shown as

$$\begin{aligned}\Delta' &= R^2(D(3BD - 2C - 2BCm)(KC + D) + KC(C - BD)(D - mC) + C^2(KC + D)m)^2 \\ &\quad - 4RDC^2(C - BD)(D - mC)(KC + D)^2.\end{aligned}$$

Applying Lemma A2, we notice that the conditions required in Theorem 8 are fulfilled. $\square$

4.2. Bifurcations of Model (3)

The local bifurcation theory is used in this part to analyze the bifurcation behavior of model (3) at the fixed point M'_2 , taking into account its practical biological significance.

4.2.1. Neimark–Sacker Bifurcation

Lemma (A3) is first invoked to study the Neimark–Sacker bifurcation of model (3) under parameter perturbations taken in a neighborhood of

$$\Delta'_{NS} = \left\{ (R, A, B, C, D, m, h) \in \mathbb{R}^7 \left| \begin{aligned} h &= h'_0 = \frac{R(V - KC(C - BD)(D - mC) - C^2(KC + D)m)}{RD(C - BD)(D - mC)(KC + D)}, \\ \Delta' &= R^2(D(3BD - 2C - 2BCm)(KC + D) + KC(C - BD)(D - mC) + C^2(KC + D)m)^2 \\ &\quad - 4RDC^2(C - BD)(D - mC)(KC + D)^2 < 0, \text{ and } V = D(2C - 3BD + 2BCm)(KC + D) \end{aligned} \right. \right\},$$

where the following result holds.

Theorem 10. Model (3) undergoes a Neimark–Sacker bifurcation at the fixed point M'_2 provided that the following conditions hold:

$$\begin{aligned}1 - \text{Det}(\mathcal{J}'(M'_2)) &= 0, \\ 1 + \text{Det}(\mathcal{J}'(M'_2)) &> 0, \\ 1 - \text{Tr}(\mathcal{J}'(M'_2)) + \text{Det}(\mathcal{J}'(M'_2)) &> 0, \\ 1 + \text{Tr}(\mathcal{J}'(M'_2)) + \text{Det}(\mathcal{J}'(M'_2)) &> 0.\end{aligned}$$

As a result, a Neimark–Sacker bifurcation occurs at h when the parameters R, A, B, C, D, m , and h vary in a neighborhood of the set Δ'_{NS} .

Proof. Using Lemma A3 and Theorem 9, and evaluating equation (14) at the fixed point M'_2 , we obtain

$$\begin{aligned} D_0^\mp(h) &= 1 > 0, \\ D_1^-(h) &= 1 - \text{Det}(\mathcal{J}'(M'_2)) = 0, \\ D_1^+(h) &= 1 + \text{Tr}(\mathcal{J}'(M'_2)) > 0, \\ \Lambda_{\mathcal{J}'}(1) &= 1 - \text{Tr}(\mathcal{J}'(M'_2)) + \text{Det}(\mathcal{J}'(M'_2)) > 0, \\ (-1)^2 \Lambda_{\mathcal{J}'}(-1) &= 1 + \text{Tr}(\mathcal{J}'(M'_2)) + \text{Det}(\mathcal{J}'(M'_2)) > 0, \end{aligned}$$

which hold if and only if

$$h = h'_0 = \frac{R(D(2C - 3BD + 2BCm)(KC + D) - KC(C - BD)(D - mC) - C^2(KC + D)m)}{RD(C - BD)(D - mC)(KC + D)}, \Delta' < 0.$$

Furthermore, the transversality condition is given by

$$\left. \frac{dD_1^-(h)}{dh} \right|_{h=h'_0} = \left. \frac{d(1 - \text{Det})}{dh} \right|_{h=h'_0} \neq 0.$$

As a consequence, a Neimark–Sacker bifurcation takes place at $h = h'_0$. $\square$

4.2.2. Period-Doubling Bifurcation

This part utilizes Lemma (A4) to seek the existence of the period-doubling bifurcation of system (3) under parameter perturbations taken in a neighborhood of Δ'_{PD} , where

$$\Delta'_{PD} = \left\{ \begin{aligned} &(R, A, B, C, D, m, h) \in \mathbb{R}^7 \mid h = h'_{1,2} = \frac{R(V - KC(C - BD)(D - mC) - C^2(KC + D)m) \mp \sqrt{\Delta'}}{RD(C - BD)(D - mC)(KC + D)}, \\ &\Delta' = R^2(D(3BD - 2C - 2BCm)(KC + D) + KC(C - BD)(D - mC) + C^2(KC + D)m)^2 \\ &\quad - 4RDC^2(C - BD)(D - mC)(KC + D)^2 > 0, \text{ and } V = D(2C - 3BD + 2BCm)(KC + D) \end{aligned} \right\}.$$

Theorem 11. Model (3) undergoes a period-doubling bifurcation at the fixed point M'_2 if the following conditions hold:

$$\begin{aligned} 1 + \text{Det}(\mathcal{J}'(M'_2)) &> 0, \\ 1 + \text{Tr}(\mathcal{J}'(M'_2)) + \text{Det}(\mathcal{J}'(M'_2)) &= 0, \\ 1 - \text{Tr}(\mathcal{J}'(M'_2)) + \text{Det}(\mathcal{J}'(M'_2)) &> 0. \end{aligned} \tag{15}$$

Consequently, a period-doubling bifurcation takes place at M'_2 when the parameters (R, A, B, C, D, m, h) vary within a neighborhood of the set Δ'_{PD} .

Proof. Applying Lemma A4 and Theorem 9, and evaluating the characteristic Equation (14) of system (3) at the fixed point M'_2 , we obtain the following conditions:

$$\begin{aligned} D_0^\mp(h) &= 1 > 0, \\ D_1^+(h) &= 1 + \text{Det}(\mathcal{J}'(M'_2)) > 0, \\ (-1)^2 \Lambda_{\mathcal{J}'}(-1) &= 1 + \text{Tr}(\mathcal{J}'(M'_2)) + \text{Det}(\mathcal{J}'(M'_2)) = 0, \\ \Lambda_{\mathcal{J}'}(1) &= 1 - \text{Tr}(\mathcal{J}'(M'_2)) + \text{Det}(\mathcal{J}'(M'_2)) > 0, \end{aligned}$$

which hold if and only if

$$h = h'_{1,2} = \frac{R(D(2C - 3BD + 2BCm)(KC + D) - KC(C - BD)(D - mC) - C^2(KC + D)m) \mp \sqrt{\Delta'}}{RD(C - BD)(D - mC)(KC + D)}.$$

In addition, the transversality condition is given by

$$\frac{\text{Tr}' + \text{Det}'}{\text{Tr} + 2} \neq 0,$$

where

$$\text{Tr}' = \left. \frac{d\text{Tr}}{dh} \right|_{h=h'_{1,2}} \quad \text{and} \quad \text{Det}' = \left. \frac{d\text{Det}}{dh} \right|_{h=h'_{1,2}}.$$

Hence, the considered system undergoes a period-doubling bifurcation at $h = h'_1$ and $h = h'_2$. $\square$

5. Numerical Simulations

This part presents some numerical examples to confirm the obtained theoretical results.

5.1. Neimark–Sacker Bifurcation Simulation

Based on the previous outcomes, we have illustrated that the parameter h plays a powerful role in generating a Neimark–Sacker bifurcation in the system. The dynamics of the system will now be examined. In particular, we concentrate on how the qualitative behavior changes as the parameters h , C , D or m vary while the remaining parameters are fixed. This technique provides an affective understanding of the dynamical patterns and their biological implications.

- Assume that the parameter values $R = 10.5181$, $A = 01.7$, $B = 01.91$, $C = 06.2$, $D = 01.9121$, $K = 01.2$, and $h = [0, 0.6]$, and the initial condition $(T_0, E_0) = (0.353, 0.97)$, then model (4) undergoes a Neimark–Sacker bifurcation at the fixed point M_2 , with $h_0 = 0.1197$ serving as the bifurcation parameter. This phenomenon can be clearly seen in Figure 1. Moreover, the bifurcation graphs of model (4) are presented in Figure 1a, while Figure 1b presents the maximum Lyapunov exponent (MLE). A stable fixed point is reached by the tumor and effector cell populations for modest values of h . Quasiperiodic oscillations occur as h ($h = h_0$) approaches a crucial value, producing an invariant closed curve in the phase space. Additional increases in h generate secondary bifurcations, which finally lead to chaotic attractors and periodic orbits. Positive Lyapunov exponents and exceptionally dispersed points in the bifurcation diagrams are symptomatic of these complex dynamics. The Jacobian matrix at the bifurcation point is provided by

$$\mathcal{J}_{M_2} = \begin{pmatrix} 0.9635 & -0.0416 \\ 0.8780 & 1.0000 \end{pmatrix}.$$

Note that the corresponding characteristic equation is

$$Z_{\mathcal{J}}(\zeta) = \zeta^2 - 1.9635\zeta + 1.0000. \quad (16)$$

Hence, the eigenvalues of this polynomial are

$$\zeta_{1,2} = 0.9817 \mp 0.1903i,$$

with magnitude $|\zeta_{1,2}| = 1$. This verifies that when $h = h_0$, the system experiences a Neimark–Sacker bifurcation.

The phase portraits and time series plots of system (4) for various values of the bifurcation parameter h are shown in Figures 2 and 3. For $h = 0.024$, the system exhibits a stable fixed point M_2 , as illustrated in Figure 2a,b. When $h = h_0 = 0.1197$, an attracting invariant closed curve forms around the fixed point (Figure 2c,d), indicating the onset of a Neimark–Sacker bifurcation. For $h = 0.807$, this closed curve persists and continues to attract nearby trajectories, as seen in Figure 2e,f. As h increases further, periodic orbits appear for instance, at $h = 0.45$ and $h = 0.5$, illustrated in Figure 3a–d. Eventually, for $h = 0.52$ and $h = 0.6$, the system enters a chaotic regime, giving rise to chaotic attractors, as shown in Figure 3e–h. The phase portrait representation in Figure 3g further highlights the coupled interactions between tumor and effector cell populations and illustrates how their dynamics evolve as the system approaches chaotic regimes.

Figure 4a depicts the bifurcation diagram, while the maximum Lyapunov exponent (MLE) is shown in Figure 4b, with C chosen as the bifurcation parameter. These results are obtained using the parameter set ($R = 10.5181$, $A = 01.7$, $B = 01.91$, $D = 01.9121$, $K = 01.2$, and $C \in [0, 12.9]$) and $h = 0.1197$. It is observed that for $C < 3.6642$, model (4) is asymptotically stable at $M_1 = (1/B, 0)$. Moreover, model (4) undergoes a transcritical bifurcation as C passes through the critical value $C_0 = 3.6642$. At this point, the fixed point M_1 loses stability and a new stable fixed point M_2 appears. It is worth noting that as C increases, model (4) undergoes a supercritical Neimark–Sacker bifurcation at $C_1 = 6.1823$, beyond which the fixed point M_2 loses stability and an attracting invariant closed curve emerges. In addition, the bifurcation diagrams of model (4) are plotted in Figure 5a, and the corresponding maximum Lyapunov exponent (MLE) is demonstrated in Figure 5b. The initial conditions $S_0 = 25.5$ and $I_0 = 1.95$ are used to generate these numerical results, and the parameter D is varied over the interval $D \in [0.87, 5]$. Furthermore, Figure 6a,c,e illustrate the phase portraits, and Figure 6b,d,f display the time series of model (4) for several representative values of D . We also observe that Figure 6c,e demonstrate that for parameter values $1.9216 < D < 3.238$, the fixed point M_2 remains locally asymptotically stable. At the critical value $D = 3.238$, M_2 loses stability and vanishes, while a new stable fixed M_1 appears (see Figure 6e,f, indicating that the model (4) undergoes a transcritical bifurcation. For $D \leq 1.9216$, an invariant closed curve emerges around M_2 , whose radius increases as D moves away from the bifurcation value. This phenomenon arises from a subcritical Neimark–Sacker bifurcation occurring at $D = D'$ (see Figure 6a,b).

- Here, we utilize the following set of parameter values for the dynamical system described in model (3):

$$R = 8.412, A = 02.7, B = 02.31, C = 07.2, D = 01.876, K = 0.92, m = 0.021,$$

together with the initial conditions $(T_0, E_0) = (0.253, 0.397)$. The bifurcation parameter h is varied within the interval $[0, 0.7]$. At $h'_0 = 0.1079$, the system undergoes a Neimark–Sacker bifurcation. At this critical value, the fixed point $M'_2 = (0.2606, 0.3669)$ loses stability, as confirmed by the characteristic polynomial of the Jacobian matrix,

$$\Lambda(\zeta) = \zeta^2 - 1.9773\zeta + 1.$$

The eigenvalues are

$$\zeta_{1,2} = 0.9886 \mp 0.1349 i.$$

These eigenvalues lie on the unit circle at the bifurcation phenomenon. This confirms the existence of a Neimark–Sacker bifurcation.

Note that Figure 7 illustrates the bifurcation diagrams of the tumor and effector cell populations together with the corresponding maximum Lyapunov exponent over the interval $h \in [0, 0.7]$. It can be seen from Figure 7a that the fixed point M'_2 of model (3) is locally asymptotically stable for $0 < h < 0.1079$. When the parameter exceeds $h = 0.1079$, the fixed point becomes unstable, and a closed invariant curve emerges for $0.1079 \leq h < 0.7$. This emergence indicates the existence of a Neimark–Sacker bifurcation. This transition is further supported by Figure 7b, where the maximum Lyapunov exponent alternates between negative and positive values, reflecting shifts from stable dynamics to quasi-periodic or chaotic regimes. These results exhibit that even small modifications in the parameter h can have a substantial effect on the behavior of the system, leading to intricate oscillating patterns that could have an effect on the population dynamics over a long time. We show that the strong Allee effect m is a potent bifurcation parameter in Figure 8a,b, producing well-organized and instructive bifurcation diagrams.

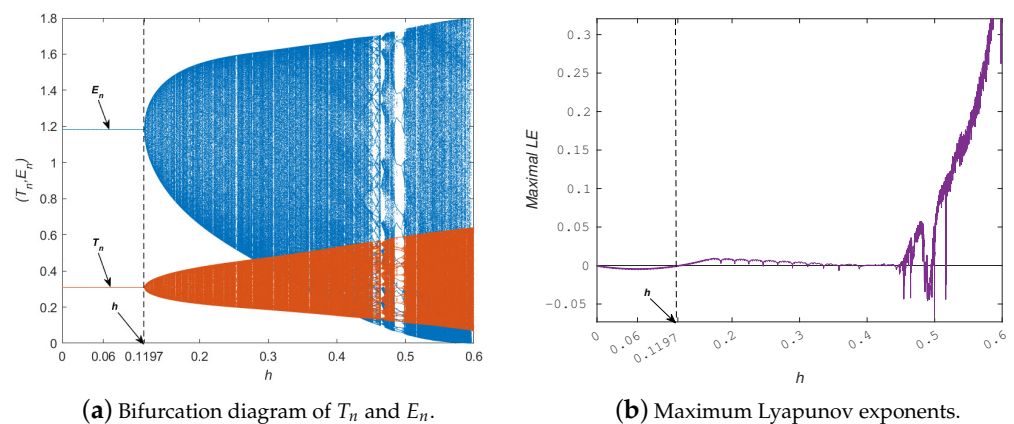


Figure 1. (a) The Neimark–Sacker bifurcation for Tumor and Effector cells of model (4) and (b) the corresponding maximum Lyapunov exponents for specific parameters as $R = 10.5181$, $A = 1.7$, $B = 1.91$, $C = 6.2$, $D = 1.9121$, $K = 1.2$, and $h \in [0, 0.6]$ with initial conditions $(T_0, E_0) = (0.353, 0.97)$.

These chaotic dynamics illustrate the system’s pronounced sensitivity to initial conditions and the loss of periodic structure. Overall, the analysis illustrates how variations in the bifurcation parameter can drive the system from regular periodic dynamics to complex and unpredictable chaos (or, conversely, from chaotic motion back to orderly behavior) thereby revealing the rich bifurcation structure and dynamic complexity inherent in the model.

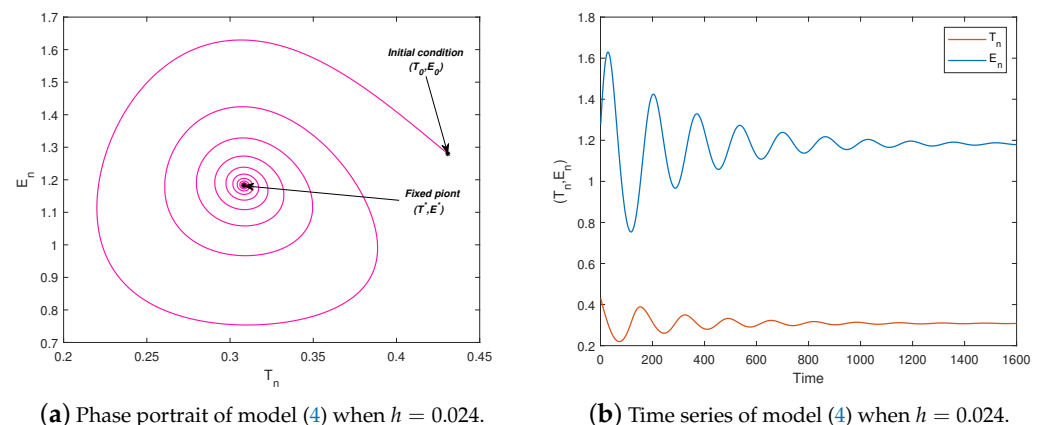


Figure 2. Cont.

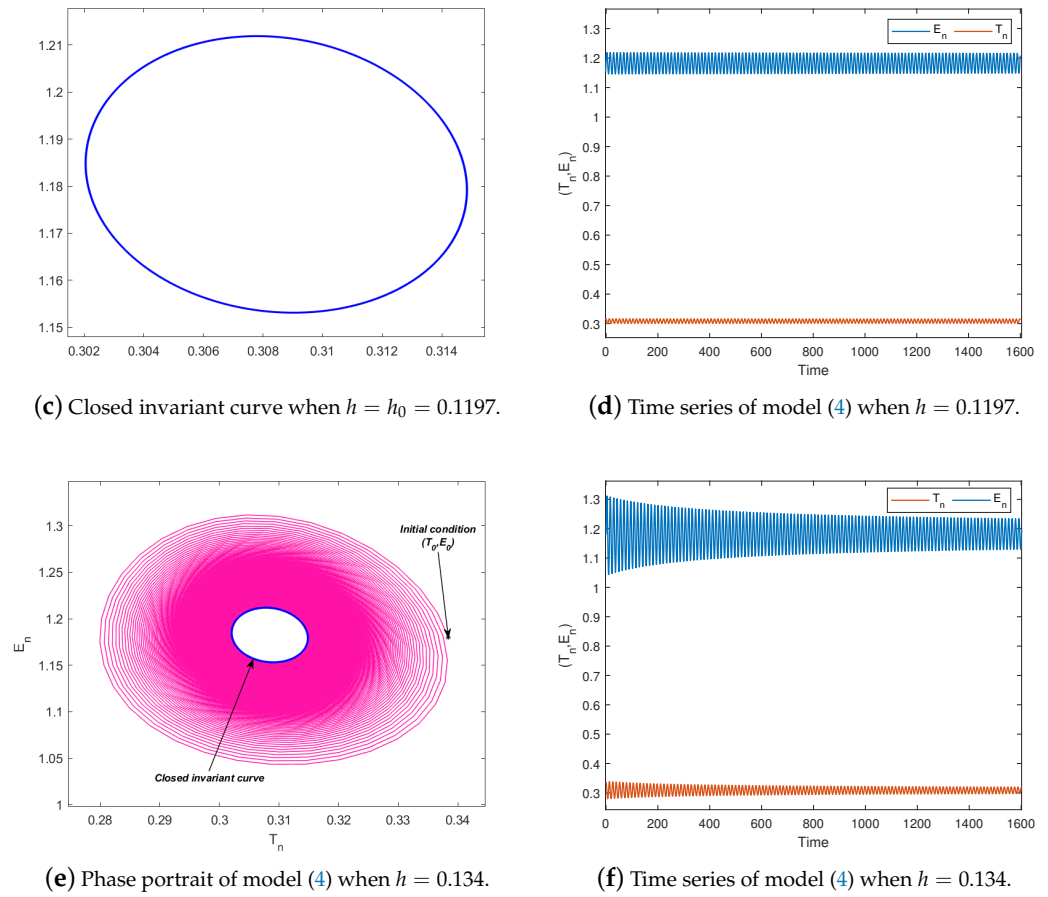


Figure 2. Phase portraits and time evolutions for different values of h are shown: (a,b) $h = 0.024$, (c,d) $h = 0.1197$, and (e,f) $h = 0.134$, using the parameter values $R = 10.5181$, $A = 1.7$, $B = 1.91$, $C = 6.2$, $D = 1.9121$, $K = 1.2$, and $h \in [0, 0.6]$ with initial conditions $(T_0, E_0) = (0.353, 0.97)$.

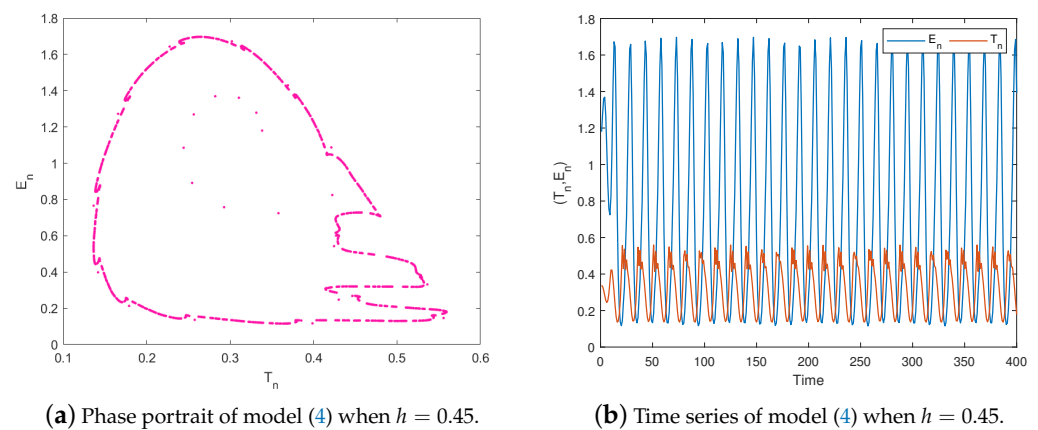


Figure 3. Cont.

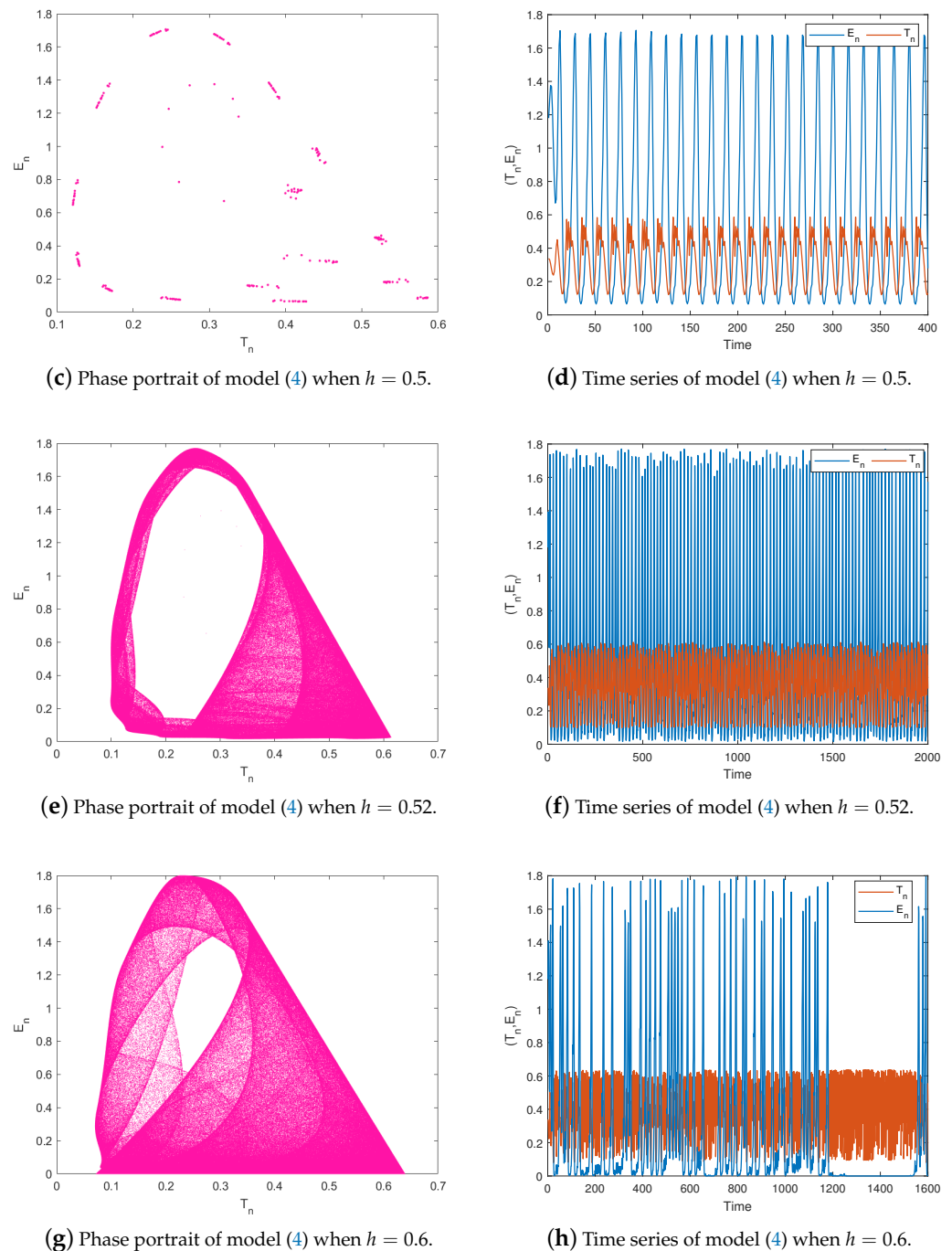


Figure 3. Phase portraits and time evolutions for different values of h are shown: (a,b) $h = 0.45$, (c,d) $h = 0.5$, (e,f) $h = 0.53$, and (g,h) $h = 0.6$, using the parameter values $R = 10.5181$, $A = 1.7$, $B = 1.91$, $C = 6.2$, $D = 1.9121$, and $K = 1.2$ with initial conditions $(T_0, E_0) = (0.353, 0.97)$.

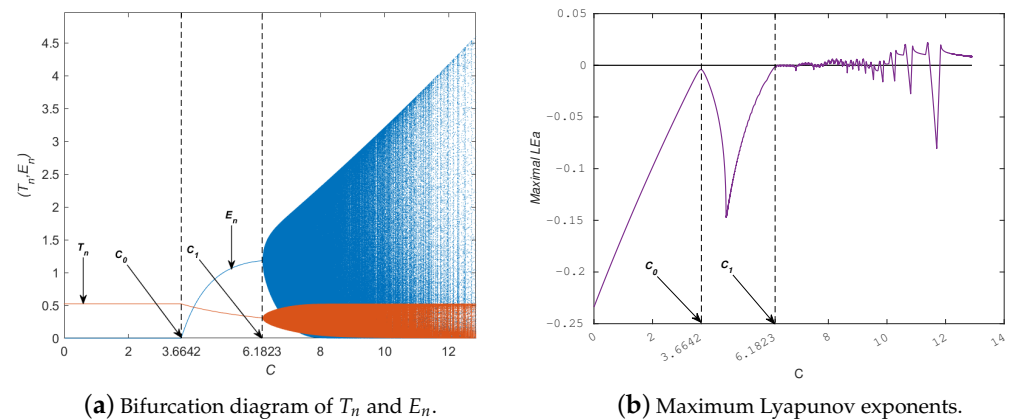


Figure 4. (a) The Neimark–Sacker bifurcation for Tumor and Effector cells of model (4) and (b) the corresponding maximum Lyapunov exponents for specific parameters as $R = 10.5181$, $A = 1.7$, $B = 1.91$, $D = 1.9121$, $K = 1.2$, $h = 0.1197$, and $C \in [0, 12.9]$ with initial conditions $(T_0, E_0) = (0.353, 0.97)$.

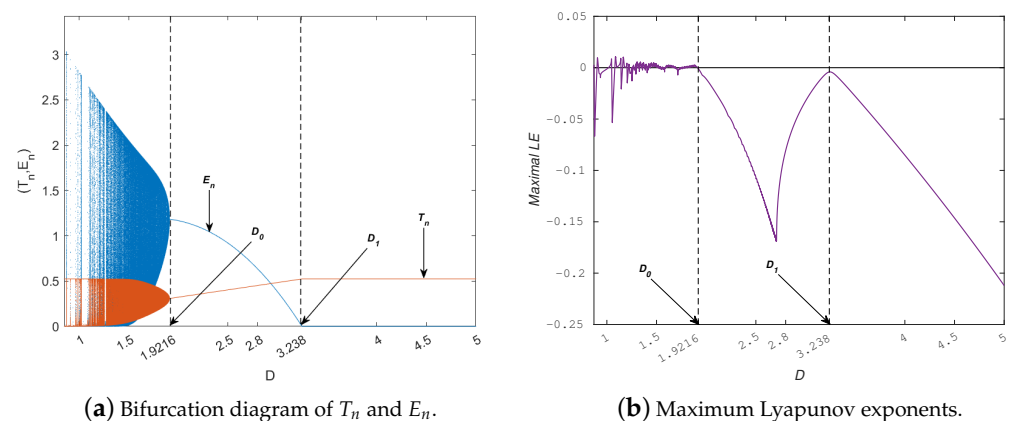


Figure 5. (a) The Neimark–Sacker bifurcation for Tumor and Effector cells of model (4) and (b) the corresponding maximum Lyapunov exponents for specific parameters as $R = 10.5181$, $A = 1.7$, $B = 1.91$, $C = 6.2$, $K = 1.2$, $h = 0.1197$, and $D \in [0, 5]$ with initial conditions $(T_0, E_0) = (0.353, 0.97)$.

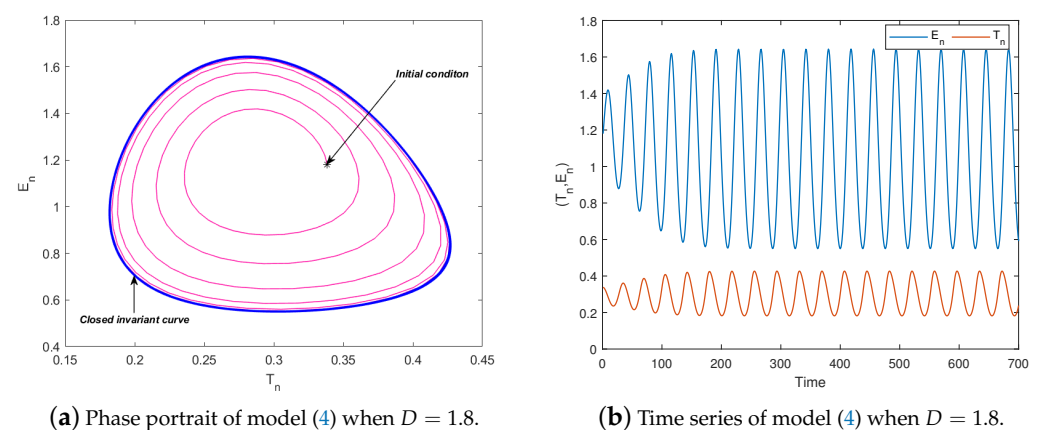


Figure 6. Cont.

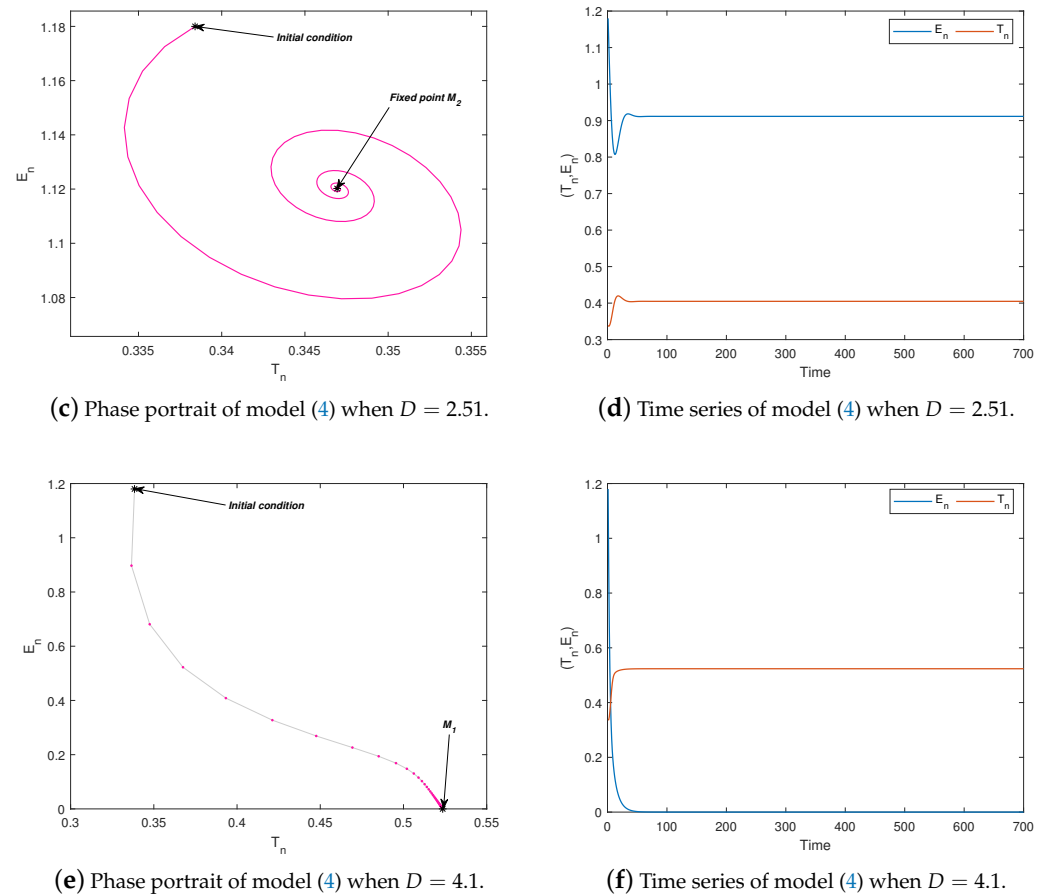


Figure 6. Phase portraits and time evolutions of model (4) for different values of D are shown: (a,b) $D = 1.8$, (c,d) $D = 2.51$, and (e,f) $D = 4.1$, using the parameter values $R = 10.5181$, $A = 1.7$, $B = 1.91$, $C = 6.2$, $K = 1.2$, $h = 0.1197$, and $D \in [0.87, 5]$ with initial conditions $(T_0, E_0) = (0.353, 0.97)$.

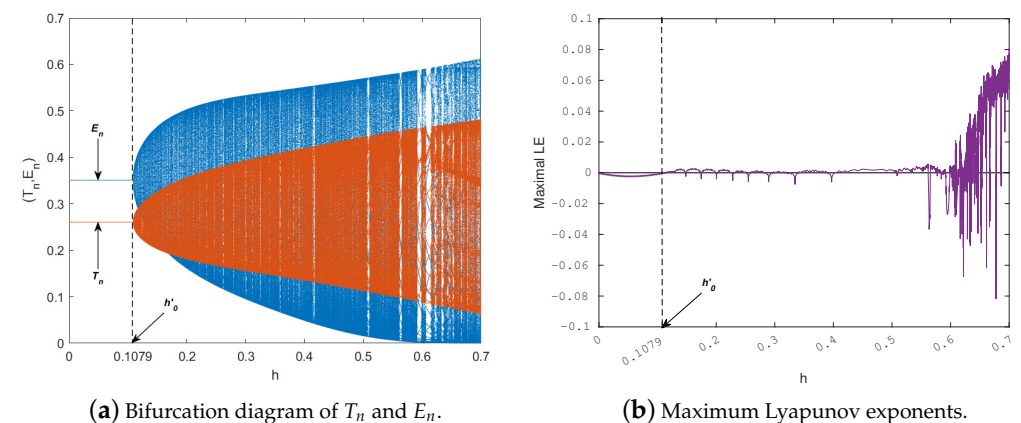


Figure 7. (a) The Neimark–Sacker bifurcation for Tumor and Effector cells of model (3) and (b) the corresponding maximum Lyapunov exponents for specific parameters as $R = 8.412$, $A = 2.7$, $B = 2.31$, $C = 7.2$, $D = 1.876$, $K = 0.92$, $m = 0.021$, and $h \in [0, 0.7]$ with initial conditions $(T_0, E_0) = (0.253, 0.397)$.

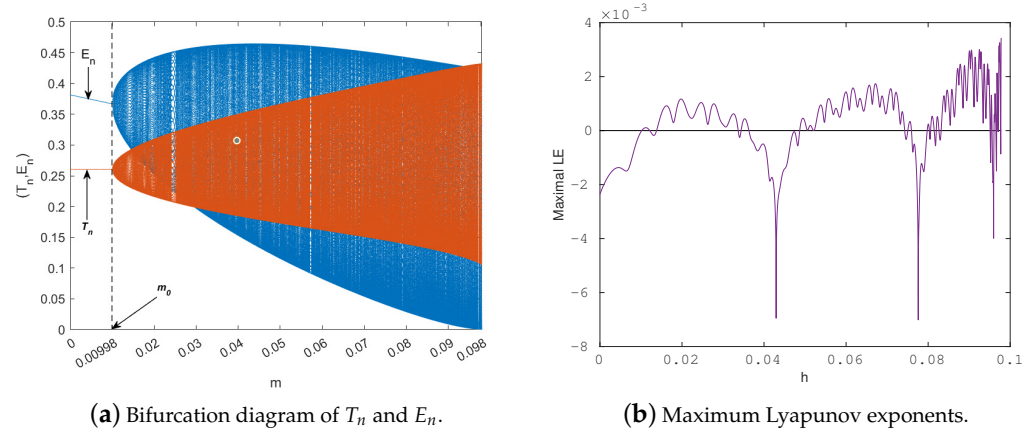


Figure 8. (a) The Neimark–Sacker bifurcation for Tumor and Effector cells of model (3) and (b) the corresponding maximum Lyapunov exponents for specific parameters as $R = 8.412$, $A = 2.7$, $B = 2.31$, $C = 7.2$, $D = 1.876$, $K = 0.92$, $h = 0.1339$, and $m \in [0, 0.098]$ with initial conditions $(T_0, E_0) = (0.253, 0.397)$.

5.2. Period-Doubling Bifurcation Simulation

One of the basic mechanisms by which discrete dynamical systems change from stable dynamics to chaos as system parameters alter is period-doubling bifurcation. This subsection utilizes computational simulations that demonstrate that period-doubling bifurcations exist and to investigate their significance in the shift to chaotic behavior.

First, the discrete tumor-immune model is considered with fixed parameter values $R = 18.481$, $A = 2.97$, $B = 1.98$, $C = 3.52$, $D = 1.768$ and $K = 0.372$, and initial conditions $(T_0, E_0) = (0.4, 0.07)$. Note that the interval $[0, 0.33]$ is successfully used to control the bifurcation parameter h . At $h = 0.2184$, the conditions of Lemma 1 are satisfied, leading to the existence of a fixed point $M_2 = (0.5023, 0.0151)$. Figure 9 illustrates the bifurcation diagrams of the tumor and effector cell populations, together with the corresponding maximum Lyapunov exponent. The tumor cell population experiences a number of period-doubling bifurcations as the bifurcation parameter h increases, as shown in Figure 9a. This surely indicates a progressive change from regular periodic oscillations to more complex dynamics. The effector cell population has a similar evolution, demonstrating that both populations react similarly to variations in h . The maximum Lyapunov exponent in Figure 9b provides quantitative evidence of these transitions: negative values correspond to stable or periodic dynamics, while positive values signal the onset of chaos. Additionally, the phase portraits depicted in Figure 10 for selected values of h visually capture the progression from simple oscillatory behavior to more intricate trajectories. Figure 9a illustrates the occurrence of a period-doubling bifurcation in the tumor cell population as the parameter h increases. Moreover, the corresponding maximum Lyapunov exponent shown in Figure 9b clearly delineates the transitions between stable and chaotic dynamical regimes. In an analogous manner, Figure 11a shows a period-doubling bifurcation in the tumor cell population with the discretization parameter set at $h = 0.213$ while the parameter R (the intrinsic tumor growth rate) grows. The bifurcation scenarios seen in the numerical simulations are further supported by the corresponding maximum Lyapunov exponent displayed in Figure 11b, which demonstrates the shift from stable behavior to chaos.

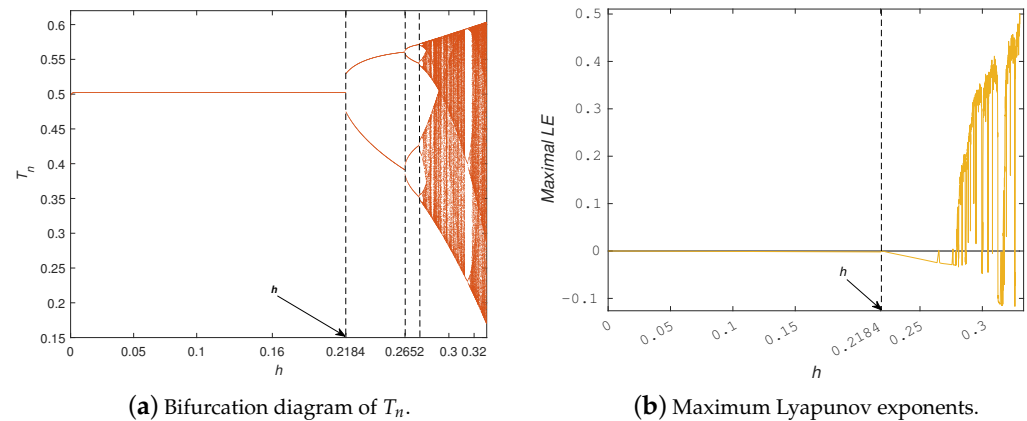


Figure 9. (a) Period-doubling bifurcation diagram of model (4). (b) Corresponding maximum Lyapunov exponents for the parameter values: $R = 18.481$, $A = 2.97$, $B = 1.98$, $C = 3.52$, $D = 1.768$, $K = 0.372$, and $h \in [0, 0.33]$ with initial conditions $(T_0, E_0) = (0.4, 0.07)$.

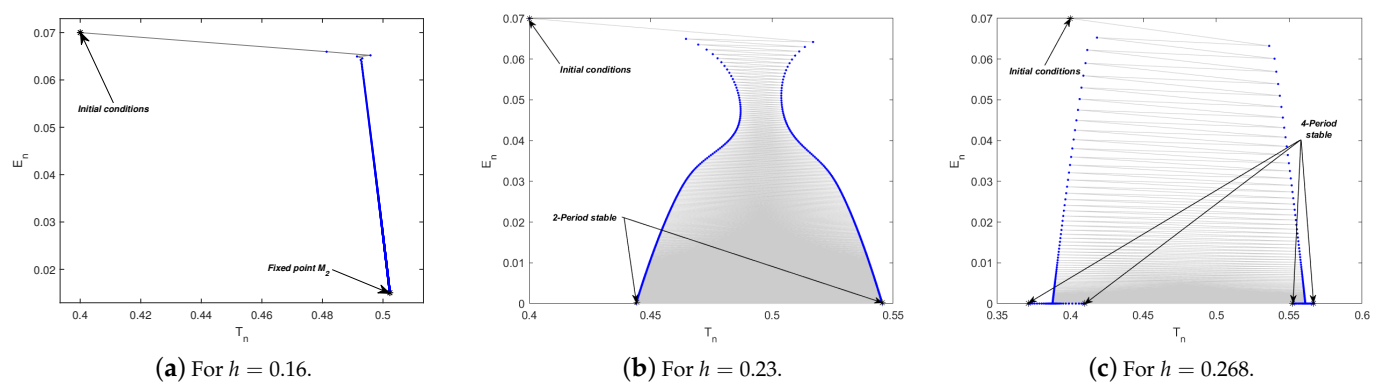


Figure 10. Phase portraits of model (4) for different values of h are shown: (a) $h = 0.16$, (b) $h = 0.23$, (c) $h = 0.268$, using the parameter values $R = 18.481$, $A = 2.97$, $B = 1.98$, $C = 3.52$, $D = 1.768$, $K = 0.372$, and $h \in [0, 0.33]$ with initial conditions $(T_0, E_0) = (0.4, 0.07)$.

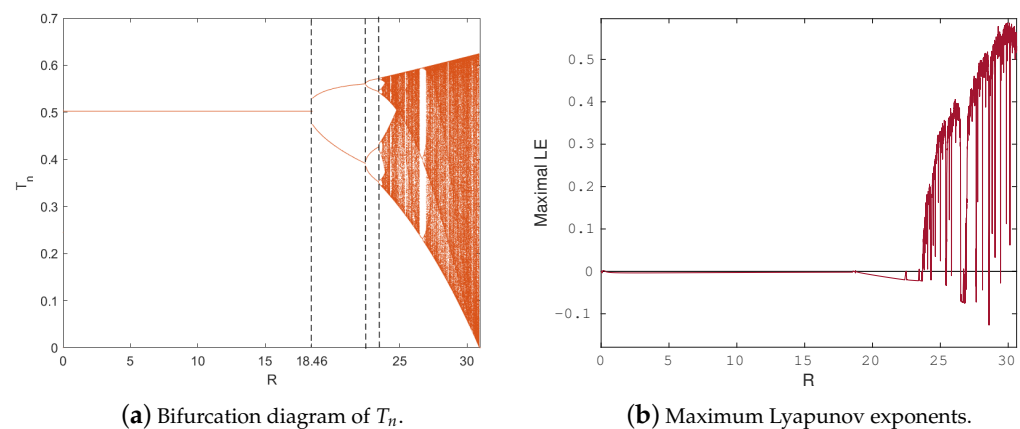


Figure 11. (a) Period-doubling bifurcation diagram of model (4). (b) Corresponding maximum Lyapunov exponents for the parameter values: $A = 2.97$, $B = 1.98$, $C = 3.52$, $D = 1.768$, $K = 0.372$, $h = 0.2184$ and $R \in [0, 31]$ with initial conditions $(T_0, E_0) = (0.54, 0.03)$.

6. 0–1 Test Method

A major problem in the study of nonlinear dynamical systems is determining chaotic behavior from time-series data, which has been thoroughly explored in fields including physics, engineering, biology, and economics. Conventional chaos detection methods, including the computation of Lyapunov exponents, correlation dimensions, and entropy-based indices, typically depend on phase-space reconstruction using Takens' embedding theorem. These techniques often require long, noise-free data sets and a careful selection of embedding parameters, which can be difficult to ensure in real-world or experimental settings.

In order to overcome these difficulties, Gottwald and Melbourne [19] developed the 0–1 test for chaos, a simple and effective method that uses simply a scalar time series to differentiate between chaotic and regular dynamics. The 0–1 test, in contrast to conventional techniques, does not require explicit knowledge of the system's governing equations or phase-space reconstruction. Alternatively, it assesses a translation process's long-term growth characteristics that are directly drawn from the observed data. The test produces a binary-type result, with values near zero denoting regular motion and values near one indicating chaotic behavior.

For the analysis of brief and noisy data sets, the 0–1 test is especially interesting due to its adaptability and simplicity of use. The strategy has gone through massive theoretical validation and practical improvement since its creation, exhibiting good consistency with recognized chaos indicators and providing notable computing benefits. It has been effectively used for a wide range of dynamical systems [15,20,21], including discrete-time maps, continuous-time models, laboratory experiments, and empirical signals.

In this section, we present the theoretical basis of the 0–1 test and describe its numerical implementation. Note that, we provide illustrative simulations to show that the technique can reliably differentiate between chaotic and regular behavior. The test operates on a given scalar time series and produces an output that approaches either 0 or 1, depending on the nature of the underlying dynamics. In particular, for a time series $\{P_1(i)\}_{i=1}^N$ and a randomly chosen parameter $e \in (0, \pi)$, the associated translation variables for $j = 1, 2, \dots, N$ are defined as follows:

$$\Phi_e(j) = \sum_{i=1}^j u(i) \cos(i e), \quad \Psi_e(j) = \sum_{i=1}^j u(i) \sin(i e). \quad (17)$$

The trajectories of the variables Φ_e and Ψ_e plotted in the (Φ_e, Ψ_e) -plane provide a visual criterion for detecting chaotic behavior. If the motion of Φ_e and Ψ_e remains bounded, the underlying dynamics of the system are regular. In contrast, if the trajectories exhibit unbounded, Brownian-like motion, the system displays chaotic dynamics. To quantify this behavior, the mean square displacement (MSD) is defined as follows:

$$\mathcal{MS}_{\mathcal{D}_e}(i) = \frac{1}{N} \sum_{j=1}^N (\Phi_e(j+i) - \Phi_e(j))^2 + (\Psi_e(j+i) - \Psi_e(j))^2, \quad i \leq \frac{N}{10}. \quad (18)$$

We also define the correlation coefficient K_e as

$$K_e = \frac{\text{cov}(\zeta_1, \zeta_2)}{\sqrt{\text{var}(\zeta_1) \text{var}(\zeta_2)}} \in [-1, 1], \quad (19)$$

where $\zeta_1 = \{1, 2, \dots, i\}$ and $\zeta_2 = \{\mathcal{MS}_{\mathcal{D}_e}(1), \mathcal{MS}_{\mathcal{D}_e}(2), \dots, \mathcal{MS}_{\mathcal{D}_e}(i)\}$. The asymptotic growth rate is then defined by

$$K = \text{median}(K_e). \quad (20)$$

This quantity serves as an indicator for distinguishing between regular and chaotic dynamics in discrete models. Specifically, values of K close to 0 indicate non-chaotic behavior, while values approaching 1 suggest chaotic dynamics.

Example 1. Using the parameter values $A = 1.7$, $B = 1.91$, $C = 6.2$, $D = 1.9121$, $K = 1.2$, and the bifurcation parameter $h \in [0, 0.6]$, we use the 0–1 test for chaos to verify the existence of chaotic dynamics in the discrete tumor-immune model (4). The phase trajectories of the translation components Φ_e and Ψ_e for different values of h are illustrated in Figure 12. For $h = 0.075$ and $h = 0.4$, the (Φ_e, Ψ_e) phase plots exhibit bounded motion, indicating periodic or quasi-periodic dynamics. In contrast, for $h = 0.6$, the trajectories become unbounded and display Brownian-like motion, which is a hallmark of chaotic behavior. These results provide additional evidence of chaotic dynamics in the system since they are in line with the behaviors shown in the bifurcation diagrams and the accompanying fluctuations of the maximal Lyapunov exponent seen in Figure 1. Similarly, for the parameter set $R = 8.412$, $A = 2.7$, $B = 2.31$, $C = 7.2$, $D = 1.876$, $K = 0.92$, $m = 0.021$, and $S \in [0, 0.7]$, the phase trajectories of the translation variables Φ_e and Ψ_e are clearly presented in Figure 13. For $h = 0.09$ and $h = 0.26$, the (Φ_e, Ψ_e) phase diagrams exhibit bounded motion, reflecting periodic or quasi-periodic dynamics. Conversely, at $h = 0.7$, the trajectories become unbounded and resemble Brownian motion, which is characteristic of chaotic behavior. This observation is in good agreement with the bifurcation analysis and the maximal Lyapunov exponent depicted in Figure 7, further validating the presence of chaos at larger values of h .

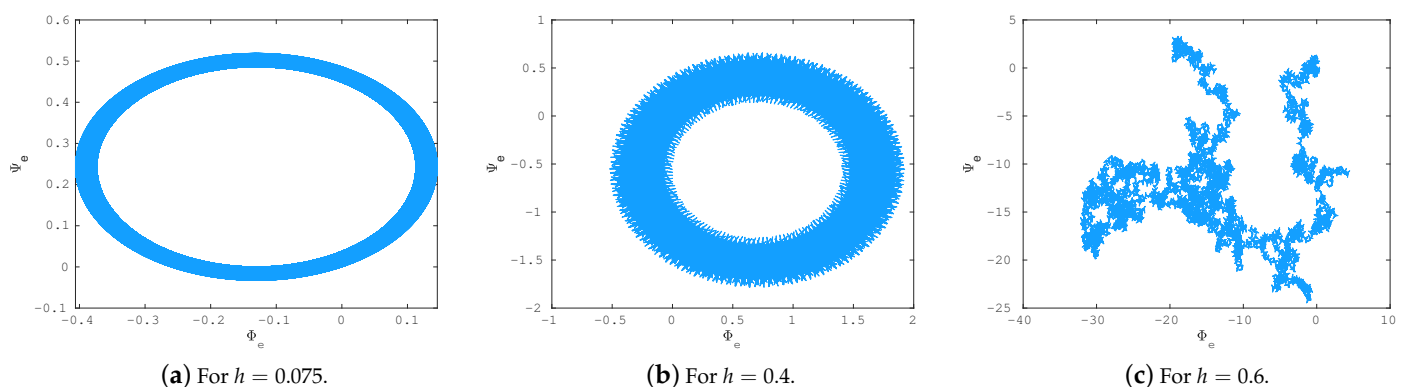


Figure 12. The 0–1 test of the discrete tumor-immune model (4) for different values of S , with parameter values: $R = 10.5181$, $A = 1.7$, $B = 1.91$, $C = 6.2$, $D = 1.9121$ and $K = 1.2$.

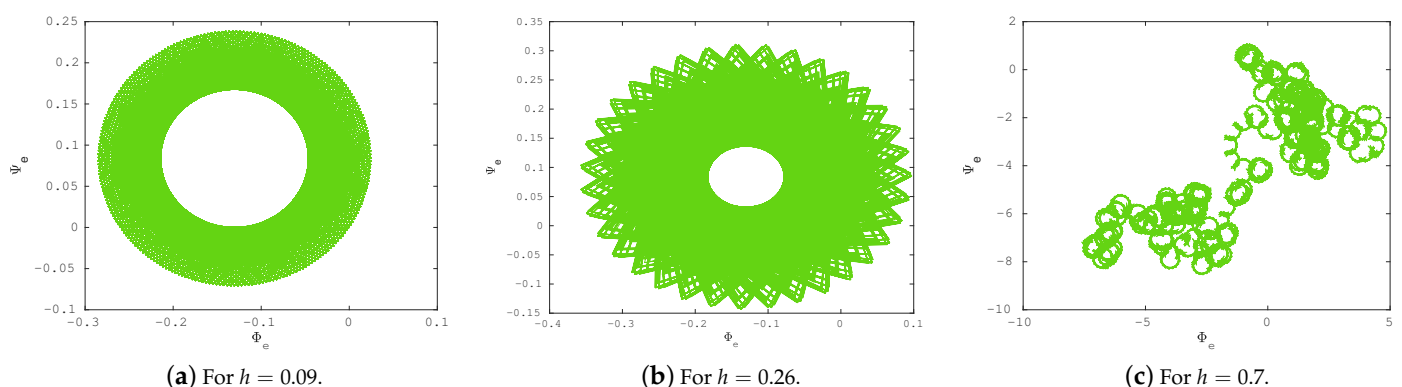


Figure 13. The 0–1 test of the discrete tumor-immune model (3) for different values of S , with parameter values: $R = 18.412$, $A = 2.7$, $B = 2.31$, $C = 7.2$, $D = 1.876$, $K = 0.92$ and $h = 0.1339$.

7. Controlling the Chaos of Model (3)

Since chaotic behavior is naturally unpredictable and highly reactive to initial conditions, chaos control in discrete dynamical systems has attracted plenty of attention. Despite their apparent randomness, chaotic systems are governed by deterministic laws and contain a dense set of unstable periodic orbits embedded within the chaotic attractor. The main objective of chaos control is to stabilize these unstable orbits with minimal actions, transforming chaotic motion into regular and predictable dynamics while maintaining the original system form. It is worth noting that the Ott–Grebogi–Yorke (OGY) technique [22], feedback control strategies [15,23], and hybrid control schemes [24] are some of the most popular methods for managing chaos in discrete-time systems. The OGY method achieves stabilization by applying small, state-dependent perturbations to system parameters when trajectories approach a desired unstable orbit. Feedback control methods, on the other hand, provide simplicity and robustness in implementation by using corrective inputs depending on the system state's divergence from a predetermined objective. It should be noticed that hybrid control methods integrate multiple control mechanisms to enhance performance, stability, and convergence speed, particularly in systems affected by uncertainties or noise.

In this section, we investigate the chaos control of model (3) using a feedback control approach. This method is simply implemented by formulating the controlled version of model (3) as follows:

$$\begin{cases} T_{n+1} = T_n + h \left(RT_n(1 - BT_n)(T_n - m) - \frac{AT_n E_n}{K + T_n} \right) - \xi_1 \left(T_n - \frac{D}{C} \right) - \xi_2 (E_n - E^*), \\ E_{n+1} = E_n + hE_n(CT_n - D), \end{cases} \quad (21)$$

where

$$E^* = \frac{R(KC + D)(C - BD)(D - mC)}{AC^3},$$

and $\xi_1, \xi_2 \in \mathbb{R}$ denote the feedback gains introduced to regulate the system dynamics.

By straightforward calculations, the Jacobian matrix of the controlled system evaluated at the fixed point M'_2 is given by

$$\begin{aligned} \mathcal{J}'_c(M'_2) &= \begin{pmatrix} \alpha'_{11} - \xi_1 & \alpha'_{12} - \xi_2 \\ \alpha'_{21} & \alpha'_{22} \end{pmatrix}, \\ &= \begin{pmatrix} 1 - \xi_1 + hR \left(-\frac{3BD^2}{C^2} + \frac{2(1 + Bm)D}{C} - \frac{K(C - BD)(D - mC)}{C(KC + D)} - m \right) & -\xi_2 - \frac{hAD}{KC + D} \\ \frac{hR(KC + D)(C - BD)(D - mC)}{AC^2} & 1 \end{pmatrix}. \end{aligned}$$

The corresponding characteristic equation associated with $\mathcal{J}'_c(M'_2)$ is

$$\Lambda_{\mathcal{J}'_c}(\zeta) = \zeta^2 - \text{Tr}(\mathcal{J}'_c)\zeta + \text{Det}(\mathcal{J}'_c), \quad (22)$$

with

$$\begin{aligned}\text{Tr}(\mathcal{J}'_c) &= \alpha'_{11} + \alpha'_{22} - \xi_1 = 2 - \xi_1 - hR \left(\frac{D(3BD - 2C - 2BCm)}{C^2} + \frac{K(C - BD)(D - mC)}{C(KC + D)} + m \right), \\ \text{Det}(\mathcal{J}'_c) &= \alpha'_{22}(\alpha_{11} - \xi_1) - \alpha_{21}(\alpha'_{12} - \xi_2), \\ &= 1 - \xi_1 - hR \left(\frac{D(3BD - 2C - 2BCm)}{C^2} + \frac{K(C - BD)(D - mC)}{C(KC + D)} + m \right) \\ &\quad + \left(\frac{hR(\xi(KC + D) + hAD)(C - BD)(D - mC)}{AC^2} \right).\end{aligned}$$

Let $\bar{\xi}_1$ and $\bar{\xi}_2$ be the solutions of Equation (22). Then

$$\bar{\xi}_1 + \bar{\xi}_2 = \alpha'_{11} + \alpha'_{22} - \xi_1, \text{ and } \bar{\xi}_1 \bar{\xi}_2 = \alpha'_{22}(\alpha_{11} - \xi_1) - \alpha_{21}(\alpha'_{12} - \xi_2). \quad (23)$$

The boundaries of marginal stability are obtained by solving the conditions $\bar{\xi}_1 \bar{\xi}_2 = 1$ and $\bar{\xi}_1 = \pm 1$. These conditions determine the parameter values at which the eigenvalues satisfy $|\bar{\xi}_{1,2}| = 1$. In particular, when the condition $\bar{\xi}_1 \bar{\xi}_2 = 1$ is imposed, Equation (23) becomes

$$B_1 : \quad \alpha'_{22} \bar{\xi}_1 - \alpha'_{21} \bar{\xi}_2 = \alpha'_{22} \alpha'_{11} - \alpha'_{12} \alpha'_{21} - 1.$$

Next, we consider the case $\bar{\xi}_1 = 1$. Substituting this condition into Equation (23), we obtain

$$B_2 : \quad (1 - \alpha_{22}) \bar{\xi}_1 + \alpha'_{21} \bar{\xi}_2 = -1 - \alpha'_{22} \alpha'_{11} + \alpha'_{21} \alpha'_{12} + \alpha'_{11} + \alpha'_{22}.$$

Finally, we consider the case $\bar{\xi}_1 = -1$. By substituting this condition into Equation (23), we obtain

$$B_3 : \quad (1 + \alpha'_{22}) \bar{\xi}_1 - \alpha'_{21} \bar{\xi}_2 = 1 + \alpha'_{22} \alpha'_{11} - \alpha'_{21} \alpha'_{12} + \alpha'_{11} + \alpha'_{22}.$$

The triangular domain defined by the lines B_1 , B_2 , and B_3 is the region of stability, where all eigenvalues meet the stability criterion.

Example 2. Let the parameters be fixed as

$$R = 10.5181, A = 1.7, B = 1.91, C = 6.2, D = 1.9121, K = 1.2, m = 0, \text{ and } h = 0.52.$$

Under this parameter configuration, model (4) admits a biologically meaningful fixed point given by

$$M_2 = (0.3084, 1.1828).$$

The stability region of the controlled system is characterized by the following linear boundary equations:

$$B_1 : \quad \xi_1 - 0.8780 \xi_2 = 0,$$

$$B_2 : \quad 0.8780 \xi_2 = -0.0365,$$

$$B_3 : \quad 2 \xi_1 - 0.8780 \xi_2 = 3.9635.$$

These constraints define a triangular domain in the (ξ_1, ξ_2) parameter space, as depicted in Figure 14a. Within this region, the eigenvalues of the controlled system described by model (21) remain inside the unit circle, ensuring local asymptotic stability of the fixed point under the feedback control strategy. By selecting the feedback gains $\xi_1 = 1.8$ and $\xi_2 = 0.15$, the equilibrium point M_2 becomes locally asymptotically stable, as demonstrated in Figure 14b. In contrast, when $\xi_1 = 0.03$

and $\xi_2 = 0.05$, the fixed point M_2 falls outside the stability region and consequently exhibits unstable behavior, as shown in Figure 14c.

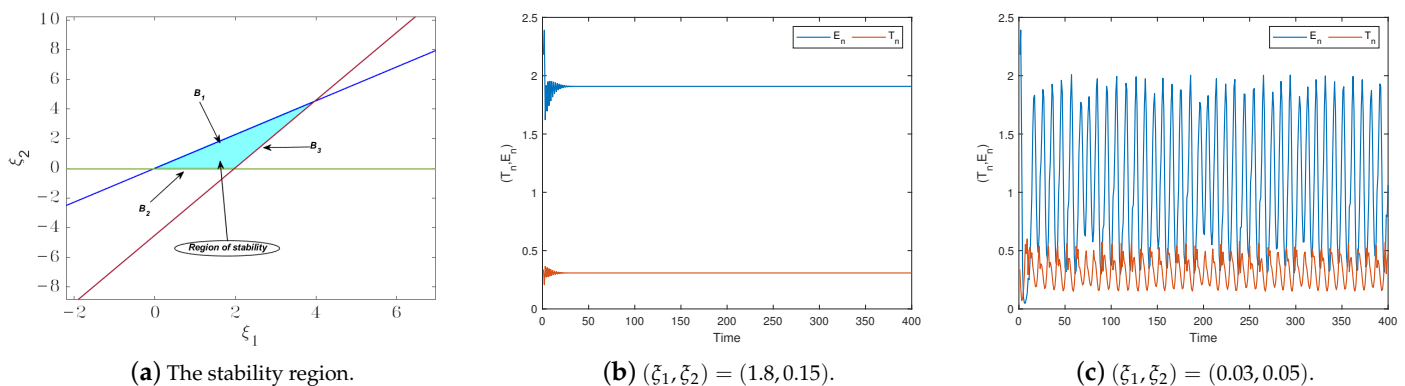


Figure 14. (a) Stability region of the controlled model (21), bounded by the lines B_1 , B_2 , and B_3 . (b,c) Time evolutions of the state variables T_n and E_n for the feedback gain pairs $(\xi_1, \xi_2) = (1.8, 0.15)$ and $(0.03, 0.05)$, respectively. The parameter values used in the simulations are $R = 10.5181$, $A = 1.7$, $B = 1.91$, $C = 6.2$, $D = 1.9121$, $K = 1.2$, $m = 0$, and $h = 0.52$.

8. Conclusions and Discussion

In this research, a discrete-time tumor-immune interaction system containing Allee effects has been clearly introduced and extensively discussed in order to demonstrate its complicated computational behavior. The incorporation of Allee effects into the tumor development process enabled the model to accurately represent important biological limits below which tumor populations may fail to survive and above which rapid growth may take place. The work established unambiguous quantitative requirements for tumor elimination, immune-controlled coexistence, and tumor dominance through identifying the existence conditions of equilibrium points and investigating their local stability qualities. We found that model (4) has three fixed points in which the coexistence point exists if and only if $C > BD$. If $h = 0.024$, model (4) has a stable fixed point M_2 , as depicted in Figure 2a,b. However, an attracting invariant closed curve emerges around the fixed point for $h = h_0 = 0.1197$. This indicates that this system has a Neimark–Sacker bifurcation (see Figure 2c,d). In particular, model (4) undergoes a Neimark–Sacker bifurcation at the fixed point M_2 . In particular, a stable invariant closed curve bifurcates from M_2 for $h > h^*$ when $\Delta < 0$ while an unstable invariant closed curve bifurcates from M_2 for $h < h^*$ when $\Delta > 0$ (see Figure 1). Biologically, the emergence of this type of bifurcation implies that the tumor-immune system loses its stable equilibrium stage and demonstrates continuous oscillatory fluctuations in the size of tumor and immune reaction. More specifically, the tumor population and immune cells reach a steady balance prior to the bifurcation. This indicates the immune system maintains the tumor under control. The tumor alternates between remission and regrowth at the Neimark–Sacker bifurcation, where this equilibrium becomes unstable. We also discovered that model (4) undergoes a period-doubling (flip) bifurcation at the positive coexistence fixed point M_2 as the parameter h varies in a small neighborhood of Δ_{PD} . Specifically, model (4) bifurcates from the coexistence fixed point M_2 to a stable period-2 orbit when $\nu_2 > 0$, while it bifurcates from the coexistence fixed point M_2 to an unstable period-2 orbit when $\nu_2 < 0$. This behavior can be clearly seen in Figure 9a which shows the occurrence of a period-doubling bifurcation in the tumor cell population as the parameter h increases. Biologically, tumor-immune dynamics vary from being regular to becoming more complicated and possibly chaotic as the tumor progresses.

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Appendix A. Main Concepts

In this section, we introduce the key concepts that form the basis of our investigation.

Lemma A1 ([25]). Let (T, E) be a fixed point of a discrete system with eigenvalues ζ_1 and ζ_2 of the Jacobian matrix $J_M(T, E)$. The stability of the fixed point is characterized as follows:

1. If $|\zeta_1| < 1$ and $|\zeta_2| < 1$, the fixed point is a sink (locally asymptotically stable).
2. If $|\zeta_1| > 1$ and $|\zeta_2| > 1$, the fixed point is a source (unstable).
3. If $|\zeta_1| < 1$ and $|\zeta_2| > 1$ or $(|\zeta_1| > 1$ and $|\zeta_2| < 1)$, the fixed point is a saddle.
4. If $|\zeta_1| = 1$ or $|\zeta_2| = 1$, the fixed point is non-hyperbolic.

Furthermore, Lemma A2 provides an alternative classification based on the characteristic polynomial.

Lemma A2 ([25]). Suppose that the polynomial $\Omega(\zeta) = \zeta^2 - \text{Tr} \zeta + \text{Det}$, where $\Omega(1) > 0$, and ζ_1 and ζ_2 are the two roots of $\Omega(\zeta) = 0$. The properties of the roots are characterized as follows:

1. $|\zeta_1| < 1$ and $|\zeta_2| < 1$ if and only if $\Omega(-1) > 0$ and $\Omega(0) < 1$.
2. $|\zeta_1| > 1$ and $|\zeta_2| > 1$ if and only if $\Omega(-1) > 0$ and $\Omega(0) > 1$.
3. $|\zeta_1| < 1$ and $|\zeta_2| > 1$ (or $|\zeta_1| > 1$ and $|\zeta_2| < 1$) if and only if $\Omega(-1) < 0$.
4. $\zeta_1 = -1$ and $|\zeta_2| \neq 1$ if and only if $\Omega(-1) = 0$ and $\text{Tr} \neq 0, 2$.
5. ζ_1 and ζ_2 are complex and $|\zeta_1| = |\zeta_2| = 1$ if and only if $\text{Tr}^2 - 4\text{Det} < 0$ and $\Omega(0) = 1$.

Lemma A3 ([23]). Assume that $X_{m+1} = \mathcal{H}_\epsilon(X_m)$ is a n -dimensional discrete dynamical system where $\epsilon \in \mathbb{R}$ is a bifurcation parameter. Let X^* be an fixed point of $\mathcal{H}_\epsilon$ and suppose that the characteristic equation of the Jacobian matrix $\mathcal{J}(X^*) = (\beta_{ij})_{n \times n}$ of n -dimensional map $\mathcal{H}_\epsilon(X_m)$ is expressed as

$$\Omega_\epsilon(\zeta) = \zeta^n + \beta_1 \zeta^{n-1} + \cdots + \beta_{n-1} \zeta + \beta_n. \quad (\text{A1})$$

Here, $\beta_i = \beta_i(\epsilon, u)$, $i = 1, 2, 3, \dots, n$ and u is a control parameter. Suppose that $D_0^\pm(\epsilon, u) = 1$, $D_1^\pm(\epsilon, u), \dots, D_n^\pm(\epsilon, u)$ are a sequence of the determinants described by

$$D_i^\pm(m, u) = \text{Det}(N_1 \pm N_2), \quad i = 1, 2, \dots, n, \quad (\text{A2})$$

where

$$\mathcal{A}_1 = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \cdots & \beta_{i-1} \\ 0 & 1 & \beta_1 & \cdots & \beta_{i-2} \\ 0 & 0 & 1 & \cdots & \beta_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} \beta_{n-i+1} & \beta_{n-i+2} & \cdots & \beta_{n-1} & \beta_n \\ \beta_{n-i+2} & \beta_{n-i+3} & \cdots & \beta_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{n-1} & \beta_n & \cdots & 0 & 0 \\ \beta_n & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (\text{A3})$$

Furthermore, suppose that the following statements hold.

- C1- Eigenvalue assignment $D_{n-1}^-(\epsilon_0, u) = 0$, $D_{n-1}^+(\epsilon_0, u) > 0$, $\Omega_{\epsilon_0}(1) > 0$, $(-1)^n \Omega_{\epsilon_0}(-1) > 0$, $D_i^\pm(\epsilon, u) > 0$, for $i = n-3, n-5, \dots, 2$ (or 1), when n is odd (or even), respectively.
- C2- Transversality condition: $\left[\frac{dD_{n-1}^-(\epsilon, u)}{d\epsilon} \right]_{\epsilon=\epsilon_0} \neq 0$.
- C3- Non-resonance condition: $\cos(2\pi/j) \neq \phi$, or resonance condition $\cos(2\pi/j) = \phi$, where $j = 3, 4, 5, \dots$, and $\psi = 1 - 0.5P_{\epsilon_0}(1)D_{n-3}^-(m_0, u)/D_{n-2}^+(\epsilon_0, u)$. Then, a Neimark–Sacker bifurcation occurs at ϵ_0 .

Lemma A4 ([23]). Let

$$X_{m+1} = \mathcal{H}_\epsilon(X_m),$$

be an n -dimensional system where $X_m \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}$ denotes the bifurcation parameter. In addition, suppose that the constraints (A1)–(A3) of Lemma A3 are satisfied and suppose that the following conditions are true:

- C1- Eigenvalue criterion: $\Omega_{\epsilon_0}(-1) = 0$, $D_{n-1}^\pm(\epsilon_0, u) > 0$, $\Omega_{\epsilon_0}(1) > 0$, $D_i^\pm(\epsilon_0, u) > 0$, $i = n-2, n-4, \dots, 1$ (or 1), when n is even (or odd), respectively.
- C2- Transversality criterion: $\frac{\sum_{i=1}^n (-1)^{n-i} \beta'_i}{\sum_{i=1}^n (-1)^{n-i} (n-i+1) \beta_{i-1}} \neq 0$, where β'_i represents the derivative of $\beta_i(\epsilon)$ at $\epsilon = \epsilon_0$. Then, a period-doubling bifurcation exists at critical value β_0 .

Appendix B. Coefficients of Taylor Expansion

$$\begin{aligned} \tilde{\mathcal{H}}'_1(\tilde{T}, \tilde{E}, \tilde{h}) = & \left(\frac{\alpha_{13}(\zeta_2 - \alpha_{11}) - \alpha_{12}\alpha_{23}}{\alpha_{12}(1 + \zeta_2)} \right) T_n^2 + \left(\frac{\alpha_{14}(\zeta_2 - \alpha_{11}) - \alpha_{12}\alpha_{24}}{\alpha_{12}(1 + \zeta_2)} \right) T_n E_n + \left(\frac{\alpha_{15}(\zeta_2 - \alpha_{11}) - \alpha_{12}\alpha_{25}}{\alpha_{12}(1 + \zeta_2)} \right) E_n^2 \\ & + \left(\frac{\alpha_{16}(\zeta_2 - \alpha_{11}) - \alpha_{12}\alpha_{26}}{\alpha_{12}(1 + \zeta_2)} \right) T_n^3 + \left(\frac{\alpha_{17}(\zeta_2 - \alpha_{11}) - \alpha_{12}\alpha_{27}}{\alpha_{12}(1 + \zeta_2)} \right) T_n^2 E_n + \left(\frac{\alpha_{18}(\zeta_2 - \alpha_{11}) - \alpha_{12}\alpha_{28}}{\alpha_{12}(1 + \zeta_2)} \right) T_n E_n^2 \\ & + \left(\frac{\alpha_{19}(\zeta_2 - \alpha_{11}) - \alpha_{12}\alpha_{29}}{\alpha_{12}(1 + \zeta_2)} \right) E_n^3 + \left(\frac{\beta_{11}(\zeta_2 - \alpha_{11}) - \alpha_{12}\beta_{21}}{\alpha_{12}(1 + \zeta_2)} \right) T_n \tilde{h} + \left(\frac{\beta_{12}(\zeta_2 - \alpha_{11}) - \alpha_{12}\beta_{22}}{\alpha_{12}(1 + \zeta_2)} \right) E_n \tilde{h} + \\ & + \left(\frac{\beta_{13}(\zeta_2 - \alpha_{11}) - \alpha_{12}\beta_{23}}{\alpha_{12}(1 + \zeta_2)} \right) \tilde{h}^2 + \left(\frac{\beta_{14}(\zeta_2 - \alpha_{11}) - \alpha_{12}\beta_{24}}{\alpha_{12}(1 + \zeta_2)} \right) T_n^2 \tilde{h} + \left(\frac{\beta_{15}(\zeta_2 - \alpha_{11}) - \alpha_{12}\beta_{25}}{\alpha_{12}(1 + \zeta_2)} \right) E_n^2 \tilde{h} \\ & + \left(\frac{\beta_{16}(\zeta_2 - \alpha_{11}) - \alpha_{12}\beta_{26}}{\alpha_{12}(1 + \zeta_2)} \right) T_n E_n \tilde{h} + \left(\frac{\beta_{17}(\zeta_2 - \alpha_{11}) - \alpha_{12}\beta_{27}}{\alpha_{12}(1 + \zeta_2)} \right) \tilde{h}^3 + \left(\frac{\beta_{18}(\zeta_2 - \alpha_{11}) - \alpha_{12}\beta_{28}}{\alpha_{12}(1 + \zeta_2)} \right) T_n \tilde{h}^2 \\ & + \left(\frac{\beta_{19}(\zeta_2 - \alpha_{11}) - \alpha_{12}\beta_{29}}{\alpha_{12}(1 + \zeta_2)} \right) E_n^2 \tilde{h} + O(|T_n|, |E_n|, |\tilde{h}|)^4, \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{H}}'_2(\tilde{T}, \tilde{E}, \tilde{h}) = & \left(\frac{\alpha_{13}(1 + \alpha_{11}) + \alpha_{12}\alpha_{23}}{\alpha_{12}(1 + \zeta_2)} \right) T_n^2 + \left(\frac{\alpha_{14}(1 + \alpha_{11}) + \alpha_{12}\alpha_{24}}{\alpha_{12}(1 + \zeta_2)} \right) T_n E_n + \left(\frac{\alpha_{15}(1 + \alpha_{11}) + \alpha_{12}\alpha_{25}}{\alpha_{12}(1 + \zeta_2)} \right) E_n^2 \\ & + \left(\frac{\alpha_{16}(1 + \alpha_{11}) + \alpha_{12}\alpha_{26}}{\alpha_{12}(1 + \zeta_2)} \right) T_n^3 + \left(\frac{\alpha_{17}(1 + \alpha_{11}) + \alpha_{12}\alpha_{27}}{\alpha_{12}(1 + \zeta_2)} \right) T_n^2 E_n + \left(\frac{\alpha_{18}(1 + \alpha_{11}) + \alpha_{12}\alpha_{28}}{\alpha_{12}(1 + \zeta_2)} \right) T_n E_n^2 \\ & + \left(\frac{\alpha_{19}(1 + \alpha_{11}) + \alpha_{12}\alpha_{29}}{\alpha_{12}(1 + \zeta_2)} \right) E_n^3 + \left(\frac{\beta_{11}(1 + \alpha_{11}) + \alpha_{12}\beta_{21}}{\alpha_{12}(1 + \zeta_2)} \right) T_n \tilde{h} + \left(\frac{\beta_{12}(1 + \alpha_{11}) + \alpha_{12}\beta_{22}}{\alpha_{12}(1 + \zeta_2)} \right) E_n \tilde{h} + \\ & + \left(\frac{\beta_{13}(1 + \alpha_{11}) + \alpha_{12}\beta_{23}}{\alpha_{12}(1 + \zeta_2)} \right) \tilde{h}^2 + \left(\frac{\beta_{14}(1 + \alpha_{11}) + \alpha_{12}\beta_{24}}{\alpha_{12}(1 + \zeta_2)} \right) T_n^2 \tilde{h} + \left(\frac{\beta_{15}(1 + \alpha_{11}) + \alpha_{12}\beta_{25}}{\alpha_{12}(1 + \zeta_2)} \right) E_n^2 \tilde{h} \\ & + \left(\frac{\beta_{16}(1 + \alpha_{11}) + \alpha_{12}\beta_{26}}{\alpha_{12}(1 + \zeta_2)} \right) T_n E_n \tilde{h} + \left(\frac{\beta_{17}(1 + \alpha_{11}) + \alpha_{12}\beta_{27}}{\alpha_{12}(1 + \zeta_2)} \right) \tilde{h}^3 + \left(\frac{\beta_{18}(1 + \alpha_{11}) + \alpha_{12}\beta_{28}}{\alpha_{12}(1 + \zeta_2)} \right) T_n \tilde{h}^2 \\ & + \left(\frac{\beta_{19}(1 + \alpha_{11}) + \alpha_{12}\beta_{29}}{\alpha_{12}(1 + \zeta_2)} \right) E_n^2 \tilde{h} + O(|T_n|, |E_n|, |\tilde{h}|)^4, \end{aligned}$$

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