

Article

Hypothesis Tests for Comparing Point Processes

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Abstract

This paper presents a comprehensive study of statistical tests for comparing temporal point processes in general, with a particular focus on Poisson processes. We explore three key approaches: (1) an intensity-based test specifically for Poisson processes, (2) general parametric tests using the notion of maximum likelihood estimation, and (3) a general nonparametric test using the Isometric Log-Ratio (ILR) transformation. The first approach adopts a three-step procedure for comparing inhomogeneous Poisson processes by testing total and normalized intensities separately and then combining the corresponding p -values using Fisher's method. The second method proposes a likelihood-based parametric test to examine the conditional intensity functions in point processes, emphasizing the application to Hawkes processes. Lastly, the third approach introduces a nonparametric test for general point processes, by transforming inter-event times into a Euclidean space via the ILR transformation, followed by conventional depth-based methods on multivariate data. We then conduct thorough studies on simulations as well as real-world data to illustrate these testing procedures and demonstrate their effectiveness.

Keywords: point process; Poisson process; Hawkes process; hypothesis testing; data depth; Isometric Log-Ratio transformation

MSC: 62G10

1. Introduction

Point processes are mathematical models which represent random events occurring in time or space. These processes are commonly used in various fields such as telecommunications, ecology, seismology, and finance, where events are randomly distributed and their temporal or spatial patterns are of particular interest. The most extensively studied class of point processes is the temporal Poisson process, known for its simplicity and utility in modeling events that occur independently over time. The homogeneous Poisson process assumes a constant event rate (intensity), while the inhomogeneous Poisson process accounts for a time-varying intensity function. By contrast, both methodological developments and inferential tools for more general temporal point processes are relatively scarce, and formal hypothesis testing procedures for comparing point processes are particularly limited.

Poisson processes have a rich history in statistical modeling, and traditional methods focus heavily on mathematical frameworks, likelihood representations, model estimation, and simulations [1]. One key feature of Poisson processes is their independence from history, which means they can be fully characterized by a deterministic intensity function $\lambda(t) \geq 0$. This intensity function dictates the rate of event occurrence over time. If $\lambda(t)$



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is constant, the process is homogeneous; otherwise, it is inhomogeneous. While the study of homogeneous Poisson processes is well established, comparing the intensities of inhomogeneous Poisson processes remains an area with many unexplored challenges.

Most classical studies focused on comparing homogeneous Poisson processes [2,3]. These approaches are equivalent to comparing the means of two Poisson distributions. More recent advancements, including the E-test [4] and parametric bootstrap tests [5], have extended these methods. However, much of the literature still concentrates on the homogeneous case, with fewer studies addressing the more complex inhomogeneous Poisson processes. Some notable exceptions include investigating the ratio of intensities [6] and normalized intensities [7]. In this paper, we propose a three-step procedure to compare two inhomogeneous Poisson processes. Our procedure addresses both the magnitude and the distribution of event times. The results from both tests are combined using Fisher's combined probability test [8], providing an overall p -value to assess whether the two intensity functions are equal.

Beyond Poisson processes, more complex temporal point processes, such as the Hawkes process, introduce additional challenges. The Hawkes process [9] is a "self-exciting" process where each event increases the probability of future events, making it highly suitable for modeling clustered or cascading events, such as earthquakes, trade orders, or bank default. The intensity function of a Hawkes process is not only time-dependent but is also influenced by past events, allowing the process to capture dependencies between events. This flexibility makes the Hawkes process a valuable tool in modeling phenomena where events are interdependent.

Parametric tests are suitable for processes with a specific form of the conditional intensity functions, such as the Poisson and Hawkes processes, since the parameters can be estimated using conventional log-likelihood. We introduce parametric tests based on maximum likelihood estimation for such processes, taking advantage of their well-defined likelihood structures to conduct parameter estimation and hypothesis testing.

In cases where conditional intensity assumptions are inappropriate or unknown, nonparametric methods provide an alternative. A prominent nonparametric approach is the Isometric Log-Ratio (ILR) transformation [10], which maps compositional data, such as inter-event times in point processes, from a constrained simplex space to an unconstrained Euclidean space. This transformation enables the application of multivariate statistical techniques to compare the transformed data, overcoming the limitations imposed by the compositional nature of the original data.

To assess the similarity of point processes based on their transformed data, we employ depth-based hypothesis testing. Several depth-based tests have been proposed for multivariate distributions, including depth versus depth plots [11] and multivariate spacings [12]. In this paper, we implement a quality index Q based on data depth [13] to compare the distributions of two groups of point processes. The quality index is defined as the probability that a random sample from one distribution has a smaller depth with respect to the other distribution. A Q -value of 0.5 indicates that the two distributions are identical, while values deviating from 0.5 suggest differences in location or scale.

The structure of this paper is as follows. Section 2 introduces the hypothesis tests for comparing Poisson processes, including the proposed three-step procedure for inhomogeneous processes. Section 3 discusses parametric tests based on maximum likelihood estimation, with a focus on point processes such as the Hawkes process. Section 4 presents nonparametric tests using the ILR transformation and depth-based methods for comparing general point processes. Simulation studies are provided throughout to demonstrate the effectiveness of each approach. In Section 5, we provide simulation studies to demonstrate the effectiveness of the proposed approaches under various scenarios. We also apply

the methods to real-world data, showcasing their practical relevance. Finally, Section 6 concludes the paper with a summary of the key findings and outlines potential directions for future research. Mathematical details are given in Appendices.

2. A Three-Step Procedure for Comparing Poisson Processes

Poisson processes are classical stochastic processes, and their intensity functions play a critical role in determining the behavior of the process. In this section, we propose a three-step hypothesis test to compare the intensity functions of two inhomogeneous Poisson processes. This method extends the conventional comparison framework by considering both the total intensities and the normalized intensities of the processes. The combined test allows for a more detailed comparison of Poisson processes in various practical scenarios.

In practice, a process is often given in a finite-time domain, and we limit our study in a finite interval $[0, T]$. Assume the intensities of two Poisson processes are $\lambda_1(t)$ and $\lambda_2(t)$, respectively. We are interested in testing

$$H_0 : \lambda_1(t) = \lambda_2(t) \quad \text{vs.} \quad H_1 : \lambda_1(t) \neq \lambda_2(t) \text{ for some } t.$$

The first step of the test examines whether the total intensities, denoted by $\Lambda_1 = \int_0^T \lambda_1(t)dt$ and $\Lambda_2 = \int_0^T \lambda_2(t)dt$, of the two processes are equal. A z-test is used to compare the two intensities based on large sample properties. If the total intensities are found to be equal, the second step compares the normalized intensities, or densities, $\frac{\lambda_1(t)}{\Lambda_1}$ and $\frac{\lambda_2(t)}{\Lambda_2}$, using the Kolmogorov–Smirnov test to assess whether the distributions of the event times are the same. Finally, the results from both tests are combined using Fisher’s combined probability test [8] to form an overall p -value, which evaluates the hypothesis that the two Poisson processes have the same intensity function.

Suppose $\{N(t), t \in [0, T]\}$ is an inhomogeneous Poisson process with an intensity function $\lambda(t)$. Given that $N(T) = K$, the unordered K arrival times $t_1, \dots, t_K$ are independent and share a common density function

$$f(t) = \frac{\lambda(t)}{\Lambda}, \quad 0 \leq t \leq T, \quad (1)$$

where $\Lambda = \int_0^T \lambda(t)dt$ is the total intensity.

Consider two inhomogeneous Poisson processes, $\{N_1(t), t \in [0, T]\}$ and $\{N_2(t), t \in [0, T]\}$, with intensity functions $\lambda_1(t)$ and $\lambda_2(t)$, respectively. The number of events in the interval $[0, T]$ for each process, denoted as $N_i(T)$, follows a Poisson distribution with mean $\Lambda_i \equiv \int_0^T \lambda_i(t)dt$, $i = 1, 2$. Specifically, for any integer $k \geq 0$, the probability mass function is

$$P\{N_i(T) = k\} = \frac{\exp(-\Lambda_i)\Lambda_i^k}{k!}, \quad i = 1, 2.$$

Let $\mathbb{S}_1 = \{s_{11}, \dots, s_{1M_1}\}$ and $\mathbb{S}_2 = \{s_{21}, \dots, s_{2M_2}\}$ be two sets of independent realizations from the first and second processes, respectively, where M_1 and M_2 denote the number of realizations. The numbers of events in these realizations are denoted as $N_{11}, N_{12}, \dots, N_{1M_1}$ and $N_{21}, N_{22}, \dots, N_{2M_2}$. Each $N_{1k} \sim \text{Poisson}(\Lambda_1)$ and $N_{2l} \sim \text{Poisson}(\Lambda_2)$ for $k = 1, \dots, M_1$ and $l = 1, \dots, M_2$. Furthermore, each realization consists of event times in strictly increasing order, represented as:

$$s_{ij} = \{t_{ij}^{(1)}, t_{ij}^{(2)}, \dots, t_{ij}^{(N_{ij})}\},$$

where $t_{ij}^{(k)}$ denotes the k -th event time in the j -th realization of the i -th process, with $k = 1, \dots, N_{ij}$, $j = 1, \dots, M_i$, and $i = 1, 2$.

To test whether the intensity functions are equal, i.e.,

$$H_0 : \lambda_1(t) = \lambda_2(t), t \in [0, T],$$

we conduct two independent subtests. The first subtest is to compare the total intensities of the two processes

$$H_0 : \Lambda_1 = \Lambda_2 \quad \text{vs.} \quad H_1 : \Lambda_1 \neq \Lambda_2. \quad (2)$$

We apply a z-test with the following test statistic

$$Z = \frac{\hat{\Lambda}_2 - \hat{\Lambda}_1}{\sqrt{\frac{\hat{\Lambda}_1}{M_1} + \frac{\hat{\Lambda}_2}{M_2}}},$$

where $\hat{\Lambda}_i = \frac{1}{M_i} \sum_{j=1}^{M_i} N_{ij}$ for $i = 1, 2$. Under the null hypothesis, Z asymptotically follows a standard normal distribution $N(0, 1)$. The p -value for this subtest is given by

$$p_1 = 2(1 - \Phi(|Z|)), \quad (3)$$

where $\Phi(\cdot)$ is the cumulative distribution function (c.d.f.) of $N(0, 1)$.

The second subtest compares the distributions of the arrival times by testing the normalized intensities:

$$H_0 : \frac{\lambda_1(t)}{\Lambda_1} = \frac{\lambda_2(t)}{\Lambda_2} \quad \text{vs.} \quad H_1 : \frac{\lambda_1(t)}{\Lambda_1} \neq \frac{\lambda_2(t)}{\Lambda_2}. \quad (4)$$

Based on the conditional property described in Equation (1), for any realization $j \in \{1, \dots, M_i\}$, the unsorted event times $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots, t_{ij}^{(N_{ij})}$ are independently distributed with density function $\lambda_i(t)/\Lambda_i, i = 1, 2$. Let

$$s_1 = \bigcup_{j=1}^{M_1} s_{1j}, s_2 = \bigcup_{j=1}^{M_2} s_{2j},$$

where each set s_i contains $K_i = \sum_{j=1}^{M_i} N_{ij}$ independent event times with density function $\lambda_i(t)/\Lambda_i$. As the sample sizes M_1 and M_2 increase, $K_i \rightarrow \infty$ almost surely. We use the two-sample Kolmogorov–Smirnov (K-S) test to compare the empirical distribution functions of s_1 and s_2 , i.e., to compare $\lambda_1(t)/\Lambda_1$ and $\lambda_2(t)/\Lambda_2$.

The test statistic for the K-S test is

$$D_{K_1, K_2} = \left(\frac{K_1 K_2}{K_1 + K_2} \right)^{\frac{1}{2}} \sup_x |F_{K_1}(x) - G_{K_2}(x)|,$$

where $F_{K_1}(x)$ and $G_{K_2}(x)$ are the empirical c.d.f.s of s_1 and s_2 , respectively. The p -values for the KS test can be derived from tables. If K_1 and K_2 are sufficiently large, D_{K_1, K_2} approximately follows the Kolmogorov–Smirnov distribution.

After obtaining the p -values from both subtests, we calculate an overall p -value to compare the two intensity functions. Note that both subtests partially address the same hypothesis—whether $\lambda_1(t) = \lambda_2(t)$. Indeed, the first subtest compares if the numbers of events in the two processes follow the same distribution, with null hypothesis $H_0 : \Lambda_1 = \Lambda_2$. The second subtest compares if the distributions of events in the two processes are the same, with null hypothesis $H_0 : \lambda_1(t)/\Lambda_1 = \lambda_2(t)/\Lambda_2$. If both subtests confirm the null hypothesis, we conclude that the two intensity functions are equal.

Since the p -values p_1 and p_2 are asymptotically independent when M_1 and M_2 are large, we apply Fisher's method to combine them:

$$X = -2(\log(p_1) + \log(p_2)),$$

where X has a chi-squared distribution with four degrees of freedom. The combined p -value, which is computed using the test statistic X , is the overall p -value for the test $H_0 : \lambda_1(t) = \lambda_2(t), t \in [0, T]$. The following algorithm outlines the test procedure for comparing two inhomogeneous Poisson processes.

The overall test procedure for the comparison of the two inhomogeneous Poisson processes is given in the following Algorithm 1.

Algorithm 1. (Given two groups of Poisson realizations $s_{11}, s_{12}, \dots, s_{1M_1}$ and $s_{21}, s_{22}, \dots, s_{2M_2}$)

1. Count the number of events of every realization: $N_{11}, \dots, N_{1M_1}$ and $N_{21}, \dots, N_{2M_2}$;
 2. Perform a z-test on the total intensities and compute the p -value p_1 ;
 3. Merge all samples in each group: $s_1 = \bigcup_{j=1}^{M_1} s_{1j}, s_2 = \bigcup_{j=1}^{M_2} s_{2j}$;
 4. Apply two-sample Kolmogorov–Smirnov test to compare distribution functions of s_1 and s_2 and compute the p -value p_2 ;
 5. Use Fisher's method to combine p_1 and p_2 , compute the overall p -value using the test statistic $X = -2(\log(p_1) + \log(p_2))$, where $X \sim \chi_4^2$.
-

Remark 1. The proposed three-step procedure can be naturally extended to compare multiple groups of inhomogeneous Poisson processes. Suppose we have K groups of independent Poisson realizations, each with an intensity function $\lambda_k(t), k = 1, \dots, K$. The first subtest compares the total intensities of the K processes:

$$H_0 : \Lambda_1 = \dots = \Lambda_K \quad \text{vs.} \quad H_1 : \Lambda_i \neq \Lambda_j \text{ for some } i, j \in \{1, \dots, K\}.$$

For this step, we can use statistical tests such as Pearson's χ^2 test or the likelihood ratio test [14].

The second subtest compares the distributions of event times among the K Poisson processes:

$$H_0 : \frac{\lambda_1(t)}{\Lambda_1} = \dots = \frac{\lambda_K(t)}{\Lambda_K} \quad \text{vs.} \quad H_1 : \frac{\lambda_i(t)}{\Lambda_i} \neq \frac{\lambda_j(t)}{\Lambda_j} \text{ for some } i, j \in \{1, \dots, K\}.$$

The K -sample Anderson–Darling test [15] is appropriate for this step.

Finally, we still apply Fisher's method to combine p -values from previous two steps to generate an overall p -value for the hypothesis:

$$H_0 : \lambda_1(t) = \dots = \lambda_K(t) \quad \text{vs.} \quad H_1 : \lambda_i(t) \neq \lambda_j(t) \text{ for some } i, j \in \{1, \dots, K\}.$$

Building upon this method for Poisson processes, we can also explore parametric approaches for more general point processes. The following section discusses the use of maximum likelihood estimation as a parametric method for comparing point processes.

3. Parametric Tests Based on the Maximum Likelihood Estimation

In this section, we explore hypothesis testing for realizations from general point processes. The maximum likelihood estimation (MLE) approach offers a robust parametric framework for comparing point processes. By estimating the unknown parameters of intensity functions, we can construct test statistics to evaluate whether two processes share the same underlying intensity.

Suppose two point processes have conditional intensity functions $\lambda(t|\mathcal{H}_t, \theta_1)$ and $\lambda(t|\mathcal{H}_t, \theta_2)$ in closed form, respectively, where $\mathcal{H}_t$ denotes history, and θ_1 and θ_2 are unknown parameters. The null hypothesis $H_0 : \theta_1 = \theta_2$ can be tested by first estimating the parameters θ_1 and θ_2 using MLE. Once the parameters are estimated, the Wald test is applied to test whether the two estimated intensities are equal. The MLE approach enables the use of optimization methods such as the Newton–Raphson method to maximize the log-likelihood functions and obtain parameter estimates.

We also present two examples: the Hawkes process [9] and the Poisson process software reliability model [16]. While these examples demonstrate the feasibility of our method, the approach is applicable to any point process as long as MLE can be performed.

3.1. Tests Using Maximum Likelihood Estimation for Point Processes

Given a point process realization with conditional intensity $\lambda^*(t|\theta) \equiv \lambda(t|\mathcal{H}_t, \theta)$, where events $t_1, \dots, t_n$ are on an finite interval $[0, T]$, the likelihood L of this point process is [1]

$$L = \left(\prod_{i=1}^n \lambda^*(t_i|\theta) \right) \exp \left(- \int_0^T \lambda^*(u|\theta) du \right).$$

Consider two independent point processes with conditional intensity functions $\lambda^*(t|\theta_1)$ and $\lambda^*(t|\theta_2)$ on $[0, T]$, where θ_1 and θ_2 are p -dimensional parameter vectors. Assume that the realizations $\tilde{\mathbb{S}}_1 = \{s_{11}, \dots, s_{1M_1}\}$ have intensity $\lambda^*(t|\theta_1)$, and $\tilde{\mathbb{S}}_2 = \{s_{21}, \dots, s_{2M_2}\}$ have intensity $\lambda^*(t|\theta_2)$. We test the null hypothesis:

$$H_0 : \theta_1 = \theta_2.$$

For set $\tilde{\mathbb{S}}_1$, suppose the likelihood of each independent realization s_{1j} is $L_{1j}(\theta_1)$, $j = 1, \dots, M_1$. Then, the likelihood of the entire set $\tilde{\mathbb{S}}_1$ is $L_1(\theta_1) = \prod_{j=1}^{M_1} L_{1j}(\theta_1)$. Similarly, the likelihood of the set $\tilde{\mathbb{S}}_2$ is denoted as $L_2(\theta_2)$. To maximize the log-likelihood, we calculate the gradients and Hessian matrices of $L_1(\theta_1)$ and $L_2(\theta_2)$. If the Hessian is well-defined and has a closed form, it is negative semi-definite, allowing us to apply the Newton–Raphson method or other appropriate optimization techniques to obtain the MLE.

Assuming MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$ exist, their asymptotic distributions are

$$\sqrt{M_1}(\hat{\theta}_1 - \theta_1) \xrightarrow{d} \mathcal{N}(0, \mathcal{I}(\theta_1)^{-1}) \quad \text{and} \quad \sqrt{M_2}(\hat{\theta}_2 - \theta_2) \xrightarrow{d} \mathcal{N}(0, \mathcal{I}(\theta_2)^{-1}),$$

as $M_1, M_2 \rightarrow \infty$. The proof of asymptotic properties follows standard techniques and is provided in Appendix A.1.

The Wald test statistic for $H_0 : \theta_1 = \theta_2$ is

$$W = M_1 M_2 (\hat{\theta}_1 - \hat{\theta}_2)^\top (M_1 \mathcal{I}^{-1}(\hat{\theta}_1) + M_2 \mathcal{I}^{-1}(\hat{\theta}_2)) (\hat{\theta}_1 - \hat{\theta}_2),$$

where $\mathcal{I}(\hat{\theta}_1)$ is the Fisher information matrix estimated by the negative Hessian

$$\mathcal{I}(\hat{\theta}_1) = - \frac{\partial^2 \log L_X(\hat{\theta}_1)}{\partial \hat{\theta}_1^2}.$$

Under the null hypothesis, W follows an asymptotic χ_r^2 distribution, where r is the dimension of θ , that is, the number of parameters being tested.

In the remainder of this section, we present two examples of point processes for which MLE can be feasibly applied.

3.2. Example: Hawkes Process

The Hawkes process models events where each occurrence increases the likelihood of subsequent events, such as earthquakes or trade orders. The conditional intensity function of a Hawkes process is [1]:

$$\lambda^*(t) = \mu + \int_0^t \nu(t-s) dN(s),$$

where $\mu > 0$ is the background intensity, and $\nu : (0, \infty) \rightarrow [0, \infty)$ is the excitation function.

A widely used choice for the excitation function is an exponentially decaying function. In this case, $\nu(t) = \alpha e^{-\beta t}$ for $\alpha, \beta > 0$, and the intensity function is given by

$$\lambda^*(t) = \mu + \int_{-\infty}^t \alpha e^{-\beta(t-s)} dN(s).$$

Given a realization of such a Hawkes process, $\{t_1, t_2, \dots, t_N\}$, on the interval $[0, T]$, the intensity function becomes

$$\lambda^*(t) = \mu + \alpha \sum_{t_i < t} e^{-\beta(t-t_i)},$$

The log-likelihood is given as [17]

$$\begin{aligned} \log L &= \sum_{i=1}^N \log(\lambda^*(t_i | \alpha, \beta)) - \int_0^T \lambda^*(u | \alpha, \beta) du \\ &= \sum_{i=1}^N \log\{\mu + \alpha A(i)\} + \sum_{i=1}^N \frac{\alpha}{\beta} (e^{-\beta(T-t_i)} - 1) - \mu T. \end{aligned}$$

We compute the partial derivatives and the Hessian matrix of the log-likelihood function to obtain the MLE of α , β and μ . The gradients and Hessians are provided in Appendix A.2. Optimization techniques, such as the Newton–Raphson method and the Berndt–Hall–Hall–Hausman (BHHE) method, are employed to maximize the log-likelihood.

3.3. Example: Poisson Process Software Reliability Model

The maximum likelihood estimators for a general nonhomogeneous Poisson process do not always exist. Therefore, in our second example we focus on a software reliability model, a nonhomogeneous Poisson process on $[0, T]$ with intensity function $\lambda(t) = \frac{\alpha(t/\beta)^k}{\beta k!} \exp(-t/\beta)$, where the unknown parameters are α and β . The maximum likelihood estimations of α and β exist as $T \rightarrow \infty$ with probability 1 [16]. For $k = 1$, the log-likelihood becomes

$$\log L = n \log \alpha - 2n \log \beta - \sum_{i=1}^n \frac{t_i}{\beta} + \sum_{i=1}^n \log t_i - \alpha \left[1 - \left(1 + \frac{T}{\beta} \right) \exp \left(-\frac{T}{\beta} \right) \right].$$

Gradients and Hessians are computed to obtain the maximum likelihood estimator using numerical optimization methods. The gradients and Hessians are given as follows:

$$\begin{aligned} \frac{\partial \log L}{\partial \alpha} &= \frac{n}{\alpha} + \left(1 + \frac{T}{\beta} \right) \exp \left(-\frac{T}{\beta} \right) - 1, \\ \frac{\partial \log L}{\partial \beta} &= -\frac{2n}{\beta} + \sum_{i=1}^n \frac{t_i}{\beta} + \frac{\alpha T^2}{\beta^3} \exp \left(-\frac{T}{\beta} \right). \end{aligned}$$

The Hessian matrix is provided by

$$\begin{aligned}\frac{\partial^2 \log L}{\partial \alpha^2} &= -\frac{n}{\alpha^2}, \\ \frac{\partial^2 \log L}{\partial \beta^2} &= \frac{2n}{\beta^2} - \sum_{i=1}^n \frac{2t_i}{\beta^3} + \alpha \left(\frac{T^3}{\beta^5} - \frac{3T^2}{\beta^4} \right) \exp \left(-\frac{T}{\beta} \right), \\ \frac{\partial^2 \log L}{\partial \alpha \partial \beta} &= \frac{T^2}{\beta^3} \exp \left(-\frac{T}{\beta} \right).\end{aligned}$$

4. Nonparametric Tests for General Point Processes

In many cases, parametric models require a known functional form for the intensity function. However, identifying an appropriate parametric form in practice can be challenging, especially when there is limited prior knowledge about the process. Even when a reasonable parametric form is available, the estimation process may encounter difficulties as an MLE for point processes may not always exist. For example, consider a Poisson process with intensity $\lambda(t) = (1 + \alpha \sin(t))$ on $[0, 2\pi]$, where α is the parameter. The likelihood of a realization $\{t_1, \dots, t_n\}$ is $L = \left(\prod_{i=1}^n (1 + \alpha \sin(t_i)) \right) \exp(-2\pi)$, the likelihood $L \rightarrow \infty$ as $\alpha \rightarrow \infty$. Hence, alternative testing methods are needed in such cases.

In general, nonparametric tests for point processes are highly desirable in cases where no appropriate assumptions can be made about the intensity functions. To address this issue, we propose to apply the well-known Isometric Log-Ratio (ILR) transformation [10] to inter-event times of the processes, mapping them into a Euclidean space. This transformation allows us to use multivariate statistical tools to compare the distributions of these transformed points. The ILR-based method is flexible and can be applied to a wide range of point processes, making it an ideal choice when parametric assumptions are not suitable. Specifically, the test compares whether the multivariate distributions of transformed inter-event times from two processes are statistically different. This method is advantageous when the intensity functions of the processes are unknown or too complicated to estimate parametrically.

We now consider tests for point processes without making assumptions about the intensity function. For a fixed time interval $[0, T]$, two groups of point processes are observed, denoted as $\tilde{\mathbb{S}}_1 = \{s_{11}, \dots, s_{1M_1}\}$ and $\tilde{\mathbb{S}}_2 = \{s_{21}, \dots, s_{2M_2}\}$. Suppose $\tilde{\mathbb{S}}_i$ has a conditional intensity function $\lambda_i^*(t) \equiv \lambda_i(t|\mathcal{H}_t)$ for $i = 1, 2$. We aim to test the null hypothesis:

$$H_0 : \tilde{\mathbb{S}}_1 \text{ and } \tilde{\mathbb{S}}_2 \text{ are from the same point process,}$$

or equivalently

$$H_0 : \lambda_1^* = \lambda_2^*. \quad (5)$$

However, directly conducting nonparametric tests to compare conditional intensity functions is challenging. To address this, we introduce the ILR transformation on inter-event times. This transformation maps data from a simplex space to Euclidean space, allowing us to transform point processes into points in Euclidean space and then compare the multivariate distributions of these points.

Given a cardinality $K \in \mathbb{N}_+$ in the time domain $[0, T]$, a point process can fully be characterized by its joint density function. If the point processes in $\tilde{\mathbb{S}}_i$ have a joint density function $f_i(t_1, \dots, t_K)$ for $i = 1, 2$, let $\tilde{\mathbb{S}}_{iK}$ denote the set of all point processes having cardinality K in $\tilde{\mathbb{S}}_i$. Then, $\tilde{\mathbb{S}}_i = \cup_{K=0}^{\infty} \tilde{\mathbb{S}}_{iK}$ for $i = 1, 2$. Let $|s_1|$ and $|s_2|$ represent the cardinalities of the point processes in $\tilde{\mathbb{S}}_1$ and $\tilde{\mathbb{S}}_2$, respectively. The hypothesis in Equation (5) can be reformulated as:

1. Condition 1. The cardinalities of point processes in both groups follow the same distribution: $P(|s_1| = K) = P(|s_2| = K)$ for all K .
2. Condition 2. For any fixed cardinality K , the point processes in $\tilde{S}_{1K}$ and $\tilde{S}_{2K}$ have the same joint density function: $f_1(t_1, \dots, t_K | |s_1| = K) = f_2(t_1, \dots, t_K | |s_2| = K)$.

Based on these conditions, we propose a series of subtests to compare $\tilde{S}_1$ and $\tilde{S}_2$:

$$\begin{aligned}
 H_{00} &: P(|s_1| = K) = P(|s_2| = K) \text{ for all } K; \\
 H_{01} &: f_1(t_1, t_2, \dots, t_{K_1} | |s_1| = K_1) = f_2(t_1, t_2, \dots, t_{K_1} | |s_2| = K_1); \\
 H_{02} &: f_1(t_1, t_2, \dots, t_{K_2} | |s_1| = K_2) = f_2(t_1, t_2, \dots, t_{K_2} | |s_2| = K_2); \\
 &\dots \\
 H_{0m} &: f_1(t_1, t_2, \dots, t_{K_m} | |s_1| = K_m) = f_2(t_1, t_2, \dots, t_{K_m} | |s_2| = K_m).
 \end{aligned} \tag{6}$$

We choose $K_1, \dots, K_m$ such that all corresponding subsets $\tilde{S}_{1K_i}$ and $\tilde{S}_{2K_i}$ have sufficiently large sample sizes.

For the first subtest H_{00} , we compare the distributions of cardinalities using Pearson's chi-square test for homogeneity. Let the cardinalities of the realizations in the two groups be

$$C_1 = \{C_{11}, C_{12}, \dots, C_{1M_1}\}, \quad C_2 = \{C_{21}, C_{22}, \dots, C_{2M_2}\}.$$

The detailed computation of this test is presented in [18].

For the subsequent subtests $H_{01} \dots H_{0m}$, we use a quality index based on data depth as proposed by [13]. However, since the point processes with a fixed cardinality K are in irregular spaces, we need to map them into regular Euclidean spaces to apply depth-based methods effectively. Let $\mathbb{S}$ denote the set of all point processes on $[0, T]$ and $\mathbb{S}_K$ denote the set of all point processes with cardinality K . Specifically, $\mathbb{S}_K = \{(t_1, t_2, \dots, t_K)^\top \in \mathbb{R}^K | 0 \leq t_1 \leq t_2 \leq \dots \leq t_K \leq T\}$ for some non-negative integer K and hence $\mathbb{S} = \cup_{K=0}^{\infty} \mathbb{S}_K$. Since depth functions are not well-defined in irregular spaces like $\mathbb{S}_K$, we apply the ILR transformation to map the point processes into regular Euclidean spaces, enabling the use of depth-based methods for hypothesis testing.

4.1. Isometric Log-Ratio Transformation

We use the notion of inter-event time (IET) representation for the ILR transformation [19]. Suppose $s = (t_1, t_2, \dots, t_K)^\top \in \mathbb{S}_K$ is a point process with given cardinality K over the time domain $[0, T]$. This process can be equivalently represented using inter-event times. Let

$$u = (u_1, u_2, \dots, u_{K+1})^\top \equiv (t_1, t_2 - t_1, \dots, t_K - t_{K-1}, T - t_K)^\top.$$

Since $\sum_{i=1}^{K+1} u_i = T$ and $u_i > 0$ almost surely for all $i = 1, \dots, K+1$, any point process with cardinality K can be viewed as a vector in the simplex

$$\mathcal{S}^{K+1} = \left\{ u = (u_1, u_2, \dots, u_{K+1})^\top \in \mathbb{R}^{K+1} \mid u_i > 0, i = 1, \dots, K+1; \sum_{i=1}^{K+1} u_i = T \right\}.$$

To perform statistical analysis, we use the ILR transformation to map the IET-represented point processes from the simplex $\mathcal{S}^{K+1}$ to unconstrained Euclidean space $\mathbb{R}^K$, allowing us to apply depth-based methods.

The ILR transformation is an isometry between $\mathcal{S}^{K+1}$ and $\mathbb{R}^K$ [10]. For a vector $\mathbf{u} \in \mathcal{S}^{K+1}$, the ILR transformation is defined as [20]

$$\mathbf{u}^* = \text{ilr}(\mathbf{u}) \equiv \Phi \cdot \left[\log\left(\frac{u_1}{g(\mathbf{u})}\right), \log\left(\frac{u_2}{g(\mathbf{u})}\right), \dots, \log\left(\frac{u_{K+1}}{g(\mathbf{u})}\right) \right]^\top, \quad (7)$$

where $g(\mathbf{u}) = (\prod_{i=1}^{K+1} u_i)^{1/(K+1)}$ is the component-wise geometric mean of $\mathbf{u}$, and Φ is a matrix in $\mathbb{R}^{K \times (K+1)}$ satisfying $\Phi\Phi^\top = I_K$ and $\Phi^\top\Phi = I_{K+1} - \frac{1}{K+1}\mathbf{1}_{K+1}\mathbf{1}_{K+1}^\top$, where I_K and I_{K+1} are identity matrices, and $\mathbf{1}_{K+1}$ is a $(K+1)$ -dimensional vector of ones. The inverse of the ILR transformation, given as $\mathbf{u}^* \in \mathbb{R}^K$, is

$$\mathbf{u} = \text{ilr}^{-1}(\mathbf{u}^*) \equiv \frac{T \exp(\mathbf{u}^{*\top} \Phi)^\top}{\exp(\mathbf{u}^{*\top} \Phi) \cdot \mathbf{1}_{K+1}}.$$

The matrix Φ is not unique. A common form, obtained via the Gram–Schmidt process, is given by [20]

$$\Phi_{ij} = \begin{cases} \sqrt{\frac{1}{(K+1-i)(K+2-i)}}, & j \leq K+1-i \\ -\sqrt{\frac{(K+1-i)}{(K+2-i)}}, & j = K+2-i \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

Using the IET representation and ILR transformation, we establish bijective relationships between point process space $\mathbb{S}_K$, the simplex $\mathcal{S}^{K+1}$ and Euclidean space $\mathbb{R}^K$. We can now map two groups of point process realizations to vectors in $\mathbb{R}^K$, denoted as

$$\mathbb{X}_1 = \{X_{s_{11}}, \dots, X_{s_{1M_1}}\}, \quad \mathbb{X}_2 = \{X_{s_{21}}, \dots, X_{s_{2M_2}}\},$$

Our goal is to compare the multivariate distributions of $\mathbb{X}_1$ and $\mathbb{X}_2$. Given a cardinality K , assume that the random vectors in $\mathbb{X}_1$ have density function f_1^K , and those in $\mathbb{X}_2$ have f_2^K . Then, the hypothesis in Equation (6) can be reformulated as

$$\begin{aligned} H_{00} &: P(|s_1| = K) = P(|s_2| = K) \text{ for all } K; \\ H'_{01} &: f_1^{K_1}(x|s_1 = K_1) = f_2^{K_1}(x|s_2 = K_1); \\ H'_{02} &: f_1^{K_2}(x|s_1 = K_2) = f_2^{K_2}(x|s_2 = K_2); \\ &\dots \\ H'_{0m} &: f_1^{K_m}(x|s_1 = K_m) = f_2^{K_m}(x|s_2 = K_m), \end{aligned} \quad (9)$$

where $s_1 \in \tilde{\mathbb{S}}_1$ and $s_2 \in \tilde{\mathbb{S}}_2$. Next, we compare the multivariate distributions in $\mathbb{R}^K$ applying depth-based methods.

4.2. Multivariate Distribution Test Based on Data Depth

Several depth-based tests have been proposed for multivariate distribution, including depth versus depth plots [11] and multivariate spacings [12]. In our approach, we implement a quality index Q , defined in [13], as follows:

$$Q(F, G) = P\{D(F; X) \leq D(F; Y) \mid X \sim F, Y \sim G\},$$

where F and G are two distribution functions on $\mathbb{R}^p$ for some $p \geq 1$, and $D(F; \cdot)$ is a data depth function with respect to distribution F . The value of Q lies in the interval $[0, 1]$,

with $Q = \frac{1}{2}$ indicating that $F = G$, and $Q < \frac{1}{2}$ suggesting a location shift or scale difference. We use this index to test the hypothesis

$$H_0 : Q = \frac{1}{2}$$

to determine whether F and G are statistically identical.

Given two samples $\mathbb{X} = \{X_1, \dots, X_m\}$ from F and $\mathbb{Y} = \{Y_1, \dots, Y_n\}$ from G , the empirical estimate of Q is

$$\hat{Q} \equiv \hat{Q}(F_m, G_n) \equiv \hat{Q}(\mathbb{X}, \mathbb{Y}) = \frac{1}{n} \sum_{i=1}^n R(F_m; Y_i),$$

where $R(F_m; Y_i)$ is the proportion of X_j s that satisfy $D(F_m; X_j) \leq D(F_m; Y_i)$, and $D(F_m; \cdot)$ is the empirical depth function with respect to the empirical distribution F_m . When $\hat{Q}$ is calculated using the Mahalanobis depth, and under certain conditions, the asymptotic behavior of $\hat{Q}$ is given by [13]

$$\left[\frac{1}{12} \left(\frac{1}{m} + \frac{1}{n} \right) \right]^{-\frac{1}{2}} \left[\hat{Q} - \frac{1}{2} \right] \rightarrow N(0, 1) \quad (10)$$

as $m, n \rightarrow \infty$.

Now, consider hypothesis in Equation (9). Given two point process samples $\tilde{\mathbb{S}}_1$ and $\tilde{\mathbb{S}}_2$ and a cardinality K , we extract two subsets $\tilde{\mathbb{S}}_{1_K}$ and $\tilde{\mathbb{S}}_{2_K}$. Applying the ILR transformation to these subsets yields two groups of vectors in $\mathbb{R}^K$, denoted as $\mathbb{X}_1^K$ and $\mathbb{X}_2^K$. Rather than directly comparing the point processes in $\tilde{\mathbb{S}}_{1_K}$ and $\tilde{\mathbb{S}}_{2_K}$, we compare the K -dim random vectors in $\mathbb{X}_{1_K}$ and $\mathbb{X}_{2_K}$ by calculating $\hat{Q}(\mathbb{X}_{1_K}, \mathbb{X}_{2_K})$.

Using the asymptotic distribution of $\hat{Q}$ from Equation (10), we can perform a large-sample z-test to determine whether $\hat{Q} = \frac{1}{2}$. In practice, we recommend selecting $K_m \leq 8$, as larger cardinalities may result in inaccurate estimates of Mahalanobis depth due to covariance matrix instability. Additionally, testing in high-dimensional settings requires a large number of samples to ensure reliable statistical inference.

4.3. Test Procedure

After obtaining the p -values of all $(m + 1)$ subtests in Equation (9), the final step is to combine these multiple comparisons into a single p -value. Fisher's method [21] is a well-known approach for combining p -values, under the assumption that the individual tests are independent. However, in our context, the subtests $H'_{01}, \dots, H'_{0m}$ are not independent due to shared data structures across the transformed point processes.

To address this dependency, we consider Bonferroni correction [22], which is effective under arbitrary dependence structures. However, Bonferroni correction can be overly conservative, particularly in cases where the number of comparisons is large. To overcome this, we use Holm's modification [23], which provides a stepwise procedure to control the family-wise error rate while being less conservative than the traditional Bonferroni correction.

Assume $p_1, p_2, \dots, p_m$ are p -values for m hypotheses $H_{01}, H_{02}, \dots, H_{0m}$, and α is the significance level. Holm's method involves sorting the p -values in ascending order, denoted by $p^{(1)} \leq p^{(2)} \leq \dots \leq p^{(m)}$, with corresponding hypotheses $H_0^{(1)}, H_0^{(2)}, \dots, H_0^{(m)}$. The stepwise procedure is detailed in Algorithm 2.

Algorithm 2. Holm's Method for Combining Hypotheses

1. If $p^{(1)} \leq \frac{\alpha}{m}$, reject $H_0^{(1)}$ and proceed to Step 2; otherwise, accept all $H_0^{(1)}, H_0^{(2)}, \dots, H_0^{(m)}$ and stop;
2. If $p^{(2)} \leq \frac{\alpha}{m-1}$, reject $H_0^{(2)}$ and proceed to Step 3; otherwise, accept $H_0^{(2)}, H_0^{(3)}, \dots, H_0^{(m)}$ and stop;
3. If $p^{(3)} \leq \frac{\alpha}{m-2}$, reject $H_0^{(3)}$ and proceed to Step 4; otherwise, accept $H_0^{(3)}, H_0^{(4)}, \dots, H_0^{(m)}$ and stop;
4. Continue this process, stop if a hypothesis is accepted;
5. If $p^{(m)} \leq \alpha$, reject $H_0^{(m)}$; otherwise, accept it and stop.

Holm's method adjusts the original p -values to corrected values:

$$mp^{(1)}, (m-1)p^{(2)}, (m-2)p^{(3)}, \dots, p^{(m)}.$$

Since the global null hypothesis $H_0 : \lambda_1^* = \lambda_2^*$ holds if all sub-hypotheses $H_{00}, H'_{01}, \dots, H'_{0m}$ in Equation (9) are true, the overall p -value is

$$\min\{mp^{(1)}, (m-1)p^{(2)}, (m-2)p^{(3)}, \dots, p^{(m)}\}.$$

Our test procedure is summarized in Algorithm 3.

Algorithm 3. Input: two groups of realizations $\mathbb{S}_1 = \{s_{11}, \dots, s_{1M_1}\}$ and $\mathbb{S}_2 = \{s_{21}, \dots, s_{2M_2}\}$

1. Count the cardinality of each realization, denoted as $C_1 = \{C_{11}, C_{12}, \dots, C_{1M_1}\}$ and $C_2 = \{C_{21}, C_{22}, \dots, C_{2M_2}\}$;
2. Use Chi-square test to compare the distributions of C_1 and C_2 , and obtain the p -value p_0 ;
3. Let $\mathbb{S}_{i_K}$ denote the set of all point processes with cardinality K in $\mathbb{S}_i, i = 1, 2$, choose cardinalities $K_1, \dots, K_m$ such that the corresponding subsets $\mathbb{S}_{1_{K_j}}$ and $\mathbb{S}_{2_{K_j}}$ have sufficiently large sample sizes;
4. For each $j \in \{1, \dots, m\}$, apply the IET representation and ILR transformation to map the realizations in $\mathbb{S}_{1_{K_j}}$ and $\mathbb{S}_{2_{K_j}}$ into Euclidean space $\mathbb{R}^{K_j}$, yielding two groups of K_j -dimensional vectors, denoted as $\mathbb{X}_{1_{K_j}}$ and $\mathbb{X}_{2_{K_j}}$;
5. For each $j \in \{1, \dots, m\}$, compute the quality index $\hat{Q}(\mathbb{X}_{1_{K_j}}, \mathbb{X}_{2_{K_j}})$, test the hypothesis $\hat{Q} = \frac{1}{2}$, and obtain the p -value p_j for each comparison;
6. Collect all p -values $p_0, p_1, \dots, p_m$, and arrange them in ascending order as $p^{(1)} \leq p^{(2)} \leq \dots \leq p^{(m+1)}$. Using Holm's correction, compute the overall p -value as

$$\text{Overall } p\text{-value} = \min\{(m+1)p^{(1)}, mp^{(2)}, (m-1)p^{(3)}, \dots, p^{(m+1)}\}.$$

5. Numerical Studies

5.1. Simulations

In this subsection, we conduct extensive simulations to evaluate the performance of the proposed hypothesis tests. We present results for three different approaches: the three-step test for Poisson processes, the parametric test using maximum likelihood estimation, and the nonparametric test based on the ILR transformation.

5.1.1. Simulation for Three-Step Test for Poisson Processes

To assess the effectiveness of the three-step procedure for comparing inhomogeneous Poisson processes, we generate two groups of Poisson processes with intensity functions:

$$\lambda_1(t) = 1 + \sin(t), \quad \lambda_2(t) = d(1 + \sin(t + c\pi)), \quad t \in [0, 2\pi],$$

where M_1 and M_2 denote the respective sample sizes. This process is repeated 1000 times to obtain stable estimates of the test statistics. We analyze how the test behaves under varying values of c and d , which control the phase shift and amplitude scaling, respectively.

For the first simulation scenario, we fix $d = 1$ and vary c , and the results are summarized in Table 1. Since the total intensities are equal ($\Lambda_1 = \Lambda_2$), the p -values for the z-test remain approximately 0.5. However, as c increases, the p -values for the K-S test decrease, indicating increasing differences in the distribution of event times.

In the second scenario, we fix $c = 0$ and vary d , and the results are presented in Table 1. Here, the normalized intensities remain identical, but the total intensities differ. Consequently, the p -values for the K-S test remain close to 0.5, while the z-test p -values decrease as d increases. Overall, these results demonstrate that our method effectively detects differences in both the total intensity and the distribution of events in Poisson processes.

Table 1. Mean ($\pm$ S.D.) of p -values and empirical power ($\alpha = 0.05$) of the three-step tests in the simulation data.

M_1	M_2	c	d	p_1	p_2	p	Power
30	30	0.1	1.0	0.5206 (± 0.2902)	0.3624 (± 0.2688)	0.4162 (± 0.2858)	0.0790
50	50	0.1	1.0	0.4877 (± 0.2868)	0.2683 (± 0.2379)	0.3202 (± 0.2577)	0.1680
30	30	0.2	1.0	0.4855 (± 0.2819)	0.0781 (± 0.1019)	0.1308 (± 0.1511)	0.3990
50	50	0.2	1.0	0.5083 (± 0.2928)	0.0176 (± 0.0316)	0.0417 (± 0.0699)	0.7530
30	30	0.3	1.0	0.5079 (± 0.2853)	0.0033 (± 0.0100)	0.0094 (± 0.0254)	0.9560
50	50	0.3	1.0	0.5102 (± 0.2827)	0.0001 (± 0.0006)	0.0004 (± 0.0026)	0.9990
30	30	0.0	1.1	0.3798 (± 0.3044)	0.5168 (± 0.2909)	0.4100 (± 0.3002)	0.1140
50	50	0.0	1.1	0.3175 (± 0.2979)	0.5309 (± 0.2919)	0.3637 (± 0.2952)	0.1660
30	30	0.0	1.2	0.1664 (± 0.2297)	0.5249 (± 0.2943)	0.2145 (± 0.2438)	0.3430
50	50	0.0	1.2	0.1005 (± 0.1761)	0.5132 (± 0.2925)	0.1398 (± 0.2013)	0.5170
30	30	0.0	1.3	0.0476 (± 0.1074)	0.5194 (± 0.2923)	0.0806 (± 0.1468)	0.6590
50	50	0.0	1.3	0.0132 (± 0.0464)	0.5085 (± 0.2866)	0.0267 (± 0.0720)	0.8610

The simulations demonstrate that as the differences between the two intensity functions increase, the overall p -value decreases, indicating that the proposed three-step test effectively detects deviations in point process characteristics. Moreover, the empirical power increases steadily with the effect size and improves substantially as the sample size grows, demonstrating the reliability of the proposed procedure under both types of alternatives.

5.1.2. Simulation for MLE Parametric Tests

Next, we evaluate the performance of the MLE-based parametric test using two types of point processes: the Hawkes process and a Poisson process with an intensity function based on a software reliability model.

Hawkes Process

We generate two groups of Hawkes processes with intensity functions:

$$\lambda_i^*(t) = \mu_i + \int_{-\infty}^t \alpha_i e^{-\beta_i(t-u)} dN(u), \quad i = 1, 2, \quad t \in [0, 20].$$

Each group consists of 50 realizations. We fix $\alpha_1 = 4$, $\beta_1 = 5$, and $\mu_1 = 0.5$ and systematically alter α_2 , β_2 , and μ_2 . We repeat this simulation 1000 times. The realizations of one of these simulations are shown in Figure 1, and the results are shown in Table 2.

Table 2. Mean ($\pm$ S.D.) of p -values and empirical power ($\alpha = 0.05$) in the simulated Hawkes process for the MLE-based test.

α_2	β_2	μ_2	p	Power
4	6	0.5	0.0027 (± 0.0163)	0.9880
4	5.5	0.5	0.0924 (± 0.1584)	0.6160
3.5	5	0.5	0.0413 (± 0.0981)	0.8100
3.7	5	0.5	0.1997 (± 0.2445)	0.3900
4	5	0.35	0.0078 (± 0.0329)	0.9650
4	5	0.4	0.0967 (± 0.1654)	0.6240

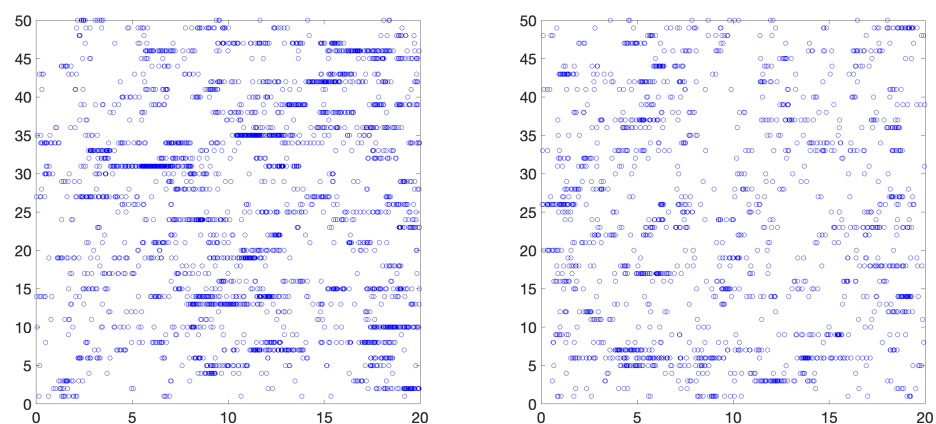


Figure 1. Simulated realizations of Hawkes processes from one representative experiment. Each point represents an event time. The x-axis denotes the observation time, and the y-axis indexes different realizations. (Left): Group 1. (Right): Group 2. The two groups are generated under different model parameters, with $\alpha_2 = 3.5$, $\beta_2 = 5$, $\mu_2 = 0.5$. The corresponding p -value is $p = 1.9868 \times 10^{-5}$.

Poisson Process

We generate two groups of Poisson processes with intensity functions:

$$\lambda_i(t) = \frac{\alpha_i(t/\beta_i)}{\beta_i} \exp(-t/\beta_i), \quad i = 1, 2, \quad t \in [0, 10].$$

Each group consists of 100 realizations. We fix $\alpha_1 = 20$, $\beta_1 = 1$ and vary α_2 and β_2 . The log-likelihood functions are maximized using numerical optimization, and hypothesis testing is performed based on the estimated parameters. This experiment is repeated 1000 times. The realizations of one of these simulations are shown in Figure 2, and the p -values (p) of each parameter setting are shown in Table 3.

Both simulations demonstrate that as the differences between the two intensity functions increase, the overall p -value decreases, indicating that the MLE-based test effectively detects deviations in point process characteristics. Additionally, the empirical power increases with the effect size and with larger sample sizes, confirming the stability of the proposed procedure under both types of alternatives.

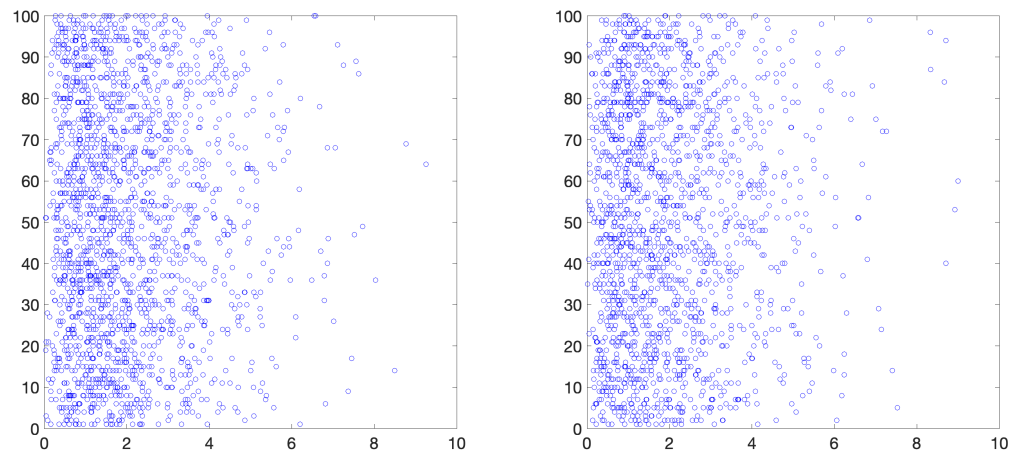


Figure 2. Simulated realizations of Poisson processes from one representative experiment. Each point represents an event time. The x-axis denotes the observation time, and the y-axis indexes different realizations. **(Left):** Group 1. **(Right):** Group 2. The two groups are generated under different intensity levels, with $\alpha_2 = 18$ and $\beta_2 = 1$. The corresponding p -value is $p = 0.0256$.

Table 3. Mean ($\pm$ S.D.) of p -values and empirical power ($\alpha = 0.05$) in the simulated Poisson process for the MLE-based test.

α_2	β_2	p	Power
20	0.90	0.0023 (± 0.0149)	0.9890
20	0.95	0.1390 (± 0.2043)	0.5040
17	1	0.0016 (± 0.0116)	0.9920
18	1	0.0325 (± 0.0926)	0.8460
19	1	0.2629 (± 0.2704)	0.2750

5.1.3. Simulation for ILR-Based Nonparametric Tests

We now evaluate the ILR transformation-based test for general point processes. We still conduct simulations on two different types of point processes: the Hawkes process and the Poisson process.

Hawkes Process

We generate two groups of Hawkes processes with intensity functions:

$$\lambda_i^*(t) = \mu_i + \int_{-\infty}^t \alpha_i e^{-\beta_i(t-u)} dN(u), \quad i = 1, 2, \quad t \in [0, 10].$$

Each group contains 5000 realizations, and we fix $\alpha_1 = 4$, $\beta_1 = 5$ and $\mu_1 = 0.5$ and alter one of the values of α_2 , β_2 and μ_2 at a time.

To conduct the ILR transformation-based test, we apply the ILR mapping on inter-event times and use multivariate depth-based hypothesis testing. $K_1, K_2, \dots, K_m$ in Equation (9) are set to $2, 3, \dots, 8$. This process is repeated 1000 times, and we record the mean and standard deviation of the p -values (p). The results are presented in Table 4.

Table 4. Mean ($\pm$ S.D.) of p_0 , overall p -values, and empirical power ($\alpha = 0.05$) in the simulated Hawkes process for the ILR-based test. When $(\alpha_2, \beta_2, \mu_2) = (\alpha_1, \beta_1, \mu_1)$, the reported power corresponds to the empirical size.

α_2	β_2	μ_2	p_0	p	Power
4.0	5.0	0.50	0.4916 (± 0.2835)	0.1073 (± 0.1310)	0.4490
3.5	5.0	0.50	0.4482 (± 0.2919)	0.1107 (± 0.1408)	0.4680
3.0	5.0	0.50	0.4093 (± 0.2843)	0.0406 (± 0.0678)	0.7530
2.8	5.0	0.50	0.3941 (± 0.2981)	0.0235 (± 0.0440)	0.8660
4.0	4.5	0.50	0.4750 (± 0.2890)	0.0779 (± 0.1039)	0.5640
4.0	4.0	0.50	0.3703 (± 0.2926)	0.0316 (± 0.0548)	0.8180
4.0	3.8	0.50	0.2898 (± 0.2675)	0.0214 (± 0.0474)	0.8860
4.0	5.0	0.45	0.3466 (± 0.2876)	0.0962 (± 0.1296)	0.5150
4.0	5.0	0.40	0.0841 (± 0.1394)	0.0387 (± 0.0677)	0.7770
4.0	5.0	0.60	0.1944 (± 0.2331)	0.0482 (± 0.0771)	0.7100
4.0	5.0	0.62	0.1474 (± 0.1975)	0.0354 (± 0.0665)	0.8040

Poisson Process

Next, we simulate two groups of Poisson processes, each characterized by the intensity function:

$$\lambda_i(t) = \frac{\alpha_i t}{\beta_i^2} \exp(-t/\beta_i), \quad i = 1, 2, \quad t \in [0, 1].$$

Each group contains 1000 realizations, and we fix $\alpha_1 = 5, \beta_1 = 0.2$, and change the values of α_2 and β_2 . This process is repeated 1000 times. The p -values (p) of each parameter setting are shown in Table 5.

Both simulation studies confirm that as the differences between the intensity functions increase, the overall p -value decreases, indicating that the ILR transformation-based test effectively detects differences in point process distributions. The empirical power increases steadily as the parameter deviation becomes larger in both the Poisson and Hawkes settings. However, due to the multi-step construction and the combination of multiple conditional subtests, the procedure can be relatively sensitive in finite samples and may exhibit elevated rejection rates under the global null.

Table 5. Mean ($\pm$ S.D.) of p_0 , overall p -values, and empirical power ($\alpha = 0.05$) in the simulated Poisson process for the ILR-based test. When $(\alpha_2, \beta_2) = (\alpha_1, \beta_1)$, the reported power corresponds to the empirical size.

α_2	β_2	p_0	p	Power
5	0.2	0.5045 (± 0.2885)	0.1241 (± 0.1521)	0.446
5	0.21	0.4937 (± 0.2871)	0.1102 (± 0.1404)	0.460
5	0.22	0.4506 (± 0.2842)	0.0965 (± 0.1276)	0.504
5	0.25	0.2618 (± 0.2582)	0.0326 (± 0.0527)	0.806
5	0.27	0.1051 (± 0.1650)	0.0119 (± 0.0210)	0.947
5	0.18	0.4783 (± 0.2947)	0.0848 (± 0.1144)	0.539
5	0.17	0.4536 (± 0.2906)	0.0558 (± 0.0767)	0.664
5	0.15	0.4077 (± 0.2903)	0.0061 (± 0.0133)	0.983
5.1	0.2	0.4787 (± 0.2924)	0.1214 (± 0.1544)	0.443
5.3	0.2	0.2237 (± 0.2377)	0.1104 (± 0.1418)	0.458
5.5	0.2	0.0589 (± 0.1241)	0.0622 (± 0.0960)	0.668
5.7	0.2	0.0076 (± 0.0298)	0.0193 (± 0.0538)	0.901

5.2. Real Data Analysis: Dopamine Spiking in Mouse VTA

We illustrate the proposed ILR-based test using a neural spiking dataset from the ventral tegmental area (VTA) of mice [24]. The data consist of spike times from optogeneti-

cally identified dopaminergic neurons recorded during classical conditioning tasks, where 1-second odor presentations served as conditioned stimuli and water deliveries as rewards (the dataset is publicly available from CRCNS.org as “vta-1”; see Starkweather et al. [25] for a detailed description). Across sessions, animals received either Salvinorin B (SalB), an agonist for Kappa Opioid Receptor DREADD expressed in the medial prefrontal cortex, or Saline injections, leading to different dopaminergic response patterns under uncertainty in reward prediction.

In this dataset, trials are organized into four experimental conditions: SalB1, SalB2, Saline1, and Saline2. SalB1 and SalB2 correspond to the two classical conditioning tasks performed under SalB injections, whereas Saline1 and Saline2 represent the same tasks performed under saline injections. The two tasks differ in their cue–reward contingencies and in the level of state uncertainty [24]. Within each condition, we observe spike trains from optogenetically identified dopaminergic neurons across many repetitions of the task, so that each group consists of a collection of long spike trains recorded on a common time window for multiple neuron–trial pairs.

To obtain many comparable point patterns, we segmented each long spike train into short point processes aligned to odor onset times. Specifically, we started from the first odor onset time t_0 and collected all spikes in the time interval $[t_0, t_0 + 1000]$ ms. We then set the next t_0 to be the first odor onset strictly after $t_0 + 1000$ and repeated this procedure, thereby constructing a sequence of non-overlapping 1000 ms windows. Each segment was then shifted so that its starting time is 0, yielding short point processes defined on the common domain $[0, 1000]$ ms in each experimental group. This procedure yields more than 8000 short point processes in each of the four groups, all defined on the common domain $[0, 1000]$. The time interval length of 1000 ms was chosen so that most segments contain at most 10 spikes while still retaining enough events for the ILR-based test.

We first compared the four groups at the between-condition level. For a given Monte Carlo sample size $S \in \{1000, 2000, 3500\}$, we repeatedly (with $N_{\text{iter}} = 1000$ iterations) drew S spike trains without replacement from each of the two groups under comparison and applied the ILR-based two-sample test on the resulting collections of point processes. Within each iteration, we evaluated the test over cardinalities $2, \dots, 8$, using the conditional framework described in Section 4, and recorded the resulting p -values. Table 6 reports the mean and standard deviation of the p -values over the 1000 repetitions for each pair of groups.

Table 6. ILR-based two-sample tests between experimental groups in the VTA dataset. Reported values are mean ($\pm$ S.D.) of p -values over $N_{\text{iter}} = 1000$ repetitions.

Comparison	$S = 1000$	$S = 2000$	$S = 3500$
SalB1 vs. SalB2	0.0159 (± 0.0401)	0.0042 (± 0.0128)	0.0004 (± 0.0021)
Saline1 vs. Saline2	0.0012 (± 0.0052)	0.00003 (± 0.00011)	0.0000 (± 0.0000)
SalB1 vs. Saline1	0.0473 (± 0.0741)	0.0281 (± 0.0503)	0.0071 (± 0.0137)
SalB2 vs. Saline2	0.0192 (± 0.0346)	0.0054 (± 0.0128)	0.0006 (± 0.0019)

Across all choices of S , the between-group comparisons yield small p -values, and the null hypothesis of equal point process distributions is consistently rejected. This indicates systematic differences in the spike timing patterns between SalB and Saline conditions, as well as between the two task-related groups. As expected for a powerful test, the ILR-based procedure is sensitive to such distributional changes and tends to reject the null whenever there are systematic differences in cardinality or temporal structure. Moreover, for each pairwise comparison, the mean p -values decrease as S increases, reflecting the gain in power with larger sample sizes.

To assess the behavior of the test under the null, we also performed within-group comparisons. Using the same resampling scheme (with $\mathcal{S} \in \{1000, 2000, 3500\}$ and $N_{\text{iter}} = 1000$), we randomly split each group into two subsamples of equal size and applied the ILR test to these artificially constructed pairs. Table 7 summarizes the mean and standard deviation of the resulting p -values.

Table 7. ILR-based two-sample tests within each experimental group in the VTA dataset. Reported values are mean ($\pm$ S.D.) of p -values over $N_{\text{iter}} = 1000$ repetitions.

Group	$\mathcal{S} = 1000$	$\mathcal{S} = 2000$	$\mathcal{S} = 3500$
SalB1	0.1445 (± 0.1623)	0.2170 (± 0.2237)	0.2652 (± 0.2564)
SalB2	0.1066 (± 0.1263)	0.1944 (± 0.2106)	0.2428 (± 0.2345)
Saline1	0.1370 (± 0.1511)	0.2066 (± 0.2094)	0.2713 (± 0.2530)
Saline2	0.1130 (± 0.1330)	0.1837 (± 0.1934)	0.2372 (± 0.2293)

The within-group p -values are moderate and do not provide evidence against the null hypothesis for any of the four groups. Although the ILR-based test is powerful and tends to reject the null when there are genuine differences between point process distributions, these internal comparisons suggest that its type I error is under reasonable control in this setting. Taken together, the results in Tables 6 and 7 demonstrate that the proposed ILR-based framework can detect meaningful changes in spike timing patterns across experimental conditions while remaining stable under repeated subsampling from the same underlying process. Moreover, for each within-group comparison, the mean p -values increase as $\mathcal{S}$ grows, which is consistent with more stable estimation under larger sample sizes when the null hypothesis holds.

6. Summary and Future Work

In this paper, we proposed and examined three statistical hypothesis testing approaches for comparing point processes, particularly focusing on Poisson processes. The three methods include: (1) a three-step test designed specifically for inhomogeneous Poisson processes, (2) a parametric test based on maximum likelihood estimation for structured point processes such as Hawkes processes, and (3) a nonparametric approach utilizing the Isometric Log-Ratio transformation for general point processes.

The three-step test for inhomogeneous Poisson processes evaluates differences in total intensity and normalized intensity separately, combining results using Fisher's method to obtain an overall p -value. Through extensive simulations, we demonstrated that this method effectively identifies both magnitude and distributional differences in event occurrences.

The MLE-based parametric test was applied to both Hawkes processes and a Poisson process model used in software reliability. By estimating intensity function parameters through numerical optimization, the test successfully detected differences between two processes in simulated experiments. The results confirmed that as the differences between intensity functions grow, the test reliably identifies deviations.

For more general point processes where a parametric form may not be suitable, we introduced a nonparametric ILR transformation-based test. By mapping inter-event times into a Euclidean space, we leveraged depth-based statistical methods to compare point process distributions. Simulation studies on both Hawkes and Poisson processes demonstrated that the ILR-based test effectively captures differences in the underlying event structure.

Additionally, we applied the ILR-based test to real spike-train recordings. We analyzed spike-time distributions across multiple experimental conditions, focusing on groups with similar or distinct pharmacological manipulations. The results showed that comparisons

between conditions with similar neural activity patterns exhibited no significant differences in their spike timing, whereas conditions involving altered excitability displayed clear and statistically significant deviations.

While the proposed methods provide effective tools for comparing point processes, there are several directions for future research:

- Extending the three-step procedure to multi-group comparisons: Our three-step test was primarily designed for comparing two Poisson processes. Extending this framework to compare multiple inhomogeneous Poisson processes simultaneously would be an interesting avenue for further study.
- Robustness of MLE-based methods in more complex point process models: The MLE-based test performed well in Poisson and Hawkes processes, but further research could explore its effectiveness in more complex models, such as renewal processes or self-correcting point processes.
- Optimization of ILR-based methods for small sample sizes: The ILR transformation provides a flexible framework for analyzing point processes, but its performance can be limited when sample sizes are small. Future research can explore ways to refine the transformation or incorporate additional statistical techniques to improve sensitivity and power in low-sample scenarios.
- Application to other real-world datasets: While we applied our methods to spiking data, similar techniques could be used to analyze other event-based data, such as financial transactions, earthquake occurrences, or neuronal spike trains.

Overall, our study provides a basic framework for statistical tests on point processes, and the proposed methods have demonstrated their effectiveness in both simulated and real-world settings. Future work will aim to refine and extend these methodologies to further improve their applicability and robustness in diverse domains.

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Appendix A. Technical Details for Maximum Likelihood Estimation of Point Processes

Appendix A.1. Asymptotic Normality of Maximum Likelihood Estimators for Point Processes

Let $\{s_1, \dots, s_M\}$ be a set of independent realizations of a point process with intensity $\lambda(t|\theta)$, where $\theta \in \Theta$ are unknown parameters. Let $\hat{\theta}_M$ be the maximum likelihood estimation of θ . Let l_M be the normalized log-likelihood function:

$$l_M(\theta) = \frac{1}{M} \log \prod_{i=1}^M L_{s_i}(\theta),$$

where L_{s_i} is the likelihood of s_i . By the definition of maximum likelihood estimation, $\hat{\theta}_M = \arg \max_{\theta \in \Theta} \log \prod_{i=1}^M L_{s_i}(\theta)$, so $l'_M(\hat{\theta}_M) = 0$. Then by mean value theorem, $l'_M(\hat{\theta}_M) = l'_M(\theta) + l''_M(\tilde{\theta})(\hat{\theta}_M - \theta_0)$ for some $\tilde{\theta}$ between θ and $\hat{\theta}_M$. It follows that $\sqrt{M}(\hat{\theta}_M - \theta) = -\frac{\sqrt{M}l'_M(\theta)}{l''_M(\tilde{\theta})}$. By the central limit theorem, $\sqrt{M}l'_M(\theta) \xrightarrow{d} \mathcal{M}(\mathbf{0}, \mathcal{I}(\theta))$. If we assume $\hat{\theta}_M \xrightarrow{p} \theta$, then by the weak law of large numbers, $l''_M(\tilde{\theta}) \xrightarrow{p} -\mathcal{I}(\theta)$. Finally, Slutsky's theorem implies the asymptotic normality

$$\hat{\theta}_M \xrightarrow{d} \mathcal{M}(\theta, \mathcal{I}(\theta)^{-1})$$

as $M \rightarrow \infty$.

Appendix A.2. Gradients and Hessians for Hawkes Process

The gradients are

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} &= \sum_{i=1}^n \frac{1}{\beta} (e^{-\beta(T-t_i)} - 1) + \sum_{i=1}^n \frac{A(i)}{\mu + \alpha A(i)}, \\ \frac{\partial \log L}{\partial \beta} &= -\alpha \sum_{i=1}^n \left[\frac{1}{\beta} (T - t_i) e^{-\beta(T-t_i)} + \frac{1}{\beta^2} (e^{-\beta(T-t_i)} - 1) \right] - \sum_{i=1}^n \frac{\alpha B(i)}{\mu + \alpha A(i)}, \\ \frac{\partial \log L}{\partial \mu} &= -T + \sum_{i=1}^n \frac{1}{\mu + \alpha A(i)},\end{aligned}$$

where $B(i) = \sum_{t_j < t_i} (t_i - t_j) e^{-\beta(t_i - t_j)}$ for $i \geq 2$ and $B(1) = 0$.

The Hessian matrix is

$$\begin{aligned}\frac{\partial^2 \log L}{\partial \alpha^2} &= -\sum_{i=1}^n \left[\frac{A(i)}{\mu + \alpha A(i)} \right]^2, \\ \frac{\partial^2 \log L}{\partial \beta \partial \alpha} &= -\sum_{i=1}^n \left[\frac{1}{\beta} (T - t_i) e^{-\beta(T-t_i)} + \frac{1}{\beta^2} (e^{-\beta(T-t_i)} - 1) \right] \\ &\quad + \sum_{i=1}^n \left[\frac{-B(i)}{\mu + \alpha A(i)} + \frac{\alpha A(i) B(i)}{(\mu + \alpha A(i))^2} \right], \\ \frac{\partial^2 \log L}{\partial \beta^2} &= \alpha \sum_{i=1}^n \left[\frac{1}{\beta} (T - t_i)^2 e^{-\beta(T-t_i)} + \frac{2}{\beta^2} (T - t_i) e^{-\beta(T-t_i)} + \frac{2}{\beta^3} (e^{-\beta(T-t_i)} - 1) \right] \\ &\quad + \sum_{i=1}^n \left[\frac{\alpha C(i)}{\mu + \alpha A(i)} - \left(\frac{\alpha B(i)}{\mu + \alpha A(i)} \right)^2 \right],\end{aligned}$$

where $C(i) = \sum_{t_j < t_i} (t_i - t_j)^2 e^{-\beta(t_i - t_j)}$ for $i \geq 2$ and $C(1) = 0$,

$$\begin{aligned}\frac{\partial^2 \log L}{\partial \mu^2} &= -\sum_{i=1}^n \frac{1}{(\mu + \alpha A(i))^2}, \\ \frac{\partial^2 \log L}{\partial \alpha \partial \mu} &= -\sum_{i=1}^n \frac{A(i)}{(\mu + \alpha A(i))^2}, \\ \frac{\partial^2 \log L}{\partial \beta \partial \mu} &= \sum_{i=1}^n \frac{\alpha B(i)}{(\mu + \alpha A(i))^2}.\end{aligned}$$

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