

Article

Optimal Control of Impulsive Systems Under State, Control, and Terminal Constraints

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Abstract

We establish a version of Pontryagin's maximum principle for optimal control problems with impulses and phase constraints. Using the Dubovitskii–Milyutin theory, we construct a conic variational framework that handles impulsive dynamics and general state constraints. The main difficulty lies in working with piecewise continuous functions, required by the impulsive nature of the system. This setting also demands an extension of the classical result on the existence of non-negative Borel measures, which leads to an adjoint equation formulated as a Stieltjes integral. Theoretical results are illustrated with examples, and key results by I. Girsanov are extended to the impulsive context.

Keywords: optimal control; impulsive systems; Pontryagin maximum principle; state constraints; Dubovitskii–Milyutin theory; Stieltjes integral

MSC: 62G10

1. Introduction

Optimal control problems with impulsive effects arise naturally in many applications, including epidemiology, economics, engineering systems, and networked dynamics. In such systems, the state trajectory is no longer smooth but exhibits discontinuities governed by impulse maps. When phase constraints and impulses are imposed simultaneously, classical optimal control techniques based on smooth trajectories and absolutely continuous adjoint variables are no longer sufficient.

The purpose of this paper is to develop a Pontryagin-type maximum principle for impulsive systems with phase constraints, formulated in spaces of piecewise continuous functions. Our approach is based on the Dubovitskii–Milyutin theory of extremum problems, which we extend to this non-smooth setting. A central difficulty lies in the fact that the natural state space is not reflexive, and classical representation theorems for linear functionals fail to apply. To overcome this, we extend Girsanov's book theorem on non-negative Borel measures to piecewise continuous functions, allowing us to explicitly characterize the dual elements associated with the constraint cones.

As a result, we derive an adjoint equation of the Volterra–Stieltjes type, which captures both continuous dynamics and impulsive effects in a unified functional form. This adjoint structure is fundamentally different from the classical ordinary adjoint differential equation and which, to the best of our knowledge, has not been derived before for impulsive systems with phase constraints.



Academic Editor: Xinguang Zhang

Received: 10 November 2025

Revised: 12 February 2026

Accepted: 13 February 2026

Published: 20 February 2026

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The Dubovitskii–Milyutin theory has long been recognized as a powerful tool in optimal control. It was originally developed in the seminal work of Dubovitskii and Milyutin [1] and was later systematized and extended within the general framework of variational analysis and optimal control (see, for instance, [2–5]). More recently, the theory has been applied to impulsive control systems without state constraints, showing its flexibility in handling hybrid dynamics [6]. In a related direction, this framework was further applied to optimal control problems governed by integral equations with both state and control constraints, though without impulses [7]. Subsequent contributions have demonstrated its effectiveness in increasingly complex settings, including infinite-dimensional systems and problems with general constraints [8]. Additional developments and applications can be found in [9–11], further illustrating the versatility and strength of conic methods in optimization.

It is important to clarify that our work differs from the results of Trélat [12] and Leiva [6]. In Trélat [12], neither state constraints nor impulses are considered, whereas in Leiva [6], impulsive effects are included but no restrictions are imposed on the state variable. In this sense, our framework provides a more comprehensive treatment by simultaneously incorporating impulses and state constraints.

To illustrate the scope and applicability of our main result, we conclude with two representative examples: a controlled epidemiological SIR model and a Bolza-type economic problem involving consumption and investment. In the epidemiological setting, impulsive effects are used to model sudden public health interventions, such as mass vaccination campaigns, mobility restrictions, or abrupt policy changes, which produce instantaneous variations in the susceptible, infected, or recovered populations. In the economic example, impulsive dynamics capture discrete financial decisions or external shocks affecting the wealth variable. In the latter case, we derive an explicit optimal control formula that is particularly suitable for numerical implementation. For both problems, we obtain explicit expressions for the optimal control in terms of the adjoint equation and the nonlinear ordinary differential equation governing the state dynamics.

Outline of The Paper

In Section 2, we formulate the optimal control problem for impulsive systems with control and terminal constraints, together with path restrictions, and present the main assumptions and results. Section 3 presents the variational tools from the Dubovitskii–Milyutin theory required for the analysis. In Section 4, we prove the Pontryagin maximum principle by constructing the associated cones and identifying the adjoint variables and measure-valued multipliers. Section 5 provides sufficient conditions under which the necessary conditions become sufficient for optimality. Section 6 illustrates the theory through epidemiological and economic models and derives explicit optimal control laws. Finally, Section 7 discusses numerical implementation aspects and concludes this paper.

2. Framework of the Optimization Problem and Main Statements

We now introduce an alternative presentation of the optimal control model considered in this work. The notation, symbols, and structure have been fully redefined to provide an equivalent but independent formulation. This setting describes the evolution of a state variable subject to continuous dynamics, impulsive effects, terminal constraints, pointwise state restrictions, and admissible control values. The problem below serves as the foundation for all subsequent analysis.

Problem 1.

$$\mathcal{F}(z, v) = F_0(z(0), z(T)) + \int_0^T L(z(t), v(t), t) dt \longrightarrow \min_{loc}, \quad (1)$$

$$(z, v) \in \mathbb{E} = \mathcal{PC}([0, T]; \mathbb{R}^m) \times L^\infty([0, T]; \mathbb{R}^\ell), \quad (2)$$

$$\dot{z}(t) = H(z(t), v(t), t), \quad z(0) = z_*, \quad (3)$$

$$C_j(z(T)) = 0, \quad j = 1, \dots, r, \quad (4)$$

$$z(\sigma_k^+) = z(\sigma_k^-) + \Gamma_k(z(\sigma_k)), \quad k = 1, \dots, N, \quad (5)$$

$$v(t) \in \mathcal{U}, \quad a.e. \ t \in [0, T], \quad (6)$$

$$\phi(z(t), t) \leq 0, \quad t \in [0, T]. \quad (7)$$

Here, $\min_{loc}$ indicates that the functional $\mathcal{F}$ attains a local minimum at (z, v) .

The dimensions $m, \ell \in \mathbb{N}$ and the horizon $T > 0$ are fixed. The mappings involved in the model are now denoted by:

$$\begin{aligned} H &: \mathbb{R}^m \times \mathbb{R}^\ell \times [0, T] \rightarrow \mathbb{R}^m, \\ L &: \mathbb{R}^m \times \mathbb{R}^\ell \times [0, T] \rightarrow \mathbb{R}, \\ \Gamma_k &: \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad k = 1, \dots, N, \\ C_j &: \mathbb{R}^m \rightarrow \mathbb{R}, \quad j = 1, \dots, r, \\ \phi &: \mathbb{R}^m \times [0, T] \rightarrow \mathbb{R}, \\ F_0 &: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}. \end{aligned}$$

The set of partition points

$$0 = \sigma_0 < \sigma_1 < \dots < \sigma_N < T$$

induces subintervals

$$I_1 = [0, \sigma_1], \quad I_{k+1} = (\sigma_k, \sigma_{k+1}], \quad k = 1, \dots, N-1.$$

The space $\mathcal{PC}([0, T]; \mathbb{R}^m)$ consists of all functions that are continuous on each I_k and possess right limits at the nodes σ_k , and it is endowed with the supremum norm

$$\|z\|_\infty = \sup_{t \in [0, T]} \|z(t)\|.$$

For $z \in \mathcal{PC}$, we define the regularized representative on each closed interval $\overline{I_k}$,

$$\tilde{z}_k(t) = \begin{cases} z(t), & t \in I_k, \\ z(\sigma_{k-1}^+), & t = \sigma_{k-1}. \end{cases}$$

The control space $L^\infty([0, T]; \mathbb{R}^\ell)$ has the usual essential supremum norm.

Assumptions:

- All functions F_0, L, H, C_j, Γ_k are continuous, and their partial derivatives $\nabla F_0, L_x, L_v, H_x, H_v, \Gamma'_k, C'_j$ exist and are sufficiently smooth.
- The control constraint set $\mathcal{U} \subset \mathbb{R}^\ell$ is nonempty, closed, and convex, with a nonempty interior.

- (c) The state constraint function ϕ is continuous and satisfies

$$\phi(z_*, 0) < 0, \quad \phi(z_T, T) < 0,$$

and its derivative ϕ_x is continuous and nonzero on the boundary where $\phi(\cdot, t) = 0$.

- (d) The impulsive mappings satisfy a uniform contraction-type bound:

$$\|\Gamma'_k(\xi)z\| \leq \rho_k \|z\|, \quad \forall z, \xi \in \mathbb{R}^m, \rho_k < 1, \quad k = 1, \dots, N.$$

Remark 1. Since $\Gamma'_k(\xi)$ is linear, this assumption provides a sufficient condition under which controllability of the linear variational system without impulses implies controllability of the corresponding impulsive variational system.

- (e) At the terminal point of the nominal solution (z°, v°) , let $C = (C_1, C_2, \dots, C_r)^\top$ and

$$\Lambda = C'(z^\circ(T)) \in \mathbb{R}^{r \times m}$$

denote the Jacobian matrix of the terminal constraint. We assume full row rank:

$$\text{rank}(\Lambda) = r.$$

Remark 2. The condition $\text{rank}(\Lambda) = r$ is equivalent to the surjectivity of $\Lambda : \mathbb{R}^m \rightarrow \mathbb{R}^r$, which ensures the existence of a right inverse $\Lambda^+ = \Lambda^\top (\Lambda \Lambda^\top)^{-1}$. Thus, the equation $\Lambda z(T) = b$ has a solution $z(T) = \Lambda^\top (\Lambda \Lambda^\top)^{-1} b$ for every $b \in \mathbb{R}^r$.

Remark 3. Let (z°, v°) be the optimal pair and consider the linearized continuous-time system

$$\dot{y}(t) = H_z(z^\circ(t), v^\circ(t), t)y(t) + H_v(z^\circ(t), v^\circ(t), t)w(t). \quad (8)$$

If system (8) is controllable and the bounds in assumption (d) hold, then the corresponding impulsive linear variational system

$$\begin{cases} \dot{y}(t) = H_x(z^\circ(t), v^\circ(t), t)y(t) + H_v(z^\circ(t), v^\circ(t), t)w(t), & t \neq \sigma_k, \\ y(\sigma_k^+) = \Gamma'_k(z^\circ(\sigma_k^-))y(\sigma_k^-) + \beta_k, \end{cases} \quad (9)$$

is also controllable.

From now on, $z^\circ(t)$ denotes the optimal trajectory and $v^\circ(t)$ the optimal control.

Theorem 1. Assume that hypotheses (a)–(e) are satisfied for an optimal pair $(z^\circ, v^\circ) \in \mathbb{E}$ associated with Problem 1. Then the following assertions hold.

1. Existence of a state–constraint measure. There exists a non-negative Borel measure

$$\nu \text{ on } [0, T],$$

whose support is contained in the active constraint set

$$\mathcal{A} = \{t \in [0, T] \mid \phi(z^\circ(t), t) = 0\}.$$

2. Adjoint equation. There exist a scalar $\alpha \geq 0$ and a function

$$p \in L^1([0, T]; \mathbb{R}^m)$$

not simultaneously zero with α , satisfying the backward integral relation

$$\begin{aligned} -p(t) = & \int_t^T \left(-H_z^\top(z^\diamond(\sigma), v^\diamond(\sigma), \sigma) p(\sigma) + \alpha L_z(z^\diamond(\sigma), v^\diamond(\sigma), \sigma) \right) d\sigma \\ & - \Lambda^\top \xi + \alpha \partial_2 F_0(z^\diamond(0), z^\diamond(T)) + \int_t^T \phi_z(z^\diamond(\sigma), \sigma) dv(\sigma), \end{aligned}$$

for some vector $\xi \in \mathbb{R}^r$.

3. Pontryagin's maximum principle.

For almost every $t \in [0, T]$ and all $w \in \mathcal{U}$,

$$\left\langle -H_v^\top(z^\diamond(t), v^\diamond(t), t) p(t) + \alpha L_v(z^\diamond(t), v^\diamond(t), t), w - v^\diamond(t) \right\rangle \geq 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^m$.

Theorem 2. Assume that structural conditions (a)–(e) hold and consider the terminally constrained problem in which the path restriction $\phi(z(t), t) \leq 0$ is replaced by the endpoint inequality

$$\phi(z(T), T) \leq 0.$$

Let $(z^\diamond, v^\diamond) \in \mathbb{E}$ be an optimal pair, and suppose that

$$\phi_z(z^\diamond(T), T) \neq 0.$$

Then, there exist non-negative multipliers

$$\alpha \geq 0, \quad \rho \geq 0,$$

and an adjoint function

$$p \in C([0, T]; \mathbb{R}^m),$$

not simultaneously zero with α , such that the relations below are satisfied.

1. Adjoint equation.

The function $p(\cdot)$ satisfies

$$\dot{p}(t) = -H_z^\top(z^\diamond(t), v^\diamond(t), t) p(t) + \alpha L_z(z^\diamond(t), v^\diamond(t), t) + \rho \phi_z(z^\diamond(T), T), \quad 0 < t < T,$$

together with the transversality condition

$$p(T) = \Lambda^\top \gamma - \alpha \partial_2 F_0(z^\diamond(0), z^\diamond(T)),$$

for some vector $\gamma \in \mathbb{R}^r$, where

$$\Lambda := C'(z^\diamond(T))$$

denotes the Jacobian matrix of the terminal constraint.

2. Pontryagin's maximum principle.

For almost every $t \in [0, T]$ and every $w \in \mathcal{U}$,

$$\left\langle -H_v^\top(z^\diamond(t), v^\diamond(t), t) p(t) + \alpha L_v(z^\diamond(t), v^\diamond(t), t), w - v^\diamond(t) \right\rangle \geq 0.$$

3. Preliminaries

This section gathers the analytical tools required for the variational approach used later in this article. Our presentation follows the classical Dubovitskii–Milyutin framework, but we

adapt the exposition to the structural features of optimal control problems with impulsive dynamics and state constraints. Only the ingredients relevant for the derivation of the necessary conditions are included here; a comprehensive treatment can be found in [5,6,13].

3.1. Cones, Dual Cones, and Separation Mechanisms

Let $\mathbb{E}$ be a locally convex topological vector space with dual $\mathbb{E}^*$. A subset $\mathcal{K} \subset \mathbb{E}$ is called a cone when $\lambda\mathcal{K} = \mathcal{K}$ for every $\lambda > 0$. The corresponding dual cone is

$$\mathcal{K}^+ := \{ \ell \in \mathbb{E}^* : \ell(x) \geq 0 \text{ for all } x \in \mathcal{K} \}.$$

A fundamental operation is the passage to dual cones under intersections. For any family of convex, weakly closed cones $\{\mathcal{K}_\alpha\}_{\alpha \in A} \subset \mathbb{E}$, one has

$$\left(\bigcap_{\alpha \in A} \mathcal{K}_\alpha \right)^+ = \overline{\sum_{\alpha \in A} \mathcal{K}_\alpha^+}^{w*}.$$

If a cone $\mathcal{K}$ has a nonempty interior relative to a linear subspace $\mathbb{L}$, then

$$(\mathcal{K} \cap \mathbb{L})^+ = \mathcal{K}^+ + \mathbb{L}^+.$$

For a finite family of open convex cones $\mathcal{K}_1, \dots, \mathcal{K}_m$ with a nonempty intersection,

$$\left(\bigcap_{i=1}^m \mathcal{K}_i \right)^+ = \sum_{i=1}^m \mathcal{K}_i^+.$$

Finally, if $\mathcal{K}_1, \dots, \mathcal{K}_{m+1}$ are convex cones with an apex at the origin and the first m are open, then

$$\bigcap_{i=1}^{m+1} \mathcal{K}_i = \emptyset \iff \exists \ell_i \in \mathcal{K}_i^+ \ (i = 1, \dots, m+1), \text{ not all zero, such that } \sum_{i=1}^{m+1} \ell_i = 0.$$

These separation properties constitute the abstract backbone of the Euler–Lagrange equation used later.

3.2. Abstract Euler–Lagrange Principle

Consider an optimization problem of the form

$$F(x) \longrightarrow \min_{\text{loc}} \quad \text{s.t.} \quad x \in Q_i, \ i = 1, \dots, m+1,$$

where $F : \mathbb{E} \rightarrow \mathbb{R}$ is the objective and the sets Q_i encode the constraints. Typically, $Q_1, \dots, Q_m$ correspond to inequality constraints (with nonempty interiors), and Q_{m+1} represents equality constraints.

Let $x^\diamond$ be a local minimizer of F . We denote by

- $\mathcal{K}_0$ the decay cone of F at $x^\diamond$,
- $\mathcal{K}_i$ the admissible cone to Q_i at $x^\diamond$ ($i = 1, \dots, m$),
- $\mathcal{K}_{m+1}$ the tangent cone to Q_{m+1} at $x^\diamond$.

Theorem 3 (Dubovitskii—Milyutin). *If each $\mathcal{K}_i$ is convex, then there exist dual elements $\ell_i \in \mathcal{K}_i^+$, not all zero, such that*

$$\ell_0 + \ell_1 + \dots + \ell_{m+1} = 0.$$

This identity is the abstract Euler–Lagrange equation.

The equation above will later be instantiated in the setting of impulsive optimal control, where each dual element is identified with a co-state, a multiplier, or a measure depending on which cone generates it.

3.3. Decay and Admissible Directions

A direction $h \in \mathbb{E}$ is a *decay direction* for the functional F at $x^\diamond$ if small perturbations of the form $x^\diamond + \varepsilon h$ decrease F to first order. When the directional derivative

$$F'(x^\diamond; h) := \lim_{\varepsilon \downarrow 0} \frac{F(x^\diamond + \varepsilon h) - F(x^\diamond)}{\varepsilon}$$

exists, this condition is equivalent to $F'(x^\diamond; h) < 0$. Thus, the decay cone is

$$\mathcal{K}_d(F, x^\diamond) = \{h \in \mathbb{E} : F'(x^\diamond; h) < 0\}.$$

Admissible directions for a constraint set Q are those perturbations that keep the perturbed point inside Q at first order. The admissible cone at $x^\diamond \in Q$ is

$$\mathcal{K}_a(Q, x^\diamond) = \{h : x^\diamond + \varepsilon h \in Q \text{ for all sufficiently small } \varepsilon > 0\}.$$

Theorem 4. (See [13] (p. 59).) *If Q is an arbitrary convex set with $\text{int}(Q) \neq \emptyset$, then*

$$\mathcal{K}_a(Q, x^\diamond) = \{h \in \mathbb{E} / h = \lambda(x - x^\diamond), x \in \text{int}(Q), \lambda > 0\}.$$

3.4. Tangent Directions and Lyusternik's Principle

A vector h is tangent to a set Q at $x^\diamond$ if one can find perturbations of the form

$$x^\diamond + \varepsilon h + o(\varepsilon)$$

that remain in Q . The set of all such directions is the tangent cone $\mathcal{K}_T(Q, x^\diamond)$.

Lyusternik's theorem provides an efficient way to compute tangent cones for equality-constrained sets of the form $Q = \{x : P(x) = 0\}$.

Theorem 5 (Lyusternik). *Let $P : \mathbb{E}_1 \rightarrow \mathbb{E}_2$ be Fréchet differentiable at $x^\diamond$, and suppose that $P'(x^\diamond)$ is surjective. Then*

$$\mathcal{K}_T(Q, x^\diamond) = \ker P'(x^\diamond).$$

This result will later be used to analyze the linearized dynamics of the impulsive control system.

3.5. Dual Characterizations for Several Cones

We now list three duality results that will be needed in the construction of the functionals in the approximation dual cones for the optimal control problem.

Proposition 1. *Let us consider $\mathbb{E} = \mathcal{PW}([0, T], \mathbb{R}^m)$, Λ a matrix of dimension $r \times m$ with $r \leq m$ and $\text{Rank}(\Lambda) = r$. Consider the following cone*

$$K = \{x \in \mathbb{E} : \Lambda x(T) = 0\}.$$

Then $l \in K^+$ if and only if there is $a \in \mathbb{R}^r$ such that

$$l(x) = \langle a, \Lambda x(T) \rangle = \langle \Lambda^T a, x(T) \rangle \quad (x \in \mathbb{E}).$$

Proposition 2. (See [6].) Let $A, \Gamma_k : [0, T] \rightarrow \mathbf{R}^{m \times m}, k = 1, 2, 3, \dots, N$ and $B : [0, T] \rightarrow \mathbf{R}^{m \times l}$ be measurable and bounded functions. Suppose the following impulsive linear system is controllable on $[0, T]$ for any $\bar{b} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_N) \in \mathbf{R}^{mN} = (\mathbf{R}^m)^N$

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)v(t), & t \in (0, T], \quad t \neq \sigma_k \\ x(\sigma_k^+) = \Gamma_k(\sigma_k)x(\sigma_k^-) + \bar{b}_k, & k = 1, 2, 3, \dots, N, \end{cases} \quad (10)$$

where $(x, v) \in \mathbb{E} := \mathcal{PW}([0, T]; \mathbf{R}^m) \times L^\infty([0, T]; \mathbf{R}^\ell)$. Let us define the following cone

$$L_1 := \{(x, v) \in \mathbb{E} / x(T) = 0, \quad x(\sigma_k^+) - \Gamma_k(\sigma_k)x(\sigma_k^-) = 0, \quad k = 1, 2, 3, \dots, N\}.$$

Then, $\dim L_1^+ = m(N + 1)$, and also for all $l \in L_1^+$ there is $a \in \mathbf{R}^{m(N+1)}$ such that

$$l(x, v) = \langle a, (x(T), x(\sigma_1^+) - \Gamma_1(\sigma_1)x(\sigma_1^-), \dots, x(\sigma_N^+) - \Gamma_N(\sigma_N)x(\sigma_N^-)) \rangle, \quad (x, v) \in \mathbb{E}.$$

A modification of example 10.3 of [13] (p. 73) leads us to the following result.

Proposition 3. Suppose $\mathbb{E}_1 = \mathcal{PW}([0, T]; \mathbf{R}^m)$ and $\emptyset \neq \mathcal{A} \subset [0, T]$ closed, $h \in \mathbb{E}_1$ such that $h(t) \neq 0$ ($t \in \mathcal{A}$). Define the following cone

$$\mathcal{K} := \{x \in \mathbb{E}_1 / \langle h(t), x(t) \rangle \geq 0, \quad \forall t \in \mathcal{A}\}.$$

Then, for all $l \in \mathcal{K}^+$, there exists a non-negative Borel measure ν in $[0, T]$, with support in $\mathcal{A}$, such that

$$l(x) = \int_{\mathcal{A}} \langle h(t), x(t) \rangle d\nu(t), \quad (x \in \mathbb{E}_1).$$

Proof. Let $l \in \mathcal{K}^+$, and consider the cones

$$\mathcal{K}_k := \{x \in \mathbb{E}_1 / \langle \bar{h}_k(t), \bar{x}_k(t) \rangle \geq 0, \quad \forall t \in \bar{J}_k \cap \mathcal{A}\}, \quad k = 1, 2, 3, \dots, N.$$

Since $\mathcal{A} = \bigcup_{k=1}^N (\bar{J}_k \cap \mathcal{A})$, we get that $\mathcal{K} = \bigcap_{k=1}^N \mathcal{K}_k$. Then, from the preliminaries, we get that

$$\mathcal{K}^+ = \sum_{k=1}^N \mathcal{K}_k^+ \quad \text{and} \quad l(x) = \sum_{k=1}^N l_k(x), \quad l_k \in \mathcal{K}_k^+.$$

Hence, from Example 10.3 in [13] (p. 73), there is a non-negative Borel measure ν_k on $[0, T]$ with support on $\bar{J}_k \cap \mathcal{A}$ such that

$$l_k(x) = \int_{\bar{J}_k \cap \mathcal{A}} \langle h(t), x(t) \rangle d\nu_k(t), \quad (x \in E_1).$$

Therefore

$$l(x) = \sum_{k=1}^N \int_{\bar{J}_k \cap \mathcal{A}} \langle h(t), x(t) \rangle d\nu_k(t) = \sum_{k=1}^N \int_{\bar{J}_k \cap \mathcal{R}} \langle h(t), x(t) \rangle d\nu(t).$$

If we put $\nu = \sum_{k=1}^N \chi_{\bar{J}_k \cap \mathcal{R}} \nu_k$, where $\chi_{\bar{J}_k \cap \mathcal{A}}$ is the characteristic function of the set $\bar{J}_k \cap \mathcal{A}$, then,

$$l(x) = \int_{\mathcal{A}} \langle h(t), x(t) \rangle d\nu(t), \quad (x \in E_1),$$

and ν is a Borel measure with support in $\mathcal{A}$. $\square$

Now, let us delve into a significant example related to support functionals.

Example 1. Let $\mathcal{U} \subset \mathbb{R}^\ell$ and $\mathcal{Q} := \{v \in L^\infty([0, T]; \mathbb{R}^\ell) / v(t) \in \mathcal{U}, t \in [0, T], \text{a.e.}\}$ and consider $v^\circ \in \mathcal{Q}$, $a \in L^\infty([0, T]; \mathbb{R}^\ell)$ and $\ell : L^\infty([0, T]; \mathbb{R}^\ell) \rightarrow \mathbb{R}$ defined as follows

$$\ell(v) := \int_0^T \langle a(t), v(t) \rangle dt, \quad (v \in L^\infty([0, T]; \mathbb{R}^\ell)).$$

Let us suppose that $\ell(v) \geq \ell(v^\circ)$ ($v \in \mathcal{Q}$). Then for all $\omega \in \mathcal{U}$ and almost all $t \in [0, T]$

$$\langle a(t), \omega - v^\circ(t) \rangle \geq 0.$$

For details of this example, see [13] (p. 76).

4. Proof of the Main Theorem 1

Proof. Let us

$$\mathcal{F} : \mathbb{E} \rightarrow \mathbb{R}$$

be the objective functional defined by

$$\mathcal{F}(z, v) = F_0(z(0), z(T)) + \int_0^T L(z(t), v(t), t) dt.$$

Consider

$$\mathcal{Q} := \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \mathcal{Q}_3,$$

where each subset of feasible pairs $(z, v) \in \mathbb{E}$ is given in the following description:

Constraint	Description
$\mathcal{Q}_1$	Pairs $(z, v) \in \mathbb{E}$ satisfying the continuous dynamics $\dot{z}(t) = H(z(t), v(t), t), \quad z(0) = z^*,$ together with the impulsive conditions $z(\sigma_k^+) = z(\sigma_k^-) + \Gamma_k(z(\sigma_k)), \quad k = 1, \dots, N.$
$\mathcal{Q}_2$	Pairs $(z, v) \in \mathbb{E}$ such that the control satisfies the admissibility constraint $v(t) \in \mathcal{U}, \quad \text{for a.e. } t \in [0, T].$
$\mathcal{Q}_3$	Pairs $(z, v) \in \mathbb{E}$ satisfying the path (state) constraint $\phi(z(t), t) \leq 0, \quad \forall t \in [0, T].$

Thus, the optimal control problem is equivalent to

$$\begin{cases} \mathcal{F}(z, v) \rightarrow \text{loc min}, \\ (z, v) \in \mathcal{Q}. \end{cases}$$

(a) **Decay cone of the functional $\mathcal{F}$.**

Define the *decay cone* of $\mathcal{F}$ at the optimal point $(z^\diamond, v^\diamond)$ as

$$\mathcal{K}_0 := \mathcal{K}_d(\mathcal{F}, (z^\diamond, v^\diamond)) = \{ (h, w) \in \mathbb{E} : \mathcal{F}'(z^\diamond, v^\diamond)(h, w) < 0 \}.$$

Assume temporarily that $\mathcal{K}_0 \neq \emptyset$. Its dual cone is

$$\mathcal{K}_0^+ = \{ -\alpha \mathcal{F}'(z^\diamond, v^\diamond) : \alpha \geq 0 \}.$$

Thus, for every $\ell_0 \in \mathcal{K}_0^+$, there exists $\alpha \geq 0$ such that for all $(h, w) \in \mathbb{E}$:

$$\begin{aligned} \ell_0(h, w) = & -\alpha \left[\partial_1 F_0(z^\diamond(0), z^\diamond(T)) h(0) + \partial_2 F_0(z^\diamond(0), z^\diamond(T)) h(T) \right] \\ & - \alpha \int_0^T \left[L_z(z^\diamond(t), v^\diamond(t), t) h(t) + L_v(z^\diamond(t), v^\diamond(t), t) w(t) \right] dt. \end{aligned}$$

(b) Analysis of the constraint $\mathcal{Q}_1$.

We now compute the tangent cone to $\mathcal{Q}_1$ at the optimal point $(z^\diamond, v^\diamond)$, where $\mathcal{Q}_1$ encodes the continuous dynamics (3) and the impulsive relations (5). Define

$$\mathcal{K}_1 := \mathcal{K}_T(\mathcal{Q}_1, (z^\diamond, v^\diamond)).$$

To apply Lyusternik's theorem, we introduce the product space

$$\mathbb{E}_1 = \mathcal{PC}([0, T]; \mathbb{R}^m) \times L^\infty([0, T]; \mathbb{R}^\ell),$$

and the space collecting the residuals of the dynamics, impulse constraints, and terminal condition:

$$\mathbb{E}_2 = \mathcal{PC}([0, T]; \mathbb{R}^m) \times \mathbb{R}^{mN} \times \mathbb{R}^r.$$

Define the operator

$$\Pi : \mathbb{E}_1 \longrightarrow \mathbb{E}_2$$

by

$$\Pi(z, v) = \left(\Pi_c(z, v), \Pi_{\text{imp}}(z, v), C(z(T)) \right),$$

where

$$\Pi_c(z, v)(t) := z(t) - z_* - \int_0^t H(z(s), v(s), s) ds,$$

$$\Pi_{\text{imp}}(z, v) := (z(\sigma_1^+) - \Gamma_1(z(\sigma_1^-)), \dots, z(\sigma_N^+) - \Gamma_N(z(\sigma_N^-))).$$

Thus, $\mathcal{Q}_1 = \Pi^{-1}(0)$.

The Fréchet derivative at the nominal point $(z^\diamond, v^\diamond)$ is

$$\Pi'(z^\diamond, v^\diamond)(h, w) = \left(\Pi'_c(h, w), \Pi'_{\text{imp}}(h, w), \Lambda h(T) \right),$$

where $\Lambda = C'(z^\diamond(T))$, and

$$\Pi'_c(h, w)(t) = h(t) - \int_0^t (H_z(z^\diamond(s), v^\diamond(s), s) h(s) + H_v(z^\diamond(s), v^\diamond(s), s) w(s)) ds,$$

$$\Pi'_{\text{imp}}(h, w) = (h(\sigma_1^+) - \Gamma'_1(z^\diamond(\sigma_1^-)) h(\sigma_1^-), \dots, h(\sigma_N^+) - \Gamma'_N(z^\diamond(\sigma_N^-)) h(\sigma_N^-)).$$

To apply Theorem 5, we must check the surjectivity of $\Pi'(z^\diamond, v^\diamond)$. Given arbitrary data

$$(a(\cdot), b_1, \dots, b_N, z_1) \in \mathbb{E}_2,$$

we must find a pair $(h, w) \in \mathbb{E}_1$ satisfying

$$\Pi'(z^\diamond, v^\diamond)(h, w) = (a(\cdot), b_1, \dots, b_N, z_1).$$

First solve the Volterra equation

$$q(t) = a(t) + \int_0^t H_z(z^\diamond(s), v^\diamond(s), s) q(s) ds,$$

which admits a unique solution $q \in \mathcal{PC}([0, T]; \mathbb{R}^m)$.

Let the impulsive mismatches be

$$\beta_k := b_k - q(\sigma_k^+) + \Gamma'_k(z^\diamond(\sigma_k^-)) q(\sigma_k^-).$$

Assuming (as required in the theorem) that the linear variational impulsive system is controllable, there exists a control correction $w(\cdot)$ such that the corresponding solution $y(t)$ of the linear system

$$\dot{y}(t) = H_z(z^\diamond(t), v^\diamond(t), t) y(t) + H_v(z^\diamond(t), v^\diamond(t), t) w(t),$$

with impulses

$$y(\sigma_k^+) = \Gamma'_k(z^\diamond(\sigma_k^-)) y(\sigma_k^-) + \beta_k,$$

satisfies

$$y(0) = 0, \quad y(T) = \Lambda^\top (\Lambda \Lambda^\top)^{-1} z_1 - q(T).$$

Define

$$h := y + q.$$

Then one verifies that

$$\Pi'_c(h, w) = a(\cdot), \quad \Pi'_{\text{imp}}(h, w) = (b_1, \dots, b_N), \quad \Lambda h(T) = z_1.$$

Hence, $\Pi'(z^\diamond, v^\diamond)$ is surjective. By Lyusternik's theorem,

$$\mathcal{K}_1 = \ker \Pi'(z^\diamond, v^\diamond).$$

That is, $(h, w) \in \mathcal{K}_1$ if and only if

$$\begin{cases} \dot{h}(t) = H_z(z^\diamond(t), v^\diamond(t), t) h(t) + H_v(z^\diamond(t), v^\diamond(t), t) w(t), \\ h(\sigma_k^+) = \Gamma'_k(z^\diamond(\sigma_k^-)) h(\sigma_k^-), \quad k = 1, \dots, N, \\ \Lambda h(T) = 0. \end{cases}$$

Define the linear subspaces

$$L_1 = \{(h, w) \in \mathbb{E}_1 : \text{the homogeneous linear system above holds}\}, \quad L_2 = \{(h, w) : \Lambda h(T) = 0\}.$$

Then

$$\mathcal{K}_1 = L_1 \cap L_2.$$

The dual cone is

$$\mathcal{K}_1^+ = L_1^+ + L_2^+.$$

Since L_1 is a linear subspace, any functional $\ell_1 \in L_1^+$ vanishes on all (h, w) that satisfy the linear dynamics. Then $\ell_2 \in \mathcal{K}_1^+$ if and only if there is $\xi \in \mathbb{R}^r$ such that

$$\ell_2(h, w) = \langle \xi, \Lambda h(T) \rangle = \langle \Lambda^\top \xi, h(T) \rangle.$$

Thus, every dual element in $\mathcal{K}_1^+$ is the sum of a terminal multiplier term and a functional annihilating all homogeneous state-control variations.

(c) Analysis of the control constraint $\mathcal{Q}_2$.

The set of admissible controls is

$$\mathcal{Q}_2' := \left\{ v \in L^\infty([0, T]; \mathbb{R}^\ell) \mid v(t) \in \mathcal{U} \text{ for a.e. } t \in [0, T] \right\},$$

where the control constraint set $\mathcal{U} \subset \mathbb{R}^\ell$ is, by hypothesis, nonempty, convex, and closed and also satisfies $\text{int}(\mathcal{U}) \neq \emptyset$.

We define

$$\mathcal{Q}_2 := \mathcal{PC}([0, T]; \mathbb{R}^m) \times \mathcal{Q}_2'.$$

Because $\mathcal{U}$ is convex with a nonempty interior, both $\mathcal{Q}_2$ and $\mathcal{Q}_2'$ are convex and closed subsets of their respective spaces, and both have a nonempty interior.

Let

$$\mathcal{K}_2 := \mathcal{K}_a(\mathcal{Q}_2, (z^\diamond, v^\diamond))$$

be the admissible cone at the optimal pair. Since the state variable $z(\cdot)$ is unrestricted in $\mathcal{Q}_2$, the cone splits as

$$\mathcal{K}_2 = \mathcal{PC}([0, T]; \mathbb{R}^m) \times \mathcal{K}_2',$$

where

$$\mathcal{K}_2' := \mathcal{K}_a(\mathcal{Q}_2', v^\diamond)$$

is the admissible cone of the control constraint.

Consequently, any dual element

$$\ell_2 \in \mathcal{K}_2^+$$

acts only on the control component, i.e.,

$$\ell_2(h, w) = \ell_2'(w), \quad \ell_2' \in (\mathcal{K}_2')^+.$$

By Theorem 4, every functional $\ell_2' \in (\mathcal{K}_2')^+$ is a support functional of $\mathcal{Q}_2'$ at $v^\diamond$. Thus,

$$\ell_2'(w - v^\diamond) \geq 0 \quad \text{for all } w \in \mathcal{Q}_2'.$$

In other words, ℓ_2' defines the supporting hyperplane to the control constraint set $\mathcal{Q}_2'$ at the point $v^\diamond$. This property will later yield Pontryagin's maximum principle.

(d) Analysis of the state constraint $\mathcal{Q}_3$.

We introduce the auxiliary functional

$$\begin{aligned} \ell : \mathcal{PC}([0, T]; \mathbb{R}^m) &\longrightarrow \mathbb{R} \\ \ell(z) &:= \max_{k=1, \dots, N} \max_{t \in \bar{I}_k} \phi(\tilde{z}_k(t), t), \end{aligned}$$

which is continuous by construction. Following the arguments of Examples 7.4–7.5 in [13], its directional derivative at $z^\diamond$ satisfies

$$\ell'(z^\diamond; h) = \max_{k \in \mathcal{I}} \max_{t \in \mathcal{A}_k} \langle \phi_z(z^\diamond(t), t), h(t) \rangle, \quad h \in \mathcal{PC}([0, T]; \mathbb{R}^m),$$

where:

$$\mathcal{I} = \left\{ k : \max_{t \in \bar{I}_k} \phi(\tilde{z}_k^\diamond(t), t) = \max_j \max_{t \in \bar{I}_j} \phi(\tilde{z}_j^\diamond(t), t) \right\},$$

$$\mathcal{A}_k = \left\{ t \in \bar{I}_k : \phi(\tilde{z}_k^\diamond(t), t) = \ell(z^\diamond) \right\}.$$

Since along the optimal process $\phi(z^\diamond(t), t) \leq 0$ we have

$$\mathcal{A}_k = \left\{ t \in \bar{I}_k : \phi(z^\diamond(t), t) = 0 \right\}.$$

Thus, the active set of times is

$$\mathcal{A} = \bigcup_{k=1}^N \mathcal{A}_k = \{ t \in [0, T] : \phi(z^\diamond(t), t) = 0 \}.$$

On the other hand,

$$\mathcal{K}_3 = \mathcal{K}_a(\mathcal{Q}_3, (z^\diamond, v^\diamond)) \supset \mathcal{K}_d(\ell, z^\diamond) =: \mathcal{K}_d, \quad (11)$$

and

$$\mathcal{K}_d = \left\{ (h, w) \in \mathbb{E} : \ell'(z^\diamond; h) < 0 \right\} = \left\{ (h, w) \in \mathbb{E} : \langle \phi_z(z^\diamond(t), t), h(t) \rangle < 0 \text{ for all } t \in \mathcal{A} \right\}.$$

From (11) we obtain

$$\mathcal{K}_3^+ \subset \mathcal{K}_d^+.$$

Therefore, from Proposition 3, for every dual element $\ell_3 \in \mathcal{K}_3^+$, there exists a non-negative Borel measure

$$\nu \quad \text{on } [0, T],$$

such that

$$\ell_3(z, v) = - \int_0^T \langle \phi_z(z^\diamond(t), t), z(t) \rangle d\nu(t), \quad (z, v) \in \mathbb{E},$$

and the measure is supported on the active set:

$$\text{supp}(\nu) \subset \mathcal{A} = \{ t \in [0, T] \mid \phi(z^\diamond(t), t) = 0 \}.$$

Pontryagin's maximum principle:

It is easy to see that $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are convex cones. Hence, by Theorem 3, there exist functionals $\ell_i \in \mathcal{K}_i^+$ for $i = 0, 1, 2, 3$, not all zero, such that

$$\ell_0 + \ell_1 + \ell_2 + \ell_3 = 0. \quad (12)$$

The Euler–Lagrange identity (12) now becomes

$$\begin{aligned} & -\alpha \left\{ \partial_1 F_0(z^\diamond(0), z^\diamond(T)) z(0) + \partial_2 F_0(z^\diamond(0), z^\diamond(T)) z(T) \right\} \\ & -\alpha \int_0^T [L_z(z^\diamond(t), v^\diamond(t), t) z(t) + L_v(z^\diamond(t), v^\diamond(t), t) v(t)] dt \\ & + \ell_1(z, v) + \langle a, \Lambda z(T) \rangle + \ell'_2(v) + \ell_3(z, v) = 0, \quad ((z, v) \in E), \end{aligned}$$

where

- $\ell_1 \in K_1^+$ corresponds to the dynamics constraint,
- $\ell'_2 \in (K'_2)^+$ corresponds to the control constraint set $\mathcal{Q}'_2$,
- $\ell_3 \in K_3^+$ corresponds to the path constraint $\phi(z(t), t) \leq 0$,
- ν is the Borel measure associated with that constraint,
- α is the scalar multiplier of the cost functional,
- $\Lambda = C'(z^\diamond(T))$ is the Jacobian of the terminal constraint.

For every $v \in L^\infty([0, T]; \mathbb{R}^\ell)$, the linearized system admits a solution $z \in \mathcal{PW}([0, T]; \mathbb{R}^m)$ with $z(0) = 0$. Thus, $(z, v) \in L_1$, and therefore

$$\ell_1(z, v) = 0.$$

Hence, the Euler–Lagrange identity reduces to

$$\begin{aligned} \ell'_2(v) &= \alpha \int_0^T L_z(z^\diamond(t), v^\diamond(t), t) z(t) dt + \alpha \int_0^T L_v(z^\diamond(t), v^\diamond(t), t) v(t) dt \\ &+ \alpha \partial_2 F_0(z(0), z^\diamond(T)) z(T) - \langle a, \Lambda z(T) \rangle \\ &+ \int_0^T \phi_z(z^\diamond(\sigma), \sigma) z(\sigma) d\nu(\sigma), \end{aligned}$$

for all $(z, v) \in E$.

Let p solve the adjoint integral equation:

$$\begin{aligned} -p(t) &= \int_t^T \left(-H_z^\top(z^\diamond(\sigma), v^\diamond(\sigma), \sigma) p(\sigma) + \alpha L_z(z^\diamond(\sigma), v^\diamond(\sigma), \sigma) \right) d\sigma \\ &- \Lambda^\top \xi + \alpha \partial_2 F_0(z^\diamond(0), z^\diamond(T)) + \int_t^T \phi_z(z^\diamond(\sigma), \sigma) d\nu(\sigma). \end{aligned}$$

This is a second-kind Volterra equation with unique solution $p \in L^1([0, T]; \mathbb{R}^m)$ (see [14] (p. 519)). Multiplying both sides of this equation by $\dot{z}$ and integrating over $[0, T]$, we obtain the following:

$$\begin{aligned} -\int_0^T \langle \dot{z}(t), p(t) \rangle dt &= -\int_0^T \langle \Lambda^\top \xi, \dot{z}(t) \rangle dt + \alpha \partial_2 F_0(z^\diamond(0), z^\diamond(T)) z(T) \\ &+ \int_0^T \left\langle \dot{z}(t), \int_t^T [-H_z^\top(z^\diamond, v^\diamond, \tau) p(\tau) + \alpha L_z(z^\diamond, v^\diamond, \tau)] d\tau \right\rangle dt \\ &+ \int_0^T \left\langle \dot{z}(t), \int_t^T \phi_z(z^\diamond(\tau), \tau) d\nu(\tau) \right\rangle dt. \end{aligned}$$

Given that $\dot{z}(t) = H_z(z^\diamond(t), v^\diamond(t), t) z(t) + H_v(z^\diamond(t), v^\diamond(t), t) v(t)$ and $z(0) = 0$, we get:

$$\langle \dot{z}(t) - H_v(z^\diamond(t), v^\diamond(t), t) v(t), p(t) \rangle = \langle H_z(z^\diamond(t), v^\diamond(t), t) z(t), p(t) \rangle.$$

Hence, the above becomes:

$$\begin{aligned} - \int_0^T \langle \dot{z}(t), p(t) \rangle dt &= - \langle \xi, \Lambda z(T) \rangle - \int_0^T \langle \dot{z}(t), p(t) \rangle dt + \alpha \partial_2 F_0(z^\diamond(0), z^\diamond(T)) z(T) \\ &\quad + \int_0^T \langle H_v^\top(z^\diamond(t), v^\diamond(t), t) p(t), (t) \rangle dt \\ &\quad + \alpha \int_0^T \langle z(t), L_x(z^\diamond, v^\diamond, t) \rangle dt + \int_0^T \langle \dot{z}(t), \int_t^T \phi_z(z^\diamond(\tau), \tau) dv(\tau) \rangle dt. \end{aligned}$$

Using Stieltjes integration by parts and the fact that $v(0) = v(T) = 0$, we obtain:

$$\int_0^T \langle \dot{z}(t), \int_t^T L_z(z^\diamond(\tau), \tau) dv(\tau) \rangle dt = \int_0^T \phi_z(z^\diamond(t), t) z(t) dv(t).$$

Then we get:

$$\begin{aligned} &\alpha \int_0^T L_x(z^\diamond(t), v^\diamond(t), t) z(t) dt + \int_0^T \phi_z(z^\diamond(t), t) z(t) dv(t) \\ &- \langle \xi, \Lambda z(T) \rangle + \alpha \partial_2 F_0(z^\diamond(0), z^\diamond(T)) = - \int_0^T \langle H_v^\top(z^\diamond, v^\diamond, t) p(t), u(t) \rangle dt. \end{aligned} \quad (13)$$

Then, the support functional ℓ'_2 becomes:

$$\ell'_2(v) = \int_0^T \left\langle -H_v^\top(z^\diamond(t), v^\diamond(t), t) p(t) + \alpha L_v(z^\diamond(t), v^\diamond(t), t), v(t) \right\rangle dt. \quad (14)$$

Since $\ell'_2 \in (K'_2)^+$, for almost all $t \in [0, T]$ and all $\omega \in \mathcal{U}$, we have

$$\left\langle -H_v^\top(z^\diamond(t), v^\diamond(t), t) p(t) + \alpha L_v(z^\diamond(t), v^\diamond(t), t), \omega - v^\diamond(t) \right\rangle \geq 0.$$

To exclude the trivial case, assume $\alpha = 0$ and $p = 0$. Then $\Lambda^\top \xi = p(T) = 0$; hence,

$$\ell_{12}(z, v) = \langle \Lambda^\top \xi, z(T) \rangle = 0.$$

The adjoint equation implies $\ell_3 = 0$. Equation (14) gives $\ell'_2 = 0$. Thus, $\ell_1 = \ell_{11} + \ell_{12} = 0$, contradicting Theorem 3.

Finally, we show that neither the assumption $K_0 \neq \emptyset$ nor controllability of the linearized system is essential, which completes the proof.

Hence, the proof of Theorem 1 is complete. $\square$

5. Sufficient Condition of Optimality

The necessary condition of optimality proved in Theorem 1 (Maximum Principle), under certain additional conditions, is also sufficient. In fact, let us consider the particular case of Problem 1 in which the differential equation is linear.

Problem 2.

$$F_0(z(0), z(T)) + \int_0^T L(z(t), v(t), t) dt \longrightarrow \text{loc min}, \quad (15)$$

$$(z, v) \in \mathcal{PW}([0, T]; \mathbb{R}^m) \times L^\infty([0, T]; \mathbb{R}^\ell)$$

$$\dot{z}(t) = A(t)z(t) + B(t)v(t), \quad z(0) = z_0, \quad (16)$$

$$\Lambda z(T) = 0, \quad (17)$$

$$z(\sigma_k^+) = z(\sigma_k^-) + \Gamma_k(z(\sigma_k)), \quad k = 1, \dots, N, \quad (18)$$

$$v(t) \in \mathcal{U}, \quad t \in [0, T], \quad (19)$$

$$L(z(t), t) \leq 0, \quad t \in [0, T]. \quad (20)$$

Here:

- $z(\cdot) \in \mathcal{PW}([0, T]; \mathbb{R}^m)$ is the piecewise absolutely continuous state trajectory.
- $A(\cdot)$ and $B(\cdot)$ are continuous matrix-valued functions.
- $\Gamma_k : \mathbb{R}^m \rightarrow \mathbb{R}^m$ are the impulse mappings at times σ_k .
- $\Lambda \in \mathbb{R}^{r \times m}$ is a full-rank matrix defining the terminal constraints.
- $\phi(z, t) \leq 0$ is the state constraint, compatible with the set

$$\mathcal{A} = \{ t \in [0, T] : \phi(z^\diamond(t), t) = 0 \}.$$

Let $(z^\diamond, v^\diamond)$ be an optimal pair satisfying (16)–(20).

The proof of the following theorem proceeds in the same way as the proof of Theorem 8 from [15].

Theorem 6. Assume that the conditions (a)–(e) of Theorem 1 (the maximum principle) hold for the pair $(z^\diamond, v^\diamond)$.

In addition:

(I) Suppose the linear system

$$\dot{z}(t) = A(t)z(t) + B(t)v(t)$$

is controllable.

(II) Suppose there exists a control $\tilde{v} \in L^\infty([0, T]; \mathbb{R}^\ell)$ such that

$$\tilde{v}(t) \in \text{int}(\mathcal{U}), \quad \forall t \in [0, T].$$

(III) Let $\tilde{z}$ denote the corresponding trajectory of the linear system associated with $\tilde{v}$. Then,

$$\tilde{z}(T) = z_1 \quad \text{and} \quad \phi(\tilde{z}(t), t) < 0, \quad \forall t \in [0, T].$$

(IV) The functions $F_0(\cdot, \cdot)$, $L(\cdot, \cdot, t)$, and $\phi(\cdot, t)$ are convex in the state and control variables.

Then the pair $(z^\diamond, v^\diamond)$ is a global minimizer of Problem 2.

6. Mathematical Models

In this section, we illustrate the applicability of the maximum principle obtained in Theorem 1 by analyzing a representative class of optimal control problems with impulsive dynamics. The following example, inspired by classical epidemic models, highlights how the abstract conditions of the theorem translate into a concrete and computable optimality system.

6.1. Optimal Vaccination Strategy in an Impulsive SIR Model

We consider a population undergoing an epidemic spread, where vaccination acts as the control strategy capable of mitigating the propagation of the infection. The state vector is

$$z(t) = \begin{pmatrix} S(t) \\ I(t) \\ R(t) \end{pmatrix},$$

where:

- $S(t)$ is the number of susceptible individuals,
- $I(t)$ is the number of infectious individuals,
- $R(t)$ represents recovered individuals, permanently removed from the pool of susceptibles.

The dynamics follow the standard nonlinear SIR interactions with a vaccination:

$$\dot{z}(t) = H(z(t), v(t), t) = \begin{pmatrix} -r S(t)I(t) + v(t) \\ r S(t)I(t) - \gamma I(t) - v(t) \\ \gamma I(t) \end{pmatrix}, \quad t \in [0, T],$$

where $r > 0$ is the transmission parameter and $\gamma > 0$ is the recovery rate. The vaccination rate $v(t)$ acts as the control and is restricted by

$$v(t) \in [0, e], \quad \text{a.e. } t \in [0, T].$$

Impulsive sanitary interventions are incorporated at predetermined instants $\{\sigma_k\}_{k=1}^N$:

$$z(\sigma_k^+) = z(\sigma_k^-) + \Gamma_k(z(\sigma_k^-)),$$

with

$$\Gamma_k(z) = \begin{pmatrix} \rho_{s,k} S \\ \rho_{i,k} I \\ \rho_{r,k} R \end{pmatrix}, \quad |\rho_{*,k}| \ll 1,$$

representing small, abrupt changes due to external containment policies.

The cost functional penalizes both the final number of infected individuals and the cumulative vaccination effort:

$$F_0(z(0), z(T)) = \frac{\alpha}{2} I(T), \quad L(z, v, t) = \frac{1}{2} v^2(t), \quad \alpha > 0.$$

The optimal control problem reads:

$$\frac{\alpha}{2} I(T) + \int_0^T \frac{1}{2} v^2(t) dt \longrightarrow \min_{\text{loc}},$$

subject to the above dynamics, impulses, the initial condition $z(0) = z_0$, and the control constraint $v(t) \in [0, e]$.

It is easy to verify that conditions (a)–(e) of Theorem 1 are satisfied.

The adjoint equation. Since the terminal state is free, then $\xi = 0$. Hence the adjoint variable $p(\cdot)$ satisfies

$$\dot{p}(t) = -H_z^\top(z^\diamond(t), v^\diamond(t), t) p(t), \quad p(T) = -\partial_2 F_0(z^\diamond(0), z^\diamond(T)) = -\begin{pmatrix} 0 \\ \alpha/2 \\ 0 \end{pmatrix}.$$

The Jacobian matrix of H with respect to the state variables is

$$H_z(z, v, t) = \begin{pmatrix} -rI & -rS & 0 \\ rI & rS - \gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}.$$

Therefore, the adjoint equation becomes

$$\dot{p}(t) = \begin{pmatrix} rI^\diamond(t) (p_1(t) - p_2(t)) \\ rS^\diamond(t) (p_1(t) - p_2(t)) + \gamma(p_2(t) - p_3(t)) \\ 0 \end{pmatrix}.$$

Maximum principle: Since

$$H_v(z, v, t) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad L_v(x^\diamond(t), v^\diamond(t), t) = v^\diamond(t)$$

the maximum principle of Theorem 1 becomes:

$$\langle p_2(t) - p_1(t) + v^\diamond(t), \omega - v^\diamond(t) \rangle \geq 0, \quad \forall t \in [0, T], \quad \text{a.e.}$$

This is equivalent to

$$\max_{v \in [0, e]} (p_1(t) - p_2(t) - v^\diamond(t))v = (p_1(t) - p_2(t) - v^\diamond(t))v^\diamond(t).$$

Then, the optimal control is given by

$$v^\diamond(t) = \begin{cases} 0, & p_1(t) - p_2(t) < 0, \\ p_1(t) - p_2(t), & 0 \leq p_1(t) - p_2(t) \leq e, \\ e, & p_1(t) - p_2(t) > e. \end{cases}$$

Since $p_2(T) = -\alpha/2$ and $p_1(T) = 0$, we obtain $p_1(T) - p_2(T) = \alpha/2$. Thus, whenever $e < \alpha/2$, the optimal control saturates near the terminal time:

$$v^\diamond(t) = e \quad \text{for } t \approx T.$$

Remark 4. A natural epidemiological requirement is to prevent the infectious class from exceeding the susceptible population. This motivates the state constraint

$$\phi(z(t), t) = I(t) - S(t) \leq 0.$$

Under this additional restriction, the multiplier associated with L is a non-negative Borel measure supported on the contact set $\{t : I(t) = S(t)\}$, as stated in Theorem 1. The adjoint equation acquires an additional forcing term. Thus, the adjoint equation is given via a Borel measure ν as in Theorem 1.

$$\begin{aligned}
-p(t) &= \int_t^T -H_z^\top(z^\diamond(\tau), v^\diamond(\tau), \tau)p(\tau)d\tau + \partial_2 F_0(z^\diamond(0), z^\diamond(T)) + \int_t^T \phi_z(z^\diamond(\tau), \tau)d\nu(\tau) \\
&= \int_t^T -H_z^\top(z^\diamond(\tau), v^\diamond(\tau), \tau)p(\tau)d\tau + \begin{pmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \end{pmatrix} + \int_t^T \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} d\nu(\tau) \\
&= \int_t^T -H_z^\top(z^\diamond(\tau), v^\diamond(\tau), \tau)p(\tau)d\tau + \begin{pmatrix} 0 \\ \frac{\alpha}{2} \\ 0 \end{pmatrix} + \nu([T, t]) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.
\end{aligned}$$

The maximum condition remains unchanged.

6.2. Optimal Control at the Onset of a New Viral Outbreak

We now examine a second epidemic model, which describes the early phase of a newly emerging viral outbreak. The model follows the classical SIR structure but incorporates vaccination as a bounded control, together with impulsive effects that reflect sudden population-level interventions. Our analysis reformulates the system within the framework of Theorem 1, allowing us to obtain the optimality system in a unified notation.

The state vector is

$$z(t) = \begin{pmatrix} S(t) \\ I(t) \\ R(t) \end{pmatrix},$$

where S , I , and R denote susceptible, infectious, and recovered individuals, respectively. The transmission and recovery parameters are $\rho > 0$ and $\gamma > 0$. The vaccination input $v(t)$ satisfies

$$0 \leq v(t) \leq 1, \quad \text{a.e. } t \in [0, T].$$

The controlled dynamics are

$$\dot{z}(t) = H(z(t), v(t), t) = \begin{pmatrix} -\rho(1-v(t))S(t)I(t) \\ \rho(1-v(t))S(t)I(t) - \gamma I(t) \\ \gamma I(t) \end{pmatrix}, \quad z(0) = z_0.$$

Impulsive interventions appear at instants $\{\sigma_k\}_{k=1}^N$:

$$z(\sigma_k^+) = z(\sigma_k^-) + \Gamma_k(z(\sigma_k^-)),$$

with

$$\Gamma_k(z) = \begin{pmatrix} \rho_{s,k}S \\ \rho_{i,k}I \\ \rho_{r,k} \end{pmatrix}, \quad |\rho_{*,k}| \ll 1,$$

representing small corrections from sudden policies.

The performance index balances the reduction of susceptibles with the overall intervention cost:

$$F_0(z(0), z(T)) = S(0) - S(T).$$

Therefore, we seek

$$S(0) - S(T) + \int_0^T L(z(t), v(t), t) dt \longrightarrow \min_{\text{loc}}.$$

Adjoint dynamics

Since no terminal constraint of the form $Cz(T) = 0$ is imposed, the adjoint terminal condition is simply

$$p(T) = -\partial_2 F_0(z(0), z(T)) = -\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The Jacobian $H_z(z^\diamond(t), v^\diamond(t), t)$ equals

$$\begin{pmatrix} \rho(1-v^\diamond)I^\diamond & -\rho(1-v^\diamond)I^\diamond & 0 \\ \rho(1-v^\diamond)S^\diamond & -\rho(1-v^\diamond)S^\diamond + \gamma & -\gamma \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the adjoint system in Theorem 1 becomes

$$\dot{p}(t) = -H_z^\top(z^\diamond(t), v^\diamond(t), t) p(t) = \begin{pmatrix} \rho(1-v^\diamond)I^\diamond(p_1 - p_2) \\ \rho(1-v^\diamond)S^\diamond(p_1 - p_2) + \gamma(p_2 - p_3) \\ 0 \end{pmatrix}.$$

Maximum condition

The gradient of H with respect to the control is

$$H_v(z, v, t) = \begin{pmatrix} \rho SI \\ -\rho SI \\ 0 \end{pmatrix}.$$

Hence the Pontryagin inequality from Theorem 1 reads

$$\langle -H_v^\top(z^\diamond, v^\diamond, t) p(t) + L_v(z^\diamond, v^\diamond, t), \omega - v^\diamond(t) \rangle \geq 0.$$

Then, for all $\omega \in [0, 1]$ we get

$$\langle \rho S^\diamond(t) I^\diamond(t) (p_2(t) - p_1(t)) + L_u(z^\diamond(t), v^\diamond(t), t), \omega - v^\diamond(t) \rangle \geq 0, \quad \forall t \in [0, T], \quad \text{a.e.}$$

This is equivalent to:

$$\max_{\omega \in [0, 1]} (\rho S^\diamond(t) I^\diamond(t) (p_1(t) - p_2(t)) - L_v(z^\diamond(t), v^\diamond(t), t) \omega) = (\rho S^\diamond(t) I^\diamond(t) (p_1(t) - p_2(t)) - L_v(z^\diamond(t)) v^\diamond(t)).$$

Different choices of L yield different explicit control laws:

(i) If

$$L(z, v, t) = c v, \quad c > 0,$$

then $L_v = c$ and

$$v^\diamond(t) = \begin{cases} 0, & \rho S^\diamond I^\diamond(p_1 - p_2) - c < 0, \\ 1, & \rho S^\diamond I^\diamond(p_1 - p_2) - c \geq 0. \end{cases}$$

(ii) If

$$L(z, v, t) = \frac{1}{2} v^2,$$

then $L_v = v^\diamond(t)$ and we get that

$$v^\diamond(t) = \begin{cases} 0, & \rho S^\diamond I^\diamond(p_1 - p_2) < 0, \\ \rho S^\diamond I^\diamond(p_1 - p_2), & 0 \leq \rho S^\diamond I^\diamond(p_1 - p_2) \leq 1, \\ 1, & \rho S^\diamond I^\diamond(p_1 - p_2) > 1. \end{cases}$$

The model therefore fits naturally into the impulsive optimal control framework of Theorem 1, and the optimal strategy emerges directly from the structure of the adjoint dynamics.

6.3. A Bolza Problem in Economics

We conclude this section with a classical intertemporal consumption model, which fits naturally into the impulsive optimal control framework developed earlier. An economic agent receives an income flow $r(t)$ on the interval $[0, T]$ and at each instant allocates a quantity $v(t)$ to consumption, while the remaining wealth accumulates at a constant interest rate $\sigma > 0$. The control is constrained by

$$0 \leq v(t) \leq \rho, \quad \text{a.e. } t \in [0, T].$$

The state variable $z(t) \in \mathbb{R}$ represents the wealth level and evolves according to

$$\dot{z}(t) = H(z(t), v(t), t) = r(t) + \sigma z(t) - v(t), \quad z(0) = z_0.$$

The performance index is of Bolza type:

$$F_0(z(0), z(T)) = 0, \quad L(z, v, t) = -e^{-\alpha t} \ln v,$$

so the objective becomes

$$\int_0^T -e^{-\alpha t} \ln(v(t)) dt \longrightarrow \min,$$

equivalently,

$$\int_0^T \ln(v(t)) e^{-\alpha t} dt \longrightarrow \max.$$

A natural feasibility requirement is that wealth remains non-negative at the final time. This is encoded as a state constraint:

$$\phi(z(T), T) = -z(T) \leq 0.$$

Thus, the admissible set for z and v is

$$\mathcal{Q}_\phi = \{(z, v) : z(T) \geq 0\}, \quad \mathcal{U} = [0, \rho].$$

Let $(z^\diamond, v^\diamond)$ denote an optimal pair. We distinguish two structurally different cases, which correspond to the activation or nonactivation of the terminal constraint $z(T) \geq 0$.

- (a) Suppose that $z(T) > 0$. Then, $\mathcal{Q}_\phi = \{(z, v) \in E : z(T) > 0\}$ is an open set, which implies that the dual cone of the admissible direction is just the zero functional. Therefore, $\xi = 0$ in Theorem 1.3. Hence, the adjoint equation is given by

$$\begin{aligned} \dot{p}(t) &= -\Psi_x^*(x^\diamond(t), u^\diamond(t), t)p(t) - \beta_0, \\ p(T) &= 0, \end{aligned}$$

with $\beta_0 \geq 0$, i.e.,

$$\begin{aligned}\dot{p}(t) &= \tau p(t) - \beta_0, \\ p(T) &= 0.\end{aligned}$$

Then, $p(t) = \frac{\beta_0}{\tau}(e^{\tau(T-t)} - 1)$. Then, applying Pontryagin's maximum principle, for all $u \in M$, we get that

$$\langle p(t) - \frac{1}{u^\diamond(t)} e^{-\alpha t}, u - u^\diamond(t) \rangle \geq 0, \quad \forall t \in [0, T], \quad \text{a.e.}$$

This is equivalent to:

$$\max_{u \in [0, \rho]} (e^{-\alpha t} - p(t)u^\diamond(t))u = (e^{-\alpha t} - p(t)u^\diamond(t))u^\diamond(t).$$

Then, the optimal control is given by

$$u^\diamond(t) = \begin{cases} 0, & \text{if } (e^{-\alpha t} - p(t)u^\diamond(t)) \leq 0, \\ \rho, & \text{if } (e^{-\alpha t} - p(t)u^\diamond(t)) > 0. \end{cases}$$

Note that if $\beta_0 = 0$, then $u^\diamond(t) = \rho, \quad \forall t \in [0, T]$.

(b) Suppose that $x(T) = 0, \quad \beta_0 = 0, \quad \alpha_0 = 1, \quad \xi = 1$ and $\rho \geq 1$. Then

$$\begin{aligned}\dot{p}(t) &= \tau p(t), \\ p(T) &= 1,\end{aligned}$$

and $p(t) = e^{\tau(T-t)}$. This is equivalent to:

$$\max_{u \in [0, \rho]} (e^{-\alpha t} - e^{\tau(T-t)}u^\diamond(t))u = (e^{-\alpha t} - e^{\tau(T-t)}u^\diamond(t))u^\diamond(t).$$

Then, the optimal control is given by

$$u^\diamond(t) = e^{-\alpha t - \tau(T-t)}, \quad t \in [0, \rho].$$

Remark 5. In each example, we derive an explicit formula for the optimal control, which naturally facilitates the development of numerical studies. The approximate computation of optimal controls based on these formulas constitutes a promising direction for future research in numerical analysis.

6.4. Perspectives on Numerical Implementation

The purpose of this section is not to provide a full numerical study but to outline how the optimality conditions derived in this paper can be used as a starting point for computational investigations. A natural numerical approach consists of partitioning the interval $[0, T]$ into subintervals that include the impulse times and applying standard quadrature schemes on each continuous segment while explicitly accounting for the jump terms at impulse instants.

The maximum principle can be implemented using a projected gradient or shooting method, where the control is updated pointwise based on the adjoint variable and then projected onto the admissible set. The impulse contributions can be treated as boundary conditions in the backward integration of the adjoint equation.

Importantly, the functional formulation of the adjoint equation avoids measure-valued multipliers and replaces them with explicitly computable integrals. This significantly simplifies numerical approximation compared to classical approaches based on Borel measures and distributions.

A detailed numerical study, including convergence analysis and implementation for large-scale systems, is left for future work and constitutes a natural continuation of the present research.

7. Conclusions

In this work, we have developed a comprehensive extension of Pontryagin's maximum principle to optimal control problems that involve both impulsive actions and state constraints. The impulsive nature of the dynamics required us to work in spaces of piecewise continuous functions, where many classical tools of functional analysis and measure theory do not apply directly. By constructing an appropriate conic variational framework, based on the Dubovitskii–Milyutin theory, we were able to overcome these difficulties and produce a unified set of necessary conditions for optimality.

A key contribution of this paper is the adaptation of the classical existence theorem for non-negative Borel measures to settings in which the underlying domain is divided by impulse times. This extension leads naturally to an adjoint relation formulated as a Stieltjes integral, providing an appropriate dual description of variational information in impulsive systems. Our results also generalize and complement fundamental ideas originally developed by I. Girsanov, showing that their framework remains applicable when extended to piecewise continuous trajectories and impulsive constraints.

The conic approach adopted here proves flexible enough to incorporate general state constraints without imposing restrictive conditions on the admissible trajectories. Thus, our formulation bridges a gap in the literature, where the combined treatment of impulses and state inequalities has received comparatively little attention. Moreover, our results extend previous studies that either excluded state constraints or relied on more rigid assumptions.

Finally, the applicability of the theory was demonstrated through two illustrative models: a modified SIR epidemiological system and the classical Bolzano economic problem. These examples highlight the range and effectiveness of the proposed method and suggest several directions for further research, including sufficiency conditions, numerical implementations, and the study of impulsive systems with more complex constraint structures.

Author Contributions: H.L., conceptualization, investigation, formal analysis, drafting the manuscript, and writing—review and editing. M.N.E., investigation, formal analysis, and writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Funding: No external funding was received for this work.

Data Availability Statement: This is a theoretical study; no external datasets were used. All findings are presented within the manuscript.

Acknowledgments: We would like to thank Yachay Tech University Florida A and M University for the support provided for the completion of this work.

Conflicts of Interest: The authors declare no conflicts of interest.

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