

Article

On Expectation Measures for Failure Processes in Multiple Populations: Mathematical Theory and Applications on Two Lines

Rashad M. EL-Sagheer ¹, Mohamed F. Abouelenein ^{2,*}, Mohamed H. El-Menshawey ³ and Mahmoud M. Ramadan ⁴

¹ High Institute of Computers and Management Information Systems, First Statement, New Cairo 11865, Egypt; rashadmath27@gmail.com

² Department of Insurance and Risk Management, College of Business, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia

³ Mathematics Department, Faculty of Science, Al-Azhar University, Cairo 11884, Egypt; m.h.elmenshawey@azhar.edu.eg

⁴ Department of Mathematics, Faculty of Education, Ain Shams University, Cairo 11341, Egypt; mahmoudramadan@edu.asu.edu.eg

* Correspondence: mabouelenein@imamu.edu.sa

Abstract

This paper develops classical and Bayesian inferential procedures for Weibull exponential lifetime models under joint progressive Type-II censoring, motivated by comparative reliability analysis of products manufactured across multiple production lines. The theoretical framework is formulated for a general setting involving k independent Weibull exponential populations, allowing for flexible modeling of heterogeneous lifetime behaviors under a common censoring scheme. Maximum likelihood estimators and their asymptotic confidence intervals are derived, and Bayesian estimation is conducted using Markov chain Monte Carlo methods under both squared-error and LINEX loss functions. For numerical illustration and practical interpretability, the primary emphasis of the simulation study, expected-failure analysis, and real-data applications is placed on the two-population case ($k = 2$), which commonly arises in comparative life-testing scenarios such as the evaluation of two production lines or systems. Explicit expressions for the expected number of failures are presented for two populations, and their performance is examined through Monte Carlo simulations under various censoring schemes. The proposed methods are further illustrated using real datasets, demonstrating their applicability and effectiveness in reliability assessment. Overall, the results show that the proposed inferential procedures perform well under joint progressive censoring and provide a useful statistical framework for comparative reliability analysis, with methodology that naturally extends to general k -population settings.

Keywords: Weibull–exponential model; joint progressive censoring; Bayesian estimation; Markov chain Monte Carlo (MCMC); reliability analysis; expected number of failures

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1. Introduction

In numerous industrial areas, items are frequently produced by multiple production lines functioning concurrently under comparable or differing process conditions. Despite being engineered to conform to identical technical specifications, inherent discrepancies

in machinery, raw materials, operational conditions, and human elements may result in fluctuations in product quality and reliability. Consequently, it is imperative to evaluate and contrast the longevity performance of products from various manufacturing lines to assure uniformity, enhance production processes, and uphold superior quality standards. Comparative life testing has become an efficient statistical method for assessing and comparing the reliability attributes of products manufactured under varying situations. Through the application of controlled life tests on samples from various production lines, producers and researchers can discern substantial variations in longevity performance, identify substandard processes, and facilitate data-informed decisions pertaining to process enhancement and quality assurance. Such tests are especially pertinent in sectors where product failure may result in significant financial losses, safety issues, or reputational harm. From a statistical standpoint, comparative life testing entails modeling lifespan data with suitable probability distributions and utilizing inferential methods to compare relevant parameters across manufacturing lines. These processes may encompass hypothesis testing, parameter estimation, and goodness-of-fit evaluation, frequently under practical limitations such as censoring, restricted sample numbers, or expedited testing circumstances. The selection of a suitable statistical model and testing framework is essential for accurately representing the underlying lifetime behavior and assuring meaningful comparisons. This work addresses the analysis of comparative life test data generated from items produced across several production lines, motivated by industrial and statistical constraints. The suggested methodology seeks to establish a robust and efficient framework for comparing lifetime characteristics across product lines, thus delivering useful insights for reliability evaluation, quality control, and production enhancement in contemporary industrial systems.

Numerous studies in the reliability and life-testing literature have tackled the issue of comparing lifetime characteristics across various production lines by analyzing jointly censored samples. Initial work by Balakrishnan and Rasouli [1] formulated precise likelihood-based inference methods for two exponential populations within a combined Type-II censoring framework, creating a robust foundation for parameter estimation and hypothesis testing in comparative life assessments. Ashour and Abo-Kasem [2] expanded this research by examining both Bayesian and classical estimation techniques for two generalized exponential populations, subsequently analyzing Weibull populations within the same joint Type-II censoring framework [3], thereby illustrating the adaptability of comparative life testing methodologies across various lifetime distributions. Balakrishnan and Feng [4] offered additional generalization to intricate industrial environments with many production lines, proposing precise likelihood inference for k exponential populations under joint Type-II censoring. Furthermore, Abu-Zinadah [5] investigated statistical inference for two Pareto populations utilizing simultaneously Type-II censored samples, emphasizing the relevance of these methods to heavy-tailed lifespan models. These studies highlight the significance of joint censoring schemes and comparative inference techniques in evaluating and comparing the reliability performance of products from various production lines, thereby directly motivating the methodological advancements discussed in this paper. Rasouli and Balakrishnan [6] provided a significant methodological development in comparative life testing by formally proposing the joint progressive Type-II censoring scheme, to the best of current knowledge. This censoring framework enhances the traditional joint Type-II censoring mechanism by permitting the exclusion of surviving units from one or both populations at predetermined failure times, thus offering a more adaptable and realistic structure for life-testing experiments conducted within practical industrial limitations. Rasouli and Balakrishnan [6] established precise likelihood-based inference methodologies for two exponential populations, encompassing point and interval estimates, along with hypothesis testing, within the framework of this newly presented

censoring technique. Their research established the theoretical basis for extensive subsequent investigations into progressive censoring models and markedly improved the relevance of comparative life tests in multi-production-line contexts, where premature termination or intentional withdrawal of test units is frequently inevitable. This innovative addition immediately encourages the application of joint progressive Type-II censoring in the current investigation.

According to Rasouli and Balakrishnan [6], the joint progressive Type-II censoring (JPT2C) can be described as follows: Suppose that products are being produced by k different lines within the same facility. An independent random samples of size n_h are selected from these k lines and placed at the same time on a life testing experiment. Also, suppose $X_{h1}, X_{h2}, \dots, X_{hn_h}$ are the lifetimes of n_h units from line L_h and they are assumed to be independent and identically distributed (iid) random variables. Suppose that the k populations are Weibull exponential distribution (WED) with probability density function (pdf) $f_h(x)$ and cumulative distribution function (cdf) $F_h(x)$, $h = 1, \dots, k$, can be written as

$$f_h(\alpha_h, \gamma_h, \beta_h, x) = \alpha_h \gamma_h \beta_h (e^{\gamma_h x} - 1)^{\beta_h - 1} e^{\gamma_h x} e^{-\alpha_h (e^{\gamma_h x} - 1)^{\beta_h}}, x > 0; \alpha_h, \gamma_h, \beta_h > 0, \quad (1)$$

and

$$F_h(\alpha_h, \gamma_h, \beta_h, x) = 1 - e^{-\alpha_h (e^{\gamma_h x} - 1)^{\beta_h}}, x > 0; \alpha_h, \gamma_h, \beta_h > 0. \quad (2)$$

Let $N = \sum_{h=1}^k n_h$ denote the total sample size, r denote a pre-fixed total number of failures to be observed and $w_1 \leq w_2 \leq \dots \leq w_N$ represent the order statistics of N random variables $\{X_{h1}, X_{h2}, \dots, X_{hn_h}\}$. Therefore, under the JPT2C scheme for the k samples, the observable data consist of (Z, W) , where $W = \{w_1 \leq w_2 \leq \dots \leq w_r\}$, $w_i \in \{X_{h_{i1}}, X_{h_{i2}}, \dots, X_{h_{ir}}\}$, for $1 \leq h_i \leq k, i = 1, \dots, r$. Also, according to $h_i, i = 1, \dots, r$, let Z can be defined as: $Z = (Z_1(h), \dots, Z_r(h))$ where $Z_i(h) = \begin{cases} 1, & \text{if } h = h_i \\ 0, & \text{otherwise} \end{cases}$.

At the first failure time w_1 , R_1 units are chosen randomly from the remaining $(N - 1)$ units and they are withdrawn from the experiment. Then, at the second failure time w_2 , R_2 units are chosen randomly from the remaining $N - R_1 - 2$ units and they are withdrawn from the experiment. Finally, at the r th failure time w_r , all the remaining $R_r = N - r - \sum_{j=1}^{r-1} R_j$ units are withdrawn from the experiment and the experiment terminates. Also, let $s_i(h)$ denote the number of units withdrawn at the time of the i th failure from the sample of h th population and these are unknown random variables. Let $R_i = \sum_{h=1}^k s_i(h)$ are chosen before the experiment. The total number of failures r is defined in advance as a fixed design parameter of the JPT2C scheme. Subsequently, $M_r(h)$ is defined as the number of failures contributed by population h among the predetermined total of r failures, i.e., $M_r(h) = \sum_{i=1}^r Z_i(h)$ denote the number of failures in W which belonging to the sample from h th population.

Recent advancements in comparative life testing have increasingly emphasized JPT2C schemes, which provide more flexibility and realism in industrial reliability trials where units may be withdrawn at various phases of the test. Parsi et al. [7] introduced conditional maximum likelihood and interval estimation for two Weibull populations within a unified framework. Doostparast et al. [8] and Shafay et al. [9] further investigated Bayesian methodologies under progressive censoring, emphasizing the influence of prior knowledge and asymmetric loss functions on reliability estimates. Recent methodological advancements have encompassed a diverse array of lifetime distributions, including Weibull, weighted exponential, Lindley, inverted exponentiated Rayleigh, generalized inverted exponential, gamma, power Rayleigh, Lomax, Nadarajah–Haghighi, Gompertz, inverse Weibull, and Gumbel Type-II distributions, illustrating the extensive applicability of joint progressive censoring models; see [10–21]. Furthermore, Lin et al. [22] have examined optimal test

planning and design within these censoring frameworks, while Vali Ahmadi et al. [23] and Panahi [24] have explored extensions to competitive risks and order-restricted inference, respectively. Recent studies have underscored the practical significance of these methods in industrial production environments by evaluating the reliability of units manufactured by various production lines [25], alongside applied Bayesian analysis [26]. Alongside fundamental findings regarding anticipated failure rates under joint progressive censoring [27], these contributions underscore the pivotal significance of JPT2C in comparative life testing and establish a robust methodological framework for the reliability analysis of products derived from multiple production lines, as examined in this study. This research aims to propose and assess robust statistical inference procedures for two independent WEDs inside a JPT2C framework. The research examines the mathematical derivation of maximum likelihood and Bayesian estimators, the formulation of corresponding confidence and credible intervals under various loss functions, and the theoretical assessment of the predicted number of failures from each population. The finite-sample efficacy and numerical characteristics of the proposed estimators are examined via Monte Carlo simulations, and the practical significance of the theoretical findings is illustrated using actual lifespan data sets.

Motivation for using the WED: The WED provides a flexible and practically meaningful extension of classical lifetime models, making it particularly attractive for reliability and survival analysis. One of its main advantages lies in the increased flexibility of its hazard rate function. While the Weibull distribution can accommodate only monotone hazard rates (increasing, decreasing, or constant), many real-world lifetime data sets, especially in engineering reliability and biomedical studies, exhibit more complex hazard patterns, such as initially decreasing hazards followed by an increasing phase or non-linear monotone behavior. The WED introduces an additional shape parameter through the exponential component, allowing it to capture a wider range of hazard-rate shapes while retaining interpretability and analytical tractability.

The hazard rate function of the WED exhibits considerable flexibility and can accommodate a wide range of failure-rate behaviors depending on the value of the shape parameter β :

1. **Initial behavior:**
 - If $0 < \beta < 1$, the hazard rate tends to be high at early times and may decrease initially, representing early-life or infant-mortality failures.
 - If $\beta = 1$, the hazard rate starts at a constant level for small t .
 - If $\beta > 1$, the hazard rate starts near zero and increases with time.
2. **Monotonicity:** The exponential term $e^{\gamma\beta t}$ ensures that the hazard rate eventually increases with time for all $\beta > 0$. As a result, the WED can model monotonically increasing hazard rates, which are commonly observed in aging and wear-out phenomena.
3. **Flexibility and reliability interpretation:** By jointly controlling the scale parameter α , the acceleration parameter γ , and the shape parameter β , the WED hazard rate can represent diverse reliability patterns ranging from early failures to accelerated aging. This flexibility makes the WED particularly suitable for lifetime data where classical Weibull or exponential models may be inadequate.

In general, the WED provides a rich hazard-rate structure while maintaining analytical tractability, which is especially advantageous in censored life-testing and comparative reliability studies. Additional information on the WED and a detailed discussion of its statistical properties can be found in Bourguignon et al. [28] and Oguntunde et al. [29].

The remainder of this paper is organized as follows: Section 2 develops maximum likelihood estimation procedures and constructs the corresponding confidence intervals.

Section 3 presents the Bayesian inference framework, including prior specifications, Bayes estimators under different loss functions, and the implementation of MCMC methods. Section 4 investigates the expected number of failures for each population and discusses its practical implications. A comprehensive Monte Carlo simulation study is performed in Section 5 to compare the expected values of the number of failures before the test procedure, and the exact values of the number of failures after the test procedure at $k = 2$. Section 6 presents an illustrative example to explain the theoretical results at $k = 2$, in addition to the applicability of the methods through analyses of real data sets. Finally, Section 7 concludes the paper with a summary of the main findings and possible directions for future research.

2. Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is a prevalent inferential technique in statistical modeling, especially in reliability analysis and life-testing research. The core concept of MLE is to ascertain unknown model parameters by maximizing the likelihood function, which denotes the joint probability of witnessing the specified sample data as a function of the parameters. Under broadly applicable regularity requirements, maximum likelihood estimators have various advantageous asymptotic features, including consistency, asymptotic normality, and efficiency, rendering them appealing for both theoretical and practical applications. In the realm of lifetime data, MLE offers a cohesive framework for parameter estimates across several censoring methodologies, including Type-I, Type-II, and progressive censoring, where incomplete observations frequently occur. Furthermore, likelihood-based methods enable the formulation of confidence intervals and hypothesis testing by utilizing observed or expected Fisher information matrices. Due to these benefits, MLE has emerged as a common estimation method in comparative life testing and reliability modeling, particularly in the context of intricate censoring structures. According to Rasouli and Balakrishnan [6], the likelihood function of the observed k samples based on JPT2C, is formulated as

$$L(\alpha_{1,\dots,k}, \gamma_{1,\dots,k}, \beta_{1,\dots,k}, \mathbf{Z}, \mathbf{s}, \mathbf{W}) = c_r \prod_{i=1}^r \prod_{h=1}^k (f_h(\alpha_h, \gamma_h, \beta_h, w_i)^{Z_i(h)}) \prod_{i=1}^r \prod_{h=1}^k (\bar{F}_h(\alpha_h, \gamma_h, \beta_h, w_i))^{s_i(h)}, \quad (3)$$

where

$$c_r = D_1 D_2, \quad D_1 = \prod_{i=1}^r \left[\sum_{h=1}^k \left(n_h - M_{i-1}(h) - \sum_{j=1}^{i-1} s_j(h) \right) Z_i(h) \right],$$

and

$$D_2 = \prod_{i=1}^{r-1} \left\{ \frac{\prod_{h=1}^k \binom{n_h - M_i(h) - \sum_{j=1}^{i-1} s_j(h)}{s_j(h)}}{\binom{N - i - \sum_{j=1}^{i-1} R_j}{R_i}} \right\}.$$

Upon using (1) and (2) $L(\alpha_h, \gamma_h, \beta_h, \mathbf{Z}, \mathbf{s}, \mathbf{W})$ can be written as follows:

$$L(\alpha_h, \gamma_h, \beta_h, \mathbf{Z}, \mathbf{s}, \mathbf{w}) = c_r \prod_{h=1}^k (\alpha_h \gamma_h \beta_h)^{M_r(h)} \prod_{i=1}^r \left[(e^{\gamma_h w_i} - 1)^{\beta_h - 1} \times e^{\gamma_h w_i} \times e^{-\alpha_h (e^{\gamma_h w_i} - 1)^{\beta_h}} \right]^{Z_i(h)} \\ \times \prod_{i=1}^r \left[e^{-\alpha_h (e^{\gamma_h w_i} - 1)^{\beta_h}} \right]^{s_i(h)}. \quad (4)$$

Then, the log-likelihood function $\ell(\alpha_h, \gamma_h, \beta_h, \mathbf{Z}, \mathbf{s}, \mathbf{w})$ can be written as:

$$\begin{aligned}\ell(\alpha_h, \gamma_h, \beta_h, \mathbf{Z}, \mathbf{s}, \mathbf{w}) &= \log c_r + \sum_{h=1}^k M_r(h) (\log \alpha_h + \log \gamma_h + \log \beta_h) \\ &\quad + \sum_{i=1}^r \sum_{h=1}^k [(\beta_h - 1) Z_i(h) \log(e^{\gamma_h w_i} - 1) + \gamma_h w_i Z_i(h)] \\ &\quad - \sum_{i=1}^r \sum_{h=1}^k (\alpha_h (e^{\gamma_h w_i} - 1)^{\beta_h}) (Z_i(h) + s_i(h)).\end{aligned}$$

Equating the partial derivatives of the log-likelihood function with respect to α_h , γ_h and β_h , $h = 1, \dots, k$, respectively, by zero, gives:

$$\frac{\partial \ell}{\partial \alpha_h} = \frac{M_r(h)}{\alpha_h} - \sum_{i=1}^r ((e^{\gamma_h w_i} - 1)^{\beta_h}) (Z_i(h) + s_i(h)) = 0, \quad (5)$$

$$\begin{aligned}\frac{\partial \ell}{\partial \gamma_h} &= \frac{M_r(h)}{\gamma_h} + \sum_{i=1}^r [(\beta_h - 1) Z_i(h) \frac{w_i e^{\gamma_h w_i}}{(e^{\gamma_h w_i} - 1)} + w_i Z_i(h)] \\ &\quad - \sum_{i=1}^r (\alpha_h \beta_h w_i e^{\gamma_h w_i} (e^{\gamma_h w_i} - 1)^{\beta_h - 1}) (Z_i(h) + s_i(h)) = 0, \quad (6)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \beta_h} &= \frac{M_r(h)}{\beta_h} + \sum_{i=1}^r [Z_i(h) \log(e^{\gamma_h w_i} - 1)] \\ &\quad - \sum_{i=1}^r (\alpha_h (e^{\gamma_h w_i} - 1)^{\beta_h} \log(e^{\gamma_h w_i} - 1)) (Z_i(h) + s_i(h)) = 0. \quad (7)\end{aligned}$$

From Equations (5)–(7), a system of nonlinear equations is obtained. Closed-form analytical solutions to this system are generally not available. Consequently, the parameter estimates are computed using numerical techniques, specifically the Newton–Raphson (NR) iterative. The algorithm is carried out according to the following sequence of steps:

- (1) To initiate the iterative procedure, suitable preliminary estimates of the model parameters are first obtained by applying the method of moments or other appropriate estimation techniques. These initial values are denoted by $(\alpha_h^{(0)}, \gamma_h^{(0)})$ and $\beta_h^{(0)}$ for the parameters (α_h, γ_h) and β_h , respectively. The iteration counter is then initialized by setting $j = 0$.
- (2) At the j th iteration, evaluate the score functions $\frac{\partial \ell}{\partial \alpha_h}$, $\frac{\partial \ell}{\partial \gamma_h}$, and $\frac{\partial \ell}{\partial \beta_h}$ at the current parameter estimates $\alpha_h^{(j)}, \gamma_h^{(j)}$ and $\beta_h^{(j)}$. In addition, compute the observed asymptotic Fisher information matrix $I^{-1}(\alpha_h, \gamma_h, \beta_h)$, as defined in the subsequent section.
- (3) Update $(\alpha_h, \gamma_h, \beta_h)$ as

$$(\alpha_h^{(j+1)}, \gamma_h^{(j+1)}, \beta_h^{(j+1)}) = (\alpha_h^{(j)}, \gamma_h^{(j)}, \beta_h^{(j)}) + \left(\frac{\partial \ell}{\partial \alpha_h}, \frac{\partial \ell}{\partial \gamma_h}, \frac{\partial \ell}{\partial \beta_h} \right)_{(\alpha_h^{(j)}, \gamma_h^{(j)}, \beta_h^{(j)})} \times I^{-1}(\alpha_h, \gamma_h, \beta_h).$$

- (4) Increment the iteration index by setting $j = j + 1$, and then return to Step 2.
- (5) Repeat the successive steps until the stopping criterion is satisfied

$$\left| (\alpha_h^{(j+1)}, \gamma_h^{(j+1)}, \beta_h^{(j+1)}) - (\alpha_h^{(j)}, \gamma_h^{(j)}, \beta_h^{(j)}) \right| \leq \epsilon \rightarrow 0,$$

The resulting values of $(\alpha_h, \gamma_h, \beta_h)$ obtained at convergence constitute the MLEs of the parameters and are denoted by $(\hat{\alpha}_h, \hat{\gamma}_h, \hat{\beta}_h)$ for $h = 1, \dots, k$.

To achieve stable and reliable convergence of the NR algorithm, appropriate starting values for the parameters $(\alpha_h, \gamma_h, \beta_h)$ were judiciously chosen using practical estimation strategies. Specifically, two initialization methods were employed: First, method-of-moments estimators obtained from complete or uncensored samples when available; and second, an exploratory grid search across the parameter space to ascertain initial values linked to comparatively high values of the log-likelihood function. The initial selections demonstrated similar convergence behavior across various sample sizes and censoring setups. Supplementary sensitivity analyses demonstrated that, for modest sample sizes, the NR algorithm displayed robustness concerning the choice of initial values. However, in the case of small samples or significant censoring, both the speed of convergence and the accuracy of estimation were more influenced by the beginning points, underscoring the importance of informed initialization. Furthermore, the intricacy of the log-likelihood function resulting from the three-parameter model and the combined progressive Type-II censoring method may cause the NR algorithm to occasionally yield invalid estimates when parameters cannot be separately updated. Consequently, it is advisable to validate the numerical outcomes by utilizing various sets of beginning values.

Furthermore, to ensure numerical stability and enforce the positivity constraints on all parameters, the NR algorithm is implemented on the log-transformed scale, that is, $\eta_h = (\log \alpha_h, \log \gamma_h, \log \beta_h)^T$. At iteration j , the update is given by

$$\eta_h^{(j+1)} = \eta_h^{(j)} - H(\eta_h^{(j)})^{-1} U(\eta_h^{(j)}),$$

where $U(\cdot)$ and $H(\cdot)$ denote the score vector and the observed Hessian matrix, respectively. The algorithm is initialized using method-of-moments estimators or grid-based exploratory values that yield relatively large likelihood values. Iterations are continued until convergence, defined by

$$\|\eta_h^{(j+1)} - \eta_h^{(j)}\| < \varepsilon$$

where $\varepsilon = 10^{-6}$ and $\|\cdot\|$ denotes the Euclidean norm, or until a maximum of 1000 iterations is reached. To handle cases where the Hessian matrix is not positive definite or when large steps lead to likelihood decreases, a step-halving (damping) strategy is employed. Specifically, the step size is reduced iteratively until the updated parameter vector increases the log-likelihood. Upon convergence, the estimates are transformed back to the original scale. The observed Fisher information matrix is evaluated at the final estimates and is used to compute standard errors and asymptotic confidence intervals.

Approximate Confidence Intervals

Confidence intervals (ACIs) for the model parameters can be approximated by utilizing the asymptotic normality of the MLEs alongside the observed Fisher information matrix (FIM). The inverse of the observed FIM yields an estimate of the asymptotic variance-covariance matrix of the estimators. Under regularity conditions, each parameter estimator is approximately normally distributed, with a mean equal to the true parameter value and a variance represented by the corresponding diagonal member of this matrix. Consequently, a two-sided $(1 - \delta)100\%$ ACI for a parameter can be obtained by adding and subtracting the appropriate normal quantile multiplied by the estimated standard error. This likelihood-based approach is widely used in reliability and life-testing studies, particularly when closed-form exact intervals are unavailable, as it offers a practical and theoretically justified method for quantifying estimation uncertainty. The asymptotic Fisher information matrix I

of MLEs is the symmetric matrix of negative second order partial derivatives of the log-likelihood function with respect to α_h , γ_h and β_h . Let $\eta_{1h} = \alpha_h$, $\eta_{2h} = \gamma_h$ and $\eta_{3h} = \beta_h$, then

$$I = - \left(\frac{\partial^2 \ell}{\partial \eta_{ih} \partial \eta_{jh}} \right)_{\downarrow(\hat{\alpha}_h, \hat{\gamma}_h, \hat{\beta}_h)}, i, j = 1, 2, 3, \forall h. \quad (8)$$

Therefore, the asymptotic variance–covariance matrix of the MLEs can subsequently be obtained as follows:

$$I^{-1} = \left[- \left(\frac{\partial^2 \ell}{\partial \eta_{ih} \partial \eta_{jh}} \right)_{\downarrow(\hat{\alpha}_h, \hat{\gamma}_h, \hat{\beta}_h)} \right]^{-1}. \quad (9)$$

Then, $(1 - \delta)100\%$ ACIs for the parameters α_h , γ_h and β_h , for $h = 1, \dots, k$ are given by

$$(\hat{\alpha}_{hL}, \hat{\alpha}_{hU}) = \hat{\alpha}_h \mp z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\alpha}_h)}, (\hat{\gamma}_{hL}, \hat{\gamma}_{hU}) = \hat{\gamma}_h \mp z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\gamma}_h)}, (\hat{\beta}_{hL}, \hat{\beta}_{hU}) = \hat{\beta}_h \mp z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\beta}_h)}.$$

In this expression, $z_{(\delta/2)}$ denotes the upper $\delta/2$ quantile of the standard normal distribution, corresponding to the point beyond which the right-tail area is $\delta/2$. The terms $\text{var}(\hat{\alpha}_h)$, $\text{var}(\hat{\gamma}_h)$ and $\text{var}(\hat{\beta}_h)$ represent the asymptotic variances of the MLEs for the parameters α_h , γ_h and β_h , respectively, which can be obtained from the diagonal elements of the inverse of the observed FIM, for more details see Cohen [30].

A well-known limitation of the standard $100(1 - \gamma)\%$ ACIs is that it may yield a negative lower limit, despite the fact that the parameter of interest is constrained to be positive. To overcome this shortcoming, Meeker and Escobar [31] recommended constructing confidence intervals based on a logarithmic transformation of the maximum likelihood estimators. Accordingly, the two-sided $100(1 - \gamma)\%$ ACIs for the parameter vector $\vartheta_h = (\alpha_h, \gamma_h, \beta_h)$ are obtained using the log-transformed estimators as follows:

$$\left[\hat{\vartheta}_h \cdot \exp \left\{ - \frac{z_{\frac{\gamma}{2}} \sqrt{\text{var}(\hat{\vartheta}_h)}}{\hat{\vartheta}} \right\}, \hat{\vartheta}_h \cdot \exp \left\{ \frac{z_{\frac{\gamma}{2}} \sqrt{\text{var}(\hat{\vartheta}_h)}}{\hat{\vartheta}} \right\} \right]. \quad (10)$$

where $\hat{\vartheta}_h = (\hat{\alpha}_h, \hat{\gamma}_h, \hat{\beta}_h)$. The ACIs derived in this study are based on asymptotic normality of the MLEs and therefore rely on large-sample approximations. As a consequence, their accuracy is expected to improve as the effective sample size increases and the degree of censoring decreases. In contrast, when the sample size is small or the censoring level is high, the normal approximation may be less reliable, potentially leading to wider intervals or coverage probabilities that deviate from their nominal levels. These limitations are particularly relevant under JPT2C, where the amount of observed failure information can be substantially reduced. For this reason, Bayesian CRIs are used as a complementary inferential tool, as they do not depend on asymptotic arguments and can provide more stable and coherent uncertainty quantification, especially in heavily censored settings.

3. Bayesian Estimation

Bayesian estimation utilizing prior and posterior distributions can be modified to accommodate various decision-making criteria by selecting an appropriate loss function. Utilizing a defined prior distribution that encapsulates previous knowledge regarding the parameters, the likelihood function generated from the observed data facilitates the derivation of the posterior distribution by Bayes' theorem. The Bayes estimator is defined as the value that minimizes the predicted posterior loss. When utilizing a symmetric loss function, such as squared error (SE) loss, the Bayes estimator generally aligns with the posterior mean, equally weighing overestimation and underestimation. Conversely, with an

asymmetric loss function, such the linear exponential (LINEX)-based loss, the estimator adjusts to reflect scenarios where overestimation and underestimation have disparate repercussions, resulting in biased posterior summaries. Specifically, the LINEX loss for estimating a scalar parameter ϑ by $\hat{\vartheta}$ is defined as

$$L(\hat{\vartheta}, \vartheta) = \exp\{C(\hat{\vartheta} - \vartheta)\} - C(\hat{\vartheta} - \vartheta) - 1$$

where $C \neq 0$ controls the direction and degree of asymmetry. When $C > 0$, overestimation of ϑ is penalized more heavily than underestimation, whereas $C < 0$ imposes a higher penalty on underestimation. The magnitude $|C|$ determines the strength of the asymmetry, with larger values corresponding to greater departure from symmetric loss. This adaptability enables Bayes' estimation to yield resilient and customized parameter estimates, particularly in reliability and life-testing contexts where the cost of errors may be asymmetric or where prior information exists.

In the Bayesian analysis, asymptotic variance estimates obtained from the observed FIM are employed to construct and tune the proposal distributions in the Metropolis–Hastings steps of the MCMC algorithm, thereby improving sampling efficiency and achieving reasonable acceptance rates. Convergence of the generated Markov chains is assessed using standard diagnostic tools, including visual inspection of trace plots, examination of autocorrelation functions, and evaluation of effective sample sizes, ensuring that the posterior samples adequately represent the target distributions. In addition, issues of parameter identifiability under JPT2C are carefully considered, as heavy censoring may reduce the amount of information available for certain parameters. To address this, the analysis relies on proper and weakly informative prior distributions and includes a brief sensitivity assessment, demonstrating that the posterior inferences are stable with respect to moderate changes in prior specifications.

In Bayesian estimation, a vital initial step is to evaluate the impact of the selected prior distribution on the resultant conclusion. This study selects the gamma distribution as the prior due to its suitability for modeling positive, continuous characteristics, especially those related to rates or accuracy parameters in exponential-family models. The gamma prior is defined by two hyperparameters, known as shape and scale (or rate), which collectively regulate its flexibility and the concentration of prior knowledge. The gamma distribution's primary advantage is its conjugacy with several commonly utilized probability functions, such as the exponential and Poisson distributions, guaranteeing that the posterior distribution remains within the same family as the prior. This conjugate characteristic significantly streamlines analytical derivations and improves computational efficiency when integrating observed data. The gamma distribution effectively represents a wide range of prior knowledge, from weakly informative to highly informative contexts, rendering it a viable and versatile option in Bayesian inference. Its mathematical convenience and adaptability have therefore contributed to its widespread use in Bayesian analyses involving non-negative parameters. In this section, Bayesian estimators and corresponding credible intervals are derived for the parameter vector $\omega_h = (\alpha_h, \gamma_h, \beta_h)$ under JPT2C samples from the WED, while accounting for both symmetric and asymmetric loss functions. It is assumed that the parameters α_h , γ_h and β_h are independent random variables and each follows a gamma prior distribution, with probability density functions specified as follows:

$$\begin{aligned}\pi_1(\alpha_h) &\propto \alpha_h^{a_h-1} e^{-\alpha_h b_h}, \quad \alpha_h > 0, a_h, b_h > 0, h = 1, 2, \dots, k \\ \pi_2(\gamma_h) &\propto \gamma_h^{c_h-1} e^{-\gamma_h d_h}, \quad \gamma_h > 0, c_h, d_h > 0, h = 1, 2, \dots, k \\ \pi_3(\beta_h) &\propto \beta_h^{u_h-1} e^{-\beta_h v_h}, \quad \beta_h > 0, u_h, v_h > 0, h = 1, 2, \dots, k\end{aligned}\tag{11}$$

Here, the hyperparameters a_h , b_h , c_h , d_h , u_h and v_h , for $h = 1, 2, \dots, k$ are assumed to be known positive constants that reflect prior information and beliefs regarding the parameter

vector $\omega_h = (\alpha_h, \gamma_h, \beta_h)$. Under this assumption, the joint prior probability density function of α_h, γ_h and β_h can be written as follows:

$$\pi(\omega_h) \propto \alpha_h^{a_h-1} \gamma_h^{c_h-1} \beta_h^{u_h-1} e^{-\alpha_h b_h - \gamma_h d_h - \beta_h v_h}, a_h, b_h, c_h, d_h, u_h, v_h > 0, h = 1, 2, \dots, k. \quad (12)$$

By applying Bayes' theorem in conjunction with Equations (4) and (12), the joint posterior probability density function of the model parameters can be derived as

$$\begin{aligned} \pi^*(\omega_h | \mathbf{Z}, \mathbf{s}, \mathbf{w}) &= \frac{L(\omega_h, \mathbf{Z}, \mathbf{s}, \mathbf{w}) \times \pi(\omega_h)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\omega_h, \mathbf{Z}, \mathbf{s}, \mathbf{w}) \times \pi(\omega_h) d\alpha_h d\gamma_h d\beta_h} \\ &\propto \prod_{h=1}^k \alpha_h^{M_r(h)+a_h-1} \gamma_h^{M_r(h)+c_h-1} \beta_h^{M_r(h)+u_h-1} \times e^{-\alpha_h b_h - \gamma_h d_h - \beta_h v_h} \\ &\quad \times \prod_{i=1}^r \left[(e^{\gamma_h w_i} - 1)^{\beta_h-1} \times e^{\gamma_h w_i} \right]^{Z_i(h)} \times \left(e^{-\alpha_h (e^{\gamma_h w_i} - 1)^{\beta_h}} \right)^{Z_i(h)+s_i(h)}. \end{aligned} \quad (13)$$

Thus, Bayesian estimators for any parametric function $\psi(\omega_h)$ can be obtained under different loss functions according to the underlying decision framework. In particular, the estimators corresponding to the SE and LINEX loss functions are derived using the respective formulations given below.

$$\begin{aligned} \hat{\psi}_{SE}(\omega_h) &= E_{\omega_h | \mathbf{Z}, \mathbf{s}, \mathbf{w}}[\psi(\omega_h)] \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \psi(\omega_h) \times \pi^*(\omega_h | \mathbf{Z}, \mathbf{s}, \mathbf{w}) d\alpha_h d\gamma_h d\beta_h, \end{aligned} \quad (14)$$

and

$$\begin{aligned} \hat{\psi}_{LINEXSE}(\omega_h) &= -\frac{1}{C} \log E_{\omega_h | \mathbf{Z}, \mathbf{s}, \mathbf{w}} \left[e^{-C\psi(\omega_h)} \right] \\ &= -\frac{1}{C} \log \left[\int_0^\infty \int_0^\infty \int_0^\infty e^{-C\psi(\omega_h)} \times \pi^*(\omega_h | \mathbf{Z}, \mathbf{s}, \mathbf{w}) d\alpha_h d\gamma_h d\beta_h \right]. \end{aligned} \quad (15)$$

For comprehensive derivations and the underlying theoretical framework, the reader is referred to Calabria and Pulcini [32]. It is observed that the integrals appearing in Equations (14) and (15) do not admit closed-form solutions. Accordingly, MCMC methods are employed to simulate samples from the corresponding posterior distributions, which are subsequently used to compute the Bayes estimates of the parameters α_h, γ_h and β_h for $h = 1, 2, \dots, k$. The primary objective of MCMC techniques is to obtain accurate numerical approximations of the integrals involved in (14) and (15) through stochastic simulation. Among the most widely used classes of MCMC algorithms are Gibbs sampling and the more general Metropolis-within-Gibbs schemes. In particular, the Metropolis algorithm generates a random walk over the parameter space and employs an acceptance–rejection mechanism to ensure convergence to the target posterior distribution. This algorithm was originally introduced by Metropolis et al. [33] and later extended by Hastings [34]. Based on Equation (13), the conditional posterior density functions of α_h, γ_h and β_h can be expressed as follows:

$$\pi_1^*(\alpha_h | \gamma_h, \beta_h, \mathbf{w}) \sim \text{Gamma}[M_r(h) + a_h, b_h + \sum_{i=1}^r (e^{\gamma_h w_i} - 1)^{\beta_h (Z_i(h) + s_i(h))}], \quad (16)$$

$$\begin{aligned} \pi_2^*(\gamma_h | \alpha_h, \beta_h, \mathbf{w}) &\propto \prod_{h=1}^k \gamma_h^{M_r(h)+c_h-1} e^{-\gamma_h d_h} \times \prod_{i=1}^r \left[(e^{\gamma_h w_i} - 1)^{\beta_h-1} \times e^{\gamma_h w_i} \right]^{Z_i(h)} \\ &\quad \times \left[e^{-\alpha_h (e^{\gamma_h w_i} - 1)^{\beta_h}} \right]^{Z_i(h)+s_i(h)}, \end{aligned} \quad (17)$$

and

$$\pi_3^*(\beta_h | \alpha_h, \gamma_h, \mathbf{w}) \propto \prod_{h=1}^k \beta_h^{M_r(h)+u_h-1} e^{-\beta_h v_h} \times \prod_{i=1}^r \left[(e^{\gamma_h w_i} - 1)^{\beta_h-1} \times e^{\gamma_h w_i} \right]^{Z_i(h)} \times \left[e^{-\alpha_h (e^{\gamma_h w_i} - 1)^{\beta_h}} \right]^{Z_i(h)+s_i(h)}. \quad (18)$$

Now, the sampling procedure integrates the Gibbs sampling (GS) approach with the Metropolis–Hastings (MH) algorithm, as described by Tierney [35]. This hybrid technique employs a normal proposal distribution to efficiently generate samples from the posterior distribution and follows the iterative sequence of steps outlined below.

- (1) Select initial values for the parameters and set $(\alpha_h^{(0)}, \gamma_h^{(0)}, \beta_h^{(0)}) = (\hat{\alpha}_h, \hat{\gamma}_h, \hat{\beta}_h)$.
- (2) Initialize the iteration counter by letting $i = 1$.
- (3) Generate a new value $\alpha_h^{(i)}$ from the gamma distribution with parameters $(M_r(h) + a_h, b_h + \sum_{i=1}^r (e^{\gamma_h w_i} - 1)^{\beta_h (Z_i(h)+s_i(h))})$.
- (4) Using the MH scheme, generate updated values $\gamma_h^{(i)}$ and $\beta_h^{(i)}$ from their conditional posterior densities (Equations (17) and (18)), employing normal proposal distributions $N(\gamma_h^{(i-1)}, \text{var}(\gamma_h))$ and $N(\beta_h^{(i-1)}, \text{var}(\beta_h))$, respectively, where $\text{var}(\gamma_h)$ and $\text{var}(\beta_h)$ are obtained from the diagonal elements of the inverse asymptotic FIM.

(a) Generate proposal values γ_h^* and β_h^* from the corresponding normal proposal distributions.

(b) Compute the acceptance probabilities Λ_{γ_h} and Λ_{β_h} .

$$\Lambda_{\gamma_h} = \min \left[1, \frac{\pi_2^*(\gamma_h^* | \alpha_h^{(i)}, \beta_h^{(i-1)}, w)}{\pi_2^*(\gamma_h^{(i-1)} | \alpha_h^{(i)}, \beta_h^{(i-1)}, \gamma_h^{(i-1)}, \mathbf{w})} \right], \Lambda_{\beta_h} = \min \left[1, \frac{\pi_3^*(\beta_h^* | \alpha_h^{(i)}, \gamma_h^{(i)}, w)}{\pi_3^*(\beta_h^{(i-1)} | \alpha_h^{(i)}, \gamma_h^{(i)}, \mathbf{w})} \right],$$

(c) Draw two independent random variables Ω_1 and Ω_2 from the Uniform(0,1) distribution.

(d) Accept γ_h^* and set $\gamma_h^{(i)} = \gamma_h^*$ if $\Omega_1 \leq \Lambda_{\gamma_h}$; otherwise, retain the previous value $\gamma_h^{(i)} = \gamma_h^{(i-1)}$.

(e) Accept β_h^* and set $\beta_h^{(i)} = \beta_h^*$ if $\Omega_2 \leq \Lambda_{\beta_h}$; otherwise, set $\beta_h^{(i)} = \beta_h^{(i-1)}$.

(5) Record the updated values of $\gamma_h^{(i)}$ and $\beta_h^{(i)}$.

(6) Increase the iteration index by setting $i = i + 1$.

(7) Repeat Steps 2 – 6 for a total of Q iterations.

(8) Discard the first Q_0 simulated samples as burn-in. The remaining samples $\{\alpha_h^{(i)}, \gamma_h^{(i)}, \beta_h^{(i)}\}$, for $i = Q_0 + 1, \dots, Q$, are treated as approximate draws from the posterior distributions.

The Bayes estimators under the SE loss function are computed as the posterior means of these samples, while the Bayes estimators under the LINEX loss function are obtained using the corresponding exponential-based averages as follows:

$$\hat{\omega}_h^{SE} = \frac{1}{Q - Q_0} \sum_{i=Q_0+1}^Q \omega_h^{(i)}, \quad \hat{\omega}_h^{LINEX} = \frac{-1}{C} \log \left[\frac{1}{Q - Q_0} \sum_{i=Q_0+1}^Q e^{-C \omega_h^{(i)}} \right],$$

where, $\omega_h = (\alpha_h, \gamma_h, \beta_h)$. To construct credible intervals (CRIs) for ω_{jh} , where $\omega_{1h} = \alpha_h$, $\omega_{2h} = \gamma_h$ and $\omega_{3h} = \beta_h$, the retained posterior samples are sorted in ascending order $\{\omega_{jh}^{Q_0+1}, \omega_{jh}^{Q_0+2}, \dots, \omega_{jh}^Q\}$ as $\{\omega_{jh}^{[1]}, \omega_{jh}^{[2]}, \dots, \omega_{jh}^{[Q-Q_0]}\}$, $j = 1, 2, 3$; for $h = 1, 2, \dots, k$. The $100(1 - \delta)\%$ symmetric credible interval is then formed by taking the empirical quantiles at levels $\delta/2$ and $1 - \delta/2$ as the lower and upper bounds, respectively as

$$\left[\omega_{jh} \left(\frac{\delta}{2} (Q - Q_0) \right), \omega_{jh} \left(\left(1 - \frac{\delta}{2} \right) (Q - Q_0) \right) \right]. \quad (19)$$

The acceptance probabilities are computed in the usual manner using ratios of the corresponding conditional posterior densities. The total number of MCMC iterations is set to $Q = 12,000$, of which the first $Q_0 = 2000$ samples are discarded as burn-in. Convergence of the Markov chains is assessed using multiple diagnostics, including trace plots, autocorrelation functions, and effective sample sizes. In addition, multiple chains with dispersed initial values are run to verify stability and convergence to the same posterior distribution.

4. Expected Values of the Number of Failures M_r

The expected number of failures plays an important role in the subsequent analyses by providing a quantitative summary of how failures are distributed across populations under the JPT2C scheme. In the simulation study, this measure is used to compare the theoretically approximated expected number of failures with their empirical counterparts, thereby assessing the adequacy of the proposed model and the validity of the censoring mechanism. In the real-data applications, the expected number of failures offers an interpretable reliability measure that complements parameter estimation by illustrating the relative contribution of each population to the observed failures. Moreover, this quantity supports test planning and design by indicating how different censoring schemes and sample configurations affect the information obtained from each population. In particular, this quantity serves three main purposes:

- (i) Supporting test planning and design by anticipating the number of failures expected from each population under a given censoring scheme.
- (ii) Validating the adopted censoring structure by comparing theoretical expectations with empirical averages obtained from simulation studies
- (iii) Complementing the inferential results by offering an additional reliability-based interpretation of the estimated model parameters.

In this section, we consider a framework in which JPT2C is applied simultaneously to two samples originating from distinct populations, and we examine the associated failure probabilities. This setting is particularly relevant in practical situations where the experimenter is interested in determining the expected number of failures contributed by each population. Such information is crucial for designing suitable sampling and censoring schemes and for developing efficient parameter estimation procedures. Accordingly, we discuss an explicit expression that enables the numerical evaluation of the expected number of failures for each of the two populations based on Parsi and Bairamov [27] theorem. Consider two samples of sizes m and n from WE populations X and Y with

$$Z_i = \begin{cases} 1 & \text{if } W_i \text{ is from } X \text{ failure} \\ 0 & \text{if } W_i \text{ is from } Y \text{ failure} \end{cases}.$$

Then, the expected values of the number of X failures M_r in W_i under JPT2C can be calculated from the formula

$$E(M_r) \approx \sum_{i=1}^r \mu_{Z_i}, i = 1, 2, \dots, r \quad (20)$$

where μ_{Z_i} and $\mu_{R'_i}$ for $i = 1, 2, \dots, r$ can be approximate as follows, for $i = 1$:

$$\mu_{Z_1} = E(Z_1) = \frac{m}{mp + nq} p, q = 1 - p, \quad (21)$$

$$\mu_{R'_1|Z_1} = E(R'_1 | Z_1 = z_1) = \frac{r_1 \left(m - \frac{m}{mp + nq} p \right)}{(N - 1)}, N = m + n, \quad (22)$$

and for $i = 2$ with respect to Z_1 and Z_2 , we get

$$\begin{aligned}\mu_{Z_2|Z_1,R'_1} &= E(Z_2 | Z_1 = z_1, R'_1 = r'_1) \\ &= \frac{\binom{m-\mu_{Z_1}-\mu_{R'_1}}{r'_1}}{\binom{m-\mu_{Z_1}-\mu_{R'_1}}{r'_1}p + \binom{n-(1-\mu_{Z_1})-(r_1-\mu_{R'_1})}{r'_1}q}p,\end{aligned}\quad (23)$$

$$\mu_{R'_2|Z_2,R'_1} = E(R'_2 | Z_1 = z_1, Z_2 = z_2, R'_1 = r'_1) = \frac{r_2 \left(m - \sum_{j=1}^2 \mu_{Z_j} - \mu_{R'_1} \right)}{(N - \mu_{R'_1} - 2)}, \quad (24)$$

By repeated the conditional expectation formula for $i = 3, \dots, r$, we obtain

$$\mu_{Z_i} = \frac{\binom{m - \sum_{j=1}^{i-1} \mu_{Z_j} - \sum_{j=1}^{i-1} \mu_{R'_j}}{r'_i}}{\binom{m - \sum_{j=1}^{i-1} \mu_{Z_j} - \sum_{j=1}^{i-1} \mu_{R'_j}}{r'_i}p + \binom{n - \sum_{j=1}^{i-1} (1 - \mu_{Z_j}) - \sum_{j=1}^{i-1} (r_j - \mu_{R'_j})}{r'_i}q}p, \quad (25)$$

and

$$\mu_{R'_i} = \frac{r_i \left(m - \sum_{j=1}^i \mu_{Z_j} - \sum_{j=1}^{i-1} \mu_{R'_j} \right)}{\left(N - \sum_{j=1}^{i-1} \mu_{R'_j} - i \right)}, \quad (26)$$

where

$$p = P(Z = 1) = P(X < Y) = \int_0^\infty f(x, \alpha_1, \gamma_1, \beta_1) \bar{F}(x, \alpha_2, \gamma_2, \beta_2) dx. \quad (27)$$

The approximations for the expected number of failures are derived under the assumed WED and the JPT2C scheme, and their validity depends on having a sufficiently large effective sample size and a moderate level of censoring. When the number of observed failures is relatively large, these approximations tend to be accurate, whereas heavy censoring or very small samples may introduce noticeable approximation error due to increased variability and reduced information content. Potential sources of error include deviations from the assumed model, extreme censoring patterns, and the use of asymptotic arguments in the derivation of the expectation measures. From a practical standpoint, the expected number of failures provides an interpretable summary of how failures are anticipated to be distributed across populations under a given test design. It should be emphasized that the expected number of failures “prior to the test” represents a theoretical, model-based quantity that is useful for planning, design, and comparative analysis of censoring schemes, rather than an observable outcome available before conducting the experiment.

5. Simulation Study

Prior information regarding the expected number of failures before the execution of a life test is of substantial importance in selecting suitable sampling designs and in developing efficient parameter estimators. To investigate this aspect, a simulation study is carried out under various JPT2C schemes, where 1000 JPT2C samples are generated from two WE populations for different sample sizes. The primary objective is to compute the approximated expected number of failures from the first production line prior to the test (A.E.B) and to compare it with the mean of the observed number of failures after the test (M.E.A) obtained from the simulated data. This comparison is essential for assessing the accuracy of the approximation proposed by Parsi and Bairamov [27], which

was originally developed for JPT2C samples arising from two Weibull, Gamma and two Pareto populations. Their findings indicated that the approximated and empirical expected values were generally very close. Applying the same approximation to JPT2C samples generated from two WEDs yields similar conclusions, with A.E.B and M.E.A exhibiting close agreement in most scenarios, as reported in Table 1. Moreover, for sufficiently large sample sizes N , the two measures become nearly identical. The computations are conducted under specific parameter settings for the two WEDs based on the following assumptions, X_1 is chosen from WE(1.5, 2, 2.5) and X_2 from WE(0.05, 6, 1.5), $p = 0.74$. Once again X_1 is chosen from WE(1, 2.5, 3) and X_2 from WE(0.07, 5, 2.4), $p = 0.26$, where $p = P(X_1 < X_2) = \int_0^\infty \bar{F}(x, \alpha_2, \gamma_2, \beta_2) dG(x, \alpha_1, \gamma_1, \beta_1)$, X_1 and X_2 denote lifetimes from the first and second production lines, respectively. Also, $\bar{F}(x, \alpha_2, \gamma_2, \beta_2)$ is the survival function of X_2 , while $G(x, \alpha_1, \gamma_1, \beta_1)$ is the cdf of X_1 . The results demonstrate strong agreement between the approximated and simulated expected numbers of failures across most censoring schemes (SCs), with only a few exceptions (SCs No. 4–6, 31–33, and 45–51), further supporting the effectiveness of the approximation method.

Table 1. The comparison between A.E.B and M.E.A.

SCs	N	r	SCs (R)	(n_1, n_2)	$p = 0.74$		$p = 0.26$	
					A.E.B	M.E.A	A.E.B	M.E.A
1	30	10	(20, 0, ..., 0)	(20, 10)	6.8392	6.7303	6.5183	6.5745
2				(15, 15)	5.0881	5.0632	4.9117	4.9091
3				(10, 20)	3.4824	3.4097	3.1608	3.2905
4			(0, 0, 0, 0, 10, 10, 0, 0, 0, 0)	(20, 10)	8.7112	7.3603	6.5466	5.7224
5				(15, 15)	7.6405	5.9196	5.4462	4.0873
6				(10, 20)	5.5752	4.2723	3.8555	2.6262
7			(0, 0, ..., 20)	(20, 10)	8.2943	8.1894	4.6552	4.7401
8				(15, 15)	7.0345	6.9251	2.9652	3.0914
9				(10, 20)	5.3445	5.2802	1.7057	1.8119
10	30	15	(15, 0, ..., 0)	(20, 10)	10.0332	10.0181	9.8772	9.9415
11				(15, 15)	7.5835	7.5265	7.4173	7.4574
12				(10, 20)	5.1224	5.0833	4.9665	4.9526
13			(0, ..., 0, 15, 0, ..., 0)	(20, 10)	10.9162	10.8246	8.6857	8.9056
14				(15, 15)	8.5265	8.5364	6.5345	6.4354
15				(10, 20)	6.1427	6.0664	4.1534	4.1835
16			(0, 0, ..., 15)	(20, 10)	12.2057	12.2176	7.5335	7.4586
17				(15, 15)	10.1476	10.1845	4.8525	4.8084
18				(10, 20)	7.4666	7.4975	2.7944	2.7873
19	30	20	(10, 0, ..., 0)	(20, 10)	13.3192	13.3771	13.2492	13.3103
20				(15, 15)	10.0632	10.0362	9.9363	9.9661
21				(10, 20)	6.7505	6.7014	6.6802	6.6356
22			(0, ..., 0, 5, 5, 0, ..., 0)	(20, 10)	14.5431	14.1535	12.3664	12.2625
23				(15, 15)	11.2582	11.0492	9.4474	8.9493
24				(10, 20)	8.0325	7.7037	6.3317	5.8025
25			(0, 0, ..., 10)	(20, 10)	15.8242	15.9033	10.9645	10.9203
26				(15, 15)	12.7872	12.8761	7.2132	7.1134
27				(10, 20)	9.0355	9.0763	4.1767	4.1005
28	50	20	(30, 0, ..., 0)	(30, 20)	12.0741	12.0432	11.8512	11.9451
29				(25, 25)	10.1261	10.0614	9.8734	9.9742
30				(20, 30)	8.1485	8.0482	7.9252	7.9693

Table 1. Cont.

SCs	N	r	SCs (R)	(n ₁ , n ₂)	p = 0.74		p = 0.26	
					A.E.B	M.E.A	A.E.B	M.E.A
31	50	25	(0, . . . , 0, 15, 15, 0, . . . , 0)	(30, 20)	15.3992	13.3921	11.7572	10.3552
32			(25, 25)	13.4694	11.5944	10.2262	8.4473	
33			(20, 30)	11.2395	9.6422	8.2452	6.6291	
34			(0, 0, . . . , 30)	(30, 20)	15.4963	15.4042	8.119	8.1334
35			(25, 25)	13.8413	13.7844	6.1585	6.2202	
36			(20, 30)	11.8801	11.8422	4.5042	4.6061	
37			(25, 0, . . . , 0)	(30, 20)	15.0736	15.0185	14.8741	14.9832
38			(25, 25)	12.6112	12.5362	12.3883	12.4614	
39			(20, 30)	10.1254	10.0603	9.9265	9.9734	
40			(0, . . . , 0, 25, 0, . . . , 0)	(30, 20)	16.4714	16.4795	13.0886	13.2417
41	70	30	(0, . . . , 0, 25, 0, . . . , 0)	(25, 25)	14.3395	14.1606	10.7376	10.7945
42				(20, 30)	12.0364	11.7566	8.4615	8.5214
43			(0, 0, . . . , 25)	(30, 20)	19.0484	19.0785	10.6763	10.5492
44			(25, 25)	16.8723	16.9383	8.1275	8.0684	
45			(20, 30)	14.3237	14.4456	5.9515	5.9156	
46			(40, 0, . . . , 0)	(40, 30)	17.2541	17.1352	17.0002	17.1152
47			(35, 35)	15.1314	15.0103	14.8681	14.9653	
48			(30, 40)	12.9995	12.8916	12.7457	12.8348	
49			(0, . . . , 0, 20, 20, 0, . . . , 0)	(40, 30)	21.5802	19.1942	16.7672	14.8475
50			(35, 35)	19.5654	17.2115	14.8445	12.7733	
51	70	35	(0, . . . , 0, 20, 20, 0, . . . , 0)	(30, 40)	17.3424	15.1725	12.9102	10.8313
52				(40, 30)	22.4473	22.3484	11.4714	11.4432
53			(35, 35)	20.6045	20.5926	9.3964	9.4135	
54			(30, 40)	18.5282	18.5453	7.5524	7.6254	
55			(35, 0, . . . , 0)	(40, 30)	20.1015	20.0635	19.8754	19.9825
56			(35, 35)	17.6164	17.5206	17.3834	17.4796	
57			(30, 40)	15.1256	14.0393	14.8987	14.9878	
58			(0, 0, . . . , 35)	(40, 30)	25.8263	25.9203	13.8975	13.7275
59			(35, 35)	23.5973	23.7352	11.4024	11.3033	
60			(30, 40)	21.1034	21.2812	9.1737	9.1056	
61	100	20	(80, 0, . . . , 0)	(70, 30)	14.0643	13.9894	13.8025	13.9785
62				(50, 50)	10.1724	10.0063	9.8273	10.0192
63			(30, 70)	6.1975	6.0056	5.9363	6.0164	
64			(0, 0, . . . , 80)	(70, 30)	17.1594	16.6603	9.6791	10.2002
65			(50, 50)	14.3604	13.7184	5.6394	6.2865	
66			(30, 70)	10.3203	9.7615	2.8403	3.3224	
67			(50, 0, . . . , 0)	(70, 30)	35.0634	34.9644	34.8753	35.0084
68			(50, 50)	25.1183	24.9682	24.8817	24.9865	
69			(30, 70)	15.1243	15.0314	14.9365	15.0014	
70			(0, 0, . . . , 50)	(70, 30)	41.7683	41.7982	27.343	27.0245
71	100	50	(0, 0, . . . , 50)	(50, 50)	33.6852	33.8924	16.3144	16.1045
72				(30, 70)	22.6562	22.9623	8.23113	8.23402

Table 1 compares the A.E.B with the M.E.A under various JPT2C schemes, sample sizes, and allocation patterns for two populations, considering two values of the mixing probability p . Overall, the results demonstrate a close agreement between A.E.B and M.E.A across most censoring schemes, indicating that the proposed approximation provides a reliable representation of the empirical performance under joint progressive censoring. This agreement is particularly evident for moderate censoring schemes and balanced sample allocations, where the absolute discrepancies between A.E.B and M.E.A remain small for both values of p .

As expected, larger discrepancies are observed under extreme censoring scenarios, especially when a substantial number of units are removed early in the experiment or when censoring is highly concentrated at later stages. These effects are more pronounced for smaller effective sample sizes and unbalanced allocations between the two populations. In such cases, the reduction in information due to aggressive removals leads to increased variability, which explains the observed deviation between the theoretical approximation (A.E.B) and the empirical measure (M.E.A). Nevertheless, even in these challenging settings, the overall trend of M.E.A closely follows that of A.E.B, suggesting that the approximation remains qualitatively informative. Additionally, increasing the total sample size from $N = 30$ to $N = 50$ substantially improves agreement between A.E.B and M.E.A across nearly all censoring schemes, highlighting the stabilizing effect of larger samples. Similar patterns are observed for both values of p , although slightly larger discrepancies tend to occur when $p = 0.26$, reflecting increased heterogeneity in the population composition. Furthermore, the approximation performs poorly under specific censoring schemes (SCs No. 4–6, 31–33, and 45–51), attributing this behavior to extreme removal patterns and limited effective sample sizes.

Comparison of Estimation Methods

In this subsection, a Monte Carlo simulation study is conducted for two WEDs under varying sample sizes and different numbers of observed failures r . The simulation framework is designed to assess and compare the performance of the maximum likelihood and Bayesian estimation procedures. The estimators of the parameters α_h , γ_h , and β_h are evaluated using the mean squared error (MSE) criterion, which serves as a measure of their estimation accuracy, where $MSE = \frac{1}{M} \sum_{i=1}^M \left(\hat{\Omega}_{jh}^{(i)} - \Omega_{jh} \right)^2$, $j = 1, 2, 3$. The simulation study considers three different censoring schemes (CSs). In Scheme I, all remaining $N - r$ units are removed at the time of the first observed failure, that is, $R_1 = N - r$, $R_i = 0$ for $i \neq 1$. In Scheme II, the removals occur at the midpoint of the experiment: if r is even, $R_{\frac{r}{2}} = N - r$, with $R_i = 0$ for $i \neq \frac{r}{2}$; if r is odd, $R_{\frac{r}{2}+1} = N - r$, and $R_i = 0$, for all other indices. In Scheme III, all remaining units are withdrawn at the time of the last observed failure, that is, $R_r = N - r$, and $R_i = 0$ for $i \neq r$. Under each of these censoring schemes, and for various choices of m , n , r , p and R , a total of 1000 JPT2C samples are generated from two WEDs. The outcomes of the Monte Carlo simulation study are summarized in Tables 2–4. Based on these numerical results, several key observations and conclusions can be derived, as discussed below.

1. The MSEs of the parameter estimates decrease as the number of observed failures r increases.
2. Bayesian estimators outperform the ML estimators in terms of accuracy, as they exhibit smaller MSEs.
3. Bayesian estimates obtained under the SE loss function are nearly identical to those derived under the LINEX loss function when $C = 0.0001$.
4. Bayesian estimation under the LINEX loss function with $C = 2$ yields more accurate estimates, as reflected by comparatively smaller MSEs.

Table 2. MSE of estimates for the parameters with $(\alpha_1, \alpha_2, \gamma_1, \gamma_2, \beta_1, \beta_2) = (1.5, 0.05, 2, 6, 2.5, 1.5)$, $m = 15, n = 15$ at $r = 10$.

SC	()	MLE	Bayesian (MCMC)			
			SE	LINEX		
				$C = 0.0001$	$C = -2$	$C = 2$
I	α_1	0.6319	0.7207	0.7207	0.5941	0.8442
	α_2	0.0047	0.0034	0.0034	0.0039	0.0031
	γ_1	0.6061	0.6102	0.6102	0.6102	0.6102
	γ_2	0.9505	0.9610	0.9610	0.9584	0.9626
	β_1	0.3552	0.3541	0.3541	0.3540	0.3542
	β_2	0.4656	0.4665	0.4665	0.4663	0.4667
II	α_1	0.7105	0.7880	0.7880	0.6842	0.8923
	α_2	0.0028	0.0026	0.0026	0.0028	0.0023
	γ_1	0.7564	0.7571	0.7571	0.7571	0.7570
	γ_2	1.0757	1.0935	1.0935	1.0877	1.0973
	β_1	0.4022	0.4022	0.4022	0.4022	0.4022
	β_2	0.4710	0.4730	0.4730	0.4725	0.4736
III	α_1	0.4640	0.5062	0.5062	0.4029	0.6190
	α_2	0.0019	0.0017	0.0017	0.0019	0.0016
	γ_1	0.3612	0.3615	0.3615	0.3615	0.3615
	γ_2	1.0076	1.0339	1.0339	1.0270	1.0384
	β_1	0.4178	0.4178	0.4178	0.4178	0.4178
	β_2	0.5939	0.6009	0.6009	0.5996	0.6020

Table 3. MSE of estimates for the parameters with $(\alpha_1, \alpha_2, \gamma_1, \gamma_2, \beta_1, \beta_2) = (1.5, 0.05, 2, 6, 2.5, 1.5)$, $m = 25, n = 25$ at $r = 10$.

SC	()	MLE	Bayesian (MCMC)			
			SE	LINEX		
				$C = 0.0001$	$C = -2$	$C = 2$
I	α_1	0.6017	0.7081	0.7081	0.5759	0.8349
	α_2	0.0039	0.0031	0.0031	0.0035	0.0028
	γ_1	0.5752	0.5751	0.5750	0.5751	0.575
	γ_2	1.0884	1.0996	1.0996	1.0967	1.1016
	β_1	0.3505	0.3504	0.3504	0.3505	0.3503
	β_2	0.5141	0.5151	0.5151	0.5149	0.5154
II	α_1	0.7238	0.7859	0.7859	0.6840	0.8897
	α_2	0.0023	0.0020	0.0020	0.0022	0.0019
	γ_1	0.8087	0.8084	0.8084	0.8084	0.8084
	γ_2	1.2041	1.2182	1.2182	1.2141	1.2215
	β_1	0.3377	0.3377	0.3377	0.3377	0.3377
	β_2	0.4856	0.4871	0.4871	0.4867	0.4876
III	α_1	0.4328	0.4849	0.4849	0.3790	0.5995
	α_2	0.0019	0.0016	0.0016	0.0018	0.0015
	γ_1	0.4188	0.4193	0.4193	0.4193	0.4193
	γ_2	1.1614	1.1721	1.1721	1.1695	1.1739
	β_1	0.4149	0.4147	0.4147	0.4147	0.4147
	β_2	0.6247	0.6261	0.6261	0.6257	0.6265

Table 4. MSE of estimates for the parameters with $(\alpha_1, \alpha_2, \gamma_1, \gamma_2, \beta_1, \beta_2) = (1.5, 0.05, 2, 6, 2.5, 1.5)$, $m = 40, n = 40$ at $r = 10$.

SC	()	MLE	Bayesian (MCMC)			
			SE	LINEX		
				$C = 0.0001$	$C = -2$	$C = 2$
I	α_1	0.6098	0.6941	0.6941	0.5650	0.8205
	α_2	0.0040	0.0030	0.0030	0.0034	0.0027
	γ_1	0.5967	0.5968	0.5968	0.5968	0.5967
	γ_2	1.2525	1.2604	1.2604	1.2584	1.2618
	β_1	0.3267	0.3266	0.3266	0.3266	0.3266
	β_2	0.5626	0.5629	0.5629	0.5628	0.5630
II	α_1	0.7343	0.8242	0.8242	0.7225	0.9256
	α_2	0.0030	0.0025	0.0025	0.0028	0.0023
	γ_1	0.9161	0.9161	0.9161	0.9162	0.9161
	γ_2	1.1770	1.1926	1.1926	1.1889	1.1951
	β_1	0.3037	0.3037	0.3037	0.3037	0.3037
	β_2	0.4772	0.4781	0.4781	0.4778	0.4784
III	α_1	0.4076	0.4617	0.4617	0.3451	0.5853
	α_2	0.0022	0.0019	0.0019	0.0020	0.0017
	γ_1	0.4498	0.4497	0.4497	0.4497	0.4497
	γ_2	1.0504	1.0587	1.0587	1.0564	1.0605
	β_1	0.4345	0.4345	0.4345	0.4345	0.4345
	β_2	0.6852	0.6863	0.6863	0.6860	0.6867

6. Applications and Data Analysis

In this section, the proposed model is implemented for two WE populations. The results are first illustrated using a simulated example, after which the model is applied to two real datasets to assess and demonstrate the performance and effectiveness of the proposed methods.

6.1. Illustrative Example

An approximate approach is employed to estimate the expected number of failures occurring prior to the commencement of the test, while the exact numbers of failures are recorded after the completion of the test procedure. To assess the accuracy of this approximation, the estimated values are compared with the corresponding exact values using a simulation-based example. In this setting, simulated data are generated for the case $k = 2$. Table 5 presents the simulated failure times corresponding to the first production line, generated using the parameter values $\alpha_1 = 1.5$, $\gamma_1 = 2$, and $\beta_1 = 2.5$ with sample size $n_1 = 25$. Similarly, Table 6 reports the simulated failure times for the second production line, where the parameters are set to $\alpha_2 = 0.05$, $\gamma_2 = 5$, and $\beta_2 = 1.5$ with sample size $n_2 = 20$.

Table 5. Simulated data from the first production line.

0.159	0.171	0.189	0.202	0.213	0.219	0.222	0.223	0.229	0.238	0.258	0.259
0.263	0.264	0.280	0.294	0.297	0.310	0.311	0.314	0.316	0.323	0.355	0.385
0.412											

Table 6. Simulated data from the second production line.

0.051	0.053	0.151	0.204	0.231	0.258	0.301	0.313	0.361	0.367	0.401	0.428
0.438	0.454	0.462	0.468	0.494	0.515	0.565	0.601				

Table 7 displays the simulated JPT2C data obtained from the two production lines under a joint progressive censoring scheme with total sample size $N = n_1 + n_2$, censoring point $r = 10$ and censoring vector $R = (2, 2, 0, 2, 2, 1, 1, 2, 0, 23)$. This scheme is constructed based on the selections $s(1) = (2, 0, 0, 0, 2, 0, 1, 2, 0, 8)$ and $s(2) = (0, 2, 0, 2, 0, 1, 0, 0, 0, 15)$, where $R_i = \sum_{h=1}^2 s_i(h)$, $i = 1, 2, \dots, 10$. In addition, Tables 8 and 9 report various point estimates of the parameters $\alpha_1, \gamma_1, \beta_1; \alpha_2, \gamma_2, \beta_2$, along with their corresponding 95% confidence intervals. Within the MCMC framework, the Markov chain is generated for 12,000 iterations, with the initial 2000 samples removed as a burn-in period. The hyperparameters representing prior information are specified as $a = c = u = 0.01$ and $b = d = v = 2.0$. By applying the approximation proposed by Parsi and Bairamov [27] and evaluating its performance, the expected number of failures is estimated as 6.8324 for the first production line and 4.1395 for the second production line. In comparison, the observed numbers of failures after the test procedure are 7 and 3 for the first and second production lines, respectively.

Table 7. Simulated JPT2C data from the two production lines.

w	0.051	0.053	0.151	0.159	0.171	0.189	0.202	0.213	0.222	0.223
Z	0	0	0	1	1	1	1	1	1	1

Table 8. Different point estimates for $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \beta_1$, and β_2 .

	$(\cdot)_{ML}$	$(\cdot)_{SE}$	$(\cdot)_{LINEX}$		
			$C = 0.0001$	$C = -2$	$C = 2$
$\hat{\alpha}_1$	1.0675	1.6383	1.6384	1.8602	1.4667
$\hat{\alpha}_2$	0.0597	0.0675	0.0665	0.0686	0.0663
$\hat{\gamma}_1$	1.7541	1.7826	1.7824	1.7825	1.7824
$\hat{\gamma}_2$	5.1103	5.0845	5.0845	5.0854	5.0836
$\hat{\beta}_1$	2.7724	2.8507	2.8507	2.8513	2.8498
$\hat{\beta}_2$	1.1876	1.1798	1.1799	1.1798	1.1796

Table 9. 95% confidence intervals for $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \beta_1$ and β_2 .

Method	α_1	γ_1	β_1
ACI	[0.7351, 2.0342]	[0.9213, 2.6880]	[1.2547, 4.5428]
CRI	[0.6952, 2.7944]	[1.7648, 1.7999]	[2.7998, 2.8843]
	α_2	γ_2	β_2
ACI	[0.0189, 0.1019]	[2.8242, 7.2351]	[0.5862, 1.9912]
CRI	[0.0135, 0.1624]	[5.0191, 5.1356]	[1.1736, 1.1867]

6.2. Melanoma Example

In this subsection, we analyze a dataset consisting of survival times, measured in months, for stage-4 melanoma patients classified by gender who were treated at the University of Oklahoma Health Sciences Center during the period 1974–1978. These data were first presented by Lee et al. [36], and additional details can be found in Lee and Wang [37]. Moreover, this data is used in joint progressive censoring by Boulkeroua et al. [38]. The datasets are organized as follows: Data set 1 corresponds to female patients:

1.3 3.8 2.7 4.2 7.4 9.3 10.5 11.4 13.3 13.8 13.8 20 22.2

Data set 2 corresponds to male patients:

0.4	0.9	1.2	1.5	1.6	1.7	2.5	2.5	3.9	3.9	4	4.2
4.5	5.8	5.9	6.3	7.3	7.4	8.3	9.8	11	11.1	16.1	20.5

To assess the adequacy of the fitted models, the Kolmogorov–Smirnov (KS) statistic was computed by comparing the empirical distributions with the corresponding fitted distribution functions. For Data Set 1, the KS distance between the empirical distribution and the WED cumulative distribution function is 0.13336, with an associated p -value of 0.9749, based on the estimated parameters $(\hat{\alpha}_1, \hat{\gamma}_1, \hat{\beta}_1) = (0.592641, 0.0776059, 1.14272)$. This result indicates an excellent fit of the WED to Data Set 1. Similarly, for Data Set 2, the KS distance is 0.077296 with a p -value of 0.9988 under the parameter estimates $(\hat{\alpha}_2, \hat{\gamma}_2, \hat{\beta}_2) = (14.9896, 0.0148996, 1.18764)$, demonstrating a strong agreement between the WED and the data. Furthermore, to further strengthen the goodness-of-fit analysis, we complement the KS test with additional diagnostic tools. In particular, bootstrap-based Q–Q and P–P plots are provided to visually assess the agreement between the fitted model and the observed data; see Figures 1 and 2.

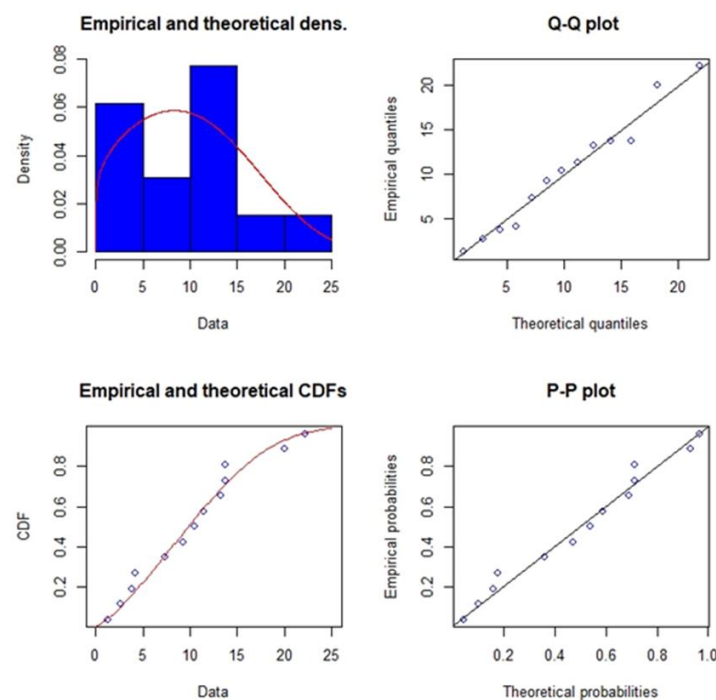


Figure 1. The empirical, Q–Q, and P–P plots for Data Set 1.

Also, to further support the choice of the WED, we compare its goodness-of-fit with several simpler competing lifetime distributions, namely the Weibull, gamma, and lognormal models. Model comparison is conducted using the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC).

Table 10 reports the AIC and BIC values for the female and male patient datasets. For the female dataset, the Weibull distribution yields the smallest AIC and BIC, followed closely by the WED. The differences between these two models are relatively small, indicating that the WED provides a comparable fit while offering additional flexibility in modeling hazard-rate behavior. The gamma and lognormal models show larger AIC and BIC values, suggesting inferior fit.

For the male dataset, all models perform similarly in terms of AIC, with the gamma and Weibull models yielding slightly smaller values than the WED. However, once model

complexity is more heavily penalized through BIC, the simpler Weibull and gamma distributions are favored, as expected for moderate sample sizes. Nevertheless, the WED remains competitive, with AIC differences well within commonly accepted thresholds, indicating no substantial loss of fit.

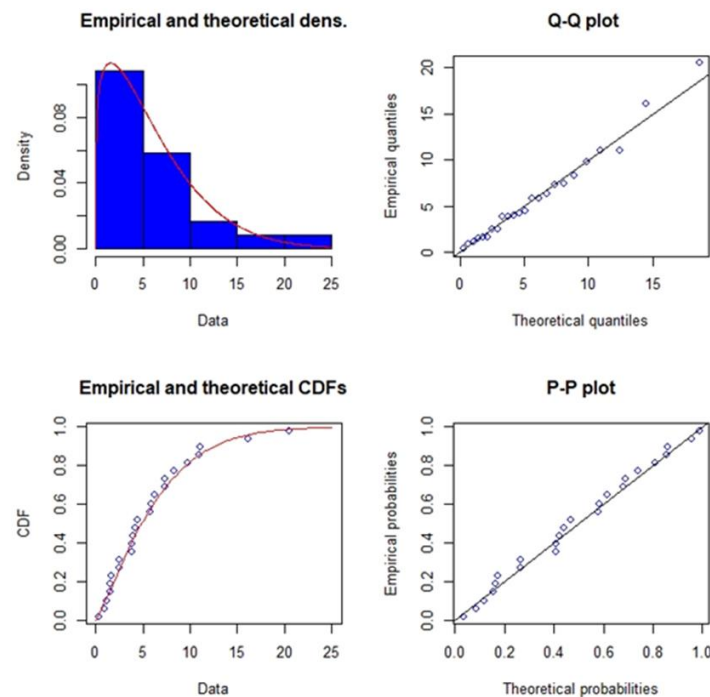


Figure 2. The empirical, Q-Q, and P-P plots for Data Set 2.

Table 10. AIC and BIC values for competing lifetime models.

Dataset	Model	AIC	BIC
Female patients	Weibull	86.55	87.67
	Gamma	87.18	88.31
	Lognormal	89.19	90.32
	WE	88.47	90.17
Male patients	Weibull	135.56	137.92
	Gamma	135.35	137.70
	Lognormal	136.76	139.12
	WE	135.69	139.22

Overall, these results suggest that while simpler models may be adequate in some cases, the WED provides a flexible and competitive alternative, particularly when additional hazard-rate flexibility is required. When considered alongside the bootstrap-based goodness-of-fit tests and graphical diagnostics, the information-criterion analysis supports the use of the WED for the analyzed datasets.

Table 11 presents the JPT2C sample constructed from Data Sets 1 and 2 under a joint progressive censoring scheme with total sample size $N = n_1 + n_2$ and $r = 10$. The censoring vector is specified as $R = (2, 2, 0, 2, 2, 1, 1, 2, 0, 15)$, which is obtained from the corresponding allocations $s(1)$ and $s(2)$, where $s(1) = (0, 0, 0, 2, 2, 0, 1, 2, 0, 2)$ and $s(2) = (2, 2, 0, 0, 0, 1, 0, 0, 0, 13)$, such that $R_i = \sum_{h=1}^2 s_i(h)$, $i = 1, 2, \dots, 10$. The point estimates of the parameters $(\alpha_1, \gamma_1, \beta_1)$ and $(\alpha_2, \gamma_2, \beta_2)$, along with their corresponding 95% confidence intervals, are reported in Tables 12 and 13, respectively. A comparison of the results obtained from the Bayesian and maximum likelihood approaches indicates that

both methods yield very similar estimates in the melanoma data application, supporting the adequacy of the proposed procedures. Moreover, the expected numbers of failures are estimated as 3.0849 for the first dataset and 7.8255 for the second dataset, which are close to the observed post-test failure counts of 3 and 7, respectively.

Table 11. JPT2C data from the two data sets.

w	0.4	0.9	1.3	1.5	2.5	2.5	2.7	3.8	3.9	4
Z	0	0	1	0	0	0	1	1	0	0

From Table 12, it is observed that the Bayesian estimates under the SE loss function are very close to those obtained under the LINEX loss function when $C = 0.0001$, which is consistent with the theoretical property that the LINEX loss approximates the SE loss for values of C close to zero. In contrast, the LINEX estimates with $C = -2$ and $C = 2$ exhibit noticeable asymmetry, particularly for the scale parameters α_1 and α_2 , reflecting the asymmetric penalization of overestimation and underestimation inherent in the LINEX loss. Overall, the Bayesian estimates demonstrate stability across loss functions and, in several cases, provide shrinkage toward more plausible parameter values compared to the MLEs.

Table 12. Point estimates for $(\alpha_1, \gamma_1, \beta_1)$ and $(\alpha_2, \gamma_2, \beta_2)$.

	$(\cdot)_{ML}$	$(\cdot)_{SE}$	$(\cdot)_{LINEX}$		
			$C = 0.0001$	$C = -2$	$C = 2$
$\hat{\alpha}_1$	0.1296	0.0525	0.0525	0.0537	0.0514
$\hat{\alpha}_2$	1.6348	2.0800	2.0800	3.1957	1.6295
$\hat{\gamma}_1$	0.2797	0.2804	0.2803	0.2805	0.2802
$\hat{\gamma}_2$	0.0735	0.0651	0.0652	0.0651	0.0651
$\hat{\beta}_1$	1.8821	1.8795	1.8795	1.8799	1.8792
$\hat{\beta}_2$	1.3013	1.2995	1.2995	1.4995	1.2999

Table 13 presents the corresponding 95% ACIs and CRIs. The CRIs are generally shorter and more concentrated than the ACIs, especially for the shape parameters β_1 and β_2 , indicating reduced estimation uncertainty under the Bayesian framework. Moreover, the Bayesian intervals respect the positivity constraints of the parameters and are centered around the Bayesian point estimates, whereas the ACIs are wider due to their reliance on asymptotic normality. These findings illustrate the practical advantage of Bayesian inference in censored reliability data, particularly under JPT2C, where sample information may be limited and asymptotic approximations can be less precise.

Table 13. 95% CIs for $(\alpha_1, \gamma_1, \beta_1)$ and $(\alpha_2, \gamma_2, \beta_2)$.

Method	α_1	γ_1	β_1
ACI	[0.0854, 0.1755]	[0.1543, 0.4129]	[1.0547, 2.7054]
CRI	[0.0094, 0.1349]	[0.2656, 0.3104]	[1.852, 1.9248]
	α_2	γ_2	β_2
ACI	[0.6584, 2.651]	[0.0100, 0.1270]	[0.6141, 2.0473]
CRI	[0.8392, 3.9326]	[0.0595, 0.0722]	[1.1964, 1.5927]

6.3. An Insulating Fluid Example

In this subsection, the theoretical results are demonstrated through a numerical illustration based on Nelson's dataset (Nelson [39]), which records the breakdown times, measured in minutes, of an insulating fluid exposed to high-voltage stress. This dataset

has been widely analyzed in the literature, including studies by Abo-Kasem et al. [40] on jointly type-I progressive hybrid censored exponential populations, by Krishna and Goel [41] on jointly type-II censored Lindley populations and by Cetinkaya et. al. [42] on jointly generalized progressive hybrid censored exponential populations. In the present analysis, the observed failure times are denoted by X_1 and X_2 , respectively as follows

$$\begin{array}{l} X_1: \quad 1.99 \quad 0.64 \quad 2.15 \quad 1.08 \quad 2.57 \quad 0.93 \quad 4.75 \quad 0.82 \quad 2.06 \quad 0.49 \\ X_2: \quad 8.11 \quad 3.17 \quad 5.55 \quad 0.80 \quad 0.20 \quad 1.13 \quad 6.63 \quad 1.08 \quad 2.44 \quad 0.78 \end{array}$$

To evaluate how well the proposed models fit the data, the KS test was applied by contrasting the empirical distributions with the fitted cumulative distribution functions. The KS statistic and the corresponding p -value with the parameter estimates are reported in Table 14, which confirms a strong consistency between the WED and the observed data. In addition, the empirical distribution functions, along with the bootstrap-based Q-Q and P-P plots displayed in Figures 3 and 4, further support the conclusion that the WE model provides an adequate fit for both datasets.

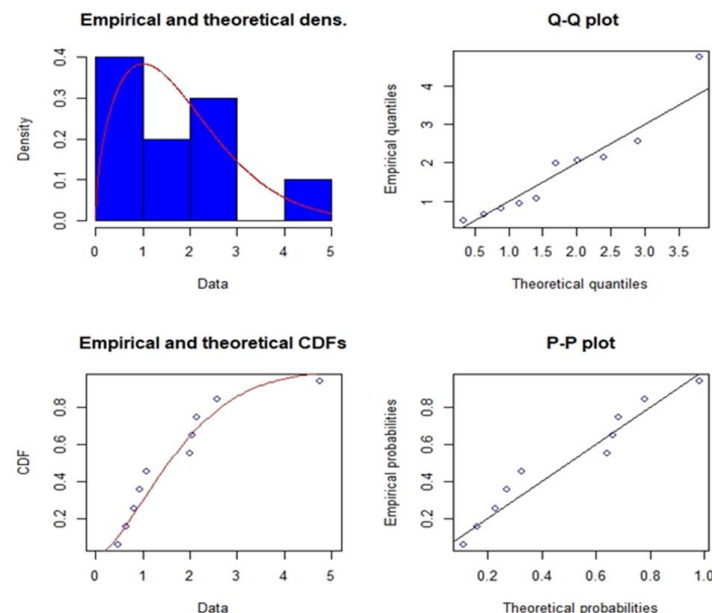


Figure 3. The empirical, Q-Q, and P-P plots for X_1 .

Table 14. The fit test values for X_1 and X_2 .

Data	KS Statistic	p -Value	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\beta}$
X_1	0.17276	0.87882	38.9339	0.05629	1.55232
X_2	0.21330	0.67873	1.94252	0.11869	0.90105

To further assess the adequacy of the WED, we compare its fit with several simpler competing lifetime distributions, namely the Weibull, gamma, and lognormal models. The comparison is based on AIC and BIC, where smaller values indicate a better balance between model fit and complexity, see Table 15.

For both datasets X_1 and X_2 , the WED yields substantially smaller AIC and BIC values than the Weibull, gamma, and lognormal models. The improvements are particularly pronounced, indicating a markedly superior fit even after accounting for the additional model parameter. These large differences far exceed commonly used thresholds for meaningful improvement, providing strong evidence in favor of the WED. The superior performance of the WED suggests that the underlying lifetime data exhibit hazard-rate features that cannot

be adequately captured by simpler two-parameter distributions. In particular, the added flexibility of the WED allows it to accommodate more complex hazard shapes, which appears to be essential for accurately modeling the observed data in both datasets.

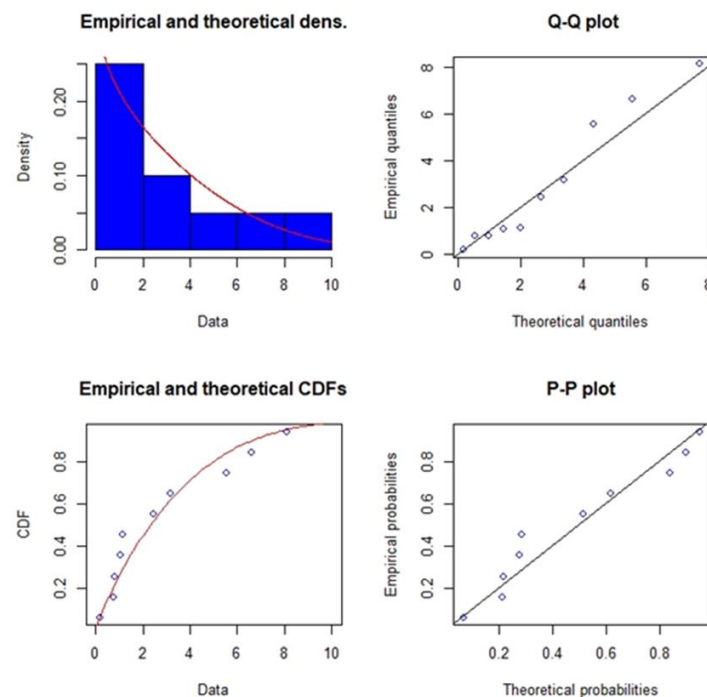


Figure 4. The empirical, Q–Q, and P–P plots for X_2 .

Table 15. AIC and BIC values for for X_1 and X_2 .

Dataset	Model	AIC	BIC
X_1	Weibull	32.41	33.02
	Gamma	31.81	32.42
	Lognormal	31.16	31.77
	WE	22.79	23.70
X_2	Weibull	45.83	46.44
	Gamma	45.82	46.43
	Lognormal	46.18	46.79
	WE	16.74	17.65

Table 16 displays the JPT2C sample derived from X_1 and X_2 using a joint progressive censoring scheme with a total sample size $N = n_1 + n_2$ and $r = 10$. The associated censoring scheme is given by $R = (2, 0, 2, 0, 2, 0, 0, 2, 0, 2)$, which is constructed from the component vectors $s(1) = (0, 0, 2, 0, 2, 0, 0, 0, 0, 2)$ and $s(2) = (2, 0, 0, 0, 0, 0, 0, 2, 0, 0)$, satisfying $R_i = \sum_{h=1}^2 s_i(h)$, $i = 1, 2, \dots, 10$. The corresponding point estimates of the parameters $(\alpha_1, \gamma_1, \beta_1)$ and $(\alpha_2, \gamma_2, \beta_2)$, together with their 95% confidence intervals, are summarized in Tables 17 and 18, respectively. Comparing the Bayesian and maximum likelihood results reveals that both approaches produce closely matching estimates in an insulating fluid data analysis, thereby confirming the reliability of the proposed methods. Furthermore, the estimated expected numbers of failures are 6.1736 for X_1 and 4.4543 for X_2 , which are in good agreement with the observed numbers of failures after testing, namely 6 and 4.

Table 16. JPT2C data from X_1 and X_2 .

w	0.20	0.49	0.64	0.80	0.82	0.93	1.08	1.08	1.13	4.75
Z	0	1	1	0	1	1	1	0	0	1

Tables 17 and 18 present the ML and Bayesian point and interval estimates of the WE parameters for Nelson’s insulating fluid breakdown data. From Table 17, it is evident that the Bayesian estimates obtained under the SE loss function are nearly identical to those derived under the LINEX loss with $C = 0.0001$, confirming the close relationship between these two loss functions when the asymmetry parameter is close to zero. The LINEX estimates with $C = -2$ and $C = 2$ exhibit asymmetric behavior, particularly for the scale parameters α_1 and α_2 , reflecting the differential penalization of overestimation and underestimation inherent in the LINEX loss. Overall, the Bayesian estimates demonstrate greater stability than the MLEs and provide interpretable parameter values consistent with the underlying failure mechanism of the insulating fluid.

Table 17. Point estimates for $(\alpha_1, \gamma_1, \beta_1)$ and $(\alpha_2, \gamma_2, \beta_2)$.

	$(\cdot)_{ML}$	$(\cdot)_{SE}$	$(\cdot)_{LINEX}$		
			$C = 0.0001$	$C = -2$	$C = 2$
$\hat{\alpha}_1$	2.7718	1.9882	1.9881	2.8710	1.5728
$\hat{\alpha}_2$	0.1472	0.0582	0.0582	0.0594	0.0571
$\hat{\gamma}_1$	0.0881	0.0864	0.0864	0.0866	0.0846
$\hat{\gamma}_2$	1.4808	1.5283	1.5282	1.5289	1.5276
$\hat{\beta}_1$	0.9503	0.9428	0.9427	0.9429	0.9428
$\hat{\beta}_2$	1.2209	1.2528	1.2529	1.2531	1.2527

Table 18. 95% CIs for $\alpha_1, \gamma_1, \beta_1, \alpha_2, \gamma_2$ and β_2 .

Method	α_1	γ_1	β_1
ACI	[1.4992, 4.0825]	[0.0204, 0.1534]	[0.4257, 1.4947]
CRI	[0.7944, 3.7367]	[0.0812, 0.0973]	[0.9354, 0.9792]
	α_2	γ_2	β_2
ACI	[0.0831, 0.1957]	[1.0036, 2.0335]	[0.4514, 2.1254]
CRI	[0.0114, 0.1445]	[1.4774, 1.5685]	[1.1415, 1.4639]

Table 18 reports the corresponding 95% ACIs and Bayesian CRIs. The CRIs are generally narrower and more concentrated around the Bayesian point estimates than the ACIs, especially for the shape parameters β_1 and β_2 , indicating reduced estimation uncertainty under the Bayesian framework. In addition, the Bayesian intervals respect the positivity constraints of the parameters and exhibit improved coherence with the point estimates, whereas the ACIs are wider due to their reliance on asymptotic normal approximations. These results further support the effectiveness of the Bayesian approach for analyzing breakdown time data under JPT2C.

7. Conclusions

The JPT2C sampling approach estimates the k WE population parameters by both Bayesian and non-Bayesian methodologies. The Metropolis–Hastings method for Bayesian estimation underscored the significance of the MCMC technique. An example is presented to illustrate the functionality of the MLE and MCMC algorithms. The invariance property of the MLEs of the parameters is utilized to assess the reliability of the units generated by the k production lines. Preliminary experimental data regarding the failure rate of the

initial production line is estimated prior to the testing methodology and juxtaposed with the actual failure count post-test operation. By analyzing the failure rates of units produced by different production lines, the proposed model may be utilized to assess the reliability of those lines. These manufacturing facilities may be located within the same building or in a separate structure. The proposed model is recommended for evaluating the quality of manufacturing processes and can be a valuable tool for engineering research focused on failure analysis. When the study's selected samples were somewhat large, it yielded more reliable results. The suggested model relies on a parametric framework. The efficacy of the proposed strategies outlined in this research is evaluated in the concluding Monte Carlo simulation analysis offered. The data indicate that the A.E.B are comparable to the M.E.A of M_r across many censoring scenarios, with the exception of schemes No. 4, 5, 6, 31, 32, 33, 49, 50, and 51. The applicability and robustness of the proposed approaches were further demonstrated by analysis of actual datasets. The empirical results validated that the WED adequately fits the data, and that the suggested inferential methods produce consistent and equivalent outcomes in both classical and Bayesian frameworks.

The findings from the simulation study provide important insight into the empirical performance of the proposed methods observed in the real-data applications. In particular, the simulation results demonstrate that the ML and Bayesian estimators exhibit stable behavior and satisfactory accuracy under varying sample sizes and censoring schemes, with Bayesian estimators especially under the LINEX loss function, often yielding smaller MSEs. These patterns are consistent with the real-data analyses, where Bayesian point estimates and CRIs tend to be more stable and informative than their asymptotic counterparts, particularly in the presence of heavy censoring. Moreover, the close agreement between the approximated and empirical expected numbers of failures observed in the simulations helps explain the reliable fit of the WED to the real datasets. Together, the simulation outcomes validate the robustness of the proposed inferential procedures and provide theoretical support for the empirical findings, thereby strengthening confidence in the practical applicability of the methods under JPT2C.

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