

Article

Strong Laws of Large Numbers for General Random Variables Under Conditional Sub-Additive Expectation and Capacity

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Abstract

We study strong laws of large numbers in a non-linear framework based on conditional sub-additive expectations and conditional sub-additive capacities. Using an axiomatic approach to conditional sub-additive expectation, we establish a conditional Hájek–Rényi-type maximal inequality assuming a general conditional Kolmogorov-type maximal inequality but without imposing any weak dependence structure on the underlying sequence. As a consequence, we derive a general conditional strong law of large numbers. Finally, we introduce a notion of conditional negative dependence under sub-additive expectations and obtain the corresponding conditional Kolmogorov-type maximal inequality, leading to a conditional strong law of large numbers for conditionally negatively dependent random variables.

Keywords: sub-additive expectation; sub-additive probability; conditional sub-additive expectation; strong laws of large numbers; conditional negative dependence

MSC: 60C05; 60F15

1. Introduction

The strong laws of large numbers (SLLN) is a fundamental result in probability theory that guarantees that sample averages converge to the expected value under appropriate assumptions. However, in many complex applications, the standard assumptions of additivity of probability measures and linearity of expectations may fail. To address such situations, alternative probabilistic frameworks have been developed, particularly those involving sub-additive probabilities (capacities) and sub-additive expectations.

One of the most famous results in probability theory is the following: Kolmogorov's SLLN. Let $\xi_1, \xi_2, \dots$ be independent identically distributed random variables with finite expectation $\mu = \mathbb{E}\xi_i$ and with partial sum $S_n = \sum_{i=1}^n \xi_i$. Then $S_n/n \rightarrow \mu$ almost surely as $n \rightarrow \infty$, see [1]. The original proof of the above SLLN is based on the well-known Kolmogorov's maximal inequality. Another essential tool for proving the SLLN is the Hájek–Rényi inequality (see [2]). Both the Kolmogorov and the Hájek–Rényi inequalities have numerous extensions to non-independent random variables.

Fazekas and Klesov in [3] presented a general approach to SLLNs for sequences of possibly non-independent random variables. They proved that a Kolmogorov-type inequality implies a Hájek–Rényi-type inequality, which in turn implies an SLLN directly; see also [4]. Their method imposes no restriction on the underlying dependence structure of



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the random variables. Therefore, the approach of [3] was applied and extended by several authors, see, e.g., [5].

Conditional versions of numerous SLLNs were obtained, see, e.g., [6]. The conditional versions of some results of [3] were obtained in [7]. Another way to extend the scope of the usual SLLNs is to obtain their appropriate version for sub-additive probabilities and for sub-additive expectations. Huang and Wu in [8] employed the method of [3] to obtain a general SLLN of the form (12) in sub-additive expectation spaces that does not require independence of the random variables.

A novel approach is to obtain SLLNs for conditional sub-additive probabilities and conditional sub-additive expectations; see [9].

In this paper, we want to extend some results of [3,4] to conditional sub-additive expectations and conditional sub-additive probabilities. To this end, we shall find some plausible axioms of conditional sub-additive expectations and conditional sub-additive probabilities that guarantee a certain general SLLN. We aim to understand which assumptions are necessary to prove the SLLN and which are superfluous.

In the setting of sub-additive expectation and sub-additive probability, the Kolmogorov-type SLLN takes a generalized form: it asserts that every cluster point of the sequence of empirical averages lies between the lower and upper expectations, with lower capacity equal to 1. Formally,

$$v\left(\mathcal{E}[X_1] \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k \leq \hat{\mathbb{E}}[X_1]\right) = 1. \quad (1)$$

Here and in what follows $\hat{\mathbb{E}}[X]$ is the upper expectation (sub-linear expectation) of X , $\mathcal{E}[X] = -\hat{\mathbb{E}}[-X]$ is the lower expectation, $\hat{\mathbb{V}}(A)$ is the upper probability (upper capacity) of A , $v(A) = 1 - \hat{\mathbb{V}}(A^c)$ is the lower probability (lower capacity), where A^c is the complementary set of A , see [10–12] and Section 2 of this paper. For the precise details of the SLLN in (1), see [11,13,14].

Chen [11] summarized key properties and lemmas concerning upper and lower probabilities (capacities), providing a basis for limit theorems under non-additive measures. Then, in [11], an SLLN was proved for independent random variables having uniformly bounded second moments. Subsequently, Chen, Wu, and Li [14] established a strong law of large numbers for independent random variables having uniformly bounded $1 + \alpha$ moments ($\alpha > 0$) under non-additive probabilities. More recently, Zhang, Tang, and Xiong [9] extended the results of [14] to the G-expectation framework by proving conditional SLLNs, using the theory of conditional G-expectations stated in Hu and Peng [15]. A main result of [9] is of the shape

$$\hat{\mathbb{V}}\left(\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_i = \hat{\mathbb{E}}[X_1 | \mathcal{F}]\right) = 1, \quad (2)$$

where the random variables X_i are conditionally independent and identically distributed having finite $1 + \alpha$ moments ($\alpha > 0$). The result of [9] can be viewed as a conditional version of the non-additive SLLN obtained in [11,14].

A well-established constructive framework for sub-additive probability and expectation, and their conditional versions, is given by the theory of G-Brownian motion. This theory was introduced and developed by Peng and co-authors; see, e.g., [12,16]. This framework has drawn considerable interest from the research community, leading to numerous contributions that have advanced this field. Zhang, Tang, and Xiong used this framework in [9] to obtain conditional versions of the non-additive SLLN. Another starting point for conditional non-linear expectation and probability is the use of several probability measures on the same space. Then the upper probability and upper expectation are defined as

the supremum of probabilities and expectations, respectively. To define conditional upper probability and upper expectation, one can apply the essential supremum. In this paper, we do not use these two approaches directly. Instead, we summarize specific properties and establish them as axioms. Then, based on the axioms of the sub-additive probability and expectation, and their conditional versions, we prove our results. We emphasize that the usual probability and expectation are additive, which is a significant difference between the classical and sub-additive frameworks.

This paper is organized as follows. Section 2 introduces some basic concepts about sub-additive probabilities and expectations. Building on the ideas of several previous papers, we introduce the axioms we need in the subsequent sections. In Section 3, we fix our axioms of conditional sub-additive probabilities and expectations. In Section 4, we prove the fundamental inequalities. We show that a Kolmogorov-type inequality implies a Hájek–Rényi-type inequality in the conditional non-linear setting. Here, we follow the ideas of [3,4], see also [7,8]. In Section 5, we prove our SLLN in the conditional non-linear setting. It shows that a Kolmogorov-type maximal inequality implies the SLLN directly. Section 6 gives an application to conditional negatively dependent random variables.

2. Sub-Additive Probability and Expectation

In this section, we present the fundamental concepts and results of sub-additive expectation spaces, which provide the framework for conditional sub-additive expectations. Let Ω be a non-empty set and let $\mathcal{A}$ be a σ -algebra of subsets of Ω . Ω is the sample space, $\mathcal{A}$ is the family of events. Let $\hat{\mathbb{V}}$ be a real-valued function on $\mathcal{A}$ which we call sub-additive probability (upper probability or capacity).

Here,, we list the properties of the sub-additive probability.

Definition 1. $\hat{\mathbb{V}}$ is called a sub-additive probability if it satisfies the following properties.

1. $\hat{\mathbb{V}}(\Omega) = 1, \hat{\mathbb{V}}(\emptyset) = 0$.
2. (Monotonicity.) If $A \subset B$, then $\hat{\mathbb{V}}(A) \leq \hat{\mathbb{V}}(B)$ for any $A, B \in \mathcal{A}$.
3. (Sub-additivity.) If $A, B \in \mathcal{A}$, then

$$\hat{\mathbb{V}}(A \cup B) \leq \hat{\mathbb{V}}(A) + \hat{\mathbb{V}}(B).$$

4. (Lower-continuity.) If $A_n \uparrow A, A_n \in \mathcal{A}$, then $\hat{\mathbb{V}}(A_n) \uparrow \hat{\mathbb{V}}(A)$.

We mention that sub-additivity and lower-continuity imply the following σ -sub-additivity: For any sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$,

$$\hat{\mathbb{V}}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \hat{\mathbb{V}}(A_n).$$

An event A will be called a quasi-sure (for short q.s.) event if $\hat{\mathbb{V}}(A^c) = 0$, where A^c is the complementary set of A .

Example 1. A well-known method for obtaining a sub-additive probability is the following (see, e.g., Choquet [10]). Let $\mathcal{P}$ be a family of usual probabilities on $(\Omega, \mathcal{A})$. Let

$$\hat{\mathbb{V}}(A) = \sup_{Q \in \mathcal{P}} Q(A), \quad \forall A \in \mathcal{A}. \quad (3)$$

Then $\hat{\mathbb{V}}$ satisfies the properties listed in Definition 1.

Now, we turn to the notion of sub-additive (sometimes called sub-linear) expectation $\hat{\mathbb{E}}$. We shall apply the usual notion of a random variable. So an extended real-valued function X on Ω will be called a random variable if $X^{-1}(A) \in \mathcal{A}$ for any Borel set A . We assume that there exists a subset $\mathcal{H}$ of the random variables and there exists an extended real-valued function $\hat{\mathbb{E}}[X]$ of $X \in \mathcal{H}$. We also assume that any non-negative random variable belongs to $\mathcal{H}$. We call attention to the fact that, in Definition 2, the operation $\hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ is undefined if it were $\hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y] = \infty + (-\infty)$.

Definition 2. An extended real-valued function $\hat{\mathbb{E}}[X]$ of $X \in \mathcal{H}$ is called a sub-additive expectation if it satisfies the following properties.

1. *Monotonicity:* If $X \leq Y$, then $\hat{\mathbb{E}}[X] \leq \hat{\mathbb{E}}[Y]$.
2. *Constant preserving:* $\hat{\mathbb{E}}[c] = c$, for any $c \in \mathbb{R}$.
3. *Sub-additivity:* $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$.
4. *Positive homogeneity:* $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for any constant $\lambda \geq 0$.
5. *Monotone convergence:* If $X_n \uparrow X$, and $X_1 \geq 0$, then $\hat{\mathbb{E}}[X_n] \uparrow \hat{\mathbb{E}}[X]$.

$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-additive expectation space in contrast with a probability space. Given $X = (X_1, X_2, \dots, X_n)$, we say $X \in \mathcal{H}^n$ if $X_i \in \mathcal{H}$ for all $i = 1, \dots, n$.

If a sub-additive probability $\hat{\mathbb{V}}$ is given in advance on $(\Omega, \mathcal{A})$, then we assume that $\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[Y]$ if $X = Y$ q.s.

If the sub-additive expectation $\hat{\mathbb{E}}[\cdot]$ is given in advance, then we can introduce a sub-additive probability $\hat{\mathbb{V}}$ by $\hat{\mathbb{V}}(A) = \hat{\mathbb{E}}[\mathbb{I}_A]$, for any event A , where $\mathbb{I}_A$ denotes the indicator of A . Then this $\hat{\mathbb{V}}$ satisfies the properties given in Definition 1.

Example 2. A well-known method to obtain a sub-additive expectation is the following. Let $\mathcal{P}$ be a family of probabilities on $(\Omega, \mathcal{A})$. Let

$$\hat{\mathbb{E}}[X] = \sup_{Q \in \mathcal{P}} \mathbb{E}_Q[X], \quad (4)$$

where $\mathbb{E}_Q$ is the usual expectation corresponding to $Q \in \mathcal{P}$ such that

$$\mathbb{E}_Q[\mathbb{I}_A] = Q(A), \quad \forall A \in \mathcal{A}.$$

A simple calculation shows that the properties listed in Definition 2 are satisfied. If $\hat{\mathbb{E}}[X]$ is defined by (4), then relations among random variables in Definition 2 are considered as quasi-sure relations with respect to $\hat{\mathbb{V}}$ defined by (3).

Remark 1. Assume that an abstract sub-additive expectation $\hat{\mathbb{E}}[\cdot]$ is given, which satisfies the properties listed in Definition 2. Under certain conditions, $\hat{\mathbb{E}}$ has a representation (4). For precise formulations of the statement, see [12,17].

3. Conditional Sub-Additive Probability and Expectation

In a classical probability space $(\Omega, \mathcal{A}, Q)$, the conditional expectation $\mathbb{E}_Q[X|\mathcal{F}]$ of a random variable X with respect to a sub- σ -algebra $\mathcal{F} \subset \mathcal{A}$ is itself an $\mathcal{F}$ -measurable random variable which is simpler than X . The existence and uniqueness (up to Q -a.s. equivalence) of conditional expectation are guaranteed by the Radon–Nikodym theorem.

Here, we shall use the following abstract concept of sub-additive conditional expectation. Let $(\Omega, \mathcal{A})$ be a measurable space, and let $\mathcal{F} \subset \mathcal{A}$ be a sub- σ -algebra. We shall assume that there exists a subset $\mathcal{H}$ of the random variables on $(\Omega, \mathcal{A})$ and for any $X \in \mathcal{H}$ there exists an extended real-valued $\mathcal{F}$ -measurable random variable $\hat{\mathbb{E}}[X|\mathcal{F}]$ so that the axioms

in Definition 3 are satisfied. We call attention to the fact that, in Definition 3, we exclude the case of $\infty + (-\infty)$. We assume that any constant random variable belongs to $\mathcal{H}$.

Definition 3. $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ is called a sub-additive conditional expectation operator if $\hat{\mathbb{E}}[X|\mathcal{F}]$ is an $\mathcal{F}$ -measurable random variable for any $X \in \mathcal{H}$, which satisfies the following properties.

1. *Non-triviality:* $-\infty < \hat{\mathbb{E}}[0|\mathcal{F}] < +\infty$;
2. *Monotonicity:* $\hat{\mathbb{E}}[X|\mathcal{F}] \geq \hat{\mathbb{E}}[Y|\mathcal{F}]$ if $X \geq Y$;
3. *Sub-additivity:* $\hat{\mathbb{E}}[X + Y|\mathcal{F}] \leq \hat{\mathbb{E}}[X|\mathcal{F}] + \hat{\mathbb{E}}[Y|\mathcal{F}]$;
4. *Measurability:* $\hat{\mathbb{E}}[X + Y|\mathcal{F}] = X + \hat{\mathbb{E}}[Y|\mathcal{F}]$ if X is $\mathcal{F}$ -measurable;
5. *Positive homogeneity:* $\hat{\mathbb{E}}[cX|\mathcal{F}] = c\hat{\mathbb{E}}[X|\mathcal{F}]$ if $c \geq 0$ is a constant;
6. *Monotone convergence:* If $X_n \uparrow X$, and $X_1 \geq 0$, then $\hat{\mathbb{E}}[X_n|\mathcal{F}] \uparrow \hat{\mathbb{E}}[X|\mathcal{F}]$.

The non-triviality axiom of Definition 3 excludes the cases $\hat{\mathbb{E}}[X|\mathcal{F}] \equiv +\infty$ and $\hat{\mathbb{E}}[X|\mathcal{F}] \equiv -\infty$. Then, by positive homogeneity, $\hat{\mathbb{E}}[0|\mathcal{F}] = \hat{\mathbb{E}}[0 \cdot 0|\mathcal{F}] = 0\hat{\mathbb{E}}[0|\mathcal{F}] = 0$. Then, by the measurability axiom, $\hat{\mathbb{E}}[X|\mathcal{F}] = \hat{\mathbb{E}}[X + 0|\mathcal{F}] = X + \hat{\mathbb{E}}[0|\mathcal{F}] = X$ if X is $\mathcal{F}$ -measurable.

Now, we recall the well-known definition of the essential supremum of a family of random variables, see [18,19]. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{K}$ be an arbitrary set of extended real-valued random variables on Ω . Then there exists a countable subset $\mathcal{K}_0$ of $\mathcal{K}$ such that $\xi(\omega) = \sup\{\eta(\omega) : \eta \in \mathcal{K}_0\}$ is an extended real-valued random variable, and

- (a) $P(\xi \geq \eta) = 1$ for each $\eta \in \mathcal{K}$,
- (b) if τ is another extended real valued random variable for which $P(\tau \geq \eta) = 1$ for each $\eta \in \mathcal{K}$, then $P(\tau \geq \xi) = 1$.

ξ is called the essential supremum of $\mathcal{K}$ and it is denoted by

$$\xi = \operatorname{ess\,sup}_P \{\eta : \eta \in \mathcal{K}\}.$$

Example 3. A simple way to obtain a sub-additive conditional expectation is as follows. Let $\mathcal{P}$ be a family of usual probabilities on $(\Omega, \mathcal{A})$. Let $\mathcal{F} \subset \mathcal{A}$ be a sub- σ -algebra and let P be a fixed probability measure on $(\Omega, \mathcal{F})$. Assume that, for every $Q \in \mathcal{P}$, the restriction of Q to the σ -algebra $\mathcal{F}$ and the measure P are mutually absolutely continuous.

Let $\mathcal{H}$ be the set of random variables on $(\Omega, \mathcal{A})$ such that $\mathbb{E}_Q[X|\mathcal{F}]$, i.e., the usual conditional expectation according to Q , which is an extended real valued $\mathcal{F}$ -measurable function, exists for any $Q \in \mathcal{P}$. Let

$$\hat{\mathbb{E}}[X|\mathcal{F}] = \operatorname{ess\,sup}_P \{\mathbb{E}_Q[X|\mathcal{F}] : Q \in \mathcal{P}\}. \quad (5)$$

A simple calculation shows that the properties listed in Definition 3 are satisfied. If $\hat{\mathbb{E}}[X|\mathcal{F}]$ is defined by (5), then relations among random variables in Definition 3 are considered as quasi-sure relations with respect to $\hat{\mathbb{V}}$ defined by (3).

Now, we turn to the abstract notion of conditional sub-additive probability (in other words, conditional capacity). Let $(\Omega, \mathcal{A})$ be a measurable space, and let $\mathcal{F} \subset \mathcal{A}$ be a sub- σ -algebra. We shall assume that for any $A \in \mathcal{A}$ there exists a real valued $\mathcal{F}$ -measurable random variable $\hat{\mathbb{V}}[A|\mathcal{F}]$ so that the axioms of Definition 4 are satisfied.

Definition 4. $\hat{\mathbb{V}}[\cdot|\mathcal{F}]$ is called a sub-additive conditional probability operator if $\hat{\mathbb{V}}[A|\mathcal{F}]$ is an $\mathcal{F}$ -measurable random variable for any $A \in \mathcal{A}$, which satisfies the following properties.

1. *Normalized:* $\hat{\mathbb{V}}[\emptyset|\mathcal{F}] = 0$, $\hat{\mathbb{V}}[\Omega|\mathcal{F}] = 1$;
2. *Monotonicity:* $\hat{\mathbb{V}}[A|\mathcal{F}] \leq \hat{\mathbb{V}}[B|\mathcal{F}]$ if $A \subseteq B$;
3. *Sub-additivity:* $\hat{\mathbb{V}}[A \cup B|\mathcal{F}] \leq \hat{\mathbb{V}}[A|\mathcal{F}] + \hat{\mathbb{V}}[B|\mathcal{F}]$;

4. Lower continuity: $\hat{\mathbb{V}}[A_n|\mathcal{F}] \uparrow \hat{\mathbb{V}}[A|\mathcal{F}]$ if $A_n \uparrow A$.

Example 4. If the conditional sub-additive expectation $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ is defined according to Definition 3 and the indicator $\mathbb{I}_A$ belongs to $\mathcal{H}$ for any event from $\mathcal{A}$, then $\hat{\mathbb{V}}[A|\mathcal{F}] = \hat{\mathbb{E}}[\mathbb{I}_A|\mathcal{F}]$ defines a sub-additive conditional probability satisfying the properties in Definition 4.

Now, we consider the case of Example 3.

Example 5. Let $\mathcal{P}$ be a family of usual probabilities on $(\Omega, \mathcal{A})$. Let $\mathcal{F} \subset \mathcal{A}$ be a sub- σ -algebra and let P be a fixed probability measure on $(\Omega, \mathcal{F})$. Assume that, for every $Q \in \mathcal{P}$, the restriction of Q to the σ -algebra $\mathcal{F}$ and the measure P are mutually absolutely continuous.

For any $Q \in \mathcal{P}$, let $Q[A|\mathcal{F}]$, be the usual conditional probability given $\mathcal{F}$ according to the usual probability Q . $Q[A|\mathcal{F}]$ is a real-valued $\mathcal{F}$ -measurable random variable for any $Q \in \mathcal{P}$. Let

$$\hat{\mathbb{V}}[A|\mathcal{F}] = \operatorname{ess\,sup}_P \{Q[A|\mathcal{F}] : Q \in \mathcal{P}\}. \quad (6)$$

Of course, $\hat{\mathbb{V}}[A|\mathcal{F}] = \hat{\mathbb{E}}[\mathbb{I}_A|\mathcal{F}]$ if $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ is defined by (5). A simple calculation shows that the properties listed in Definition 4 are satisfied. If $\hat{\mathbb{V}}[A|\mathcal{F}]$ is defined by (6), then relations among random variables in Definition 4 is considered as a quasi-sure relation with respect to $\hat{\mathbb{V}}$ defined by (3): $\hat{\mathbb{V}}(A) = \sup_{Q \in \mathcal{P}} Q(A)$.

We see that in the situation of this example, each of our four operators is defined: $\hat{\mathbb{V}}(\cdot)$ is defined by equation (3), $\hat{\mathbb{V}}[\cdot|\mathcal{F}]$ is defined by (6), $\hat{\mathbb{E}}[\cdot]$ is defined by (4), and $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ is defined by (5).

We shall need information concerning the joint behavior of $\hat{\mathbb{V}}$ and $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$. So we introduce a further plausible axiom.

Definition 5. Let $(\Omega, \mathcal{A})$ be a measurable space, and let $\mathcal{F} \subset \mathcal{A}$ be a sub- σ -algebra. Let $\hat{\mathbb{V}}$ be a sub-additive probability on $\mathcal{F}$ satisfying the axioms given in Definition 1. Let $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ be a sub-additive conditional expectation satisfying the $\hat{\mathbb{V}}$ quasi-sure versions of the axioms in Definition 3. (It means that in Definition 3, any relation among random variables is understood as a $\hat{\mathbb{V}}$ quasi-sure relation.) Then the finiteness axiom is the following:

1. Finiteness: If $\hat{\mathbb{E}}[X|\mathcal{F}] < \infty$ $\hat{\mathbb{V}}$ -quasi-surely, then $X < \infty$ $\hat{\mathbb{V}}$ -quasi-surely.

The above Finiteness axiom excludes certain non-conventional cases, as the example below shows.

Example 6. Let B_1 and B_2 be non-empty disjoint sets, $\Omega = B_1 \cup B_2$, $\mathcal{A} = \{\Omega, \emptyset, B_1, B_2\}$, $\mathcal{F} = \{\Omega, \emptyset\}$. Let $\hat{\mathbb{V}}$ be a usual probability on $\mathcal{A}$ with $\hat{\mathbb{V}}(B_1) > 0$, $\hat{\mathbb{V}}(B_2) > 0$. For a random variable $X = b_1 \mathbb{I}_{B_1} + b_2 \mathbb{I}_{B_2}$, where b_1 and b_2 are fixed numbers (possibly $\pm\infty$) define $\hat{\mathbb{E}}[X|\mathcal{F}] = b_2$. Then the axioms in Definition 3 are satisfied, but the Finiteness axiom in Definition 5 fails with $X = \infty \cdot \mathbb{I}_{B_1} + 0 \cdot \mathbb{I}_{B_2}$ because $\hat{\mathbb{V}}(B_1) > 0$.

For the proof of some SLLNs, we shall need an upper continuity property. However, it may fail even in the case of a non-conditional sub-additive probability.

Example 7. Let $\Omega = [-1, 1]$, let $\mathcal{A}$ be the family of its Borel sets, and let Q_n be the uniform distribution on $[-1/n, +1/n]$. Let $\hat{\mathbb{V}}(A) = \sup_n Q_n(A)$ for any Borel set of Ω . Then for any $-1 < a < 0$ and $0 < b < 1$, we have $\hat{\mathbb{V}}([a, b]) = 1$, but $\hat{\mathbb{V}}(\{0\}) = 0$. Now, let $A_\infty = \{0\}$ and $A_n = [-1/n, +1/n]$. Therefore, $A_n \downarrow A_\infty$ but $\hat{\mathbb{V}}(A_n) = 1$ so it does not converge to $\hat{\mathbb{V}}(A_\infty) = 0$. We remark that, in this case, the family of probability measures Q_n , $n = 1, 2, \dots$ is not compact.

However, in the case of $A_n \downarrow A_\infty$ and $\hat{V}(A_n) \downarrow 0$, we have $\hat{V}(A_\infty) = 0$. It is a consequence of the non-negativity and the monotonicity axioms: $0 \leq \hat{V}(A_\infty) \leq \hat{V}(A_n) \downarrow 0$. So in this particular case, $\hat{V}(A_n) \downarrow \hat{V}(A_\infty)$. We shall need a similar upper continuity for the conditional sub-additive probability. It turns out that the following plausible recursivity axiom does the job.

Let us introduce the recursivity axiom concerning the joint behavior of $\hat{V}$ and $\hat{V}[\cdot|\mathcal{F}]$.

Definition 6. Let $(\Omega, \mathcal{A})$ be a measurable space, and let $\mathcal{F} \subset \mathcal{A}$ be a sub- σ -algebra. Let $\hat{V}$ be a sub-additive probability on $\mathcal{F}$ satisfying the axioms given in Definition 1. Let $\hat{V}[\cdot|\mathcal{F}]$ be a sub-additive conditional probability satisfying the $\hat{V}$ quasi-sure versions of the axioms in Definition 4. (It means that in Definition 4, any relation among random variables are understood as a $\hat{V}$ quasi-sure relation.) Then the recursivity axiom is the following:

1. Recursivity: If $\hat{V}[A|\mathcal{F}] = 0$ $\hat{V}$ -quasi-surely, then $\hat{V}(A) = 0$.

The Recursivity axiom excludes unconventional cases. For example, when both $\hat{V}[A|\mathcal{F}] = P_1(A)$ and $\hat{V}(A) = P_2(A)$ are usual probabilities and P_2 is not absolutely continuous with respect to P_1 .

Remark 2. The recursivity axiom implies the following upper continuity property. If $\hat{V}[A_n|\mathcal{F}] \rightarrow 0$ $\hat{V}$ -quasi-surely, and $A_n \downarrow A_\infty$, then $\hat{V}(A_\infty) = 0$. To show it, we apply the monotonicity

$$0 \leq \hat{V}[A_\infty|\mathcal{F}] \leq \hat{V}[A_n|\mathcal{F}] \rightarrow 0 \quad \hat{V}\text{-quasi-surely},$$

so $\hat{V}[A_\infty|\mathcal{F}] = 0$ $\hat{V}$ -quasi-surely, so the recursivity axiom gives $\hat{V}(A_\infty) = 0$.

The finiteness and the recursivity axioms are satisfied in the following cases.

Example 8. Consider the setting of Example 5. That is, let $\mathcal{P}$ be a family of usual probabilities on $(\Omega, \mathcal{A})$. Let $\mathcal{F} \subset \mathcal{A}$ be a sub- σ -algebra and let P be a fixed probability measure on $(\Omega, \mathcal{F})$. Assume that, for every $Q \in \mathcal{P}$, the restriction of Q to the σ -algebra $\mathcal{F}$ and the measure P are mutually absolutely continuous. Let $\hat{V}(A) = \sup_{Q \in \mathcal{P}} Q(A)$ be the sub-additive probability.

1. Let $\hat{\mathbb{E}}[X|\mathcal{F}] = \text{ess sup}_P\{E_Q[X|\mathcal{F}] : Q \in \mathcal{P}\}$ be the sub-additive conditional expectation. Then, the finiteness axiom from Definition 5 is satisfied.
2. Let $\hat{V}[A|\mathcal{F}] = \text{ess sup}_P\{Q[A|\mathcal{F}] : Q \in \mathcal{P}\}$ be the sub-additive conditional probability. Then, the recursivity axiom from Definition 6 is satisfied.

Remark 3. A well-known example of sub-additive (conditional) probability and expectation is served by the theory of G -Brownian motion elaborated by Peng and his co-authors, see [12,16]. Within this theory, the sub-additive expectation and conditional expectation are known as the G -expectation and the conditional G -expectation, respectively. The properties we want to apply are known for those notions. A representation like (5) is also known for the conditional G -expectation, see [20].

Remark 4. In [21], Cohen applied an axiomatic approach to sub-additive expectation and the corresponding sub-additive conditional expectation and looked for a (5)-like representation. Under the so-called Hahn-property, he proved that $\hat{\mathbb{E}}[X|\mathcal{F}]$ is equal to the generalized essential supremum of certain usual conditional expectations.

4. Hájek–Rényi-Type Maximal Inequalities for Conditional Sub-Additive Expectations and Capacities

In this section, we follow the approach presented in [3]. First, we prove that the conditional Kolmogorov inequality for sub-additive expectation implies the conditional Hájek–Rényi inequality for sub-additive expectation. In the following theorem, we consider the setting of Definition 3. We assume that all random variables studied belong to the space $\mathcal{H}$. Let $\{X_i\}_{i \geq 1}$ denote a sequence of random variables in the space $\mathcal{H}$. Let the partial sums of the random variables be $S_k = \sum_{i=1}^k X_i$, for all $k \in \mathbb{N}$, and let $S_0 = 0$.

Theorem 1. Let $X_1, X_2, \dots, X_n$ be random variables belonging to the space $\mathcal{H}$. Assume that the conditional expectation operator $\hat{\mathbb{E}}[\cdot | \mathcal{F}]$ on space $\mathcal{H}$ satisfies the monotonicity, sub-additivity and positive homogeneity axioms of Definition 3. Let $\alpha_1, \dots, \alpha_n$ be non-negative $\mathcal{F}$ -measurable random variables, and $r > 0$ be real number. Assume that the general conditional Kolmogorov-type inequality is true, that is,

$$\hat{\mathbb{E}} \left[\left(\max_{1 \leq l \leq m} |S_l| \right)^r | \mathcal{F} \right] \leq \sum_{l=1}^m \alpha_l \quad \text{for all } 1 \leq m \leq n. \quad (7)$$

Then the conditional Hájek–Rényi inequality is true, that is,

$$\hat{\mathbb{E}} \left[\left(\max_{1 \leq l \leq n} \left| \frac{S_l}{\beta_l} \right| \right)^r | \mathcal{F} \right] \leq 4 \sum_{l=1}^n \frac{\alpha_l}{\beta_l^r} \quad (8)$$

for $\mathcal{F}$ -measurable random variables $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n$ with $\beta_1 \geq \beta_0$, where β_0 is a positive constant.

Proof. Multiplying both sides of inequality (8) by β_0^r , we see that we can assume $\beta_1 \geq 1$ during the proof. Let $c = 2^{\frac{1}{r}}$. Let K_i be the set of subscripts k for which $c^i \leq \beta_k < c^{i+1}$, i.e., $K_i = \{k : c^i \leq \beta_k < c^{i+1}\}$, $i = 0, 1, 2, \dots$. Then, K_i is $\mathcal{F}$ -measurable because β_k is $\mathcal{F}$ -measurable. Let $i(n)$ be the index of the last non-empty K_i . Then $i(n)$ is an $\mathcal{F}$ -measurable random variable (possibly having value ∞). Let $k(i)$ be the maximal index in K_i . More precisely, $k(i) = \max\{k : k \in K_i\}$, if K_i is non-empty, but $k(i) = k(i-1)$ if K_i is empty ($k(-1) = 0$ by definition).

Let

$$\delta_l = \sum_{j=k(l-1)+1}^{k(l)} \alpha_j, \quad l = 0, 1, 2, \dots$$

be the sum of α_j values in K_l . Then $k(i)$ and δ_l are $\mathcal{F}$ -measurable, $k(i) \leq n$. Then, by using monotonicity, sub-additivity, and positive homogeneity of $\hat{\mathbb{E}}[\cdot | \mathcal{F}]$, and inequality (7), we obtain the following sequence of inequalities.

$$\begin{aligned} \hat{\mathbb{E}} \left[\left(\max_{1 \leq l \leq n} \frac{|S_l|}{\beta_l} \right)^r | \mathcal{F} \right] &\leq \hat{\mathbb{E}} \left[\sum_{i=0}^{i(n)} \left(\max_{l \in K_i} \frac{|S_l|}{\beta_l} \right)^r | \mathcal{F} \right] \quad (\text{by monotonicity}) \\ &\leq \sum_{i=0}^{i(n)} \hat{\mathbb{E}} \left[\left(\max_{l \in K_i} \frac{|S_l|}{\beta_l} \right)^r | \mathcal{F} \right] \quad (\text{by sub-additivity}) \\ &\leq \sum_{i=0}^{i(n)} c^{-ir} \hat{\mathbb{E}} \left[\left(\max_{l \in K_i} |S_l| \right)^r | \mathcal{F} \right] \quad (\text{by positive homogeneity}) \\ &\leq \sum_{i=0}^{i(n)} c^{-ir} \hat{\mathbb{E}} \left[\left(\max_{k \leq k(i)} |S_k| \right)^r | \mathcal{F} \right] \quad (\text{by monotonicity}) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{i(n)} c^{-ir} \sum_{k=1}^{k(i)} \alpha_k = \sum_{i=0}^{i(n)} c^{-ir} \sum_{l=0}^i \delta_l \quad (\text{by (7)}) \\
&= \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{i(n)} c^{-ir} \leq \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{\infty} c^{-ir} \\
&= \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \delta_l = \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \sum_{k=k(l-1)+1}^{k(l)} \alpha_k \\
&\leq \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \sum_{k=k(l-1)+1}^{k(l)} \alpha_k \frac{c^{lr+r}}{\beta_k^r} \\
&= \frac{c^r}{1-c^{-r}} \sum_{l=0}^{i(n)} \sum_{k=k(l-1)+1}^{k(l)} \frac{\alpha_k}{\beta_k^r} = 4 \sum_{k=1}^n \frac{\alpha_k}{\beta_k^r}.
\end{aligned}$$

So, we obtained inequality (8). $\square$

Remark 5. Another version of Theorem 1 can be obtained as follows. Assume that there is a sub-additive probability $\hat{\mathbb{V}}$ on the space $(\Omega, \mathcal{A})$. Suppose that all assumptions of Theorem 1 are satisfied $\hat{\mathbb{V}}$ -quasi-surely, including the axioms listed in Definition 3. That is, in the axioms of monotonicity, sub-additivity, and positive homogeneity, all relations among random variables are understood in the $\hat{\mathbb{V}}$ -quasi-sure sense. Then inequality (8) is true $\hat{\mathbb{V}}$ -quasi-surely.

Our next theorem shows that the conditional Kolmogorov inequality for sub-additive probability implies the conditional Hájek–Rényi inequality for sub-additive probability. In the following theorem, we consider the setting of Definition 4.

Theorem 2. Let $X_1, X_2, \dots, X_n$ be random variables, $S_k = X_1 + \dots + X_k$. Let $\hat{\mathbb{V}}[\cdot | \mathcal{F}]$ be a conditional sub-additive probability satisfying the axioms normalization, monotonicity, and sub-additivity of Definition 4. Let r be a positive real number. Let $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n$ be $\mathcal{F}$ -measurable, $\alpha_1, \dots, \alpha_n$ non-negative $\mathcal{F}$ -measurable random variables. Assume that $\beta_1 \geq \beta_0 > 0$, where β_0 is non random. If

$$\hat{\mathbb{V}} \left[\max_{1 \leq l \leq m} |S_l| \geq \varepsilon | \mathcal{F} \right] \leq \frac{1}{\varepsilon^r} \sum_{l=1}^m \alpha_l \quad \text{for all } 1 \leq m \leq n \quad (9)$$

and for all $\varepsilon > 0$, then

$$\hat{\mathbb{V}} \left[\max_{1 \leq l \leq n} \left| \frac{S_l}{\beta_l} \right| \geq \varepsilon | \mathcal{F} \right] \leq \frac{4}{\varepsilon^r} \sum_{k=1}^n \frac{\alpha_k}{\beta_k^r} \quad (10)$$

for all $\varepsilon > 0$.

Proof. We use the same notation as in the proof of Theorem 1. Then, by sub-additivity and monotonicity of $\hat{\mathbb{V}}[\cdot | \mathcal{F}]$ and inequality (9),

$$\begin{aligned}
\hat{\mathbb{V}} \left[\max_{1 \leq l \leq n} \frac{|S_l|}{\beta_l} \geq \varepsilon | \mathcal{F} \right] &\leq \sum_{i=0}^{i(n)} \hat{\mathbb{V}} \left[\max_{l \in K_i} \frac{|S_l|}{\beta_l} \geq \varepsilon | \mathcal{F} \right] \quad (\text{by sub-additivity}) \\
&\leq \sum_{i=0}^{i(n)} \hat{\mathbb{V}} \left[\max_{l \in K_i} \frac{|S_l|}{c^i} \geq \varepsilon | \mathcal{F} \right] \quad (\text{by monotonicity}) \\
&\leq \sum_{i=0}^{i(n)} \hat{\mathbb{V}} \left[\max_{k \leq k(i)} \frac{|S_k|}{c^i} \geq \varepsilon | \mathcal{F} \right] \quad (\text{by monotonicity})
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{i(n)} (\varepsilon c^i)^{-r} \sum_{k=1}^{k(i)} \alpha_k = \sum_{i=0}^{i(n)} (\varepsilon c^i)^{-r} \sum_{l=0}^i \delta_l \quad (\text{by (9)}) \\
&= \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{i(n)} (\varepsilon c^i)^{-r} \leq \sum_{l=0}^{i(n)} \delta_l \sum_{i=l}^{\infty} (\varepsilon c^i)^{-r} \\
&= \varepsilon^{-r} \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \delta_l = \varepsilon^{-r} \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \sum_{k=k(l-1)+1}^{k(l)} \alpha_k \\
&\leq \varepsilon^{-r} \frac{1}{1-c^{-r}} \sum_{l=0}^{i(n)} c^{-lr} \sum_{k=k(l-1)+1}^{k(l)} \alpha_k \frac{c^{lr+r}}{\beta_k^r} \\
&= \varepsilon^{-r} \frac{c^r}{1-c^{-r}} \sum_{l=0}^{i(n)} \sum_{k=k(l-1)+1}^{k(l)} \frac{\alpha_k}{\beta_k^r} = 4\varepsilon^{-r} \sum_{k=1}^n \frac{\alpha_k}{\beta_k^r}.
\end{aligned}$$

So, we obtained inequality (10). $\square$

Remark 6. Another version of Theorem 2 can be obtained as follows. Assume that there is a sub-additive probability $\hat{\mathbb{V}}$ of the space $(\Omega, \mathcal{A})$. Suppose that all assumptions of Theorem 2 are satisfied $\hat{\mathbb{V}}$ -quasi-surely, including the axioms listed in Definition 4. That is, in Definition 4, in the axioms of monotonicity, sub-additivity, and positive homogeneity, all relations among random variables are understood in the $\hat{\mathbb{V}}$ -quasi-sure sense. Then inequality (10) is true $\hat{\mathbb{V}}$ quasi-surely.

5. Strong Laws of Large Numbers in Terms of Conditional Sub-Additive Expectations and Capacities

The intuitive background of our next theorem is the well-known Kolmogorov SLLN (see [22], p. 288). If we consider a usual probability space and independent, zero-mean random variables $X_1, X_2, \dots$ and $\alpha_l = \text{Cvar}(X_l)$, then assumption (11) is satisfied by the Kolmogorov–Doob inequality (see [22], p. 505), and our Theorem 3 is the same as the above mentioned Kolmogorov SLLN.

To prove convergence in our major results, we shall use the well-known theorem of Abel and Dini for real-valued non-random sequences.

Proposition 1 (The Abel-Dini theorem). *Let $b_1, b_2, \dots$ be positive real numbers. If $\sum_{k=1}^{\infty} b_k$ converges, then with $T_n = \sum_{k=n}^{\infty} b_k$ as the n^{th} tail sum, then $\sum_{n=1}^{\infty} \frac{b_n}{T_n^{1+\alpha}}$ converges if and only if $\alpha < 0$. For the proof, see [23].*

We now prove a general strong law of large numbers under assumptions formulated in terms of conditional sub-additive expectations. Let $(\Omega, \mathcal{A})$ be a measurable space, and assume that there exists a sub-additive probability $\hat{\mathbb{V}}$ on $(\Omega, \mathcal{A})$ satisfying the axioms given in Definition 1. Throughout the next theorem, a quasi-sure event is understood in the sense of $\hat{\mathbb{V}}$. Let $\mathcal{F}$ be a σ -sub-algebra of $\mathcal{A}$, and let $\hat{\mathbb{E}}[\cdot | \mathcal{F}]$ denote a sub-additive conditional expectation. In the next theorem, all random variables will be defined on $(\Omega, \mathcal{A})$.

Theorem 3. *Let $X_1, X_2, \dots$ be random variables, $S_n = X_1 + \dots + X_n$ for any n . Let $b_1, b_2, \dots$ be q.s. finite, $\mathcal{F}$ -measurable random variables with $b_0 \leq b_1 \leq b_2 \leq \dots$ q.s., $b_n \rightarrow \infty$ q.s., where b_0 is a positive constant. Let $\alpha_1, \alpha_2, \dots$ be non-negative $\mathcal{F}$ -measurable random variables. Assume that for the conditional expectation $\hat{\mathbb{E}}[\cdot | \mathcal{F}]$ the axioms in Definition 3 are satisfied, where all relations among random variables are understood in the $\hat{\mathbb{V}}$ -quasi-sure sense. Assume further that*

the finiteness axiom in Definition 5 holds. Let $r > 0$ be a fixed number and suppose that, for any $n \geq 1$

$$\hat{\mathbb{E}} \left[\left(\max_{1 \leq l \leq n} |S_l| \right)^r \middle| \mathcal{F} \right] \leq \sum_{l=1}^n \alpha_l \quad \text{quasi-surely.} \quad (11)$$

If $\sum_{l=1}^{\infty} \frac{\alpha_l}{b_l^r} < \infty$ quasi-surely, then

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = 0 \quad \text{quasi-surely} \quad (12)$$

and $\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right)$, where β_n is defined by (13).

Proof. We shall apply the method of [3]. We can assume that $\alpha_n \geq \alpha'_n > 0$ and α'_n is non random for any n . To see it, let a non random $\alpha'_n > 0$, for any n so that $\sum_n \alpha'_l < \infty$. Then, instead of α_n , we can consider $\max\{\alpha_n, \alpha'_n\}$. Let

$$v_n = \sum_{k=n}^{\infty} \frac{\alpha_k}{b_k^r}, \quad \beta_n = \max_{1 \leq k \leq n} b_k v_k^{\frac{1}{2r}}. \quad (13)$$

Then, the sequence β_n is increasing, $\beta_1 > \beta_0 > 0$ where β_0 is non random. Then, because of the assumption $\sum_{l=1}^{\infty} \frac{\alpha_l}{b_l^r} < \infty$ q.s., we have

$$0 < v_n < \infty \quad \text{for all } n \text{ quasi-surely, } v_n \rightarrow 0 \quad \text{quasi-surely,}$$

and v_n is a decreasing sequence. Then, using the Abel-Dini theorem,

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{b_n^r v_n^{\frac{1}{2r}}} < \infty \quad \text{quasi-surely.}$$

Therefore, we have

$$0 < \beta_0 \leq \beta_1 \leq \beta_2 \leq \dots, \quad \beta_0 \text{ is non random,}$$

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{\beta_k^r} < \infty \quad \text{quasi-surely,}$$

$$\lim_{k \rightarrow \infty} \frac{\beta_k}{b_k} = 0 \quad \text{quasi-surely.}$$

Then, our Theorem 1 implies

$$\hat{\mathbb{E}} \left[\max_{1 \leq l \leq n} \left| \frac{S_l}{\beta_l} \right|^r \middle| \mathcal{F} \right] \leq 4 \sum_{l=1}^n \frac{\alpha_l}{\beta_l^r} \quad \text{q.s. for all } n.$$

So, by the monotone convergence axiom from Definition 3,

$$\hat{\mathbb{E}} \left[\sup_{1 \leq l \leq \infty} \left| \frac{S_l}{\beta_l} \right|^r \middle| \mathcal{F} \right] \leq 4 \sum_{l=1}^{\infty} \frac{\alpha_l}{\beta_l^r} < \infty \quad \text{q.s.}$$

So, by the finiteness axiom in Definition 5,

$$\sup_{1 \leq l \leq \infty} \left| \frac{S_l}{\beta_l} \right|^r < \infty \quad \text{q.s.}$$

Therefore,

$$0 \leq \left| \frac{S_n}{b_n} \right| = \left| \frac{S_n}{\beta_n} \right| \frac{\beta_n}{b_n} \leq \left(\sup_{1 \leq l \leq \infty} \left| \frac{S_l}{\beta_l} \right| \right) \frac{\beta_n}{b_n} \rightarrow 0 \quad \text{q.s. as } n \rightarrow \infty.$$

Hence, $\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = 0$ quasi-surely and $\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right)$ quasi-surely. $\square$

We now turn to a general strong law of large numbers in which the assumptions are formulated in terms of conditional sub-additive probability. In the following theorem, all random variables are defined on the measurable space $(\Omega, \mathcal{A})$, which is equipped with a sub-additive probability $\hat{\mathbb{V}}$ satisfying the axioms of Definition 1. As before, quasi-sure events are understood in the sense of $\hat{\mathbb{V}}$. Let $\mathcal{F}$ be a sub- σ -algebra of $\mathcal{A}$ and let $\hat{\mathbb{V}}[\cdot | \mathcal{F}]$ denote a sub-additive conditional probability.

Theorem 4. Let $X_1, X_2, \dots$ be random variables, $S_n = X_1 + \dots + X_n$ for any n . Let $b_1, b_2, \dots$ be q.s. finite, $\mathcal{F}$ -measurable random variables with $b_0 \leq b_1 \leq b_2 \leq \dots$ q.s., $b_n \rightarrow \infty$ q.s., where b_0 is a positive constant. Let $\alpha_1, \alpha_2, \dots$ be non-negative $\mathcal{F}$ -measurable random variables. Assume that the conditional sub-additive probability $\hat{\mathbb{V}}[\cdot | \mathcal{F}]$ satisfies the axioms in Definition 4, where all relations among random variables are understood in the $\hat{\mathbb{V}}$ -quasi-sure sense. Assume further that the recursivity axiom in Definition 6 holds. Let $r > 0$ be a fixed number and suppose that, for all $n \geq 1$ and all $\varepsilon > 0$,

$$\hat{\mathbb{V}} \left[\max_{1 \leq l \leq n} |S_l| \geq \varepsilon | \mathcal{F} \right] \leq \frac{1}{\varepsilon^r} \sum_{l=1}^n \alpha_l \quad \text{quasi-surely.} \quad (14)$$

If $\sum_{l=1}^{\infty} \frac{\alpha_l}{b_l^r} < \infty$ quasi-surely, then

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = 0 \quad \text{quasi-surely} \quad (15)$$

with the convergence rate $\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right)$ quasi-surely.

Proof. As in the proof of Theorem 3, we can assume that $\alpha_n \geq \alpha'_n > 0$, where α'_n is non random for any n . Let

$$v_n = \sum_{k=n}^{\infty} \frac{\alpha_k}{b_k^r}, \quad \beta_n = \max_{1 \leq k \leq n} b_k v_k^{\frac{1}{2r}}.$$

Then, because of the assumption $\sum_{l=1}^{\infty} \frac{\alpha_l}{b_l^r} < \infty$ q.s., we have

$$0 < v_n < \infty \quad \text{for all } n \geq 1 \quad \text{q.s. and } v_n \rightarrow 0 \quad \text{q.s.}$$

Moreover, the Abel-Dini theorem implies

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{b_n^r v_n^{\frac{1}{2r}}} < \infty \quad \text{q.s.}$$

Therefore, $\beta_1, \beta_2, \dots$ is an increasing sequence, $\beta_1 \geq \beta_0 > 0$, where β_0 is non random,

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{\beta_k^r} < \infty, \quad \text{q.s.}$$

$$\lim_{k \rightarrow \infty} \frac{\beta_k}{b_k} = 0 \quad \text{q.s.}$$

Then our Theorem 2 implies

$$\hat{\mathbb{V}}\left[\max_{1 \leq l \leq n} \frac{|S_l|}{\beta_l} \geq \varepsilon | \mathcal{F}\right] \leq \frac{4}{\varepsilon^r} \sum_{l=1}^n \frac{\alpha_l}{\beta_l^r} \quad \text{q.s. for all } n \text{ and } \varepsilon > 0.$$

So, by the lower continuity axiom,

$$\hat{\mathbb{V}}\left[\sup_{1 \leq l < \infty} \frac{|S_l|}{\beta_l} \geq \varepsilon | \mathcal{F}\right] \leq \frac{4}{\varepsilon^r} \sum_{l=1}^{\infty} \frac{\alpha_l}{\beta_l^r} \quad \text{q.s.}$$

Let $\varepsilon \rightarrow \infty$, then by the recursivity axiom, we have

$$\sup_{1 \leq l < \infty} \frac{|S_l|}{\beta_l} < \infty \quad \text{q.s.}$$

Now,

$$0 \leq \left| \frac{S_n}{b_n} \right| = \left| \frac{S_n}{\beta_n} \right| \frac{\beta_n}{b_n} \leq \left(\sup_{1 \leq l < \infty} \frac{|S_l|}{\beta_l} \right) \frac{\beta_n}{b_n} \rightarrow 0 \quad \text{q.s. as } n \rightarrow \infty$$

because $\frac{\beta_n}{b_n} \rightarrow 0$ q.s. Therefore,

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = 0 \quad \text{q.s.}$$

So we obtained (15). $\square$

6. Application to Conditionally Negatively Dependent Random Variables

In this section, we assume the following setting. All random variables are defined on a measurable space $(\Omega, \mathcal{A})$ equipped with a sub-additive probability $\hat{\mathbb{V}}$ satisfying the axioms given in Definition 1. A quasi-sure event will be understood as a $\hat{\mathbb{V}}$ quasi-sure (q.s.) event. Let $\mathcal{F}$ be a σ -sub-algebra of $\mathcal{A}$. Let $\hat{\mathbb{E}}[\cdot | \mathcal{F}]$ denote a sub-additive conditional expectation. Assume that in Definition 3, any relation among random variables is a quasi-sure relation. Moreover, in this section, we understand any relation among random variables as a quasi-sure relation.

Negative dependence plays an important role in probability theory, as it constitutes a dependence structure that is strictly weaker than independence. In the classical framework of additive probabilities and linear expectations, various aspects of negatively dependent random variables have been extensively studied; see, for example, refs. [24–26].

Within the framework of sub-additive expectations, Zhang LiXin [27] introduced a notion of negative dependence and established useful maximal inequalities under this setting. These inequalities were subsequently applied by Huang [8] to derive strong laws of large numbers for negatively dependent random variables under sub-additive expectations.

In this section, we introduce a definition of negative dependence for random variables under conditional sub-additive expectations, which naturally extends the concept proposed in [27]. By adapting the approach of [27], we establish corresponding maximal inequalities in this conditional framework. Finally, combining these inequalities with our general results developed earlier, we derive a strong law of large numbers for conditionally negatively dependent random variables under sub-additive expectations. Throughout this section, we follow the notation of [27].

First, we recall the notion of a local Lipschitz function, see, e.g., [27]. $C_{l, \text{Lip}}(\mathbb{R}^n)$ denotes the linear space of real-valued functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + \|x\|^m + \|y\|^m)\|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n,$$

for some $C > 0$, and $m \in \mathbb{N}$ depending on φ . Here, $\|x\|$ denotes the norm of $x \in \mathbb{R}^n$.

Definition 7 (Conditional negative dependence under sub-additive expectation). Let X be an m -dimensional random vector and Y be an n -dimensional random vector.

We say that Y is negatively dependent on X under the conditional sub-additive expectation $\hat{\mathbb{E}}[\cdot | \mathcal{F}]$ if, for every pair of test functions $\psi_1 \in C_{l,\text{Lip}}(\mathbb{R}^m)$ and $\psi_2 \in C_{l,\text{Lip}}(\mathbb{R}^n)$, we have

$$\hat{\mathbb{E}}[\psi_1(X)\psi_2(Y) | \mathcal{F}] \leq \hat{\mathbb{E}}[\psi_1(X) | \mathcal{F}] \hat{\mathbb{E}}[\psi_2(Y) | \mathcal{F}],$$

whenever $\psi_1(X) \geq 0$, $\hat{\mathbb{E}}[\psi_2(Y) | \mathcal{F}] \geq 0$, and $\hat{\mathbb{E}}[|\psi_1(X)\psi_2(Y)| | \mathcal{F}] < \infty$, $\hat{\mathbb{E}}[|\psi_1(X)| | \mathcal{F}] < \infty$, $\hat{\mathbb{E}}[|\psi_2(Y)| | \mathcal{F}] < \infty$, and either both ψ_1 and ψ_2 are coordinate-wise non-decreasing or both are coordinate-wise non-increasing.

We give a simple example for conditional negative dependence under sub-additive expectation. We apply the idea used in the non-conditional case, see [27,28].

Example 9. Let $P_1, P_2, \dots$ be usual probabilities on the space $(\Omega, \mathcal{A})$. Let X, Y, Z be random variables, let $\mathcal{F}$ be the sub- σ -algebra generated by Z . Let $\hat{\mathbb{E}}[\cdot | \mathcal{F}] = \sup_i \mathbb{E}_i[\cdot | \mathcal{F}]$, where $\mathbb{E}_i[\cdot | \mathcal{F}]$ is the usual conditional expectation under the probability P_i . Assume that X and Y are conditionally independent given Z under each P_i . If the joint density functions exist, it means $f_i(x, y|z) = f_i(x|z)f_i(y|z)$ for any i , where the subscript i refers to P_i . This equality gives us

$$\mathbb{E}_i[\psi_1(X)\psi_2(Y)|Z=z] = \mathbb{E}_i[\psi_1(X)|Z=z]\mathbb{E}_i[\psi_2(Y)|Z=z]$$

for any i and $\psi_1, \psi_2 \in C_{l,\text{Lip}}(\mathbb{R}^1)$. We assume also $\psi_1(X) \geq 0$, $\hat{\mathbb{E}}[\psi_2(Y) | \mathcal{F}] \geq 0$. From the above equation, we have

$$\mathbb{E}_i[\psi_1(X)\psi_2(Y) | \mathcal{F}] = \mathbb{E}_i[\psi_1(X) | \mathcal{F}]\mathbb{E}_i[\psi_2(Y) | \mathcal{F}].$$

In this equation taking the supremum for all i , and using the assumption $\psi_1(X) \geq 0$, we obtain

$$\hat{\mathbb{E}}[\psi_1(X)\psi_2(Y) | \mathcal{F}] \leq \hat{\mathbb{E}}[\psi_1(X) | \mathcal{F}]\hat{\mathbb{E}}[\psi_2(Y) | \mathcal{F}],$$

that is X and Y are conditionally negatively dependent under $\hat{\mathbb{E}}[\cdot | \mathcal{F}]$.

The following proposition is a straightforward extension of Proposition 2.4 of [8].

Proposition 2. Let $\{f_i\}_{i=1}^{n_1} \subset C_{l,\text{Lip}}(\mathbb{R}^n)$ and $\{g_i\}_{i=1}^{m_1} \subset C_{l,\text{Lip}}(\mathbb{R}^m)$ be coordinate-wise non-decreasing or coordinate-wise non-increasing functions. If the n -dimensional random vector Y is conditionally negatively dependent on the m -dimensional random vector X under conditional sub-additive expectation, then $(f_1(Y), \dots, f_{n_1}(Y))$ is conditionally negatively dependent on $(g_1(X), \dots, g_{m_1}(X))$. In particular, $-Y$ is conditionally negatively dependent on $-X$.

Proof. For any test functions $\psi_1 \in C_{l,\text{Lip}}(\mathbb{R}^{m_1})$ and $\psi_2 \in C_{l,\text{Lip}}(\mathbb{R}^{n_1})$ such that

$$\psi_1(g_1(X), \dots, g_{m_1}(X)) \geq 0, \quad \hat{\mathbb{E}}[\psi_2(f_1(Y), \dots, f_{n_1}(Y)) | \mathcal{F}] \geq 0,$$

and

$$\hat{\mathbb{E}}[|\psi_1(g_1(X), \dots, g_{m_1}(X))\psi_2(f_1(Y), \dots, f_{n_1}(Y))| | \mathcal{F}] < \infty,$$

$$\hat{\mathbb{E}}[|\psi_1(g_1(X), \dots, g_{m_1}(X))| | \mathcal{F}] < \infty, \quad \hat{\mathbb{E}}[|\psi_2(f_1(Y), \dots, f_{n_1}(Y))| | \mathcal{F}] < \infty,$$

and either ψ_1, ψ_2 are coordinate-wise non-decreasing, or both are coordinate-wise non-increasing, define

$$\tilde{\psi}_1(\cdot) = \psi_1(g_1(\cdot), \dots, g_{m_1}(\cdot)), \quad \tilde{\psi}_2(\cdot) = \psi_2(f_1(\cdot), \dots, f_{n_1}(\cdot)).$$

Then $\tilde{\psi}_1 \in C_{l,\text{Lip}}(\mathbb{R}^m)$, $\tilde{\psi}_2 \in C_{l,\text{Lip}}(\mathbb{R}^n)$, and they are either coordinate-wise non-decreasing or coordinate-wise non-increasing. Meanwhile, $\tilde{\psi}_1(X) \geq 0$, $\hat{\mathbb{E}}[\tilde{\psi}_2(Y)|\mathcal{F}] \geq 0$, and the integrability conditions hold.

By the definition of negative dependence, we have

$$\begin{aligned} \hat{\mathbb{E}}[\psi_1(g_1(X), \dots, g_{m_1}(X)) \psi_2(f_1(Y), \dots, f_{n_1}(Y)) | \mathcal{F}] &= \hat{\mathbb{E}}[\tilde{\psi}_1(X) \tilde{\psi}_2(Y) | \mathcal{F}] \\ &\leq \hat{\mathbb{E}}[\tilde{\psi}_1(X) | \mathcal{F}] \hat{\mathbb{E}}[\tilde{\psi}_2(Y) | \mathcal{F}] = \hat{\mathbb{E}}[\psi_1(g_1(X), \dots, g_{m_1}(X)) | \mathcal{F}] \hat{\mathbb{E}}[\psi_2(f_1(Y), \dots, f_{n_1}(Y)) | \mathcal{F}]. \end{aligned}$$

Hence, $(f_1(Y), \dots, f_{n_1}(Y))$ is negatively dependent on $(g_1(X), \dots, g_{m_1}(X))$ under $\hat{\mathbb{E}}[\cdot | \mathcal{F}]$. $\square$

Our first result is a Kolmogorov-type maximal inequality (actually, it is a Rosenthal inequality). It was presented in [8,27] for usual (i.e., not conditional) sub-additive expectation.

Theorem 5 (Kolmogorov's maximal inequality for conditional sub-additive expectations). *Let $\hat{\mathbb{E}}[\cdot | \mathcal{F}]$ satisfy the axioms of monotonicity, sub-additivity, and positive homogeneity. Let $1 \leq p \leq 2$, and let $\{X_k\}_{k=1}^n$ be a sequence of random variables with $\hat{\mathbb{E}}[X_k | \mathcal{F}] = \hat{\mathbb{E}}[-X_k | \mathcal{F}] = 0$ q.s. for all $k = 1, \dots, n$. If X_k is negatively dependent on $(X_{k+1}, \dots, X_n)$ for all $k = 1, \dots, n-1$, then*

$$\hat{\mathbb{E}}\left[\max_{1 \leq k \leq n} |S_k|^p | \mathcal{F}\right] \leq 2^{3-p} \sum_{k=1}^n \hat{\mathbb{E}}[|X_k|^p | \mathcal{F}] \quad \text{q.s.},$$

where $S_k = \sum_{i=1}^k X_i$.

Proof. We follow the lines of [27]. Define,

$$T_k := \max\{X_k, X_k + X_{k+1}, \dots, X_k + X_{k+1} + \dots + X_n\}, \quad k = 1, \dots, n.$$

We see that T_k is an increasing function of $X_k, X_{k+1}, \dots, X_n$. Then it is clear that

$$T_k = X_k + T_{k+1}^+, \quad \text{where} \quad T_{k+1}^+ := \max\{0, T_{k+1}\}, \quad (16)$$

and that at the terminal index

$$T_n = X_n. \quad (17)$$

Moreover, since $T_1 = \max_{1 \leq j \leq n} S_j$ with $S_j = X_1 + \dots + X_j$, we have

$$\hat{\mathbb{E}}\left[\left|\max_{1 \leq j \leq n} S_j\right|^p | \mathcal{F}\right] = \hat{\mathbb{E}}[|T_1|^p | \mathcal{F}].$$

The following elementary inequality can be found, e.g., in [27], one can prove it by Taylor's expansion. For $1 \leq p \leq 2$ and any $x, y \in \mathbb{R}$,

$$|x + y|^p \leq 2^{2-p}|x|^p + |y|^p + p|x||y|^{p-1} \text{sgn}(y). \quad (18)$$

Setting $x = X_k$ and $y = T_{k+1}^+$ in (18) gives

$$|T_k|^p = |X_k + T_{k+1}^+|^p \leq 2^{2-p}|X_k|^p + (T_{k+1}^+)^p + pX_k(T_{k+1}^+)^{p-1}.$$

Taking the conditional sub-additive expectation $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ of both sides yields

$$\hat{\mathbb{E}}[|T_k|^p|\mathcal{F}] \leq 2^{2-p}\hat{\mathbb{E}}[|X_k|^p|\mathcal{F}] + \hat{\mathbb{E}}[(T_{k+1}^+)^p|\mathcal{F}] + p\hat{\mathbb{E}}[X_k(T_{k+1}^+)^{p-1}|\mathcal{F}] \quad \text{q.s.} \quad (19)$$

Since X_k is negatively dependent to $(X_{k+1}, \dots, X_n)$ and T_{k+1}^+ is a coordinate-wise non-decreasing function of these variables, we have by definition of negative dependence

$$\hat{\mathbb{E}}[X_k(T_{k+1}^+)^{p-1}|\mathcal{F}] \leq \hat{\mathbb{E}}[X_k|\mathcal{F}] \hat{\mathbb{E}}[(T_{k+1}^+)^{p-1}|\mathcal{F}] \leq 0,$$

because $\hat{\mathbb{E}}[X_k|\mathcal{F}] = 0$ and $T_{k+1}^+ \geq 0$. Consequently, inequality (19) simplifies to

$$\hat{\mathbb{E}}[|T_k|^p|\mathcal{F}] \leq 2^{2-p}\hat{\mathbb{E}}[|X_k|^p|\mathcal{F}] + \hat{\mathbb{E}}[|T_{k+1}|^p|\mathcal{F}]. \quad (20)$$

Applying (20) successively for $k = n-1, n-2, \dots, 1$ and using (17), we obtain

$$\begin{aligned} \hat{\mathbb{E}}[|T_1|^p|\mathcal{F}] &\leq 2^{2-p}\hat{\mathbb{E}}[|X_1|^p|\mathcal{F}] + \hat{\mathbb{E}}[|T_2|^p|\mathcal{F}] \\ &\leq 2^{2-p}(\hat{\mathbb{E}}[|X_1|^p|\mathcal{F}] + \hat{\mathbb{E}}[|X_2|^p|\mathcal{F}]) + \hat{\mathbb{E}}[|T_3|^p|\mathcal{F}] \\ &\vdots \\ &\leq 2^{2-p}\sum_{k=1}^{n-1}\hat{\mathbb{E}}[|X_k|^p|\mathcal{F}] + \hat{\mathbb{E}}[|T_n|^p|\mathcal{F}] \\ &= 2^{2-p}\sum_{k=1}^n\hat{\mathbb{E}}[|X_k|^p|\mathcal{F}], \end{aligned}$$

where the last equality uses $T_n = X_n$.

Since $\max_{1 \leq k \leq n} S_k = T_1$, then, for $p \geq 1$, we get

$$\hat{\mathbb{E}}\left[\left|\max_{1 \leq k \leq n} S_k\right|^p|\mathcal{F}\right] = \hat{\mathbb{E}}[|T_1|^p|\mathcal{F}] \leq 2^{2-p}\sum_{k=1}^n\hat{\mathbb{E}}[|X_k|^p|\mathcal{F}] \quad \text{q.s.}$$

By Proposition 2, $-X_k$ is negatively dependent on $(-X_{k+1}, \dots, -X_n)$ for all $k = 1, \dots, n-1$. Hence,

$$\hat{\mathbb{E}}\left[\left|\max_{1 \leq k \leq n} (-S_k)\right|^p|\mathcal{F}\right] \leq 2^{2-p}\sum_{k=1}^n\hat{\mathbb{E}}[|X_k|^p|\mathcal{F}].$$

Observe that

$$\max_{1 \leq k \leq n} |S_k|^p = \left(\max_{1 \leq k \leq n} |S_k|\right)^p \leq \left|\max_{1 \leq k \leq n} S_k\right|^p + \left|\max_{1 \leq k \leq n} (-S_k)\right|^p.$$

Taking $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ on both sides and using its monotonicity and sub-additivity, we get

$$\hat{\mathbb{E}}\left[\max_{1 \leq k \leq n} |S_k|^p|\mathcal{F}\right] \leq 2 \cdot 2^{2-p}\sum_{k=1}^n\hat{\mathbb{E}}[|X_k|^p|\mathcal{F}] = 2^{3-p}\sum_{k=1}^n\hat{\mathbb{E}}[|X_k|^p|\mathcal{F}],$$

which completes the proof. $\square$

In the following theorem we assume that $\hat{\mathbb{V}}[\cdot|\mathcal{F}]$ is the conditional capacity induced by $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$.

Theorem 6 (Conditional Kolmogorov maximal inequality in capacity). *Let $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ satisfy the axioms of monotonicity, sub-additivity, and positive homogeneity. Let $\hat{\mathbb{V}}[A|\mathcal{F}] = \hat{\mathbb{E}}[\mathbb{I}_A|\mathcal{F}]$ for any event A . Let $1 \leq p \leq 2$, and let $\{X_k\}_{k=1}^n$ be a sequence of random variables with*

$\hat{\mathbb{E}}[X_k|\mathcal{F}] = \hat{\mathbb{E}}[-X_k|\mathcal{F}] = 0$ for all $k = 1, \dots, n$. If X_k is negatively dependent on $(X_{k+1}, \dots, X_n)$ for all $k = 1, \dots, n-1$, then, for any $\varepsilon > 0$,

$$\hat{\mathbb{P}}\left[\max_{1 \leq k \leq n} |S_k| > \varepsilon \mid \mathcal{F}\right] \leq 2^{3-p} \varepsilon^{-p} \sum_{k=1}^n \hat{\mathbb{E}}[|X_k|^p \mid \mathcal{F}], \quad q.s., \quad (21)$$

where $S_k = \sum_{i=1}^k X_i$.

Proof. By using Chebyshev inequality under conditional sub-additive expectation, we have

$$\hat{\mathbb{P}}\left[\max_{1 \leq k \leq n} |S_k| > \varepsilon \mid \mathcal{F}\right] \leq \varepsilon^{-p} \hat{\mathbb{E}}\left[\max_{1 \leq k \leq n} |S_k|^p \mid \mathcal{F}\right].$$

Then inequality (21) follows from Theorem 5 directly. $\square$

Theorem 7 (Conditional SLLN for negatively dependent random variables). Assume that the conditional expectation operator $\hat{\mathbb{E}}[\cdot|\mathcal{F}]$ satisfies the axioms in Definition 3, where all relations among random variables are understood in the $\hat{\mathbb{P}}$ -quasi-sure sense. Assume that the finiteness axiom in Definition 5 is also satisfied. Let $1 \leq p \leq 2$, and let $\{X_n\}_{n \geq 1}$ be a sequence of random variables with $\hat{\mathbb{E}}[X_n|\mathcal{F}] = \hat{\mathbb{E}}[-X_n|\mathcal{F}] = 0$ for all $n \in \mathbb{N}$. Let $b_1, b_2, \dots$ be q.s. finite, $\mathcal{F}$ -measurable random variables with $b_0 \leq b_1 \leq b_2 \leq \dots$ q.s., $b_n \rightarrow \infty$ q.s., where b_0 is a positive constant. If X_k is negatively dependent on $(X_{k+1}, \dots, X_{k+n})$ for all $n, k \in \mathbb{N}$ with

$$\sum_{n=1}^{\infty} \frac{\hat{\mathbb{E}}[|X_n|^p|\mathcal{F}]}{b_n^p} < \infty \quad \text{quasi-surely,}$$

then

$$\lim_{n \rightarrow \infty} \frac{S_n}{b_n} = 0 \quad \text{quasi-surely.}$$

Proof. Set $\alpha_k = \hat{\mathbb{E}}[|X_k|^p|\mathcal{F}]$ for all $k \in \mathbb{N}$. Then, by Theorem 5, we have

$$\hat{\mathbb{E}}\left[\max_{1 \leq k \leq n} |S_k|^p \mid \mathcal{F}\right] \leq 2^{3-p} \sum_{k=1}^n \alpha_k, \quad n \geq 1.$$

Since

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{b_n^p} = \sum_{n=1}^{\infty} \frac{\hat{\mathbb{E}}[|X_n|^p|\mathcal{F}]}{b_n^p} < \infty,$$

we can deduce the theorem from Theorem 3 directly. $\square$

7. Discussion

In the usual theory of probability, a probability space $(\Omega, \mathcal{A}, P)$ is a measure space, and the expectation of a random variable X is its Lebesgue integral: $\mathbb{E}X = \int_{\Omega} X dP$. The conditional expectation and the conditional probability given a sub- σ -field are random variables that can be obtained via the Radon–Nikodym theorem.

To introduce sub-linear expectation and sub-additive probability, one can start with several usual probability measures and use supremum to define these quantities; see Examples 1 and 2. Similarly, to introduce sub-linear conditional expectation and sub-additive conditional probability, one can use the essential supremum of usual conditional expectations and usual conditional probabilities, see Examples 3 and 5.

However, in this paper, we followed another approach. We aimed to find small sets of properties of the sub-linear conditional expectation and sub-additive conditional probability

and fix them as axioms. Then, during the proofs, we applied only these properties of the sub-linear conditional expectation and the sub-additive conditional probability.

Below, we visualize the main implications of this paper. For sub-linear conditional expectation, we proved that a Kolmogorov-type maximal inequality implies a Hájek–Rényi-type maximal inequality, and this implies an SLLN:

$$\{\text{Definition 3}\} \implies \{\text{Theorem 1}\} \implies \{\text{Theorem 3}\}$$

Similarly, for sub-additive conditional probability, we proved that a Kolmogorov-type maximal inequality implies a Hájek–Rényi-type maximal inequality, and this implies an SLLN:

$$\{\text{Definition 4}\} \implies \{\text{Theorem 2}\} \implies \{\text{Theorem 4}\}$$

Our assumptions were similar as those in the unconditional case, so our conclusions are also similar to that case, see [8]. For conditionally negatively dependent random variables, we obtained a conditional Kolmogorov-type maximal inequality, leading to a conditional strong law of large numbers:

$$\{\text{Theorem 5}\} \ \& \ \{\text{Theorem 3}\} \implies \{\text{Theorem 7}\}$$

Our SLLNs are extensions of known SLLNs to sub-linear conditional expectation and sub-additive conditional probability frameworks.

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