

Article

The Structure of a Prestack over the Complex Plane on the Set of All Spaces of Pseudoanalytic Functions

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Abstract

In this study, we construct a prestack over the site of open subsets of the complex plane. Its total category has the set of objects canonically isomorphic to the set of all spaces of pseudoanalytic functions, and this is over all open subsets of the complex plane with respect to all the possible generating pairs. This is carried out by defining suitable morphisms between the spaces of pseudoanalytic functions over the same open subset using pseudodifferentiation, and, accordingly, obtaining for each open subset of the complex plane a category whose objects are the spaces of pseudoanalytic functions defined on the subset. It is shown that assigning to open sets the appropriate categories of spaces of pseudoanalytic functions along with suitable restriction functors for inclusions of open subsets results in a 2-presheaf satisfying the gluing of morphisms. A sufficient condition for a descent datum to be effective is presented, relating the effectiveness of a descent datum to the admissibility of coefficients of a Carleman–Bers–Vekua equation.

Keywords: pseudoanalytic; 2-presheaf; prestack

MSC: 30G20; 32C25

1. Introduction

The main question considered in the present paper is whether the set of all spaces of pseudoanalytic functions can be naturally equipped with a unifying structure, to which one possible answer is provided by the theory of prestacks. In particular, one constructs a prestack over the site of open subsets of the complex plane whose total category has the set of objects canonically isomorphic to the set of all spaces of pseudoanalytic functions. Some of these ideas are presented cursorily and without proofs in the report [1], whose contents are expanded upon here with proofs.

In [2], Y. Rodin applies sheaf theory to the study of generalized-analytic/pseudoanalytic functions of Bers–Vekua theory (see [3–5]), using the sheaf which, to each open subset of a domain, assigns the space of generalized-analytic functions defined over it with respect to a globally given pair of coefficients. In the present work, we consider the assignment to each open subset of the complex plane of a (small) category whose set of objects consists of all the spaces of pseudoanalytic functions defined with respect to all possible generating pairs over the open subset and whose morphisms are given locally by chains of pseudodifferentiations. This assignment, along with the appropriate restriction functors for open inclusions, defines a (strict) 2-presheaf on the category of open subsets of the complex plane



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and it is shown that this 2-presheaf, or rather its associated fibred category (see [6–8]), is a prestack. A sufficient condition is also given for a descent datum to be effective, relating the effectiveness of a descent datum to the admissibility of a pair of coefficients of the Carleman–Bers–Vekua equation.

Section 2 introduces the main notations and considers some classical notions, such as characteristic coefficients of a generating pair, from the sheaf-theoretic viewpoint, and states a Lemma used later in the paper.

In Section 3, we introduce the notion of a category of spaces of pseudoanalytic functions defined on an open subset of the plane, and we interpret some classical results of the theory of pseudoanalytic functions in terms of these categories. Afterwards, the assignment of these categories to open subsets is extended to a 2-presheaf, and it is shown that for this 2-presheaf, the gluing of morphisms is satisfied and so the associated fibred category is a prestack. Next, the effectiveness of a descent datum of this prestack is related to the admissibility of coefficients of an appropriate Carleman–Bers–Vekua equation.

We do not use the sign ' to denote the (complex) derivative of a function; moreover, below, no function is assumed to be complex-differentiable.

2. Preliminaries

2.1. Notation and Terminology

For the set of generating pairs (for the definition used here, see [4], p. 3), over an open subset $U \subseteq \mathbb{C}$, the following notation is used:

$$GP(U) := \{(F, G) \in C(U, \mathbb{C}) \times C(U, \mathbb{C}) \mid (F, G) \text{ is a generating pair}\}.$$

For $(F, G) \in GP(U)$, the associated space of (F, G) -pseudoanalytic functions is denoted by $\mathcal{PA}_{F,G}(U)$ and the corresponding pseudodifferentiation with respect to this pair is denoted by $D_{F,G}$. If generating pairs $(F_1, G_1) \in GP(U)$ and $(F_2, G_2) \in GP(U)$ are equipotent (see [4,5]), meaning they define the same space of pseudoanalytic functions, this is denoted by the symbol $(F_1, G_1) \sim_{\text{ep}} (F_2, G_2)$, which is an equivalence relation, so

$$(F_1, G_1) \sim_{\text{ep}} (F_2, G_2) : \Longleftrightarrow \mathcal{PA}_{F_1,G_1}(U) = \mathcal{PA}_{F_2,G_2}(U).$$

If $(F_2, G_2) \in GP(U)$ is a successor pair to $(F_1, G_1) \in GP(U)$ (see [4]), we denote this relation by $(F_1, G_1) \preceq (F_2, G_2)$. The relation $\preceq$ on $GP(U)$ is, in general, neither reflexive nor transitive. If we have a finite ordered chain of generating pairs from $GP(U)$ with each being a successor to the previous one, that is,

$$(F_1, G_1) \preceq (F_2, G_2) \preceq \cdots \preceq (F_{n+1}, G_{n+1})$$

then we may consider the composition of the corresponding pseudodifferentiation maps between the appropriate spaces of pseudoanalytic functions

$$\mathcal{PA}_{F_1,G_1}(U) \xrightarrow{D_{F_1,G_1}} \mathcal{PA}_{F_2,G_2}(U) \xrightarrow{D_{F_2,G_2}} \cdots \xrightarrow{D_{F_n,G_n}} \mathcal{PA}_{F_{n+1},G_{n+1}}(U),$$

for which we use the following shorthand notation

$$D_{(F_1,G_1) \preceq (F_2,G_2) \preceq \cdots \preceq (F_n,G_n)} := D_{F_n,G_n} \circ \cdots \circ D_{F_2,G_2} \circ D_{F_1,G_1};$$

By definition, for any $k \in \{1, 2, \dots, n-1\}$, the following holds:

$$D_{(F_1,G_1) \preceq \cdots \preceq (F_n,G_n)} = D_{(F_{k+1},G_{k+1}) \preceq \cdots \preceq (F_n,G_n)} \circ D_{(F_1,G_1) \preceq \cdots \preceq (F_k,G_k)};$$

we use such compositions later.

2.2. Auxiliary Results

One may consider the presheaf

$$GP : \text{Open}(\mathbb{C})^{\text{op}} \longrightarrow \text{Set}, \quad U \longmapsto GP(U),$$

assigning the following map to each morphism corresponding to an open inclusion $U \subseteq V$:

$$GP(V) \longrightarrow GP(U), \quad (F, G) \longmapsto (F, G)|_U := (F|_U, G|_U);$$

Due to the fact that being a generating pair is a local property, this presheaf is a sheaf of sets (see [9]). For each open subset $U \subseteq \mathbb{C}$, consider the map assigning to generating pairs over U their characteristic coefficients (see [4,5]):

$$\begin{aligned} \text{ch}_U : GP(U) &\longrightarrow C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C}), \\ (F, G) &\longmapsto \frac{1}{F\bar{G} - \bar{F}G} \cdot \left(\frac{\partial F}{\partial \bar{z}} \bar{G} - \bar{F} \frac{\partial G}{\partial \bar{z}}, F \frac{\partial G}{\partial \bar{z}} - \frac{\partial F}{\partial \bar{z}} G, \frac{\partial F}{\partial z} \bar{G} - \bar{F} \frac{\partial G}{\partial z}, F \frac{\partial G}{\partial z} - \frac{\partial F}{\partial z} G \right), \end{aligned}$$

which is well-defined since, for a generating pair $(F, G) \in GP(U)$, the function $F\bar{G} - \bar{F}G$ is nowhere vanishing. Denote the following compositions by $\text{ch}_{U,k}, k \in \{1, \dots, 4\}$:

$$\text{ch}_{U,k} := \text{pr}_k \circ \text{ch}_U : GP(U) \longrightarrow C(U, \mathbb{C}),$$

where pr_k is the projection map onto the k -th component

$$C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C}) \longrightarrow C(U, \mathbb{C});$$

The assignment $U \mapsto C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C})$ together with natural restrictions is a sheaf on $\mathbb{C}$, and the following Proposition holds.

Proposition 1. *The collection of maps*

$$\{\text{ch}_U : GP(U) \longrightarrow C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C})\}_{U \in \text{Open}(\mathbb{C})}$$

constitutes a morphism of sheaves of sets

$$\text{ch} : GP \longrightarrow C(-, \mathbb{C}) \times C(-, \mathbb{C}) \times C(-, \mathbb{C}) \times C(-, \mathbb{C}).$$

Proof. For $U \subseteq V$ and a section $(F, G) \in GP(V)$,

$$\begin{aligned} (\text{ch}_V(F, G))|_U &= \left(\frac{\frac{\partial F}{\partial \bar{z}} \bar{G} - \bar{F} \frac{\partial G}{\partial \bar{z}}}{F\bar{G} - \bar{F}G}, \frac{F \frac{\partial G}{\partial \bar{z}} - \frac{\partial F}{\partial \bar{z}} G}{F\bar{G} - \bar{F}G}, \frac{\frac{\partial F}{\partial z} \bar{G} - \bar{F} \frac{\partial G}{\partial z}}{F\bar{G} - \bar{F}G}, \frac{F \frac{\partial G}{\partial z} - \frac{\partial F}{\partial z} G}{F\bar{G} - \bar{F}G} \right) \Big|_U \\ &= \left(\left(\frac{\frac{\partial F}{\partial \bar{z}} \bar{G} - \bar{F} \frac{\partial G}{\partial \bar{z}}}{F\bar{G} - \bar{F}G} \right) \Big|_U, \left(\frac{F \frac{\partial G}{\partial \bar{z}} - \frac{\partial F}{\partial \bar{z}} G}{F\bar{G} - \bar{F}G} \right) \Big|_U, \left(\frac{\frac{\partial F}{\partial z} \bar{G} - \bar{F} \frac{\partial G}{\partial z}}{F\bar{G} - \bar{F}G} \right) \Big|_U, \left(\frac{F \frac{\partial G}{\partial z} - \frac{\partial F}{\partial z} G}{F\bar{G} - \bar{F}G} \right) \Big|_U \right) \\ &= \text{ch}_U(F|_U, G|_U) \end{aligned}$$

due to the fact that restricting to a subset commutes with all the operations involved in defining the characteristic coefficients; hence, the diagram

$$\begin{array}{ccc}
 GP(V) & \xrightarrow{\text{ch}_V} & C(V, \mathbb{C}) \times C(V, \mathbb{C}) \times C(V, \mathbb{C}) \times C(V, \mathbb{C}) \\
 \downarrow & & \downarrow \\
 GP(U) & \xrightarrow{\text{ch}_U} & C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C}) \times C(U, \mathbb{C}),
 \end{array}$$

in which vertical maps are restrictions, is commutative. $\square$

It follows that the collections of maps $\{\text{ch}_{U,k} : GP(U) \rightarrow C(U, \mathbb{C})\}_{U \in \text{Open}(\mathbb{C})}$ for $k \in \{1, \dots, 4\}$ define sheaf morphisms $\text{ch}_k : GP \rightarrow C(-, \mathbb{C})$. The following Lemma relies on the fact that two generating pairs are equipotent if and only if they have the same first two characteristic coefficients [4,5], which follows from the fact that a function $f : U \rightarrow \mathbb{C}$ is (F, G) -pseudoanalytic if and only if it satisfies the Carleman–Bers–Equation [4,10],

$$\frac{\partial f}{\partial \bar{z}} = \text{ch}_1(F, G) \cdot f + \text{ch}_2(F, G) \cdot \bar{f},$$

and this Lemma also uses the notion of admissibility (see [5]).

Lemma 1. *Let $U = \bigcup_{j \in J} U_j$ be an open cover of $U \subseteq \mathbb{C}$, an open subset, and let $(F_j, G_j) \in GP(U_j)$ be generating pairs on the members of the cover such that, for all $j, k \in J$,*

$$(F_j, G_j)|_{U_j \cap U_k} \sim_{\text{ep}} (F_k, G_k)|_{U_j \cap U_k},$$

Also let the functions resulting from gluing together the coefficients $\text{ch}_1(F_j, G_j), \text{ch}_2(F_j, G_j), j \in J$ (for which see the proof) be admissible; then, there exists a generating pair defined on U , $(F, G) \in GP(U)$, such that, for each $j \in J$,

$$(F, G)|_{U_j} \sim_{\text{ep}} (F_j, G_j).$$

Proof. The condition

$$(F_j, G_j)|_{U_j \cap U_k} \sim_{\text{ep}} (F_k, G_k)|_{U_j \cap U_k},$$

means that for each $j, k \in J$,

$$\begin{aligned}
 \text{ch}_1(F_j|_{U_j \cap U_k}, G_j|_{U_j \cap U_k}) &= \text{ch}_1(F_k|_{U_j \cap U_k}, G_k|_{U_j \cap U_k}), \\
 \text{ch}_2(F_j|_{U_j \cap U_k}, G_j|_{U_j \cap U_k}) &= \text{ch}_2(F_k|_{U_j \cap U_k}, G_k|_{U_j \cap U_k}),
 \end{aligned}$$

from which, in virtue of the previous Proposition, follows

$$(\text{ch}_1(F_j, G_j))|_{U_j \cap U_k} = (\text{ch}_1(F_k, G_k))|_{U_j \cap U_k}, \quad (\text{ch}_2(F_j, G_j))|_{U_j \cap U_k} = (\text{ch}_2(F_k, G_k))|_{U_j \cap U_k};$$

Thus, we have collections of continuous functions

$$\text{ch}_1(F_j, G_j) : U_j \rightarrow \mathbb{C}, \quad \text{ch}_2(F_j, G_j) : U_j \rightarrow \mathbb{C}, \quad j \in J,$$

coinciding on the intersections $U_j \cap U_k, j, k \in J$; therefore, there exists a unique pair of continuous functions

$$f_1 : U \rightarrow \mathbb{C} \in C(U, \mathbb{C}), \quad f_2 : U \rightarrow \mathbb{C} \in C(U, \mathbb{C}),$$

such that

$$f_1|_{U_j} = \text{ch}_1(F_j, G_j), \quad f_2|_{U_j} = \text{ch}_2(F_j, G_j), \quad j \in J;$$

Since, by the hypothesis, they are admissible, there exists a generating pair $(F, G) \in GP(U)$ such that

$$\text{ch}_1(F, G) = f_1, \quad \text{ch}_2(F, G) = f_2,$$

according to [5]; this generating pair satisfies the following:

$$\text{ch}_1(F|_{U_j}, G|_{U_j}) = (\text{ch}_1(F, G))|_{U_j} = f_1|_{U_j} = \text{ch}_1(F_j, G_j)$$

and

$$\text{ch}_2(F|_{U_j}, G|_{U_j}) = (\text{ch}_2(F, G))|_{U_j} = f_2|_{U_j} = \text{ch}_2(F_j, G_j),$$

for each $j \in J$, which means that for each $j \in J$,

$$(F, G)|_{U_j} \sim_{\text{ep}} (F_j, G_j).$$

□

3. The 2-Presheaf of Spaces of Pseudoanalytic Functions

Section 3.1 introduces the notion of a category of spaces of pseudoanalytic functions defined on an open subset of the plane, considers some of the properties of this category, and interprets some classical results in this setting. Section 3.2 extends the assignment of these categories to an open subset to a 2-presheaf and shows that this 2-presheaf satisfies the gluing of morphisms, meaning that the associated fibred category is a prestack. Next, the effectiveness of a descent datum is brought into relation with the admissibility of the coefficients of an appropriate Carleman–Bers–Vekua equation.

3.1. The Category of Spaces of Pseudoanalytic Functions Defined on an Open Subset

For an open subset $U \subseteq \mathbb{C}$, define the category of spaces of pseudoanalytic functions defined on U $\mathcal{PA}(U)$ as follows: Let $\text{Ob}(\mathcal{PA}(U))$ consist of all the spaces of pseudoanalytic functions defined on U , i.e.,

$$\text{Ob}(\mathcal{PA}(U)) := \{\mathcal{PA}_{F,G}(U) \mid (F, G) \in GP(U)\},$$

which is the image of the following map:

$$GP(U) \longrightarrow \mathcal{P}(\text{Mor}_{\text{Set}}(U, \mathbb{C})), \quad (F, G) \longmapsto \mathcal{PA}_{F,G}(U),$$

where $\mathcal{P}(\text{Mor}_{\text{Set}}(U, \mathbb{C}))$ denotes the power set of the set $\text{Mor}_{\text{Set}}(U, \mathbb{C})$, and so $\text{Ob}(\mathcal{PA}(U))$ is a set. The morphisms in $\mathcal{PA}(U)$ are based on the notion of a map *locally given by pseudodifferentiations*, which is defined as follows: Let $\mathcal{PA}_{F,G}(U)$ and $\mathcal{PA}_{F',G'}(U)$ be sets of pseudoanalytic functions defined by, respectively, $(F, G), (F', G') \in GP(U)$, and let

$$U = \bigcup_{j \in J} U_j;$$

be an open cover. Let there be given on each U_j a finite chain of generating pairs, i.e., an ordered tuple:

$$((F_{j,1}, G_{j,1}), (F_{j,2}, G_{j,2}), \dots, (F_{j,k_j}, G_{j,k_j})) \in GP(U_j)^{k_j}$$

such that, if the sequence contains no fewer than two members,

$$(F_{j,1}, G_{j,1}) \preceq (F_{j,2}, G_{j,2}) \preceq \dots \preceq (F_{j,k_j}, G_{j,k_j});$$

These data are denoted by

$$\left\{ \left(U_j, (F_{j,1}, G_{j,1}) \preceq (F_{j,2}, G_{j,2}) \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j}) \right) \right\}_{j \in J'}$$

where

$$k : J \longrightarrow \mathbb{N} \setminus \{0\}, \quad j \longmapsto k(j) \equiv k_j;$$

Also, let the following conditions hold:

$$(F_{j,1}, G_{j,1}) \sim_{\text{ep}} (F, G)|_{U_j} \quad \text{and} \quad (F_{j,k_j}, G_{j,k_j}) \preceq (F', G')|_{U_j}, \quad j \in J,$$

and for each $j, l \in J$ and $f \in \mathcal{PA}_{F,G}(U)$,

$$D_{(F_{j,1}, G_{j,1})|_{U_j \cap U_l} \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})|_{U_j \cap U_l}}(f|_{U_j \cap U_l}) = D_{(F_{l,1}, G_{l,1})|_{U_j \cap U_l} \preceq \cdots \preceq (F_{l,k_l}, G_{l,k_l})|_{U_j \cap U_l}}(f|_{U_j \cap U_l}),$$

which allows one to glue together the maps

$$D_{(F_{j,1}, G_{j,1}) \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})} : \mathcal{PA}_{F|_{U_j}, G|_{U_j}}(U_j) \longrightarrow \mathcal{PA}_{F'|_{U_j}, G'|_{U_j}}(U_j)$$

into the map

$$\omega : \mathcal{PA}_{F,G}(U) \longrightarrow \mathcal{PA}_{F',G'}(U), \quad f \longmapsto \omega(f),$$

uniquely defined by the condition

$$(\omega(f))|_{U_j} = D_{(F_{j,1}, G_{j,1}) \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})}(f|_{U_j}), \quad j \in J;$$

This map is well-defined due to the following: the equality

$$\begin{aligned} & \left(D_{(F_{j,1}, G_{j,1}) \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})}(f|_{U_j}) \right) \Big|_{U_j \cap U_l} = D_{(F_{j,1}, G_{j,1})|_{U_j \cap U_l} \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})|_{U_j \cap U_l}}(f|_{U_j \cap U_l}) \\ & = D_{(F_{l,1}, G_{l,1})|_{U_j \cap U_l} \preceq \cdots \preceq (F_{l,k_l}, G_{l,k_l})|_{U_j \cap U_l}}(f|_{U_j \cap U_l}) = \left(D_{(F_{l,1}, G_{l,1}) \preceq \cdots \preceq (F_{l,k_l}, G_{l,k_l})}(f|_{U_l}) \right) \Big|_{U_j \cap U_l}, \end{aligned}$$

implies that the collection of maps

$$\left\{ D_{(F_{j,1}, G_{j,1}) \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})}(f|_{U_j}) \in \mathcal{PA}_{F'|_{U_j}, G'|_{U_j}}(U_j) \right\}_{j \in J}$$

define uniquely the map $\omega(f) : U \rightarrow \mathbb{C}$ and the locality of pseudoanalyticity implies that $\omega(f) \in \mathcal{PA}_{F',G'}(U)$. We will say that a map

$$\omega : \mathcal{PA}_{F,G}(U) \longrightarrow \mathcal{PA}_{F',G'}(U)$$

is locally given by pseudodifferentiations, or is l.p.d., if it can be given as above. Let the set of morphisms between the objects $\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U)$ for nonequipotent (F, G) and (F', G') consist of l.p.d. maps, i.e.,

$$\text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U)) := \{ \omega : \mathcal{PA}_{F,G}(U) \longrightarrow \mathcal{PA}_{F',G'}(U) \mid \omega \text{ is l.p.d.} \},$$

whereas for equipotent $(F, G) \sim_{\text{ep}} (F', G')$, i.e., when $\mathcal{PA}_{F,G}(U) = \mathcal{PA}_{F',G'}(U)$, let the set of morphisms contain, besides l.p.d. maps, also the identity map. Thus,

$$\text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U)) := \{\omega : \mathcal{PA}_{F,G}(U) \longrightarrow \mathcal{PA}_{F',G'}(U) \mid \omega \text{ is l.p.d.}\} \cup \{\text{id}_{\mathcal{PA}_{F,G}(U)} : \mathcal{PA}_{F,G}(U) \longrightarrow \mathcal{PA}_{F,G}(U)\};$$

and let the composition of composable morphisms in $\mathcal{PA}(U)$ be given by the usual composition of maps, which is well-defined due to the following Proposition:

Proposition 2. For the composition of

$$\omega \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U)), \quad \eta \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F',G'}(U), \mathcal{PA}_{F'',G''}(U))$$

$\eta \circ \omega \in \text{Mor}_{\text{Set}}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F'',G''}(U))$ holds:

$$\eta \circ \omega \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F'',G''}(U)) \subseteq \text{Mor}_{\text{Set}}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F'',G''}(U)).$$

and so $\mathcal{PA}(U)$ is a category.

Proof. The statement is obvious if either of the maps is an identity; so, let neither ω nor η be identities, with ω being given by an open cover $U = \bigcup_{j \in J} U_j$ and the collection

$$\left\{ \left(U_j, (F_{j,1}, G_{j,1}) \preceq (F_{j,2}, G_{j,2}) \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j}) \right) \right\}_{j \in J'}$$

and η be given by $U = \bigcup_{l \in L} V_l$ and

$$\left\{ \left(V_l, (\tilde{F}_{l,1}, \tilde{G}_{l,1}) \preceq (\tilde{F}_{l,2}, \tilde{G}_{l,2}) \preceq \cdots \preceq (\tilde{F}_{l,\tilde{k}_l}, \tilde{G}_{l,\tilde{k}_l}) \right) \right\}_{l \in L'}$$

For any $f \in \mathcal{PA}_{F,G}(U)$, one has

$$\begin{aligned} ((\eta \circ \omega)(f))|_{U_j \cap V_l} &= (\eta(\omega(f)))|_{U_j \cap V_l} = (\eta(\omega(f)))|_{V_l}|_{U_j \cap V_l} \\ &= \left(D_{(\tilde{F}_{l,1}, \tilde{G}_{l,1}) \preceq \cdots \preceq (\tilde{F}_{l,\tilde{k}_l}, \tilde{G}_{l,\tilde{k}_l})} \left(\omega(f)|_{V_l} \right) \right)|_{U_j \cap V_l} = D_{(\tilde{F}_{l,1}, \tilde{G}_{l,1})|_{U_j \cap V_l} \preceq \cdots \preceq (\tilde{F}_{l,\tilde{k}_l}, \tilde{G}_{l,\tilde{k}_l})|_{U_j \cap V_l}} \left(\omega(f)|_{U_j \cap V_l} \right) \\ &= D_{(\tilde{F}_{l,1}, \tilde{G}_{l,1})|_{U_j \cap V_l} \preceq \cdots \preceq (\tilde{F}_{l,\tilde{k}_l}, \tilde{G}_{l,\tilde{k}_l})|_{U_j \cap V_l}} \left(D_{(F_{j,1}, G_{j,1})|_{U_j \cap U_l} \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})|_{U_j \cap U_l}} (f|_{U_j \cap V_l}) \right) \\ &= D_{(\tilde{F}_{l,1}, \tilde{G}_{l,1})|_{U_j \cap V_l} \preceq \cdots \preceq (\tilde{F}_{l,\tilde{k}_l}, \tilde{G}_{l,\tilde{k}_l})|_{U_j \cap V_l}} \circ D_{(F_{j,1}, G_{j,1})|_{U_j \cap U_l} \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})|_{U_j \cap U_l}} (f|_{U_j \cap V_l}) \\ &= D_{(F_{j,1}, G_{j,1})|_{U_j \cap U_l} \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})|_{U_j \cap U_l} \preceq (\tilde{F}_{l,1}, \tilde{G}_{l,1})|_{U_j \cap V_l} \preceq \cdots \preceq (\tilde{F}_{l,\tilde{k}_l}, \tilde{G}_{l,\tilde{k}_l})|_{U_j \cap V_l}} (f|_{U_j \cap V_l}), \end{aligned}$$

Thus, the composition $\eta \circ \omega$ of the maps ω and η is representable by the data

$$U = \bigcup_{(j,l) \in J \times L} U_j \cap V_l \quad \text{and} \quad \left\{ \left(U_j \cap V_l, (F_{j,1}, G_{j,1})|_{U_j \cap V_l} \preceq \cdots \preceq (\tilde{F}_{l,\tilde{k}_l}, \tilde{G}_{l,\tilde{k}_l})|_{U_j \cap V_l} \right) \right\}_{(j,l) \in J \times L};$$

therefore,

$$\eta \circ \omega \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F'',G''}(U)).$$

□

For a fixed generating pair, the set of pseudoanalytic functions with respect to it with pointwise addition and scalar multiplication naturally is an $\mathbb{R}$ -vector space, and we always imply the sets of pseudoanalytic functions to be equipped with this structure. One has the following:

Proposition 3. *The category $\mathcal{PA}(U)$ is a subcategory of $\text{Vec}_{\mathbb{R}}$, the category of $\mathbb{R}$ -vector spaces and $\mathbb{R}$ -linear maps.*

Proof. Naturally, one has

$$\text{Ob}(\mathcal{PA}(U)) \subseteq \text{Ob}(\text{Vec}_{\mathbb{R}});$$

Let us now consider a morphism $\omega \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U))$ given by the following data,

$$U = \bigcup_{j \in J} U_j, \quad \left\{ \left(U_j, (F_{j,1}, G_{j,1}) \preccurlyeq (F_{j,2}, G_{j,2}) \preccurlyeq \cdots \preccurlyeq (F_{j,k_j}, G_{j,k_j}) \right) \right\}_{j \in J},$$

and two elements of its domain $f, g \in \mathcal{PA}_{F,G}(U)$ together with a pair of numbers $a, b \in \mathbb{R}$. For each $j \in J$, due to the $\mathbb{R}$ -linearity of pseudodifferentiation,

$$\begin{aligned} (\omega(af + bg))|_{U_j} &= D_{(F_{j,1}, G_{j,1}) \preccurlyeq \cdots \preccurlyeq (F_{j,k_j}, G_{j,k_j})}((af + bg)|_{U_j}) = D_{(F_{j,1}, G_{j,1}) \preccurlyeq \cdots \preccurlyeq (F_{j,k_j}, G_{j,k_j})}(af|_{U_j} + bg|_{U_j}) \\ &= aD_{(F_{j,1}, G_{j,1}) \preccurlyeq \cdots \preccurlyeq (F_{j,k_j}, G_{j,k_j})}(f|_{U_j}) + bD_{(F_{j,1}, G_{j,1}) \preccurlyeq \cdots \preccurlyeq (F_{j,k_j}, G_{j,k_j})}(g|_{U_j}) = a(\omega(f))|_{U_j} + b(\omega(g))|_{U_j} \\ &= (a\omega(f) + b\omega(g))|_{U_j}, \end{aligned}$$

Thus, the restrictions to each element of the cover of their domain of the two functions $\omega(af + bg)$ and $a\omega(f) + b\omega(g)$ are equal and therefore

$$\omega(af + bg) = a\omega(f) + b\omega(g);$$

(this equality would also follow from these functions being equal only on any one of the covering elements, since, by the Carleman theorem (see [4]), in that case too, they would be equal on the whole domain). It follows that the map

$$\omega : \mathcal{PA}_{F,G}(U) \longrightarrow \mathcal{PA}_{F',G'}(U)$$

is $\mathbb{R}$ -linear and, therefore,

$$\text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U)) \subseteq \text{Mor}_{\text{Vec}_{\mathbb{R}}}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U)).$$

□

Remark 1. *Naturally, these categories are small: for each open subset $U \subseteq \mathbb{C}$, there is a natural bijection*

$$\text{Ob}(\mathcal{PA}(U)) \cong GP(U) / \sim_{\text{ep}},$$

sending each equivalence class of generating pairs to the corresponding space of pseudoanalytic functions, so the collection of objects is a set, which is also clear from the definition, and so too is the collection of morphisms between any two objects a set, and so the collection of all morphisms

$$\text{Ar}(\mathcal{PA}(U)) := \bigcup_{(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U)) \in \text{Ob}(\mathcal{PA}(U)) \times \text{Ob}(\mathcal{PA}(U))} \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U))$$

is also a set.

Lemma 2. *In the category $\mathcal{PA}(U)$, only the identity morphisms*

$$\text{id}_{\mathcal{PA}_{F,G}(U)} \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F,G}(U)), \quad \mathcal{PA}_{F,G}(U) \in \text{Ob}(\mathcal{PA}(U))$$

are isomorphisms.

Proof. For any non-identity morphism

$$\omega \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U))$$

and for any (F, G) -constant pseudoanalytic function f different from 0 (such as F and G themselves) and for an open covering $U = \bigcup_{j \in J} U_j$ on the elements of which ω is representable by a chain of pseudodifferentiations, due to the $\mathbb{R}$ -linearity of pseudodifferentiation, the following holds:

$$\omega(f)|_{U_j} = D_{(F_{j,1}, G_{j,1}) \preccurlyeq \dots \preccurlyeq (F_{j,k_j}, G_{j,k_j})} (f|_{U_j}) = D_{(F_{j,2}, G_{j,2}) \preccurlyeq \dots \preccurlyeq (F_{j,k_j}, G_{j,k_j})} \left(D_{F|_{U_j}, G|_{U_j}} (f|_{U_j}) \right) = 0,$$

which means that $\omega(f) = 0 \in \mathcal{PA}_{F',G'}(U)$ and so ω has a nontrivial kernel; therefore, it is not injective and thus cannot be invertible in this category. $\square$

Using Bers's result on generating sequences [4,5], one can show that, after restricting to an appropriate open subset, for any object, there are objects such that the sets of (nontrivial) morphisms between them are nonempty.

Proposition 4. For each $\mathcal{PA}_{F,G}(U) \in \text{Ob}(\mathcal{PA}(U))$, there exists an open subset $V \subseteq U$ and sequences of generating pairs

$$((F_n, G_n) \in GP(V))_{n \in \mathbb{N}}, \quad ((F'_n, G'_n) \in GP(V))_{n \in \mathbb{N}}$$

such that, for each $n \in \mathbb{N}$,

$$\text{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F|_V, G|_V}(V), \mathcal{PA}_{F_n, G_n}(V)) \neq \emptyset \text{ and } \text{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F'_n, G'_n}(V), \mathcal{PA}_{F|_V, G|_V}(V)) \neq \emptyset.$$

Proof. A generating pair $(F, G) \in GP(U)$ restricted to an appropriately chosen open subset $V \subseteq U$ can be embedded into a generating sequence ([4], p. 41), which means the existence of a sequence

$$((F_k, G_k) \in GP(V))_{k \in \mathbb{Z}}, \quad (F_0, G_0) = (F, G)|_V,$$

such that for each $k \in \mathbb{Z}$,

$$(F_{k-1}, G_{k-1}) \preccurlyeq (F_k, G_k) \preccurlyeq (F_{k+1}, G_{k+1});$$

Correspondingly, each pseudodifferentiation D_{F_k, G_k} is a morphism in $\mathcal{PA}(V)$, and one may consider the following diagram of the shape $(\mathbb{Z}, \leq)$

$$\dots \xrightarrow{D_{F_{-2}, G_{-2}}} \mathcal{PA}_{F_{-1}, G_{-1}}(V) \xrightarrow{D_{F_{-1}, G_{-1}}} \mathcal{PA}_{F|_V, G|_V}(V) \xrightarrow{D_{F_0, G_0}} \mathcal{PA}_{F_1, G_1}(V) \xrightarrow{D_{F_1, G_1}} \dots,$$

thus, for each $n \geq 2, n \in \mathbb{N}$, there are morphisms of $\mathcal{PA}(V)$:

$$\begin{aligned} D_{(F_0, G_0) \preccurlyeq \dots \preccurlyeq (F_{n-1}, G_{n-1})} : \mathcal{PA}_{F|_V, G|_V}(V) &\longrightarrow \mathcal{PA}_{F_n, G_n}(V), \\ D_{(F_{-n}, G_{-n}) \preccurlyeq \dots \preccurlyeq (F_{-1}, G_{-1})} : \mathcal{PA}_{F_{-n}, G_{-n}}(V) &\longrightarrow \mathcal{PA}_{F|_V, G|_V}(V), \end{aligned}$$

and also morphisms for $n = 1$ given by usual pseudodifferentiation, without composition, and for $n = 0$ given by the identity map. Therefore, for the sequences of generating pairs

$$((F_n, G_n) \in GP(V))_{n \in \mathbb{N}}, \quad ((F'_n, G'_n) := (F_{-n}, G_{-n}) \in GP(V))_{n \in \mathbb{N}}$$

for each $n \in \mathbb{N}$, the following holds:

$$\text{Mor}_{\mathcal{PA}(V)} \left(\mathcal{PA}_{F|_V, G|_V}(V), \mathcal{PA}_{F_n, G_n}(V) \right) \neq \emptyset, \quad \text{Mor}_{\mathcal{PA}(V)} \left(\mathcal{PA}_{F'_n, G'_n}(V), \mathcal{PA}_{F|_V, G|_V}(V) \right) \neq \emptyset.$$

The following Proposition relates periodic generating sequences to cones and cocones over diagrams of a certain shape in the categories of spaces of pseudoanalytic functions; we denote by $\mathcal{N}$ the category associated with the totally ordered set $(\mathbb{N}, \leq)$. For the morphism corresponding to $n \leq m$, we use the notation $*_{n,m}$. Consider a generating sequence $((F_k, G_k) \in GP(U))_{k \in \mathbb{Z}}$ on an open subset $U \subseteq \mathbb{C}$ with minimal period $\nu \in \mathbb{N}$:

$$(F_{k+\nu}, G_{k+\nu}) = (F_k, G_k), k \in \mathbb{Z},$$

then each $\mathcal{PA}_{F_1, G_1}(U), \dots, \mathcal{PA}_{F_\nu, G_\nu}(U) \in \text{Ob}(\mathcal{PA}(U))$ is, together with appropriately chosen morphisms, a cone and a cocone to certain diagrams.

Proposition 5. For a periodic generating sequence $((F_k, G_k) \in GP(U))_{k \in \mathbb{Z}}$ with minimal period $\nu \in \mathbb{N}$, each object

$$\mathcal{PA}_{F_j, G_j}(U) \in \text{Ob}(\mathcal{PA}(U)), \quad j \in \{1, \dots, \nu\}$$

together with the appropriate morphisms (these are given in the proof; the dependence on j is not expressed in the notation)

$$\omega_n : \mathcal{PA}_{F_j, G_j}(U) \longrightarrow \mathcal{PA}_{F_n, G_n}(U), \quad \eta_n : \mathcal{PA}_{F_n, G_n}(U) \longrightarrow \mathcal{PA}_{F_j, G_j}(U), \quad n \in \mathbb{N}$$

is a cone to the diagram

$$\Delta : \mathcal{N} \longrightarrow \mathcal{PA}(U), \quad n \longmapsto \Delta(n) := \mathcal{PA}_{F_n, G_n}(U),$$

$$*_{n,m} \longmapsto \Delta(*_{n,m}) := D_{(F_n, G_n) \preceq \dots \preceq (F_{m-1}, G_{m-1})}$$

and a cocone to

$$\Delta' : \mathcal{N}^{\text{op}} \longrightarrow \mathcal{PA}(U), \quad n \longmapsto \Delta'(n) := \mathcal{PA}_{F_{-n}, G_{-n}}(U),$$

$$*_{m,n} \longmapsto \Delta'(*_{m,n}) := D_{(F_{-n}, G_{-n}) \preceq \dots \preceq (F_{-m-1}, G_{-m-1})}.$$

Proof. For each $n \in \mathbb{N}$, define morphisms

$$\omega_n := D_{(F_{j-\nu}, G_{j-\nu}) \preceq (F_{j-\nu+1}, G_{j-\nu+1}) \preceq \dots \preceq (F_{n-1}, G_{n-1})} : \mathcal{PA}_{F_j, G_j}(U) \longrightarrow \mathcal{PA}_{F_n, G_n}(U),$$

and

$$\eta_n := D_{(F_{-n}, G_{-n}) \preceq (F_{-n+1}, G_{-n+1}) \preceq \dots \preceq (F_{j-1}, G_{j-1})} : \mathcal{PA}_{F_{-n}, G_{-n}}(U) \longrightarrow \mathcal{PA}_{F_j, G_j}(U);$$

For each $n < m$, the following holds:

$$\begin{aligned} \omega_m &= D_{(F_{j-\nu}, G_{j-\nu}) \preceq (F_{j-\nu+1}, G_{j-\nu+1}) \preceq \dots \preceq (F_{n-1}, G_{n-1}) \preceq (F_n, G_n) \preceq (F_{n+1}, G_{n+1}) \preceq \dots \preceq (F_{m-1}, G_{m-1})} \\ &= D_{(F_n, G_n) \preceq (F_{n+1}, G_{n+1}) \preceq \dots \preceq (F_{m-1}, G_{m-1})} \circ D_{(F_{j-\nu}, G_{j-\nu}) \preceq (F_{j-\nu+1}, G_{j-\nu+1}) \preceq \dots \preceq (F_{n-1}, G_{n-1})} \\ &= \Delta(*_{n,m}) \circ \omega_n \end{aligned}$$

and

$$\begin{aligned}
\eta_m &= D_{(F_{-m}, G_{-m}) \preceq (F_{-m+1}, G_{-m+1}) \preceq \cdots \preceq (F_{-n-1}, G_{-n-1}) \preceq (F_{-n}, G_{-n}) \preceq (F_{-n+1}, G_{-n+1}) \preceq \cdots \preceq (F_{j-1}, G_{j-1})} \\
&= D_{(F_{-n}, G_{-n}) \preceq (F_{-n+1}, G_{-n+1}) \preceq \cdots \preceq (F_{j-1}, G_{j-1})} \circ D_{(F_{-m}, G_{-m}) \preceq (F_{-m+1}, G_{-m+1}) \preceq \cdots \preceq (F_{-n-1}, G_{-n-1})} \\
&= \eta_n \circ \Delta'(*_{n,m}),
\end{aligned}$$

Hence, one has the following commutative diagrams:

$$\begin{array}{ccc}
& \mathcal{PA}_{F_j, G_j}(U) & \\
\omega_n \swarrow & & \searrow \omega_m \\
\mathcal{PA}_{F_n, G_n}(U) & \xrightarrow{\Delta(*_{n,m})} & \mathcal{PA}_{F_m, G_m}(U)
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{PA}_{F_{-m}, G_{-m}}(U) & \xrightarrow{\Delta'(*_{n,m})} & \mathcal{PA}_{F_{-n}, G_{-n}}(U) \\
\eta_m \searrow & & \swarrow \eta_n \\
& \mathcal{PA}_{F_j, G_j}(U) &
\end{array}$$

the first of which form faces of a cone with the vertex $\mathcal{PA}_{F_j, G_j}(U)$ to Δ and the second of a cocone to Δ' with the same vertex. $\square$

3.2. The Prestack of Spaces of Pseudoanalytic Functions and Effectiveness of a Descent Datum

The assignment

$$U \longmapsto \mathcal{PA}(U)$$

can be extended to a (strict) 2-presheaf

$$\mathcal{PA} : \text{Open}(\mathbb{C})^{\text{op}} \longrightarrow \text{Cat}$$

as follows: to each $U \subseteq V$ (or, more precisely, the corresponding morphism), assign the functor

$$\mathcal{PA}(\iota_{U,V}) \equiv \iota_{U,V}^* : \mathcal{PA}(V) \longrightarrow \mathcal{PA}(U), \quad \mathcal{PA}_{F,G}(V) \longmapsto \iota_{U,V}^*(\mathcal{PA}_{F,G}(V)) := \mathcal{PA}_{F|_U, G|_U}(U),$$

acting on morphisms in the manner described below:

$$\begin{aligned}
\text{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F,G}(V), \mathcal{PA}_{F',G'}(V)) &\longrightarrow \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F|_U, G|_U}(U), \mathcal{PA}_{F'|_U, G'|_U}(U)), \\
\omega &\longmapsto \iota_{U,V}^*(\omega),
\end{aligned}$$

To a morphism ω given by the covering $V = \bigcup_{j \in J} V_j$ and the collection

$$\left\{ \left(V_j, (F_{j,1}, G_{j,1}) \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j}) \right) \right\}_{j \in J}$$

assign the morphism $\iota_{U,V}^*(\omega)$ given by the covering $U = \bigcup_{j \in J} U_j$, where $U_j := V_j \cap U$ and the collection

$$\left\{ \left(U_j, (F_{j,1}, G_{j,1})|_{U_j} \preceq \cdots \preceq (F_{j,k_j}, G_{j,k_j})|_{U_j} \right) \right\}_{j \in J};$$

this map is well-defined since for each $j \in J$, the implication

$$(F_{j,k_j}, G_{j,k_j}) \sim_{\text{ep}} (F', G')|_{V_j} \implies (F_{j,k_j}, G_{j,k_j})|_{U_j} \sim_{\text{ep}} (F', G')|_{U_j},$$

holds and the equality

$$D_{(F_{j,1}, G_{j,1})|_{V_j \cap V_l} \preceq \dots \preceq (F_{j,k_j}, G_{j,k_j})|_{V_j \cap V_l}} = D_{(F_{l,1}, G_{l,1})|_{V_j \cap V_l} \preceq \dots \preceq (F_{l,k_l}, G_{l,k_l})|_{V_j \cap V_l}}$$

implies

$$D_{(F_{j,1}, G_{j,1})|_{U_j \cap U_l} \preceq \dots \preceq (F_{j,k_j}, G_{j,k_j})|_{U_j \cap U_l}} = D_{(F_{l,1}, G_{l,1})|_{U_j \cap U_l} \preceq \dots \preceq (F_{l,k_l}, G_{l,k_l})|_{U_j \cap U_l}},$$

with identity maps being sent to corresponding identity maps. Therefore, one has

$$\iota_{U,V}^*(\omega) \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F|_U, G|_U}(U), \mathcal{PA}_{F'|_U, G'|_U}(U)).$$

In the following 2 Propositions, it is shown that $\mathcal{PA} : \text{Open}(\mathbb{C})^{\text{op}} \rightarrow \text{Cat}$ is a prestack.

Proposition 6.

$$\mathcal{PA} : \text{Open}(\mathbb{C})^{\text{op}} \longrightarrow \text{Cat}$$

is a strict 2-presheaf.

Proof. One has to show that for the composition of morphisms $U \subseteq V \subseteq W$ in $\text{Open}(\mathbb{C})^{\text{op}}$, the composition of the associated restriction functors

$$\iota_{U,V}^* \circ \iota_{V,W}^* : \mathcal{PA}(W) \longrightarrow \mathcal{PA}(U)$$

and the restriction functor

$$\iota_{U,W}^* : \mathcal{PA}(W) \longrightarrow \mathcal{PA}(U)$$

are equal; for objects $\mathcal{PA}_{F,G}(W)$, the following holds:

$$\begin{aligned} \iota_{U,V}^* \circ \iota_{V,W}^*(\mathcal{PA}_{F,G}(W)) &= \iota_{U,V}^*(\iota_{V,W}^*(\mathcal{PA}_{F,G}(W))) = \iota_{U,V}^*(\mathcal{PA}_{F|_V, G|_V}(V)) \\ &= \mathcal{PA}_{F|_V|_U, G|_V|_U}(U) = \mathcal{PA}_{F|_U, G|_U}(U) = \iota_{U,W}^*(\mathcal{PA}_{F,G}(W)), \end{aligned}$$

for a morphism $\omega \in \text{Mor}_{\mathcal{PA}(W)}(\mathcal{PA}_{F,G}(V), \mathcal{PA}_{F',G'}(W))$ given by $W = \bigcup_{j \in J} W_j$ and

$$\left\{ (F_{j,1}, G_{j,1}) \preceq \dots \preceq (F_{j,k_j}, G_{j,k_j}) \right\}_{j \in J}$$

The morphism $\iota_{U,V}^* \circ \iota_{V,W}^*(\omega)$ is given by

$$\left\{ (F_{j,1}, G_{j,1})|_{V_j}|_{U_j} \preceq \dots \preceq (F_{j,k_j}, G_{j,k_j})|_{V_j}|_{U_j} \right\}_{j \in J} = \left\{ (F_{j,1}, G_{j,1})|_{U_j} \preceq \dots \preceq (F_{j,k_j}, G_{j,k_j})|_{U_j} \right\}_{j \in J},$$

with $V_j := W_j \cap V$ and $U_j := W_j \cap U$, which is the same as $\iota_{U,W}^*(\omega)$; therefore,

$$\iota_{U,V}^* \circ \iota_{V,W}^* = \iota_{U,W}^*,$$

and, by definition,

$$\iota_{U,U}^* = \text{id}_{\mathcal{PA}(U)},$$

which means that $\mathcal{PA} : \text{Open}(\mathbb{C})^{\text{op}} \rightarrow \text{Cat}$ is a strict 2-presheaf. $\square$

Theorem 1. The 2-presheaf $\mathcal{PA} : \text{Open}(\mathbb{C})^{\text{op}} \rightarrow \text{Cat}$ is a prestack.

Proof. Consider two objects over $U \subseteq \mathbb{C}$: $\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U) \in \text{Ob}(\mathcal{PA}(U))$ and the associated presheaf of sets on U :

$$\underline{\text{Mor}}_{\mathcal{PA}}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U)) : \text{Open}(U)^{\text{op}} \longrightarrow \text{Set},$$

Assigning the following set to each open subset $V \subseteq U$:

$$V \longmapsto \text{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F|_V, G|_V}(V), \mathcal{PA}_{F'|_V, G'|_V}(V))$$

and assigning the following map to each inclusion (its associated morphism) of open subsets $W \subseteq V \subseteq U$:

$$\begin{aligned} \text{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F|_V, G|_V}(V), \mathcal{PA}_{F'|_V, G'|_V}(V)) &\longrightarrow \text{Mor}_{\mathcal{PA}(W)}(\mathcal{PA}_{F|_W, G|_W}(W), \mathcal{PA}_{F'|_W, G'|_W}(W)), \\ \omega &\longmapsto \iota_{W,V}^*(\omega), \end{aligned}$$

one has to show that this construction is a sheaf. For an open covering $V = \bigcup_{j \in J} V_j \subseteq U$ and a collection of compatible sections

$$\omega_j \in \text{Mor}_{\mathcal{PA}(V_j)}(\mathcal{PA}_{F|_{V_j}, G|_{V_j}}(V_j), \mathcal{PA}_{F'|_{V_j}, G'|_{V_j}}(V_j)), \quad \iota_{V_j \cap V_k, V_j}^*(\omega_j) = \iota_{V_j \cap V_k, V_k}^*(\omega_k)$$

define a map

$$\omega : \mathcal{PA}_{F|_V, G|_V}(V) \longrightarrow \mathcal{PA}_{F'|_V, G'|_V}(V)$$

as follows: for $f \in \mathcal{PA}_{F|_V, G|_V}(V)$, let $\omega(f) : V \rightarrow \mathbb{C}$ be defined by the condition

$$\omega(f) : V = \bigcup_{j \in J} V_j \longrightarrow \mathbb{C}, \quad \omega(f)|_{V_j} := \omega_j(f|_{V_j}), \quad j \in J;$$

It is well-defined due to

$$\begin{aligned} (\omega(f)|_{V_j})|_{V_j \cap V_k} &= (\omega_j(f|_{V_j}))|_{V_j \cap V_k} = \iota_{V_j \cap V_k, V_j}^*(\omega_j)(f|_{V_j \cap V_k}) = \iota_{V_j \cap V_k, V_k}^*(\omega_k)(f|_{V_j \cap V_k}) \\ &= (\omega_k(f|_{V_k}))|_{V_j \cap V_k} = (\omega(f)|_{V_k})|_{V_j \cap V_k}, \end{aligned}$$

and the fact that for each, $j \in J$ $\omega(f)|_{V_j} = \omega_j(f|_{V_j}) \in \mathcal{PA}_{F'|_{V_j}, G'|_{V_j}}(V_j)$, which, because of the locality of pseudoanalyticity, means

$$\omega(f) \in \mathcal{PA}_{F'|_V, G'|_V}(V);$$

Check that this map is a morphism in $\mathcal{PA}(V)$; for each $j \in J$, let the morphism in $\mathcal{PA}(V_j)$

$$\omega_j : \mathcal{PA}_{F|_{V_j}, G|_{V_j}}(V_j) \longrightarrow \mathcal{PA}_{F'|_{V_j}, G'|_{V_j}}(V_j)$$

be given by the open covering $V_j = \bigcup_{k \in K_j} V_{jk}$ and the collection

$$\left\{ (F_{jk,1}, G_{jk,1}) \preccurlyeq F_{jk,2}, G_{jk,2} \preccurlyeq \cdots \preccurlyeq (F_{jk,l_{jk}}, G_{jk,l_{jk}}) \right\}_{k \in K_j};$$

Denote $K := \bigcup_{j \in J} \{j\} \times K_j$; one has the covering of V :

$$V = \bigcup_{j \in J} \bigcup_{k \in K_j} V_{jk} = \bigcup_{(j,k) \in K} V_{jk},$$

Check that the collection

$$\left\{ (F_{jk,1}, G_{jk,1}) \preceq F_{jk,2}, G_{jk,2} \preceq \cdots \preceq (F_{jk,l_{jk}}, G_{jk,l_{jk}}) \right\}_{(j,k) \in K}$$

defines a morphism in $\mathcal{PA}(V)$ for $j \in J$ and the $k, k' \in K_j$ compatibility on $V_{jk} \cap V_{jk'}$ follows from the construction, since for fixed $j \in J$, the corresponding collection defines ω_j . Consider $j, j' \in J, j \neq j'$ and $k \in K_j, k' \in K_{j'}$, and for the corresponding intersections $V_{jk} \cap V_{j'k'}$, the maps

$$D_{(F_{jk,1}, G_{jk,1})|_{V_{jk} \cap V_{j'k'}} \preceq \cdots \preceq (F_{jk,l_{jk}}, G_{jk,l_{jk}})|_{V_{jk} \cap V_{j'k'}}}, D_{(F_{j'k',1}, G_{j'k',1})|_{V_{jk} \cap V_{j'k'}} \preceq \cdots \preceq (F_{j'k',l_{j'k'}}, G_{j'k',l_{j'k'}})|_{V_{jk} \cap V_{j'k'}}};$$

from the fact that

$$(F_{jk,1}, G_{jk,1})|_{V_{jk} \cap V_{j'k'}} \sim_{\text{ep}} (F, G)|_{V_{jk} \cap V_{j'k'}} \sim_{\text{ep}} (F_{j'k',1}, G_{j'k',1})|_{V_{jk} \cap V_{j'k'}}$$

and

$$(F_{jk,l_{jk}}, G_{jk,l_{jk}})|_{V_{jk} \cap V_{j'k'}} \preceq (F', G')|_{V_{jk} \cap V_{j'k'}}, \quad (F_{j'k',l_{j'k'}}, G_{j'k',l_{j'k'}})|_{V_{jk} \cap V_{j'k'}} \preceq (F', G')|_{V_{jk} \cap V_{j'k'}},$$

it follows that the domains and codomains of these chains of pseudodifferentiations coincide. For each element $f \in \mathcal{PA}_{(F,G)|_{V_{jk} \cap V_{j'k'}}}(V_{jk} \cap V_{j'k'})$, the following holds:

$$\begin{aligned} D_{(F_{jk,1}, G_{jk,1})|_{V_{jk} \cap V_{j'k'}} \preceq \cdots \preceq (F_{jk,l_{jk}}, G_{jk,l_{jk}})|_{V_{jk} \cap V_{j'k'}}}(f) &= \iota_{V_{jk} \cap V_{j'k'}, V_j}^*(\omega_j)(f) \\ &= \iota_{V_{jk} \cap V_{j'k'}, V_j \cap V_{j'}}^*(\iota_{V_j \cap V_{j'}, V_j}^*(\omega_j))(f) = \iota_{V_{jk} \cap V_{j'k'}, V_j \cap V_{j'}}^*(\iota_{V_j \cap V_{j'}, V_{j'}}^*(\omega_{j'}))(f) \\ &= \iota_{V_{jk} \cap V_{j'k'}, V_{j'}}^*(\omega_{j'})(f) = D_{(F_{j'k',1}, G_{j'k',1})|_{V_{jk} \cap V_{j'k'}} \preceq \cdots \preceq (F_{j'k',l_{j'k'}}, G_{j'k',l_{j'k'}})|_{V_{jk} \cap V_{j'k'}}}(f), \end{aligned}$$

Thus, the collection

$$\left\{ (F_{jk,1}, G_{jk,1}) \preceq F_{jk,2}, G_{jk,2} \preceq \cdots \preceq (F_{jk,l_{jk}}, G_{jk,l_{jk}}) \right\}_{(j,k) \in K}$$

defines a morphism in $\mathcal{PA}(V)$; for each element $f \in \mathcal{PA}_{F|_V, G|_V}(V)$ and each $(j, k) \in K$, one has

$$(\omega(f))|_{V_{jk}} = (\omega(f))|_{V_j}|_{V_{jk}} = (\omega_j(f|_{V_j}))|_{V_{jk}} = D_{(F_{jk,1}, G_{jk,1}) \preceq \cdots \preceq (F_{jk,l_{jk}}, G_{jk,l_{jk}})}(f|_{V_{jk}}),$$

Therefore, this morphism is equal to ω and so

$$\omega \in \text{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F|_V, G|_V}(V), \mathcal{PA}_{F|_V, G|_V}(V));$$

for each $j \in J, k \in K_j$ and a section $f \in \mathcal{PA}_{F|_{V_j}, G|_{V_j}}(V_j)$, the following holds:

$$(\iota_{V_j, V}^*(\omega)(f))|_{V_{jk}} = D_{(F_{jk,1}, G_{jk,1}) \preceq \cdots \preceq (F_{jk,l_{jk}}, G_{jk,l_{jk}})}(f|_{V_{jk}}) = (\omega_j(f))|_{V_{jk}},$$

therefore, $\iota_{V_j, V}^*(\omega)(f) = \omega_j(f)$ on the entire V_j and so

$$\iota_{V_j, V}^*(\omega) = \omega_j;$$

For any morphism having this property $\eta \in \text{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F|_V, G|_V}(V), \mathcal{PA}_{F|_V, G|_V}(V))$, a section $f \in \mathcal{PA}_{F|_V, G|_V}(V)$, and any $j \in J$:

$$(\eta(f))|_{V_j} = \iota_{V_j, V}^*(\eta)(f|_{V_j}) = \omega_j(f|_{V_j}) = (\omega(f))|_{V_j},$$

so $\eta(f) = \omega(f)$ and, therefore, a section of the presheaf $\text{Mor}_{\mathcal{PA}}(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F',G'}(U))$ having this property is unique, which means that it is a sheaf and so $\mathcal{PA} : \text{Open}(\mathbb{C})^{\text{op}} \rightarrow \text{Cat}$ is a prestack. $\square$

Below, we give a sufficient condition for a descent datum to be effective. Consider a descent datum, i.e., a covering $U = \bigcup_{j \in J} U_j$, an object $\mathcal{PA}_{F_j, G_j}(U_j) \in \text{Ob}(\mathcal{PA}(U_j))$ in the fibre over each U_j , and for each pair $j, k \in J$, consider an isomorphism in the category $\mathcal{PA}(U_j \cap U_k)$ between the appropriate restrictions

$$\begin{aligned} \iota_{U_j \cap U_k, U_j}^*(\mathcal{PA}_{F_j, G_j}(U_j)) &= \mathcal{PA}_{F_j|_{U_j \cap U_k}, G_j|_{U_j \cap U_k}}(U_j \cap U_k), \\ \iota_{U_j \cap U_k, U_k}^*(\mathcal{PA}_{F_k, G_k}(U_k)) &= \mathcal{PA}_{F_k|_{U_j \cap U_k}, G_k|_{U_j \cap U_k}}(U_j \cap U_k) \end{aligned}$$

that is, an invertible morphism in this category:

$$\theta_{jk} : \mathcal{PA}_{F_j|_{U_j \cap U_k}, G_j|_{U_j \cap U_k}}(U_j \cap U_k) \longrightarrow \mathcal{PA}_{F_k|_{U_j \cap U_k}, G_k|_{U_j \cap U_k}}(U_j \cap U_k),$$

but, by Lemma 2, each θ_{jk} must be the identity morphism, which means that for each $j, k \in J$

$$\mathcal{PA}_{F_j|_{U_j \cap U_k}, G_j|_{U_j \cap U_k}}(U_j \cap U_k) = \mathcal{PA}_{F_k|_{U_j \cap U_k}, G_k|_{U_j \cap U_k}}(U_j \cap U_k),$$

which in turn means that the generating pairs are equipotent:

$$\left(F_j|_{U_j \cap U_k}, G_j|_{U_j \cap U_k} \right) \sim_{\text{ep}} \left(F_k|_{U_j \cap U_k}, G_k|_{U_j \cap U_k} \right),$$

Therefore, if the associated first two characteristic coefficients, i.e., coefficients of the appropriate Carleman–Bers–Vekua equations, glue into admissible functions, then the conditions of Lemma 1 are fulfilled, so there is a generating pair $(F, G) \in GP(U)$ such that

$$\left(F|_{U_j}, G|_{U_j} \right) \sim_{\text{ep}} (F_j, G_j),$$

For each $j \in J$, we thus have the corresponding object $\mathcal{PA}_{F, G}(U) \in \text{Ob}(\mathcal{PA}(U))$, for which the following holds:

$$\iota_{U_j, U}^*(\mathcal{PA}_{F, G}(U)) = \mathcal{PA}_{F|_{U_j}, G|_{U_j}}(U_j) = \mathcal{PA}_{F_j, G_j}(U_j),$$

for each $j \in J$, meaning that the descent datum

$$\left\{ \left(\mathcal{PA}_{F_j, G_j}(U_j), \theta_{jk} = \text{id}_{\mathcal{PA}_{F_j|_{U_j \cap U_k}, G_j|_{U_j \cap U_k}}(U_j \cap U_k) \right) \right\}_{j, k \in J}$$

is effective.

Hence, from the fact that the 2-presheaf $\mathcal{PA} : \text{Op}(\mathbb{C})^{\text{op}} \rightarrow \text{Cat}$ is a prestack, the associated fibred category

$$p : \int_{\text{Open}(\mathbb{C})} \mathcal{PA} \longrightarrow \text{Open}(\mathbb{C})$$

is a prestack over $\text{Open}(\mathbb{C})$; for its total category, we use the notation

$$PSA := \int_{\text{Open}(\mathbb{C})} \mathcal{PA};$$

so, an object of PSA is given by an open subset $U \subseteq \mathbb{C}$ and a space of pseudoanalytic functions over it is given by $\mathcal{PA}_{F,G}(U)$, so is a pair

$$(U, \mathcal{PA}_{F,G}(U)) \in \text{Ob}(PSA), \quad U \subseteq \mathbb{C}, (F, G) \in GP(U);$$

If $U \not\subseteq V$, then the set of morphisms between $(U, \mathcal{PA}_{F,G}(U))$ and $(V, \mathcal{PA}_{F',G'}(V))$ is empty, whereas if $U \subseteq V$, then each morphism between $(U, \mathcal{PA}_{F,G}(U))$ and $(V, \mathcal{PA}_{F',G'}(V))$ is given by a morphism

$$\omega : \mathcal{PA}_{F'|_{U,G'|_U}}(U) \longrightarrow \mathcal{PA}_{F,G}(U) \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F'|_{U,G'|_U}}(U), \mathcal{PA}_{F,G}(U)),$$

so,

$$\text{Mor}_{PSA}((U, \mathcal{PA}_{F,G}(U)), (V, \mathcal{PA}_{F',G'}(V))) \cong \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F'|_{U,G'|_U}}(U), \mathcal{PA}_{F,G}(U))$$

in $\mathcal{PA}(U)$, and for morphisms between $(U, \mathcal{PA}_{F,G}(U))$ and $(V, \mathcal{PA}_{F',G'}(V))$ and morphisms between $(V, \mathcal{PA}_{F',G'}(V))$ and $(W, \mathcal{PA}_{F'',G''}(W))$, with $U \subseteq V \subseteq W$, which are given by, respectively, the morphisms:

$$\omega \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F'|_{U,G'|_U}}(U), \mathcal{PA}_{F,G}(U)) \text{ and } \eta \in \text{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F''|_{V,G''|_V}}(V), \mathcal{PA}_{F',G'}(V)),$$

the composition is given by the following composition:

$$\omega \circ \iota_{U,V}^* \eta : \mathcal{PA}_{F''|_{U,G''|_U}}(U) \longmapsto \mathcal{PA}_{F,G}(U) \in \text{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F''|_{U,G''|_U}}(U), \mathcal{PA}_{F,G}(U)).$$

A globally defined generating pair $(F, G) \in GP(\mathbb{C})$ naturally induces a section of the prestack $p : PSA \rightarrow \text{Open}(\mathbb{C})$:

$$s_{F,G} : \text{Open}(\mathbb{C}) \longrightarrow PSA, \quad U \longmapsto (U, \mathcal{PA}_{F|_U,G|_U}(U)),$$

So, for example, the sheaf of holomorphic functions naturally induces a section

$$s_{1,i} : \text{Open}(\mathbb{C}) \longrightarrow PSA, \quad U \longmapsto (U, \mathcal{O}_{\mathbb{C}}(U));$$

Equipotent generating pairs define the same section and, vice versa, if two generating pairs define the same section, then they are equipotent, so for generating pairs $(F, G), (F', G') \in GP(\mathbb{C})$,

$$(F, G) \sim_{\text{ep}} (F', G') \iff s_{F,G} = s_{F',G'},$$

hence, this assignment factors through the map

$$GP(\mathbb{C}) / \sim_{\text{ep}} \longrightarrow \Gamma(\text{Open}(\mathbb{C}), PSA), \quad [(F, G)] \longmapsto s_{F,G}.$$

4. Conclusions and Possibility of Future Research

One saw how and in what sense the set of all spaces of pseudoanalytic functions defined on open subsets of the plane can be equipped with the structure of a prestack, which can be used as a unifying structure and allows one to lift the structure of the site from the base category of open subsets of the plane with the natural Grothendieck topology to the total category of the associated fibred category. This makes it possible to consider the category of sheaves on the latter site, i.e., the associated topos. Understanding the latter may contribute to the better understanding of the introduced prestack. One may attempt to extend the assignment of pseudoanalytic functions of the second kind to pseudoanalytic functions of the first kind to a (pre)sheaf.

On the whole, the results allow a more global approach to the theory of pseudoanalytic functions whose theory is of great importance locally both for mathematics and its applications. In the future, the authors might attempt to apply this approach to certain problems of the theory of pseudoanalytic functions and its applications in mathematical physics.

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