

Article

Shewhart Control Chart for Monitoring Time Between Events with Estimated Parameters in Short Production Runs

Feifei Li *  and Guangye Xu

School of Management, Nanjing University of Posts and Telecommunications, Nanjing 210003, China;
xuguangye@njupt.edu.cn

* Correspondence: fiona@njupt.edu.cn; Tel.: +86-18796011002

Abstract

To detect upward and downward parameter changes in high-quality processes (HQP), time between events (TBE) charts have traditionally been used. However, in practice, when the process parameter is unknown and short production runs (SPRs) occur in a smart manufacturing environment, the TBE chart's characteristics need to be studied carefully. To circumvent this issue, a Shewhart TBE chart with an estimated parameter for monitoring SPR processes is studied. The unknown process parameter is estimated using both the uniformly minimum variance unbiased estimator (UMVUE) and the maximum likelihood estimator (MLE). Then, the truncated run length (RL) properties—i.e., truncated average RL (TARL), truncated standard deviation of RL (TSDRL), and percentiles of the truncated RL—of the Shewhart TBE chart in SPR for different parameter settings are obtained through extensive numerical simulations. These truncated RL properties are compared with properties of the Shewhart TBE chart with a known parameter in an SPR. The results show that the Shewhart TBE chart with an estimated parameter is more favorable in cases of an underestimation of the process parameter $\hat{\theta}_0$ for the detection of upward changes or an overestimation of $\hat{\theta}_0$ of the process parameter for the detection of downward changes.

Keywords: estimated process parameter; Shewhart chart; short production runs; time between events; truncated run length

MSC: 62P30



Academic Editor: Abdel Salam G. Abdel-Salam

Received: 27 January 2026

Revised: 14 February 2026

Accepted: 20 February 2026

Published: 28 February 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and

conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

Statistical process control (SPC) has played an important role in many fields, such as in manufacturing processes, aiming at monitoring the improvement or deterioration of product quality. Control charts are one of the most widely applied tools or monitoring schemes in SPC. In general, these monitoring schemes are divided into two different types: the first memory scheme (CUSUM, cumulative sum; EWMA, exponential weighted moving average) and the second memoryless (Shewhart) scheme. Both types of monitoring schemes have their own advantages in detecting small and large changes (see Montgomery [1]).

Today, with technological progress and product quality enhancement, the occurrence of non-conforming items in some processes has become exceedingly rare. These processes are usually defined as high-quality processes (HQP). In HQPs, two types of distributions—*discrete* and *continuous*—are utilized to model quality characteristics in a process. The discrete distribution specifically focuses on monitoring the count of non-conforming items within a sampling interval or the quantity of conforming items situated

between two non-conforming items. Then, the monitoring procedure is achieved through various charts, including the EWMA, CUSUM, and Shewhart schemes (see Rakitzis [2] and Alevizakos and Koukouvinos [3]). Apart from focusing on counting items, another research stream has emerged by concentrating on monitoring the time between events (TBE). Instead of tallying the quantity of items, TBE records the duration between consecutive non-conforming items, and then practitioners can utilize various process monitoring schemes for the recorded TBE. In studies related to TBE, a non-conforming item is designated as an “event”, and the interval between two or more events is referred to as TBE (see Gan [4]). Evidently, an increase in the time between positive/negative events signifies the deterioration/improvement of the process, whereas a decrease indicates the process is improving/deteriorating. Engineering reliability and the medical industry are two of the most prevalent domains where TBE monitoring finds application (see Dogu and Noor-Ul-Amin [5]). For example, detecting the increase or decrease in infection rate of COVID-19 was highly important during the epidemic [6], and the deterioration of regularly maintained systems over time can be monitored by TBE schemes [7]. Given that the emergence of non-conforming products often adheres to a homogeneous Poisson process, the interval between two consecutive samples typically follows an exponential distribution. This particular distribution is examined in this paper for the purpose of TBE monitoring.

In the last few decades, TBE monitoring schemes have attracted much attention from researchers. Some initial works include the Shewhart TBE scheme studied by Radaelli [8], the EWMA TBE devised by Gan [4], and the CUSUM TBE studied by Gan [9]. Inspired by these pioneering works, much attention has been paid to TBE monitoring, especially the simplest memoryless Shewhart TBE chart. For instance, Kumar et al. [10] incorporated several supplementary run rules into the Shewhart TBE control chart, and the results showed that the performance of those with certain run rules exceeded that of their competitors. Kumar and Chakraborti [11] proposed a Bayesian Shewhart t_r chart that incorporates parameter uncertainty via a prior and a posterior distribution. Singh and Kumar [12] aimed to investigate the performance of the Shewhart t_r chart when the shape (skewness) of the original distribution of TBE varies from the shape (skewness) of the exponential distribution. Dorra Rahali and Khoo [13] developed new plotting statistics based on TBE and amplitude, which is actually the change size of the event itself. They figured out that the selection of statistics relies on whether the shift is due to the amplitude or the amplitude along with TBE. Ali and Shah [14] further studied the generalized Rayleigh and exponential Pareto distributions because of their practical relevance in reliability. Ali et al. [15] extended the exponential distribution assumption of the Shewhart chart to a more general Weibull and a generalized exponential distribution using the expected quadratic loss, relative average run length (ARL), and continuous rank probability score as measurement indicators. Additionally, for a better understanding of relative complicated memory-type TBE charts, e.g., EWMA TBE and CUSUM TBE charts, one can refer to Hu et al. [16], Rasheed et al. [17], Afuecheta et al. [18], Shah et al. [19], Aslam et al. [20], Madrid-Alvarez et al. [21], Aslam et al. [22], Li et al. [23], Alevizakos et al. [24], Sparks and Hazrati-Marangaloo [25], Bourke [26], Bourke [27], Sayyed et al. [28] and Saha et al. [29]. Overviews of various TBE charts have been given by Ali et al. [30] and Inez M. Zwetsloot and Woodall [31].

However, it is noteworthy that the majority of research on TBE charts reviewed above is designed for monitoring the process changes over an infinite horizon. This is problematic due to the fact that, against the backdrop of rapid industrial and technological development, new manufacturing scenarios, such as smart manufacturing environments and just-in-time production, are becoming increasingly prevalent and significant, resulting in a demand for a finite (or short) horizon (see Khoo et al. [32] and Lu et al. [33]). Accordingly, this kind of

production process is termed a short production run (SPR) and defined as a process with a planned, finite horizon, where the total number of inspections (I) is limited and traditional infinite-horizon assumptions fail. Examples of SPR can be found in clothes/shoes manufacturing processes, where different kinds of clothes and shoes are requested by customers in small quantities. Firms that follow just-in-time principles often switch production between similar product codes due to short setup times. Job-shop manufacturing of small batches of mechanical parts using multi-purpose CNC machining centers is another typical example of limited-lot production. Such limited production runs usually have short production horizons. In addition, production and inspection rates can significantly affect the number of samples that can be inspected. As a result, the implementation of SPC may be limited to only a small number of inspections. For instance, if a short production run of M parts has a one-shift production horizon $H = 8$ h, and production and/or inspection rates allow a sampling frequency $h = 1$ h, the number of scheduled inspections is $I = 8$ (see Celano et al. [34]). To monitor the SPR processes, researchers have developed different control charts. For instance, Yang and Hillier [35] first studied the mean and variance control charts based on a small number of subgroups, while the concept of SPR was systematically studied for the first time in Ladany [36]. In addition, studies on the economic design of various SPR charts include Nenes and Tagaras [37], Ho and Trindade [38], Trovato et al. [39] and Nenes et al. [40].

To evaluate the performance of different control charts in SPR in terms of *run length* (RL) characteristics, new indicators for SPR, named the truncated ARL ($TARL$) and the truncated average time to signal ($TATS$), have been proposed (see Nenes and Tagaras [41]). Building on this, the truncated standard deviation of RL ($TSDRL$) has also been naturally inferred by other researchers (see Castagliola et al. [42] and Amdouni et al. [43]). With the truncated RL characteristics, Celano et al. [34] studied the Shewhart and EWMA t charts in SPR to monitor the process mean and evaluated their statistical properties. Nenes et al. [44], Castagliola et al. [45], Chong et al. [46] and Khatun et al. [47] extended the fixed sampling interval (FSI) circumstances to variable sampling interval (VSI) and variable sampling size (VSS) for univariate or multivariate charts in SPR. Khoo et al. [32] and Saha et al. [48] studied the EWMA median chart in SPR. Apart from monitoring the process mean and variance from (approximated) normally distributed processes in SPR, control charts with other distributions have also been of great interest to researchers. For instance, Amdouni et al. [49] and Lee et al. [50] conducted research on the improvement of coefficient of variation (CV) charts in SPR. Haq and Khoo [51] and Hu et al. [52] studied the multivariate coefficient of variation (MCV) charts in SPR and came to the conclusion that both EWMA MCV and run rules MCV charts are superior to their competitors. In addition, Tran et al. [53] examined the statistical performance of a proposed one-sided Shewhart chart for monitoring the ratio of two normal variables (RZ) in SPR. These studies recognize that modern manufacturing (like smart and just-in-time production) operates over a finite horizon. They use truncated performance metrics, such as $TARL$ and $TSDRL$, to evaluate different charts' performance. However, they also focus on monitoring the normally distributed process, CV, or RZ, but not TBE.

As is well known, Phase I and Phase II are the two stages when a control chart is implemented. In Phase I, the process parameters are estimated from a set of in-control (IC) data. These estimated parameters are used to compute the control limits of a Phase II chart (see Hu and Castagliola [54]). Then, in Phase II, the constructed control limits are used by practitioners for process monitoring. Some existing works on control charts usually assumed that the process parameters are known (see Shah et al. [19], Bourke [26] and Hu et al. [16]). In practice, this assumption is difficult to sustain. In recent years, research on TBE charts with estimated parameters has attracted much attention. For instance, Saleh

et al. [55] showed that more Phase I datasets are required for charts with a large smoothing constant. Noor et al. [56] used a uniform prior under five different symmetric and asymmetric loss functions to develop TBE charts with estimated parameters. Qiao et al. [57] investigated the effect of estimated process parameters on the properties of the proposed exponential CUSUM chart. Chang and Wu [58] offered a parameter-free Kullback–Leibler Information (KLI) control chart for monitoring TBE with various estimators. Bai et al. [59] proposed an EWMA control chart with VSI to monitor the exponentially distributed process with unknown parameters. Moreover, the review work by Ali et al. [30] emphasized the importance of estimated parameters and pointed out that they were often overlooked. These studies acknowledge that process parameters are unknown in practice and must be estimated from Phase I data. However, they evaluate charts' performance over an infinite horizon. In SPR, to evaluate a TBE chart's performance, a different truncated measure should be constructed by considering the process over an infinite horizon.

To conclude, despite the abundance of studies on control charts in SPR, there is no study on TBE charts in SPR. Furthermore, while works on TBE charts with estimated parameters exist, they have not been studied within the finite-horizon context of SPR, which requires a fundamentally different performance evaluation framework, i.e., truncated run lengths. Therefore, this paper addresses the gap of how parameter estimation affects the performance of a Shewhart TBE chart when the monitoring horizon is finite. It does not aim to show the superiority of the Shewhart TBE chart in SPR over existing TBE charts. The remainder of this article is arranged as follows: The distribution model of TBE and the proposed Shewhart TBE chart with known parameters are introduced in the next section. The proposed TBE chart with estimated parameters is constructed and implemented in Section 3. To simulate the *RL* properties of the Shewhart TBE chart in SPR, Monte Carlo simulations are performed for different parameter settings. Detailed numerical analyses and comparisons are conducted in Section 4. In Section 5, a real data example is utilized to show the proposed TBE monitoring scheme in SPR. Conclusions and recommendations are provided in Section 6.

2. The Shewhart TBE Chart in SPR

2.1. The Distribution of TBE

Let X represent the interested quality characteristic to be monitored, and X_i denote the time interval between two events at time $i = 1, 2, 3, \dots$. Based on the pioneering work in Gan [9], the events in HQP usually follow a Poisson distribution, and TBE follows an exponential distribution. Then, the variable X_i in HQP follows the exponential distribution with the probability density function (p.d.f.) as

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & , x \geq 0; \\ 0 & , x < 0. \end{cases} \quad (1)$$

The corresponding cumulative distribution function (c.d.f.) is given as

$$F(x; \theta) = \begin{cases} 1 - e^{-\frac{x}{\theta}} & , x \geq 0; \\ 0 & , x < 0. \end{cases} \quad (2)$$

When the process is in an IC status, the scale parameter $\theta = \theta_0$. Once the process alters to an out-of-control (OOC) status, the scale parameter changes to $\theta = \theta_1 = \rho\theta_0$, where $\rho > 1$ indicates the upward change and $\rho < 1$ indicates the downward change. This is to say, $\rho = 1$ naturally means the process runs in an IC status. In order to check the status of

the HQPs, different monitoring schemes for TBE are implemented to detect shifts in the parameter θ .

2.2. The Shewhart TBE Chart with Known Process Parameter in SPR

Let $X_i, i = 1, 2, 3, \dots, I$ denote the monitored Shewhart TBE chart statistic in Phase II. It follows the distribution given by Equation (2), and I refers to the scheduled inspections in the process ($I \in \{10, 20, 30\}$ for the illustration). Without loss of generality, the known parameter is predetermined to be 1, i.e., $\theta_0 = 1$. For the upper-sided chart, the process is determined to be OOC when X_i is beyond the upper control limit UCL ($X_i > UCL$) for the detection of upward shifts. On the other hand, for the lower-sided chart, the process is considered to be OOC when X_i is lower than the lower control limit LCL ($X_i < LCL$) for the detection of downward shifts.

To obtain the truncated RL properties of the TBE chart, both the $TARL$ and $TSDRL$ are obtained by following existing works in Nenes and Tagaras [41], Khoo et al. [32] and Hu et al. [52]. For simplicity, the truncated RL properties of the Shewhart TBE chart for upward shifts are elaborated in detail hereafter.

Firstly, following Nenes and Tagaras [41], the IC $TARL$ ($TARL_0$) is described as

$$TARL_0 = \alpha \sum_{i=1}^I i(1-\alpha)^{i-1} + (I+1)(1-\alpha)^I = \frac{1 - (1-\alpha)^{I+1}}{\alpha}, \quad (3)$$

where α is the Type-I error for the Shewhart TBE chart:

$$\alpha = 1 - \Pr(X_i \leq UCL | \rho = 1) = 1 - F(UCL; \theta_0), \quad (4)$$

where $F(\cdot)$ represents the c.d.f of the statistic X_i in Equation (2).

Secondly, the OOC $TARL$ ($TARL_1$) can be computed by

$$TARL_1 = (1-\beta) \sum_{i=1}^I i\beta^{i-1} + (I+1)\beta^I = \frac{1 - \beta^{I+1}}{1-\beta}, \quad (5)$$

where β stands for the probability of the Type-II error for the Shewhart TBE chart:

$$\beta = \Pr(X_i \leq UCL | \rho \neq 1) = F(UCL; \theta_1 = \rho\theta_0). \quad (6)$$

In addition, the corresponding OOC $TSDRL$ ($TSDRL_1$) can be calculated by

$$TSDRL_1 = \frac{\sqrt{\beta(1-\beta^{2I+1}) - (1-\beta)\beta^{I+1}(1+2I)}}{1-\beta}. \quad (7)$$

With the aforementioned truncated RL measures, the UCL of the proposed Shewhart TBE chart can be obtained by satisfying the following constraint:

$$TARL(\theta_0, \rho = 1, UCL) = TARL_0 \quad (8)$$

Similarly, for the detection of downward shifts, the Shewhart TBE chart can be designed by satisfying the following constraint:

$$TARL(\theta_0, \rho = 1, LCL) = TARL_0 \quad (9)$$

where $TARL_0$ is calculated with $\alpha = \Pr(X_i \leq LCL | \rho = 1) = F(LCL; \theta_0)$ and the corresponding $TARL_1$ can be calculated by substituting $\beta = \Pr(X_i \geq LCL | \rho \neq 1) = F(LCL; \theta_1 = \rho\theta_0)$ into Equation (7).

To display the implementation of the Shewhart TBE chart with known parameters, the workflow is given as follows:

- Step 1 Record the time interval $X_i, i = 1, 2, 3, \dots, I$ between two interested events.
- Step 2 If $X_i \leq UCL$ (for the upward changes) or $X_i \geq LCL$ (for the downward changes), the workflow turns to Step 1 instantly and records the RL value as $RL = RL + 1$. If X_i is larger than the UCL or smaller than the LCL , the workflow moves to the next step.
- Step 3 The proposed chart gives out an OOC signal and records the current RL value. Here, due to the SPR, the RL is truncated to be $RL = I + 1$ when $i > I$.
- Step 4 The process is restored to the IC state, and the monitoring procedure returns to Step 1 for process monitoring.

To obtain the truncated RL properties of the Shewhart TBE chart, all the computations were implemented using the MATLAB software in this study.

3. The Shewhart TBE Chart with Estimated Parameter in SPR

Let $Y_i, i = 1, 2, 3, \dots, N$ denote the Phase I data for estimating the process parameter; N represents the total number of samples. To estimate the process parameter, as recommended by Kumar and Chakraborti [60], the first estimator used here is the uniformly minimum variance unbiased estimator (UMVUE), which has been shown to be more efficient for cases where the number of Phase I samples is small to medium. It is given as

$$\hat{\theta}_{0,UMVUE} = \frac{\sum_{i=1}^N Y_i}{N-1} = \frac{N}{N-1} \bar{Y}_i. \quad (10)$$

In addition, the second commonly used estimator is the maximum likelihood estimator (MLE) (see Jensen et al. [61], Cao and Ma [62] and Hu et al. [63]). It is given by

$$\hat{\theta}_{0,MLE} = \frac{1}{N} \sum_{i=1}^N Y_i = \bar{Y}_i. \quad (11)$$

Let the random variable $w = \frac{\theta_0}{\hat{\theta}_0}$ represent the ratio of the actual IC process mean to its estimator, where $\hat{\theta}_0$ is the estimator introduced in Equations (10) and (11), respectively. When w equals 1, there is no estimation error.

To obtain the truncated RL properties of the Shewhart TBE chart with estimated process parameter, θ in Equations (4) and (6) is substituted with the corresponding estimator $\hat{\theta}_0$ in Equations (10) and (11). This leads to the estimated $\hat{\alpha}$ and $\hat{\beta}$:

$$\hat{\alpha} = 1 - \Pr(X_i \leq UCL | \rho = 1, \hat{\theta}_0) = 1 - F(UCL; \hat{\theta}_0), \quad (12)$$

$$\hat{\beta} = \Pr(X_i \leq UCL | \rho \neq 1, \hat{\theta}_0) = F(UCL; \rho \hat{\theta}_0), \quad (13)$$

Therefore, by substituting the newly obtained estimated $\hat{\alpha}$ and $\hat{\beta}$ into Equations (3), (5) and (7), we can derive novel formulations of $\widehat{TARL}$ and $\widehat{TSDRL}$ of the proposed Shewhart TBE chart with estimated parameters in SPR. Then, the IC and OOC truncated RL properties can be given as follows:

$$\widehat{TARL}_0 = TARL(\hat{\theta}_0, \rho = 1, UCL), \quad (14)$$

and

$$\widehat{TARL}_1 = TARL(\hat{\theta}_0, \rho \neq 1, UCL), \quad (15)$$

$$\widehat{TSDRL}_1 = TSDRL(\hat{\theta}_0, \rho \neq 1, UCL), \quad (16)$$

It is noted that, to show the effect of parameter estimation on the truncated RL performance of the proposed Shewhart TBE chart for upward shifts in SPR, the UCL computed in the known parameter case is used in Equations (14)–(16) (see Ozsan et al. [64]). Similarly, the truncated RL properties of the Shewhart TBE chart for downward shifts can be studied. For brevity, the derivations are not presented here.

To show the workflow of the Shewhart TBE chart with the estimated parameter in SPR, the detailed procedure is given as follows:

- Step 1 Collect N Phase I observations Y_i ($i = 1, 2, \dots, N$) from an IC process followed the exponential distribution with the scale parameter $\theta_0 = 1$ and estimate the process parameter using two estimators $\hat{\theta}_{0,UMVUE}$ or $\hat{\theta}_{0,MLE}$ in Equation (10) or Equation (11).
- Step 2 Record the Phase II time interval X_i ($i = 1, 2, 3, \dots, I$) between two interested events, which follows the exponential distribution with the scale parameter $\hat{\theta}_{0,UMVUE}$ or $\hat{\theta}_{0,MLE}$.
- Step 3 If $X_i \leq UCL$ (for the upward changes) or $X_i \geq LCL$ (for the downward changes), the workflow turns to Step 2 instantly and records the RL value as $RL + 1$. If X_i is beyond UCL or falls below LCL , the workflow moves to the next step.
- Step 4 The proposed chart gives out an OOC signal and records the current RL value. When $i > I$, the RL is truncated to be $RL = I + 1$.
- Step 5 The process is restored to the IC state, and the monitoring procedure returns to Step 1 for process monitoring.

4. Numerical Results and Comparisons

4.1. RL Properties of the Shewhart TBE Chart with Known Process Parameter in SPR

For the detection of upward/downward changes, the UCL/LCL of the upper-sided/lower-sided Shewhart TBE chart can be obtained from Equations (8) and (9). Then, the IC and OOC truncated RL properties can be computed for different changes. Here, for illustration purposes, we consider $\rho \in \{1, 1.2, 1.5, 2.5, 5, 10\}$ for upward changes and $\{1, 0.9, 0.5, 0.1, 0.05, 0.01\}$ for downward changes. The scheduled inspections $I \in \{10, 20, 30\}$ are selected for illustration. These values are consistent with prior existing SPR literature (see Celano et al. [34] and Hu et al. [52]). Moreover, these selected values may span a range of practical short-run scenarios: $I = 10$ may represent very short pilot runs or prototype testing, $I = 20$ may correspond to weekly production targets for specialized components, and $I = 30$ can reflect monthly production quotas or limited-life system testing. All the simulation results are displayed in Table 1, from which conclusions can be drawn as follows:

- As the scheduled inspections I increases, the UCL gradually increases while the LCL gradually decreases.
- In terms of the $TARL$, for the detection of both upward and downward shifts, it keeps declining as the change sizes become large.
- In terms of the $TSDRL$, it is interesting to note that the IC truncated RL of the Shewhart is more stable than the OOC truncated RL . Moreover, for moderate to large change sizes, the Shewhart TBE scheme in SPR has poor truncated RL stability. For instance, when $I = 10$ and $\rho = 1.5$, the Shewhart TBE scheme has a $TARL = 7.795$ combined with a relatively large $TSDRL = 3.673$.
- To gain insights into the bias of the truncated RL of the Shewhart TBE chart, the 10th, 50th, and 90th percentiles of the truncated RL values are calculated. Here, to obtain all the percentiles, the workflow in Section 2.2 is repeated 10^5 times, and the corresponding percentiles of these 10^5 RL values are obtained. It can be seen that for smaller changes, the 50th percentile of the truncated RL is larger than the corresponding $TARL$

value. This indicates that the RL distribution is left-skewed. For larger changes, the 50th percentile of the truncated RL is smaller than the corresponding $TARL$ value. This indicates that the RL distribution is right-skewed. For instance, when $I = 10$ and a small change $\rho = 1.2$, the $TARL = 9.152$ is smaller than the 50th percentile value of 11 of the truncated RL . If the change size is large ($\rho = 5$), the $TARL = 2.197$ is larger than the 50th percentile value of 2 of the truncated RL .

Table 1. IC and OOC truncated RL properties of the Shewhart TBE chart with a known parameter in SPR.

	UCL	ρ	$TARL$	$TSDRL$	10th	50th	90th	LCL	ρ	$TARL$	$TSDRL$	10th	50th	90th
$I = 10$	3.9501	1.0	10.000	2.473	6	11	11	0.0194	1.00	10.000	2.473	6	11	11
		1.2	9.152	3.151	3	11	11		0.90	9.894	2.580	5	11	11
		1.5	7.795	3.673	2	10	11		0.50	9.131	3.171	3	11	11
		2.5	4.454	3.236	1	3	10		0.10	4.989	3.454	1	4	11
		5.0	2.197	1.600	1	2	4		0.05	3.067	2.377	1	2	6
		10.0	1.486	0.851	1	1	3		0.01	1.167	0.443	1	1	2
$I = 20$	5.3162	1.0	20.000	3.575	21	21	21	0.0049	1.00	20.000	3.575	21	21	21
		1.2	18.666	5.201	9	21	21		0.90	19.903	3.733	20	21	21
		1.5	15.918	6.855	4	21	21		0.50	19.076	4.801	11	21	21
		2.5	7.835	6.149	1	6	19		0.10	13.410	7.371	3	15	21
		5.0	2.898	2.335	1	2	6		0.05	9.316	6.781	2	8	21
		10.0	1.706	1.097	1	1	3		0.01	2.585	2.021	1	2	5
$I = 30$	6.1209	1.0	30.000	4.410	31	31	31	0.0022	1.00	30.000	4.410	31	31	31
		1.2	28.272	6.998	17	31	31		0.90	29.883	4.657	31	31	31
		1.5	24.331	9.832	7	31	31		0.50	29.058	6.017	25	31	31
		2.5	10.836	8.875	2	8	26		0.10	22.676	10.450	5	31	31
		5.0	3.403	2.867	1	2	7		0.05	17.318	10.913	3	16	31
		10.0	1.843	1.248	1	1	3		0.01	5.071	4.522	1	4	11

4.2. RL Properties of the Shewhart TBE Chart with Estimated Parameter in SPR

4.2.1. Parameter Estimation

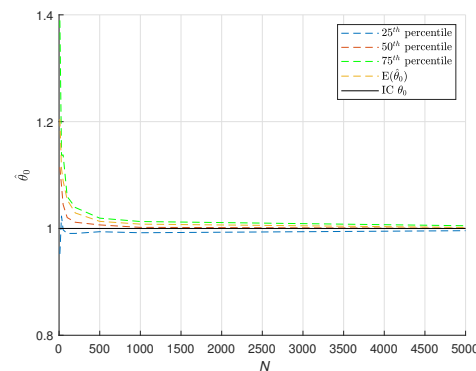
To estimate the process parameter, both UMVUE and MLE are used. Some characteristics of the estimated $\hat{\theta}_0$ are presented in Table 2 for different N . Note that in Table 2, $\hat{\theta}_0 > 1$ indicates the overestimation and $\hat{\theta}_0 < 1$ indicates the underestimation of the true θ_0 . In order to assume the overestimation, median, and underestimation of the parameter θ , 25th, 50th, and 75th percentiles of $\hat{\theta}_0$ are adopted, respectively. For a better understanding, an illustration is visualized in Figure 1. From Table 2 and Figure 1, it can be found that:

- For both UMVUE and MLE, with each Phase I sample size N , the 75th percentile of $\hat{\theta}_0$ is larger than 1 and the 25th percentile is smaller than 1. It is apparent that with the sample size N increasing, the expected value of the 25th, 50th, and 75th percentiles of $\hat{\theta}_0$ all approach 1.
- It can be seen that for the UMVUE, the $E(\hat{\theta}_0)$ is generally larger than 1, except for the case with a larger number of Phase I samples ($N = 5000$). For the MLE, the $E(\hat{\theta}_0)$ is generally larger than 1 except for the case with a smaller number of Phase I samples ($N = 10$).

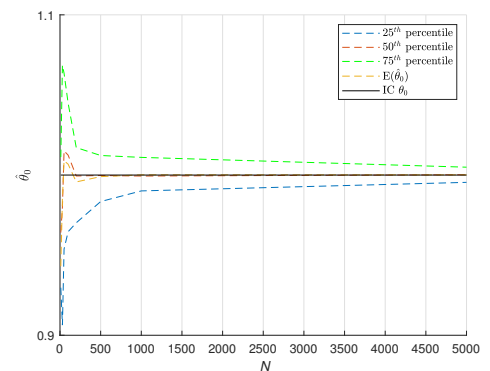
Due to the extremely similar performances of the Shewhart TBE chart with $E(\hat{\theta}_0)$ and the 50th percentile of $\hat{\theta}_0$, following Ozsan et al. [64], the 25th, 50th (instead of $E(\hat{\theta}_0)$), and 75th percentiles of $\hat{\theta}_0$ are used to estimate the process parameter and to further study the Shewhart TBE chart's performance hereafter.

Table 2. The expected and percentile values of $\hat{\theta}_0$ with UMVUE and MLE for different N .

	N	$E(\hat{\theta}_0)$	25th	50th	75th
UMVUE	10	1.109	0.859	1.071	1.322
	100	1.010	0.941	1.008	1.076
	1000	1.002	0.978	1.001	1.022
	5000	0.999	0.990	0.999	1.008
MLE	10	0.996	0.771	0.963	1.187
	100	1.000	0.931	0.995	1.065
	1000	1.001	0.979	1.001	1.024
	5000	1.001	0.993	1.000	1.009



(a) UMVUE

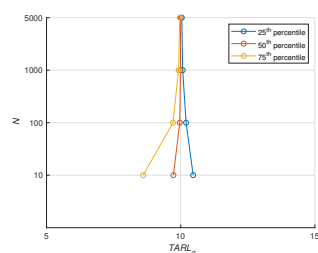


(b) MLE

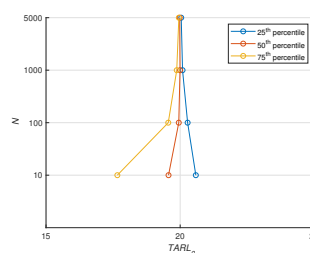
Figure 1. Percentile and expected values of $\hat{\theta}_{0,UMVUE}$ and $\hat{\theta}_{0,MLE}$ as N varies when the assumed IC $\theta_0 = 1$.

4.2.2. IC Truncated RL Comparisons of the Shewhart TBE Chart with Estimated Parameters in SPR

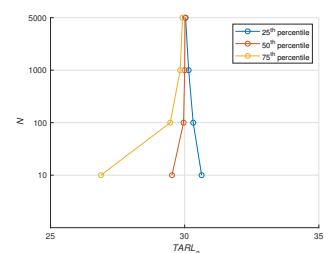
To compare the difference between the $TARL_0$ for both UMVUE and MLE with different estimation errors and the preset $TARL_0$, illustrations are displayed in Figures 2 and 3.



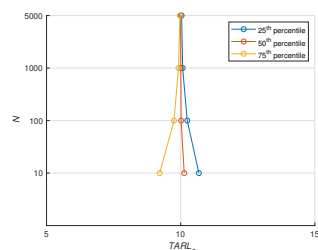
(a) I = 10



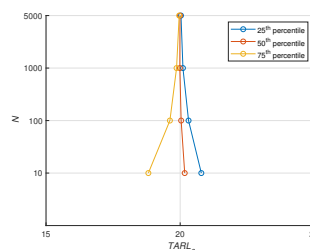
(b) I = 20



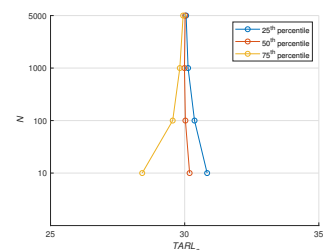
(c) I = 30



(d) I = 10



(e) I = 20



(f) I = 30

Figure 2. $TARL_0$ distribution with UMVUE in (a–c) and MLE in (d–f) for different percentiles of $\hat{\theta}_0$ for upward changes.

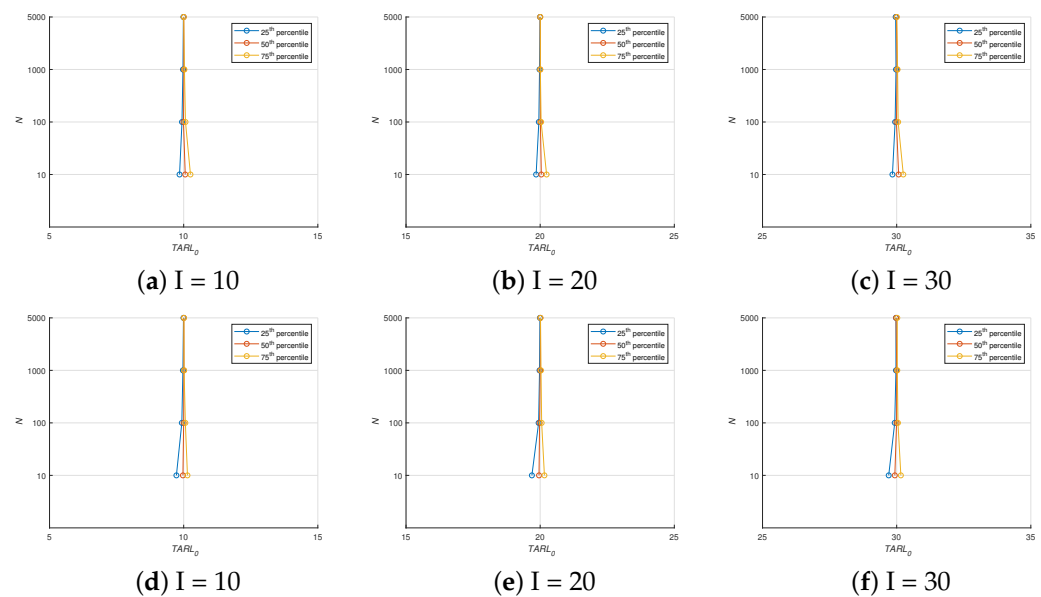


Figure 3. $TARL_0$ distribution with UMVUE in (a–c) and MLE in (d–f) for different percentiles of $\hat{\theta}_0$ for downward changes.

From Figure 2, one can see that in the upper-sided Shewhart TBE chart, an underestimation in $\hat{\theta}_0$ leads to a larger $TARL_0$ than desired. An overestimation in $\hat{\theta}_0$ causes a smaller $TARL_0$ than desired. These results are due to the fact that underestimated $\hat{\theta}_0$ leads to a smaller $\hat{\alpha}$ in Equation (12) than in the case of a known parameter. On the contrary, overestimated $\hat{\theta}_0$ leads to a larger $\hat{\alpha}$ than in the case of a known parameter. Conversely, from Figure 3, one can see that in the lower-sided Shewhart TBE chart, an underestimation in $\hat{\theta}_0$ leads to a smaller $TARL_0$ than desired, while an overestimation causes a larger $TARL_0$ than desired. These results are due to the fact that underestimated $\hat{\theta}_0$ leads to a larger $\hat{\alpha} = \Pr(X_i \leq LCL | \rho = 1) = F(LCL; \hat{\theta}_0)$ than in the case of a known parameter. On the contrary, overestimated $\hat{\theta}_0$ leads to a smaller $\hat{\alpha}$ than in the case of a known parameter. Moreover, when $N = 10$, with the 50th percentile of $\hat{\theta}_0$, the $TARL_0$ of the upper-sided Shewhart TBE chart with UMVUE becomes smaller than desired, while at the same time, the $TARL_0$ of the Shewhart TBE chart with MLE becomes a little larger than desired. This conclusion can be explained from the $\hat{\theta}_0$ shown in Table 2, where UMVUE and MLE with $N = 10$ are 1.071 (larger than 1) and 0.963 (smaller than 1), respectively. Underestimation and overestimation of $\hat{\theta}_0$ lead to a smaller and larger $\hat{\alpha}$ in Equation (12) than in the case of a known parameter, respectively.

4.2.3. OOC Truncated RL Comparisons of the Shewhart TBE Chart with Estimated Parameters in SPR

In this subsection, comparisons are first performed among the proposed Shewhart TBE chart for *upward shifts* in SPR. It is noted that to ensure the accuracy of the Monte Carlo method, 10^5 times of truncated run length simulations are considered sufficient (see Saleh et al. [55]). By computing the standard errors of the cases in Table 3, the largest value is smaller than 0.0117, ensuring the confidence intervals are centered around the mean. From the comparisons, conclusions can be drawn as follows:

- By comparing Tables 3 and 4, it can be concluded that when the underestimation of the process parameter (the 25th percentile of $\hat{\theta}_0$) is considered, the Shewhart TBE chart in SPR with UMVUE is superior to the monitoring scheme with MLE in terms of the $TARL$ when $N = 10$ and 100. For instance, when $N = 10$ and $\rho = 1.5$,

the $TARL_1 = 8.768$ of the Shewhart TBE chart with UMVUE is smaller than the $TARL_1 = 9.368$ of the Shewhart TBE chart with MLE. This may be due to the fact that UMVUE with an underestimated percentile is closer to the real mean parameter $\theta_0 = 1$ than MLE with an underestimated percentile when N is generally small. Nevertheless, the Shewhart TBE chart with two estimators performs similarly as the sample size N increases. Note that when the sample size N and the change size ρ are small, despite the smaller $TARL_1$ value of the Shewhart TBE chart with UMVUE than the chart with MLE, it may perform worse than the chart with the MLE in terms of the $TSDRL_1$. For instance, when $N = 10$ and $\rho \in \{1.2, 1.5\}$, the $TSDRL$ values of the Shewhart TBE chart with UMVUE and MLE are $\{2.606, 3.354\}$ and $\{2.167, 3.010\}$, respectively. Notably, the conclusions drawn above still apply for the cases when the 50th and 75th percentiles of $\hat{\theta}_0$ are considered. For simplicity, these analyses are not presented here. Moreover, for other values of the scheduled inspections $I \in \{20, 30\}$ in Tables S1–S4, the above conclusions still hold.

- In Tables 3 and 4 and Tables S1–S4, the 10th, 50th, and 90th percentiles of the truncated RL are also presented combined with the $TARL$ and $TSDRL$. To obtain all these percentiles, the workflow in Section 2.2 is repeated 10^5 times, and the corresponding percentiles are obtained from the 10^5 RL values. It can be seen that the 50th percentile of the truncated RL is larger than the corresponding $TARL$ value for smaller changes ($\rho \leq 1.5$), which indicates that the truncated RL distribution is left-skewed. For larger changes ($\rho > 1.5$), the 50th percentile of the truncated RL is smaller than the corresponding $TARL$, which indicates that the truncated RL distribution is right-skewed. For example, considering the Shewhart TBE chart with MLE in Table S4, when $I = 30$ and $N = 10$ in the underestimation cases, the $TARL = 28.785$ is smaller than the 50th percentile value of 31 of the truncated RL for a small change $\rho = 1.5$. When the change size is large ($\rho = 5$), the $TARL = 4.891$ is larger than the 50th percentile value of 4 of the truncated RL . It is worth noting that this conclusion is consistent with the case of the Shewhart TBE chart with a known parameter.
- From these tables, it can also be seen that when $I = 10$, an overestimation of $\hat{\theta}_0$ leads the Shewhart TBE chart with UMVUE and MLE to be better than the Shewhart TBE chart with a known parameter. This is due to the fact that overestimation of $\hat{\theta}_0$ leads to a smaller $\hat{\beta}$ in Equation (13) than in the case of a known parameter. For instance, when $I = 10$, $N = 10$, and the 75th percentile of $\hat{\theta}_0$ is used in Table 3, the $TARL$ of the Shewhart TBE chart is 3.236 for the change $\rho = 2.5$, which is smaller than the $TARL = 4.454$ of the Shewhart TBE chart in the known parameter case.

On the contrary, when an underestimation of $\hat{\theta}_0$ occurs, the Shewhart TBE chart with UMVUE and MLE is inferior to the one with a known parameter. This is due to the fact that underestimated $\hat{\theta}_0$ leads to a larger $\hat{\beta}$ in Equation (13) than in the case of a known parameter. For instance, when $I = 10$ and $N = 10$ and the 25th percentile of $\hat{\theta}_0$ is used in Table 3, the $TARL$ of the Shewhart TBE chart with UMVUE for $\rho = 2.5$ is 5.358, which is larger than the $TARL = 4.454$ of the Shewhart TBE chart in the known parameter case.

Moreover, when the 50th percentile of $\hat{\theta}_0$ is considered, the Shewhart TBE chart with UMVUE/MLE performs better/worse than the chart with a known parameter. This conclusion is due to the fact that the 50th percentiles of $\hat{\theta}_0$ with UMVUE and MLE are generally larger and smaller than the actual process parameter $\theta_0 = 1$, respectively, except for the case with a large sample size N . In fact, when N is large, the estimated process parameter $\hat{\theta}_0$ is very close to the actual $\theta_0 = 1$, leading the Shewhart TBE chart's performance to be very close to the chart with a known parameter. For instance, when sample size $N = 10$ and change size $\rho = 1.5$, the $TARL = 7.329$ of the

Shewhart TBE chart with UMVUE is smaller than the $TARL = 7.795$ of the one with a known parameter. Furthermore, it is worth noting that the above conclusions also apply for the cases with $I \in \{20, 30\}$.

In summary, for the detection of upward changes, when the 50th percentile of $\hat{\theta}_0$ is considered, the proposed Shewhart TBE chart with UMVUE performs better than the Shewhart TBE chart with MLE when $N \in \{10, 100\}$. If N becomes much larger, the performance of the Shewhart TBE chart with UMVUE and MLE is quite similar. Furthermore, if there is an underestimation of $\hat{\theta}_0$, the Shewhart TBE with a known parameter surpasses the Shewhart TBE with UMVUE and MLE. On the contrary, if there is an overestimation of $\hat{\theta}_0$, the Shewhart TBE with UMVUE and MLE outperforms the one with a known parameter at the cost of a reduced IC $TARL$.

In addition, for the detection of *downward shifts*, comparisons are performed among the Shewhart TBE chart with UMVUE and MLE. From these comparisons, the following conclusions are drawn:

- By comparing Table 5 and Table 6, it is clear that when the underestimation of the process parameter (the 25th percentile of $\hat{\theta}_0$) case is considered, the Shewhart TBE chart in SPR with UMVUE is inferior to the Shewhart TBE with MLE in terms of the $TARL$ when $N = 10$ and 100. In terms of the $TSDRL$, the $TARL$ of the Shewhart TBE chart in SPR with UMVUE performs more stably than that of the chart with MLE. As the sample size N increases, the Shewhart TBE chart with two estimators performs similarly.
Notably, these conclusions drawn above still apply for the cases when 50th and 75th percentiles of $\hat{\theta}_0$ are considered. For simplicity, these analyses are not presented here. Moreover, for other values of the scheduled inspections $I \in \{20, 30\}$ in Tables S5–S8, the above conclusions still hold.
- In Tables 5 and 6 and Tables S5–S8, the 10th, 50th, and 90th percentiles of the truncated RL are also presented combined with the $TARL$ and $TSDRL$. Similar to the conclusions drawn for the upper-sided Shewhart TBE chart in SPR, it can be seen that the truncated RL distribution is left-skewed for smaller changes ($\rho > 0.1$) and right-skewed for larger changes ($\rho < 0.1$)
- From these tables, one can see that when $I = 10$, an underestimation of $\hat{\theta}_0$ causes the Shewhart TBE chart with UMVUE and MLE to perform better than the Shewhart TBE chart with a known parameter. On the contrary, when an overestimation of $\hat{\theta}_0$ occurs, the Shewhart TBE chart with UMVUE and MLE is inferior to the one with a known parameter.

Moreover, similar to the upper-sided Shewhart TBE in SPR, when the 50th percentile of $\hat{\theta}_0$ is considered, the lower-sided Shewhart TBE chart with UMVUE or MLE performs better than the chart with a known parameter.

In summary, for the detection of downward changes, when the 50th percentile of $\hat{\theta}_0$ is considered, the Shewhart TBE chart with MLE performs better than the Shewhart TBE chart with UMVUE when sample size $N \in \{10, 100\}$. If N becomes much larger, the performance of the Shewhart TBE chart with UMVUE and MLE is quite similar. Furthermore, if there is an underestimation of $\hat{\theta}_0$, the Shewhart TBE chart with UMVUE and MLE is more favorable than the Shewhart TBE chart with a known parameter at the cost of a reduced IC $TARL$. On the contrary, if there is an overestimation of $\hat{\theta}_0$, both Shewhart TBE charts with UMVUE and MLE are inferior to the one with a known parameter.

Table 3. *RL* properties of the Shewhart TBE chart with UMVUE in SPR for upward shifts when $I = 10$.

$I = 10$																
N	ρ	25th (Underestimation)					50th					75th (Overestimation)				
		TARL	TSDRL	10th	50th	90th	TARL	TSDRL	10th	50th	90th	TARL	TSDRL	10th	50th	90th
10	1.0	10.466	1.863	11	11	11	9.729	2.733	5	11	11	8.603	3.431	3	11	11
	1.2	9.874	2.606	5	11	11	8.774	3.356	3	11	11	7.385	3.747	2	8	11
	1.5	8.768	3.354	3	11	11	7.329	3.749	2	8	11	5.873	3.687	1	5	11
	2.5	5.358	3.567	1	5	11	4.114	3.060	1	3	9	3.236	2.505	1	2	7
	5.0	2.488	1.886	1	2	5	2.085	1.490	1	2	4	1.818	1.214	1	1	3
	10.0	1.582	0.956	1	1	3	1.445	0.806	1	1	2	1.348	0.682	1	1	2
100	1.0	10.201	2.240	7	11	11	9.966	2.507	6	11	11	9.713	2.747	5	11	11
	1.2	9.485	2.932	4	11	11	9.132	3.162	3	11	11	8.723	3.374	3	11	11
	1.5	8.197	3.572	2	11	11	7.734	3.689	2	10	11	7.286	3.759	2	8	11
	2.5	4.791	3.387	1	4	11	4.433	3.223	1	3	10	4.104	3.062	1	3	9
	5.0	2.310	1.721	1	2	5	2.187	1.595	1	2	4	2.086	1.489	1	2	4
	10.0	1.520	0.885	1	1	3	1.479	0.845	1	1	3	1.443	0.798	1	1	2
1000	1.0	10.075	2.391	6	11	11	9.991	2.486	6	11	11	9.922	2.557	5	11	11
	1.2	9.271	3.080	3	11	11	9.164	3.145	3	11	11	9.053	3.210	3	11	11
	1.5	7.935	3.645	2	10	11	7.791	3.672	2	10	11	7.619	3.703	2	9	11
	2.5	4.560	3.286	1	4	11	4.473	3.248	1	4	11	4.340	3.174	1	3	10
	5.0	2.248	1.658	1	2	4	2.196	1.602	1	2	4	2.161	1.571	1	2	4
	10.0	1.491	0.857	1	1	3	1.481	0.842	1	1	3	1.470	0.833	1	1	3
5000	1.0	10.041	2.427	6	11	11	9.999	2.477	6	11	11	9.964	2.509	6	11	11
	1.2	9.212	3.116	3	11	11	9.179	3.132	3	11	11	9.123	3.169	3	11	11
	1.5	7.871	3.661	2	10	11	7.794	3.682	2	10	11	7.693	3.698	2	9	11
	2.5	4.523	3.260	1	4	11	4.476	3.238	1	4	10	4.433	3.218	1	3	10
	5.0	2.225	1.630	1	2	4	2.201	1.604	1	2	4	2.187	1.598	1	2	4
	10.0	1.492	0.853	1	1	3	1.482	0.850	1	1	3	1.479	0.839	1	1	3

Table 4. *RL* properties of the Shewhart TBE chart with MLE in SPR for upward shifts when $I = 10$.

$I = 10$																
N	ρ	25th (Underestimation)					50th					75th (Overestimation)				
		TARL	TSDRL	10th	50th	90th	TARL	TSDRL	10th	50th	90th	TARL	TSDRL	10th	50th	90th
10	1.0	10.678	1.464	11	11	11	10.132	2.325	7	11	11	9.218	3.118	3	11	11
	1.2	10.261	2.167	8	11	11	9.363	3.010	4	11	11	8.132	3.587	2	11	11
	1.5	9.368	3.010	4	11	11	8.032	3.629	2	11	11	6.629	3.766	1	7	11
	2.5	6.068	3.725	1	6	11	4.674	3.336	1	4	11	3.654	2.791	1	3	8
	5.0	2.749	2.114	1	2	6	2.270	1.680	1	2	4	1.938	1.348	1	1	4
	10.0	1.673	1.057	1	1	3	1.504	0.870	1	1	3	1.398	0.750	1	1	2
100	1.0	10.245	2.178	8	11	11	10.016	2.456	6	11	11	9.751	2.711	5	11	11
	1.2	9.516	2.917	4	11	11	9.176	3.141	3	11	11	8.810	3.337	3	11	11
	1.5	8.250	3.561	2	11	11	7.813	3.671	2	10	11	7.377	3.742	2	8	11
	2.5	4.861	3.405	1	4	11	4.504	3.258	1	4	11	4.137	3.078	1	3	9
	5.0	2.331	1.741	1	2	5	2.207	1.622	1	2	4	2.100	1.510	1	2	4
	10.0	1.526	0.898	1	1	3	1.490	0.854	1	1	3	1.450	0.808	1	1	2
1000	1.0	10.073	2.395	6	11	11	10.007	2.458	6	11	11	9.918	2.555	5	11	11
	1.2	9.275	3.074	3	11	11	9.151	3.149	3	11	11	9.048	3.217	3	11	11
	1.5	7.946	3.645	2	10	11	7.778	3.677	2	10	11	7.640	3.697	2	9	11
	2.5	4.586	3.291	1	4	11	4.470	3.242	1	3	11	4.354	3.179	1	3	10
	5.0	2.238	1.652	1	2	4	2.193	1.607	1	2	4	2.154	1.566	1	2	4
	10.0	1.494	0.860	1	1	3	1.485	0.848	1	1	3	1.473	0.837	1	1	3
5000	1.0	10.035	2.433	6	11	11	9.997	2.484	6	11	11	9.973	2.502	6	11	11
	1.2	9.209	3.116	3	11	11	9.155	3.157	3	11	11	9.101	3.184	3	11	11
	1.5	7.850	3.665	2	10	11	7.798	3.674	2	10	11	7.719	3.693	2	10	11
	2.5	4.526	3.263	1	4	11	4.473	3.243	1	3	10	4.421	3.214	1	3	10
	5.0	2.212	1.616	1	2	4	2.206	1.617	1	2	4	2.196	1.598	1	2	4
	10.0	1.490	0.857	1	1	3	1.480	0.843	1	1	3	1.482	0.842	1	1	3

Table 5. *RL* properties of the Shewhart TBE chart with UMVUE in SPR for downward shifts when $I = 10$.

$I = 10$																
N	ρ	25th (Underestimation)					50th					75th (Overestimation)				
		TARL	TSDRL	10th	50th	90th	TARL	TSDRL	10th	50th	90th	TARL	TSDRL	10th	50th	90th
10	1.00	9.846	2.628	5	11	11	10.058	2.409	6	11	11	10.253	2.175	8	11	11
	0.90	9.741	2.723	5	11	11	9.967	2.506	6	11	11	10.144	2.312	7	11	11
	0.50	8.845	3.324	3	11	11	9.235	3.103	3	11	11	9.534	2.895	4	11	11
	0.10	4.533	3.267	1	4	11	5.214	3.521	1	4	11	5.873	3.690	1	5	11
	0.05	2.725	2.089	1	2	6	3.219	2.489	1	2	7	3.766	2.859	1	3	8
	0.01	1.116	0.361	1	1	2	1.198	0.487	1	1	2	1.298	0.622	1	1	2
100	1.00	9.937	2.538	6	11	11	9.992	2.485	6	11	11	10.067	2.405	6	11	11
	0.90	9.840	2.627	5	11	11	9.902	2.573	5	11	11	9.958	2.518	6	11	11
	0.50	9.020	3.229	3	11	11	9.135	3.159	3	11	11	9.237	3.098	3	11	11
	0.10	4.816	3.389	1	4	11	5.044	3.469	1	4	11	5.219	3.523	1	4	11
	0.05	2.910	2.249	1	2	6	3.072	2.377	1	2	6	3.237	2.511	1	2	7
	0.01	1.147	0.410	1	1	2	1.170	0.447	1	1	2	1.195	0.483	1	1	2
1000	1.00	9.973	2.498	6	11	11	10.009	2.468	6	11	11	10.028	2.440	6	11	11
	0.90	9.857	2.625	5	11	11	9.908	2.567	5	11	11	9.915	2.561	5	11	11
	0.50	9.104	3.182	3	11	11	9.151	3.151	3	11	11	9.171	3.143	3	11	11
	0.10	4.919	3.432	1	4	11	5.006	3.457	1	4	11	5.035	3.474	1	4	11
	0.05	3.011	2.335	1	2	6	3.051	2.366	1	2	6	3.127	2.421	1	2	7
	0.01	1.159	0.428	1	1	2	1.167	0.442	1	1	2	1.175	0.454	1	1	2
5000	1.00	9.989	2.487	6	11	11	10.002	2.465	6	11	11	10.000	2.469	6	11	11
	0.90	9.897	2.572	5	11	11	9.901	2.576	5	11	11	9.913	2.561	5	11	11
	0.50	9.103	3.179	3	11	11	9.118	3.177	3	11	11	9.146	3.152	3	11	11
	0.10	4.965	3.431	1	4	11	4.990	3.453	1	4	11	5.022	3.458	1	4	11
	0.05	3.043	2.360	1	2	6	3.062	2.374	1	2	6	3.089	2.394	1	2	6
	0.01	1.165	0.438	1	1	2	1.170	0.445	1	1	2	1.171	0.447	1	1	2

Table 6. *RL* properties of the Shewhart TBE chart with MLE in SPR for downward shifts when $I = 10$.

$I = 10$																
N	ρ	25th (Underestimation)					50th					75th (Overestimation)				
		TARL	TSDRL	10th	50th	90th	TARL	TSDRL	10th	50th	90th	TARL	TSDRL	10th	50th	90th
10	1.00	9.729	2.739	5	11	11	9.972	2.500	6	11	11	10.141	2.315	7	11	11
	0.90	9.594	2.842	4	11	11	9.852	2.622	5	11	11	10.075	2.389	6	11	11
	0.50	8.643	3.415	3	11	11	9.075	3.193	3	11	11	9.401	2.989	4	11	11
	0.10	4.203	3.102	1	3	10	4.874	3.412	1	4	11	5.541	3.614	1	5	11
	0.05	2.508	1.902	1	2	5	2.978	2.306	1	2	6	3.484	2.673	1	3	8
	0.01	1.087	0.309	1	1	1	1.154	0.420	1	1	2	1.238	0.540	1	1	2
100	1.00	9.931	2.544	6	11	11	9.999	2.475	6	11	11	10.062	2.405	6	11	11
	0.90	9.822	2.645	5	11	11	9.894	2.583	5	11	11	9.964	2.504	6	11	11
	0.50	9.002	3.235	3	11	11	9.125	3.167	3	11	11	9.221	3.107	3	11	11
	0.10	4.784	3.375	1	4	11	4.980	3.443	1	4	11	5.195	3.511	1	4	11
	0.05	2.901	2.240	1	2	6	3.044	2.349	1	2	6	3.187	2.468	1	2	7
	0.01	1.143	0.404	1	1	2	1.164	0.436	1	1	2	1.190	0.476	1	1	2
1000	1.00	9.971	2.501	6	11	11	10.009	2.460	6	11	11	10.015	2.461	6	11	11
	0.90	9.877	2.603	5	11	11	9.901	2.571	5	11	11	9.916	2.562	5	11	11
	0.50	9.095	3.184	3	11	11	9.130	3.160	3	11	11	9.177	3.137	3	11	11
	0.10	4.919	3.432	1	4	11	4.990	3.455	1	4	11	5.068	3.478	1	4	11
	0.05	3.028	2.351	1	2	6	3.069	2.378	1	2	6	3.113	2.412	1	2	7
	0.01	1.158	0.427	1	1	2	1.168	0.446	1	1	2	1.178	0.457	1	1	2
5000	1.00	9.992	2.477	6	11	11	10.006	2.461	6	11	11	10.008	2.464	6	11	11
	0.90	9.879	2.597	5	11	11	9.886	2.589	5	11	11	9.897	2.584	5	11	11
	0.50	9.125	3.163	3	11	11	9.100	3.182	3	11	11	9.158	3.153	3	11	11
	0.10	4.962	3.450	1	4	11	5.016	3.464	1	4	11	5.020	3.459	1	4	11
	0.05	3.039	2.357	1	2	6	3.053	2.368	1	2	6	3.093	2.399	1	2	6
	0.01	1.165	0.441	1	1	2	1.169	0.442	1	1	2	1.169	0.443	1	1	2

Moreover, to conduct a systematic sensitivity analysis of the sample size N on the performance of the Shewhart TBE chart in SPR, we use a simple metric to quantify the difference,

$$Relative_{Dif} = \frac{|TARL_E - TARL_K|}{TARL_K} \times 100\% \quad (17)$$

where $TARL_E$ and $TARL_K$ are estimated and known $TARL$ values of the Shewhart TBE chart, respectively. Then, in Table 7, when $I = 10$, $\rho \in \{1.0, 1.2, 1.5, 2.0, 5.0, 10.0\}$ and $N \in \{10, 100, 1000, 5000\}$, the $Relative_{Dif}$ values of the Shewhart TBE chart in SPR with the 25th percentile estimator are presented. As can be seen from this table, as the sample size N increases, the relative $TARL$ difference becomes small, indicating the effect of the estimated parameter on the Shewhart TBE chart in SPR is more pronounced as N decreases. Moreover, when the sample size $N \in \{10, 100\}$, the relative $TARL$ difference of MLE is generally larger than that of the Shewhart TBE chart in SPR with UMVUE. This is also due to the fact that UMVUE with the 25th percentile is closer to the real process parameter $\theta_0 = 1$ than MLE with an underestimated percentile when N is generally small.

Table 7. Relative $TARL$ difference of the Shewhart TBE chart in SPR with estimated and known parameters.

ρ	$N = 10$		$N = 100$		$N = 1000$		$N = 5000$	
	UMVUE	MLE	UMVUE	MLE	UMVUE	MLE	UMVUE	MLE
1	4.66%	6.78%	2.01%	2.45%	0.75%	0.73%	0.41%	0.35%
1.2	7.89%	12.12%	3.64%	3.98%	1.30%	1.34%	0.66%	0.62%
1.5	12.48%	20.18%	5.16%	5.84%	1.80%	1.94%	0.97%	0.71%
2.5	20.30%	36.24%	7.57%	9.14%	2.38%	2.96%	1.55%	1.62%
5	13.25%	25.13%	5.14%	6.10%	2.32%	1.87%	1.27%	0.68%
10	6.46%	12.58%	2.29%	2.69%	0.34%	0.54%	0.40%	0.27%

5. A Real Data Example

In this section, a real data example of the time between component production failures is used to demonstrate the application of the Shewhart TBE charts with a known parameter or with UMVUE and MLE in SPR. Here, the component production failure is referred to as an event. As in Yuan Cheng and Guo [65] and Xie et al. [66], 30 IC Phase I TBE samples are presented in Table 8. It can be noticed that the dataset used here is divided by 1000 to standardize the known process mean. By using the “*fitdist*” function in Matlab, it is shown in Figure 4 that the Phase I TBE samples are distributed approximately along the reference line, indicating the data follow an exponential distribution. Moreover, by performing the Lilliefors test using the “*lillietest*” function in Matlab, it also validates the exponential distribution assumption of the Phase I data.

Table 8. The component production failure time in Phase I.

i	Y_i	i	Y_i	i	Y_i
1	1.0656	11	2.1918	21	0.9440
2	0.5358	12	1.0973	22	1.0845
3	0.5405	13	0.5976	23	2.3065
4	0.7162	14	0.9712	24	0.0066
5	2.5254	15	3.1573	25	3.1115
6	1.2642	16	2.9330	26	0.2839
7	0.4794	17	0.9877	27	0.6594
8	1.7832	18	1.8162	28	0.6835
9	0.4737	19	0.1172	29	0.0361
10	2.2654	20	0.1907	30	0.7542

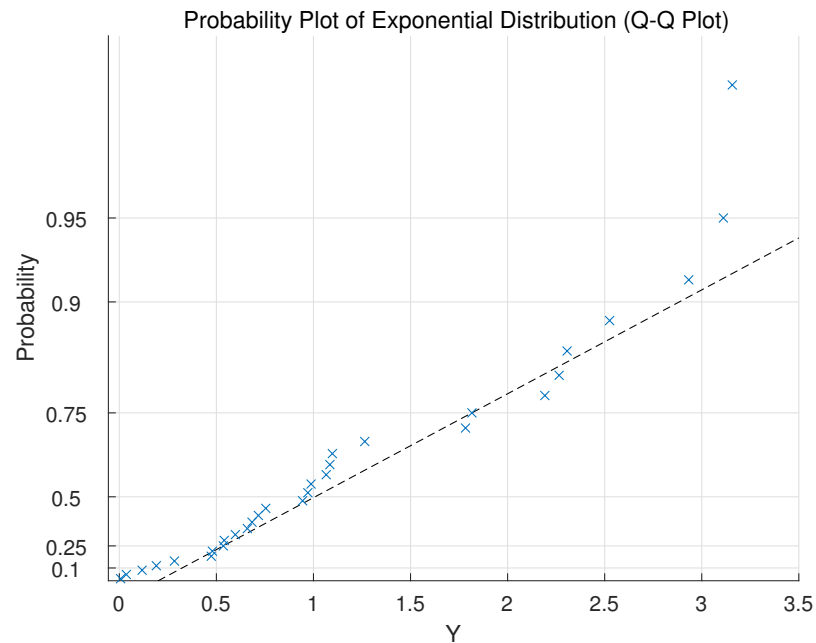


Figure 4. Probability plot for the exponential distribution using the Phase I sample data.

Here, to ensure that all the Phase I samples are from an IC process, the Shewhart TBE chart is first constructed when the process parameter is assumed $\theta_0 = 1$. Following Radaelli [8], the Phase I equal-tail probability-based two-sided Shewhart TBE scheme is designed by satisfying $\frac{1}{TARL_0} \approx \frac{1}{TARL_u} + \frac{1}{TARL_l}$, where $TARL_u$ and $TARL_l$ denote the IC $TARL_0$ of each separated one-sided Shewhart TBE scheme, respectively. Thus, assuming that $TARL_u = TARL_l = 20$, the corresponding UCL and LCL can be obtained as 5.3162 and 0.0049 from Table 1. This leads to the $TARL_0 = 10$ of the Phase I equal-tail probability-based two-sided Shewhart TBE chart. Afterwards, the 30 Phase I samples in Table 8 are plotted in Figure 5. It can be seen that these samples do not exceed the upper or lower control limits, indicating that these samples are IC and are feasible to be used for parameter estimation.

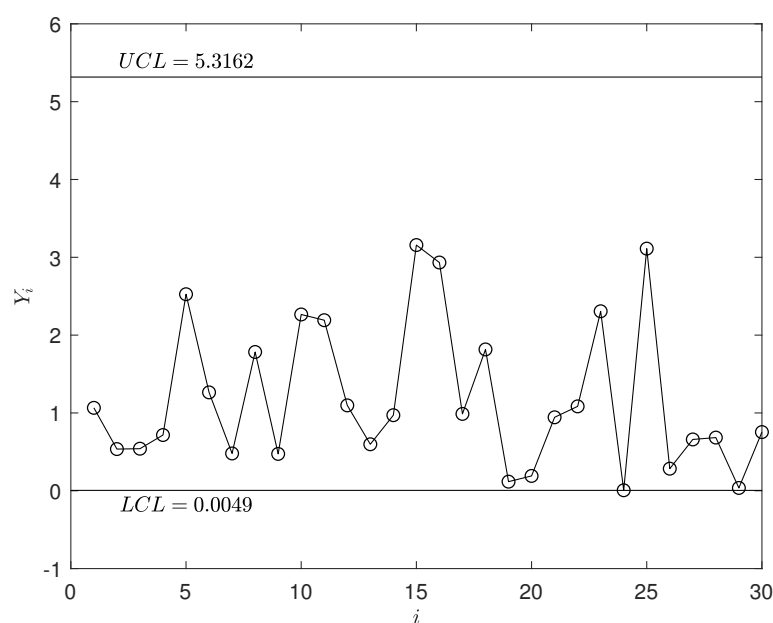


Figure 5. Phase I Shewhart scheme for 30 IC dataset in Table 8.

Since these Phase I samples are shown to be IC and follow the assumed exponential distribution with $\theta_0 = 1$, in order to show the estimation of the process parameter, different samples Y_i , each with size $N = 30$ are generated and used to estimate the process parameter $\hat{\theta}_{0,UMVUE}$ and $\hat{\theta}_{0,MLE}$ using Equations (10) and (11). Then, the 25th, 50th and 75th percentiles of these estimated $\hat{\theta}_{0,UMVUE}$ and $\hat{\theta}_{0,MLE}$ are calculated as $\{0.9020, 1.0237, 1.1548\}$ and $\{0.8711, 0.9892, 1.1169\}$, respectively. These percentiles are used to show the underestimation, median, and overestimation of the process parameter θ_0 .

To maintain the quality of component production during HQP in SPR, practitioners are often more interested in the reduction in time between failures, which implies a decrease $0 < \rho < 1$ in the process mean θ_0 . To simulate the OOC dataset of the time between component production failures, the change size $\rho = 0.1$ is selected. Moreover, the scheduled inspections I is set as 10 and thus the LCL is obtained as 0.0194 from Table 1. With the designed parameters, 10 Phase II samples X_i are generated from the exponential distribution with the change size $\rho = 0.1$ in the known parameter $\theta_0 = 1$, or in the corresponding percentiles $\{0.9020, 1.0237, 1.1548\}$ of $\hat{\theta}_{0,UMVUE}$ and percentiles $\{0.8711, 0.9892, 1.1169\}$ of $\hat{\theta}_{0,MLE}$.

These samples are then monitored by the Shewhart TBE chart with a known parameter; see Table 9, where the bold value represents the first statistic smaller than the $LCL = 0.0194$. From Table 9, it can be seen that by applying the Shewhart TBE chart with a known parameter, the first OOC signal occurs at the fourth sample for the generated Phase II samples from the exponential distribution with the change size $\rho = 0.1$ in the known parameter $\theta_0 = 1$. This is a little different from the Phase II samples generated from the 25th, 50th, and 75th percentiles of UMVUE and MLE. It can be seen that the Shewhart TBE chart gives the first signal at the third, fourth and fourth samples for the samples generated from the 25th, 50th, and 75th percentiles of UMVUE and MLE, respectively. This difference shows the effect of the estimated process parameter on the Shewhart TBE chart's performance.

Table 9. The monitoring statistic X_i corresponding to the component production failure time in Phase II.

i	Known Parameter	UMVUE			MLE		
		25th	50th	75th	25th	50th	75th
X_i							
1	0.0626	0.1692	0.0646	0.1081	0.0419	0.0478	0.0872
2	0.0245	0.0300	0.0495	0.0641	0.0234	0.1826	0.0800
3	0.1784	0.0136	0.3193	0.1518	0.0095	0.1404	0.0226
4	0.0021	0.1302	0.0067	0.0178	0.0955	0.0035	0.0194
5	0.0080	0.0564	0.0342	0.1572	0.0991	0.3801	0.0242
6	0.0380	0.3627	0.1899	0.0789	0.3607	0.1419	0.1184
7	0.0810	0.1062	0.3884	0.0370	0.1229	0.0455	0.0199
8	0.4165	0.0558	0.0426	0.0106	0.1243	0.0072	0.4568
9	0.1316	0.0321	0.1006	0.0660	0.1469	0.0172	0.1167
10	0.0004	0.0746	0.0151	0.3505	0.0991	0.0924	0.0240

6. Conclusions and Recommendations

With the goal of monitoring HQP in SPR, the Shewhart TBE chart is constructed. When the process parameter is unknown, it is estimated using both the UMVUE and MLE. The truncated RL properties of the Shewhart TBE chart are obtained by numerical simulations and then compared with the Shewhart TBE with a known process parameter. It is shown that the introduction of parameter estimation in the Shewhart TBE chart in SPR can significantly alter the actual performance relative to that of the chart with a known parameter. When an underestimation of the process parameter $\hat{\theta}_0$ occurs, for the detection

of upward changes, the Shewhart TBE chart with a known parameter is superior to the Shewhart TBE chart with UMVUE and MLE, while it is inferior to the Shewhart TBE chart with UMVUE and MLE for the detection of downward changes. Conversely, when there is an overestimation of $\hat{\theta}_0$, the Shewhart TBE chart with UMVUE and MLE outperforms the Shewhart TBE chart with a known parameter for the detection of upward changes, while it is worse than the Shewhart TBE chart with a known parameter for the detection of downward changes. In addition, as the sample size N decreases, the conclusions drawn above become more pronounced.

Future studies could be extended to other memory-type charts, such as EWMA and CUSUM charts. Furthermore, other TBE control charts for multivariate TBE distributions are promising for future research.

Supplementary Materials: The following supporting information can be downloaded at <https://www.mdpi.com/article/10.3390/math14050828/s1>, Table S1: *RL* properties of the Shewhart TBE chart with UMVUE in SPR for upward shifts when $I = 20$; Table S2: *RL* properties of the Shewhart TBE chart with UMVUE in SPR for upward shifts when $I = 30$; Table S3: *RL* properties of the Shewhart TBE chart with MLE in SPR for upward shifts when $I = 20$; Table S4: *RL* properties of the Shewhart TBE chart with MLE in SPR for upward shifts when $I = 30$. Table S5: *RL* properties of the Shewhart TBE chart with UMVUE in SPR for downward shifts when $I = 20$; Table S6: *RL* properties of the Shewhart TBE chart with UMVUE in SPR for downward shifts when $I = 30$; Table S7: *RL* properties of the Shewhart TBE chart with MLE in SPR for downward shifts when $I = 20$; Table S8: *RL* properties of the Shewhart TBE chart with MLE in SPR for downward shifts when $I = 30$.

Author Contributions: Conceptualization, F.L. and G.X.; methodology, F.L.; software, G.X.; validation, F.L. and G.X.; formal analysis, F.L.; investigation, F.L.; resources, F.L.; data curation, F.L.; writing—original draft preparation, F.L.; writing—review and editing, F.L. and G.X.; visualization, G.X.; supervision, G.X.; project administration, G.X.; funding acquisition, G.X. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the General Project of Philosophy and Social Science Research in Universities of Jiangsu Province through the project “Research on the Dynamic Management Mechanism of Jiangsu Enterprise Innovation Strategy Implementation Under the Background of High-quality Development”, grant number 2021SJJA0097.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors wish to thank the AE and reviewers for all the comments made.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Montgomery, D.C. *Introduction to Statistical Quality Control*; John Wiley & Sons: Hoboken, NJ, USA, 2020.
2. Rakitzis, A.C. Monitoring exponential data using two-sided control charts with runs rules. *J. Stat. Comput. Simul.* **2016**, *86*, 149–159.
3. Alevizakos, V.; Koukouvinos, C. A double exponentially weighted moving average chart for time between events. *Commun. Stat.-Simul. Comput.* **2020**, *49*, 2765–2784.
4. Gan, F.F. Designs of one-and two-sided exponential EWMA charts. *J. Qual. Technol.* **1998**, *30*, 55–69.
5. Dogu, E.; Noor-Ul-Amin, M. Monitoring exponentially distributed time between events data: self-starting perspective. *Commun. Stat.-Simul. Comput.* **2021**, *52*, 1–13.
6. Xie, F.; Castagliola, P.; Sun, J.; Tang, A.; Hu, X. A one-sided adaptive truncated exponentially weighted moving average scheme for time between events. *Comput. Ind. Eng.* **2022**, *168*, 108052.
7. Khoo, M.B.C.; Xie, M. *A Study of Time-Between-Events Control Chart for the Monitoring of Regularly Maintained Systems*; John Wiley & Sons: Hoboken, NJ, USA, 2009; Volume 25, pp. 805–819.
8. Radaelli, G. Planning time-between-events Shewhart control charts. *Total Qual. Manag.* **1998**, *9*, 133–140.
9. Gan, F.F. Exact run length distributions for one-sided exponential CUSUM schemes. *Stat. Sin.* **1992**, *2*, 297–312.

10. Kumar, N.; Chakraborti, S.; Rakitzis, A.C. Improved Shewhart-type charts for monitoring times between events. *J. Qual. Technol.* **2017**, *49*, 278–296.
11. Kumar, N.; Chakraborti, S. Bayesian Monitoring of Times Between Events: The Shewhart tr-Chart. *J. Qual. Technol.* **2017**, *49*, 136–154.
12. Singh, R.; Kumar, N. Sensitivity of Shewhart-type tr-chart to the departure from assumed exponential distribution. *Int. J. Agric. Stat. Sci.* **2019**, *15*, 1–9.
13. Rahali, D.; Castagliola, P.; Taleb, H.; Khoo, M.B.C. Evaluation of Shewhart time-between-events-and-amplitude control charts for several distributions. *Qual. Eng.* **2019**, *31*, 240–254.
14. Ali, S.; Shah, I. Monitoring regularly maintained systems based on the renewal process with generalized exponential distribution of time between events. *J. Test. Eval.* **2020**, *48*, 3673–3694.
15. Ali, S.; Shah, I.; Wang, L.; Yue, Z. A Comparison of Shewhart-Type Time-Between-Events Control Charts Based on the Renewal Process. *IEEE Access* **2020**, *8*, 113683–113701.
16. Hu, X.; Qiao, Y.; Zhou, P.; Zhong, J.; Wu, S. Modified one-sided EWMA charts for monitoring time between events. *Commun. Stat.-Simul. Comput.* **2023**, *52*, 1041–1056.
17. Rasheed, Z.; Zhang, H.; Anwar, S.M. Reassessment of performance evaluation of EWMA control chart for exponential process. *Qual. Reliab. Eng. Int.* **2023**, *40*, 1685–1697. <https://doi.org/10.1002/qre.3483>.
18. Afuecheta, E.; Sayyed, M.; Okorie, I.E.; Nadarajah, S. A compound exponential distribution with application to control charts. *J. Comput. Appl. Math.* **2023**, *417*, 114598.
19. Shah, M.T.; Azam, M.; Aslam, M.; Sherazi, U. Time between events control charts for gamma distribution. *Qual. Reliab. Eng. Int.* **2021**, *37*, 785–803.
20. Aslam, M.; Khan, M.; Anwar, S.M.; Zaman, B. A homogeneously weighted moving average control chart for monitoring time between events. *Qual. Reliab. Eng. Int.* **2022**, *38*, 1013–1044.
21. Madrid-Alvarez, H.M.; García-Díaz, J.C.; Tercero-Gómez, V.G. A CUSUM control chart for gamma distribution with guaranteed performance. *Qual. Reliab. Eng. Int.* **2023**, *40*, 1279–1301.
22. Aslam, M.; Khan, M.; Rasheed, Z.; Anwar, S.M.; Abbasi, S.A. An improved adaptive EWMA control chart for monitoring time between events with application in health sector. *Qual. Reliab. Eng. Int.* **2023**, *39*, 1396–1412.
23. Li, J.; Yu, D.; Song, Z.; Mukherjee, A.; Chen, R.; Zhang, J. Comparisons of some memory-type control chart for monitoring Weibull-distributed time between events and some new results. *Qual. Reliab. Eng. Int.* **2022**, *38*, 3598–3615.
24. Alevizakos, V.; Koukouvinos, C.; Lappa, A. Monitoring of time between events with a double generally weighted moving average control chart. *Qual. Reliab. Eng. Int.* **2019**, *35*, 685–710.
25. Sparks, R.S.; Hazrati-Marangaloo, H. Exponentially weighted moving averages of counting processes when the time between events is Weibull distributed. In *Quality Control-Intelligent Manufacturing, Robust Design and Charts*; IntechOpen: Rijeka, Croatia, 2020.
26. Bourke, P.D. CUSUM design for detection of event-rate increases for a Poisson process. *Qual. Eng.* **2022**, *34*, 36–51.
27. Bourke, P.D. Choosing parameter values for a geometric CUSUM chart for detecting an upward shift in a proportion. *Qual. Eng.* **2018**, *30*, 621–634.
28. Sayyed, M.; Sayyed, F.; Sharma, R.M. The extended Erlang-truncated exponential distribution under CUSUM control charts for one sided. *Life Cycle Reliab. Saf. Eng.* **2023**, *12*, 65–69.
29. Saha, S.; Mukherjee, A.; Khoo, M.B. Two CUSUM schemes for simultaneous monitoring of unknown parameters of a shifted exponential process and its application in monitoring of call durations in telemarketing. *Qual. Technol. Quant. Manag.* **2022**, *19*, 113–137.
30. Ali, S.; Pievatolo, A.; Göb, R. An overview of control charts for high-quality processes. *Qual. Reliab. Eng. Int.* **2016**, *32*, 2171–2189.
31. Inez M. Zwetsloot, T.M.; Woodall, W.H. Multivariate time-between-events monitoring: An overview and some overlooked underlying complexities. *Qual. Eng.* **2021**, *33*, 13–25.
32. Khoo, M.B.; Saha, S.; Teh, S.Y.; Haq, A.; Lee, H.C. The median control chart for process monitoring in short production runs. *Commun. Stat.-Simul. Comput.* **2022**, *51*, 5816–5831.
33. Lu, Y.; Morris, K.C.; Frechette, S. *Current Standards Landscape for Smart Manufacturing Systems*; National Institute of Standards and Technology (NISTIR): Gaithersburg, MD, USA, 2016; Volume 8107, pp. 1–39.
34. Celano, G.; Castagliola, P.; Trovato, E.; Fichera, S. Shewhart and EWMA t control charts for short production runs. *Qual. Reliab. Eng. Int.* **2011**, *27*, 313–326.
35. Yang, C.H.; Hillier, F.S. Mean and variance control chart limits based on a small number of subgroups. *J. Qual. Technol.* **1970**, *2*, 9–16.
36. Ladany, S.P. Optimal use of control charts for controlling current production. *Manag. Sci.* **1973**, *19*, 763–772.
37. Nenes, G.; Tagaras, G. The economically designed CUSUM chart for monitoring short production runs. *Int. J. Prod. Res.* **2006**, *44*, 1569–1587.
38. Ho, L.L.; Trindade, A.L.G. Economic design of an X chart for short-run production. *Int. J. Prod. Econ.* **2009**, *120*, 613–624.

39. Trovato, E.; Castagliola, P.; Celano, G.; Fichera, S. Economic design of inspection strategies to monitor dispersion in short production runs. *Comput. Ind. Eng.* **2010**, *59*, 887–897.
40. Nenes, G.; Castagliola, P.; Celano, G. Economic and statistical design of Vp control charts for finite-horizon processes. *IIE Trans.* **2017**, *49*, 110–125.
41. Nenes, G.; Tagaras, G. Evaluation of CUSUM charts for finite-horizon processes. *Commun. Stat.-Simul. Comput.* **2010**, *39*, 578–597.
42. Castagliola, P.; Amdouni, A.; Taleb, H.; Celano, G. One-sided Shewhart-type charts for monitoring the coefficient of variation in short production runs. *Qual. Technol. Quant. Manag.* **2015**, *12*, 53–67.
43. Amdouni, A.; Castagliola, P.; Taleb, H.; Celano, G. Monitoring the coefficient of variation using a variable sample size control chart in short production runs. *Int. J. Adv. Manuf. Technol.* **2015**, *81*, 1–14.
44. Nenes, G.; Castagliola, P.; Celano, G.; Panagiotidou, S. The variable sampling interval control chart for finite-horizon processes. *IIE Trans.* **2014**, *46*, 1050–1065.
45. Castagliola, P.; Celano, G.; Fichera, S.; Nenes, G. The variable sample size t control chart for monitoring short production runs. *Int. J. Adv. Manuf. Technol.* **2013**, *66*, 1353–1366.
46. Chong, N.L.; Khoo, M.B.; Haq, A.; Castagliola, P. Hotelling's T² control charts with fixed and variable sample sizes for monitoring short production runs. *Qual. Reliab. Eng. Int.* **2019**, *35*, 14–29.
47. Khatun, M.; Khoo, M.B.; Saha, S.; Castagliola, P. A new distribution-free adaptive sample size control chart for a finite production horizon and its application in monitoring fill volume of soft drink beverage bottles. *Appl. Stoch. Model. Bus. Ind.* **2021**, *37*, 84–97.
48. Saha, S.; Khoo, M.B.; Abubakar, S.S.; Das, U.K.; Lee, M.H. A new EWMA median chart with measurement errors in short production runs—an application in yogurt cup filling process. *Qual. Reliab. Eng. Int.* **2022**, *38*, 1994–2014.
49. Amdouni, A.; Castagliola, P.; Taleb, H.; Celano, G. A variable sampling interval Shewhart control chart for monitoring the coefficient of variation in short production runs. *Int. J. Prod. Res.* **2017**, *55*, 5521–5536.
50. Lee, M.H.; Khoo, M.B.; Chew, X.; Then, P.H. Effect of measurement errors on the performance of coefficient of variation chart with short production runs. *IEEE Access* **2020**, *8*, 72216–72228.
51. Haq, A.; Khoo, M.B. New adaptive EWMA control charts for monitoring univariate and multivariate coefficient of variation. *Comput. Ind. Eng.* **2019**, *131*, 28–40.
52. Hu, X.; Zhang, J.; Zhang, S.; Tang, A.; Zhou, X. EWMA monitoring schemes for MCV in short production runs with an application to the steel sleeve manufacturing process. *Comput. Ind. Eng.* **2023**, *182*, 109427.
53. Tran, K.D.; Khaliq, Q.U.A.; Nadi, A.A.; Nguyen, T.H.; Tran, K.P. One-sided Shewhart control charts for monitoring the ratio of two normal variables in short production runs. *J. Manuf. Processes* **2021**, *69*, 273–289.
54. Hu, X.; Castagliola, P. A re-evaluation of the run rules xbar chart when the process parameters are unknown. *Qual. Technol. Quant. Manag.* **2019**, *16*, 696–725.
55. Saleh, N.A.; Mahmoud, M.A.; Jones-Farmer, L.A.; Zwetsloot, I.; Woodall, W.H. Another look at the EWMA control chart with estimated parameters. *J. Qual. Technol.* **2015**, *47*, 363–382.
56. Noor, S.; Noor-ul Amin, M.; Abbasi, S.A. Bayesian EWMA control charts based on exponential and transformed exponential distributions. *Qual. Reliab. Eng. Int.* **2021**, *37*, 1678–1698.
57. Qiao, Y.; Hu, X.; Sun, J.; Xu, Q. Optimal design of one-sided exponential cumulative sum charts with known and estimated parameters based on the median run length. *Qual. Reliab. Eng. Int.* **2021**, *37*, 123–144.
58. Chang, Y.C.; Wu, Y.C. A parameter-free exponential control chart based on Kullback-Leibler information for time-between-events monitoring. *Qual. Reliab. Eng. Int.* **2022**, *38*, 733–756.
59. Bai, Y.; Chiang, J.Y.; Liu, W.; Mou, Z. An enhanced EWMA chart with variable sampling interval scheme for monitoring the exponential process with estimated parameter. *Sci. Rep.* **2024**, *14*, 7958.
60. Kumar, N.; Chakraborti, S. Phase II Shewhart-type control charts for monitoring times between events and effects of parameter estimation. *Qual. Reliab. Eng. Int.* **2016**, *32*, 315–328.
61. Jensen, W.A.; Jones-Farmer, L.A.; Champ, C.W.; Woodall, W.H. Effects of parameter estimation on control chart properties: A literature review. *J. Qual. Technol.* **2006**, *38*, 349–364.
62. Cao, C.; Ma, Y. Design of Exponential Distributed Median Chart with Unknown Parameters. In *Proceedings of the 2018 International Conference on Advanced Control, Automation and Artificial Intelligence (ACAAI 2018)*; Atlantis Press: Dordrecht, The Netherlands, 2018; Volume 155, pp. 182–187.
63. Hu, X.; Castagliola, P.; Tang, A.; Zhou, X.; Zhou, P. Conditional median run length performance of the synthetic X⁻ chart with unknown process parameters. *Qual. Reliab. Eng. Int.* **2020**, *36*, 1111–1131.
64. Ozsan, G.; Testik, M.C.; Weiß, C.H. Properties of the exponential EWMA chart with parameter estimation. *Qual. Reliab. Eng. Int.* **2010**, *26*, 555–569.

65. Yuan Cheng, L.S.; Guo, B. Phase II synthetic exponential charts and effect of parameter estimation. *Qual. Technol. Quant. Manag.* **2018**, *15*, 125–142.
66. Xie, M.; Goh, T.; Ranjan, P. Some effective control chart procedures for reliability monitoring. *Reliab. Eng. Syst. Saf.* **2002**, *77*, 143–150.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.