mathematics

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Theory and Applications of Primal Weak Structures

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Abstract

This paper presents the notion of primal weak structures as a generalized mathematical framework obtained by relaxing the axioms of classical topological spaces. We formally define primal weak structures and provide a detailed investigation of their fundamental properties. Particular attention is given to the operator $c_w^\diamond$, whose essential characteristics and related properties are analyzed to obtain a comprehensive characterization of primal weak structures. Furthermore, we introduce new constructions denoted by $\sigma(w,\mathcal{P})$, $\pi(w,\mathcal{P})$, $\beta(w,\mathcal{P})$, and $\alpha(w,\mathcal{P})$, and demonstrate that they generate generalized primal topological spaces. These results establish a unifying connection between primal weak structures and existing generalized topological frameworks. In addition, several separation axioms are proposed to distinguish between different classes of primal weak structure spaces. Overall, primal weak structures constitute a flexible and robust class of mathematical models with strong connections to classical topology and significant potential for future applications. The operators and constructions developed in this work provide a solid foundation for further research in this area.

Keywords: primal topology; weak structure; primal weak structure; $(w,\mathcal{P})$-open set; relative primal weak structure; Kuratowski's closure operator

MSC: 54A05; 54C10; 54A10

1. Introduction

The idea of generalization in topological theory has been developed progressively over time. In 2002, Császár [1] introduced the concept of a generalized open set in topological space by relaxing some of the axioms of classical topological spaces. The notion of generalized topological space (for short **GTS**) was discussed in detail in [2], where its basic definitions and main properties were clearly outlined. In addition, the study in [3] introduced the separation axioms and explained their importance within the structure of topological spaces. Moreover, the research article [4] presented further detailed and supplementary information related to the concept of generalized topological spaces, adding more depth to the existing literature.

The topic of generalized connectedness was thoroughly studied in [5], laying the foundation for further exploration. This was followed by several studies focusing on various topological concepts within this newly investigated generalized space. For instance, the notion of generalized open sets was introduced in Reference [6], while the concept of θ-modification was discussed in Reference [7]. Furthermore, Reference [8] presented the separation axioms in a more complex and refined form.

Academic Editor: Miguel Ángel Sánchez-Granero

Received: 7 February 2026
Revised: 21 February 2026
Accepted: 27 February 2026
Published: 3 March 2026

On the other hand, the concept of continuity has received considerable attention from researchers. In particular, the notion of almost continuity was introduced in Reference [9], reflecting ongoing efforts to refine and expand classical ideas within a broader generalized framework.

Additionally, Reference [10] contained another generalization of the continuous function, expanding the concept further. Meanwhile, Reference [11] introduced a stronger form of this function, offering a more robust framework for its analysis.

Then, Császár presented a more general mathematical structure in 2011 when presenting [12] the concept of "weak structures" **WS**s. Moreover, Al-Omari and Noiri [13] introduced the concept of a gw-closed set on a **WS**. Navaneethakrishnan [14] expanded the field of weak structure research by defining several subsets of X and investigating their properties in relation to a weak structure. Subsequently, Thamaraiselvi [15] presented the notion of "m-structures" and demonstrated how an m-structure induced a finer topology. Additionally, Güldürdek [16] introduced the concept of "p-staks" within the context of weak structures.

In another direction of research within pure mathematics, several mathematical tools were introduced to address complex problems. Acharjee et al. [17] presented the dual structure of a grill [18], which was subsequently termed the "primal" structure. The associated topology, known as the "primal topology", was defined on this structure. Furthermore, various operators with many desirable properties were developed over this space; for detailed discussions, see [19].

In the context of soft structures, Al-shami et al. [20] introduced soft primal structures, highlighted their properties, and demonstrated their applications within soft set theory, which offers a flexible approach to handle uncertainties in mathematical modeling. Related developments included soft weak structures introduced by Zakari, Ghareeb, and Omran [21].

Al-Omari and Alqahtani [22] introduced primal structures with closure operators and discussed their main properties and applications, which extended the theoretical framework of primal spaces in topology.

Özkoç and Köstel [23] introduced new operators in primal topological spaces, showing that one satisfied the Kuratowski closure axioms and generated a finer topology than the existing δ-topology.

Al-Omari and Alghamdi [24] explored regularity and normality in primal spaces, and presented new classes of primal Hausdorff, regular, and normal spaces, thereby extending the theoretical understanding of primal structures in topological settings.

Alghamdi [25] investigated different notions of compactness in primal topological spaces, including $\mathfrak{P}$-compactness, strongly $\mathfrak{P}$-compactness, and super $\mathfrak{P}$-compactness, provided theoretical results and illustrative examples, and contributed to the study of primal structures.

In 2023, Al-Saadi and Al-Malki [26] introduced the concept of generalized topology in the context of primal sets, thoroughly investigated its properties, and provided new examples and results. Below, we present several notions that are pertinent to our study.

Al-Saadi and Al-Malki [27] introduced new categories of open sets in generalized primal topological (for short **GPT**) spaces, explored their interrelationships, and defined (g, P)-continuity using these structures. Moreover, in [28] the authors proposed notions of strong GP-continuity and weakly GP-closed functions in **GPT** spaces, analyzed conditions where they coincided with standard GP-continuity, and investigated their preservation of GP-connectedness properties.

In this article, we present a definition of the primal weak structure in the sense of Császár and provide a detailed description of the operator $c_w^\diamond$, which possesses distinctive

https://doi.org/10.3390/math14050864

properties. Subsequently, we examine several topological properties within the context of our defined space. Furthermore, we aim to address the following questions:

Question 1: Under what conditions can the primal weak structure coincide with the primal weak structure induced by an operator satisfying Kuratowski's closure axioms?

Question 2: Is it possible to further develop this structure to generate a generalized primal topological space?

Question 3: Do the primal collection have any effect on the separation axioms of a primal weak structure space compared to their behavior in the classical weak structure setting?

This work addresses a gap in the existing literature by systematically studying the interplay between weak structures and primal collections, which has not been previously explored. We emphasize a fundamentally new structural phenomenon in PWS spaces, where the presence of the primal collection $\mathcal{P}$ modifies classical closure and separation behaviors, producing outcomes not observed in standard weak structures. This addition demonstrates the novelty of our approach and its potential to expand the understanding of generalized topological constructions.

The following provides an outline of the paper's structure.

In Section 2, we provide a formal definition of the primal weak structure, present illustrative examples, and discuss its basic properties. This section establishes the foundational understanding of the space and highlights its key characteristics in relation to classical topology.

Section 3 is devoted to the introduction and study of the operator $c_w^{\diamond}$. We examine its fundamental properties, illustrate its interaction with the primal weak structure, and discuss how it can be used to derive further mathematical constructions.

Section 4 is devoted to the introduction of several new mathematical structures based on the operator definitions given in Section 3. These structures are further developed into more advanced forms, and the interconnections among them are thoroughly examined.

In Section 5, we formulate the separation axioms within the primal weak structure space and analyze their interrelations. We also highlight certain results that distinguish this space from classical topological spaces. Finally, Section 6 summarizes the key achievements of this article and provides detailed answers to the questions posed in the Introduction.

Preliminaries

This subsection is devoted to a systematic review of the core concepts and essential results that underpin the developments presented in this article. Throughout, the power set of a non-empty set X, that is, the collection of all its subsets, will be denoted by 2^X.

Initially, we introduce the notion of weak structure, a concept originally formulated by Császár [12] to extend the classical theory of topology.

Definition 1. *Let X be a non-empty set, and let $w \subseteq 2^X$. The family w is called a weak structure (or for short* **WS***) on X if $\emptyset \in w$. The elements of w are referred to as w-open sets, and their complements in X are termed w-closed sets.*

Definition 2. *Given a* **WS** *w on X and a subset $A \subseteq X$, the union of all w-open sets of X that are contained in A is denoted by $i_w(A)$, while the intersection of every w-closed set of X that contains A is denoted by $c_w(A)$.*

Theorem 1. *Consider w as a* **WS** *defined on a non-empty set X, and $A, B \subseteq X$. Then, the following statements are satisfied:*

(i) If $A \subseteq B$, then it follows that $i_w(A) \subseteq i_w(B)$; also $c_w(A) \subseteq c_w(B)$.

https://doi.org/10.3390/math14050864

(ii) *The inclusion* $i_w(A) \subseteq A$ *holds, and similarly,* $A \subseteq c_w(A)$.
(iii) *The operators* i_w *and* c_w *are idempotent, that is,* $i_w(i_w(A)) = i_w(A)$ *and* $c_w(c_w(A)) = c_w(A)$.

Proposition 1. *Consider* w *as a* **WS** *on* X*, and let* $A \subseteq X$*. Then, the following statements hold:*
(i) *An element* x *belongs to* $i_w(A)$ *if and only if there exists* $U \in w$ *such that* $x \in U \subseteq A$.
(ii) *An element* x *belongs to* $c_w(A)$ *if and only if, for every* $U \in w$ *containing* x*, we have* $U \cap A \neq \emptyset$.
(iii) *A subset* A *is said to be w-open (respectively, w-closed) if* $i_w(A) = A$ *(respectively,* $c_w(A) = A$*).*

In a different strand of research within topology, a novel construct known as the primal collection was introduced in [17], which has played a significant role in broadening the notion of topological spaces. We proceed to present this concept in detail.

Definition 3. *Let* X *be a non-empty set. A subfamily* $\mathcal{P} \subseteq 2^X$ *is said to be a primal collection over* X *if, for every two subsets* $A, B \subseteq X$*, the following conditions are fulfilled:*
(i) *The entire set* X *is excluded from* $\mathcal{P}$*, i.e.,* $X \notin \mathcal{P}$.
(ii) *If* B *belongs to* $\mathcal{P}$ *and* $A \subseteq B$*, then* A *must also belong to* $\mathcal{P}$.
(iii) *If the intersection* $A \cap B$ *lies in* $\mathcal{P}$*, then at least one of* A *or* B *necessarily belongs to* $\mathcal{P}$.

Let (X, τ) *be a topological space and let* $\mathcal{P}$ *be a primal on* X*. When these conditions are met, the triple* $(X, \tau, \mathcal{P})$ *is referred to as a primal topological space.*

Corollary 1. *A subfamily* $\mathcal{P} \subseteq 2^X$ *qualifies as a primal structure over* X *if and only if, for all* $A, B \subseteq X$*, the following statements hold:*
(i) $X \notin \mathcal{P}$.
(ii) *If* $A \subseteq B$ *and* $A \notin \mathcal{P}$*, then* $B \notin \mathcal{P}$.
(iii) *If both* A *and* B *fail to belong to* $\mathcal{P}$*, then their intersection* $A \cap B$ *also lies outside* $\mathcal{P}$.

Császár [2] also introduced the concept of generalized topological structures, significantly broadening the classical framework.

Definition 4. *Let* X *be a non-empty set and* $\mathfrak{g}$ *a collection of subsets of* X*. The structure* $(X, \mathfrak{g})$ *is termed a generalized topological space (or for short* **GTS***) provided that the following axioms hold:*
(i) *The empty set* $\emptyset$ *is an element of* $\mathfrak{g}$.
(ii) *For any indexed family* $\{A_\beta\}_{\beta \in \Delta} \subseteq \mathfrak{g}$*, the union* $\bigcup_{\beta \in \Delta} A_\beta$ *belongs to* $\mathfrak{g}$.

More recently, a new structure was investigated that integrates the features of generalized topological spaces with those of the primal collection. This hybrid framework, introduced as a triple in Reference [26], is formulated as follows.

Definition 5. *Let* X *be a non-empty set, let* $\mathfrak{g}$ *be a generalized topology on* X*, and let* $\mathcal{P}$ *be a primal on* X*. The triple* $(X, \mathfrak{g}, \mathcal{P})$ *is said to determine a generalized primal topological space (or for short* **GPT***) space.*

2. Primal Weak Structure

In this part, we will define the weak structure in the sense of the primal sets, identify some of the operators, study their properties, and study the relationship between them. Moreover, we provide some examples and results.

Definition 6. *Consider* $\mathcal{P}$ *and* w *as primal and weak structures defined on* X*, respectively.*

https://doi.org/10.3390/math14050864

The primal weak structure space (**PWS** *space) is a pair (X, w) together with a primal $\mathcal{P}$ defined over X, that is symbolized via $(X, w, \mathcal{P})$. The member of a* **PWS** *space is mentioned as $(w, \mathcal{P})$-open and its complement is mentioned as $(w, \mathcal{P})$-closed.*

Example 1. *Let X be an infinite set and fix a point $x_0 \in X$. Define*

$$w = \{\emptyset, X\} \cup \{A \subseteq X : x_0 \in A\}.$$

Then, w is a weak structure on X. For each $n \in \mathbb{N}$, define

$$\mathcal{P}_n = \{A \subseteq X : |A| \leq n\}.$$

Then, $(X, w, \mathcal{P}_n)$ is a **PWS** *space for every $n \in \mathbb{N}$. Hence, this construction yields an infinite family of distinct primal weak structure spaces.*

Example 2. *Let $X = \mathbb{N}$ and define the weak structure*

$$w = \{\emptyset, X\} \cup \{A \subseteq X : A \text{ is finite}\}.$$

For each $k \in \mathbb{N}$, define the primal

$$\mathcal{P}_k = \{A \subseteq X : k \notin A\}.$$

Then, $(X, w, \mathcal{P}_k)$ is a **PWS** *space for every $k \in \mathbb{N}$. Thus, by varying the primal, we obtain infinitely many* **PWS** *spaces on the same weak structure.*

Example 3. *Let (X, τ) be a topological space and let $w = \tau$. For each closed set $F \subseteq X$, define*

$$\mathcal{P}_F = \{A \subseteq X : A \cap F = \emptyset\}.$$

Then, $(X, w, \mathcal{P}_F)$ is a **PWS** *space. Since a topological space generally admits infinitely many closed sets, this construction generates an infinite family of primal weak structure spaces.*

Example 4. *Let X be an infinite set and define*

$$w = \{\emptyset, X\} \cup \{X \setminus A : A \text{ is finite}\}.$$

For each infinite subset $B \subseteq X$, define the primal

$$\mathcal{P}_B = \{A \subseteq X : A \subseteq B\}.$$

Then, $(X, w, \mathcal{P}_B)$ is a **PWS** *space. By varying B over all infinite subsets of X, we obtain an uncountable family of primal weak structure spaces.*

The above examples show that primal weak structure spaces form a very rich class. Even for a fixed weak structure, infinitely many non-equivalent **PWS** spaces can be obtained by varying the primal, which highlights the essential role of the primal in this framework.

Remark 1. *Consider $\mathcal{P}$ and w as primal and weak structures defined on X, respectively. Hence, the following holds:*

(i) A primal weak structure is named a strong primal weak structure iff $X \in w$.
(ii) Every primal topological space and primal generalized space is a primal weak structure.

(iii) *Unlike the topological space, the intersection of any two* **PWS***s gives a* **PWS** *and the same is true for the union.*
(iv) *If a set A is $(w,\mathcal{P})$-open, then it coincides with its interior, i.e., $i_w(A) = A$. Similarly, A is considered $(w,\mathcal{P})$-closed if it coincides with its closure, i.e., $c_w(A) = A$. This clarifies that the openness or closedness of A is determined by equality with its corresponding interior or closure operator.*

Proof. Let $(X,\tau,\mathcal{P})$ be a primal topological space. Since every topology τ contains $\emptyset$ and is closed under arbitrary unions, it satisfies the axioms of a weak structure; hence $(X,\tau,\mathcal{P})$ is a primal weak structure space. The same argument applies to a primal generalized space, as a generalized topology also fulfills the weak structure conditions. □

Definition 7. *Consider $(X,w,\mathcal{P})$ as a* **PWS** *space with $A \subseteq X$. Hence, the following holds:*
(i) *A is named $(w,\mathcal{P})$-dense when $c_w(A) = X$.*
(ii) *A is named $(w,\mathcal{P})$-nowhere dense when $i_w(c_w(A)) = \emptyset$.*

Example 5. *Consider $(X,w,\mathcal{P})$ as a* **PWS** *space such that*

$$X = \mathbb{N}, \quad w = \{\emptyset\} \cup \{\{n\} : n \in \mathbb{N}\} \cup \{\mathbb{N} \setminus \{1\}\}, \quad \mathcal{P} = 2^X \setminus \{X\}.$$

Let

$$A = \mathbb{N} \setminus \{1\} \quad \text{and} \quad B = \{1\}.$$

Then, A is $(w,\mathcal{P})$-dense, whereas B is $(w,\mathcal{P})$-nowhere dense.

Theorem 2. *Let $(X,w,\mathcal{P})$ be a* **PWS** *space. Then, the collection of all $(w,\mathcal{P})$-nowhere dense sets forms a* **GPTS** *space.*

Proof. Let $\mathcal{N}$ be the collection of all $(w,\mathcal{P})$-nowhere dense sets. Clearly, $\emptyset \in \mathcal{N}$ and $X \notin \mathcal{N}$. For any family $\{A_i\} \subseteq \mathcal{N}$, by the axioms of a **GPTS** space, we have

$$i_w\Big(c_w\Big(\bigcup_i A_i\Big)\Big) \subseteq \bigcup_i i_w(c_w(A_i)) = \emptyset,$$

so $\bigcup_i A_i \in \mathcal{N}$. Hence, $\mathcal{N}$ itself forms a **GPTS** space. □

3. Operators in Primal Weak Structure Spaces

In this section, we introduce key operators on **PWS** spaces and explore their fundamental properties, which play a central role in studying the behavior of sets with respect to the weak structure and the primal.

Definition 8. *Consider $(X,w,\mathcal{P})$ as a* **PWS** *space. Define an operator $(.)^{\diamond}_w : 2^X \to 2^X$ by*

$$A^{\diamond}_w = \{x \in X : \forall U \in \psi_w(x),\ A^c \cup U^c \in \mathcal{P}\},$$

where $\psi_w(x) = \{U \in w : x \in U\}$ and $A \subseteq X$.

Note that $A^{\diamond}_w$ consists of all points $x \in X$ such that for every neighborhood U of x in the weak structure w, the complement of A together with the complement of U belongs to the primal collection $\mathcal{P}$. In other words, x "respects" A with respect to all its weak neighborhoods and the primal collection.

Example 6. *Consider $(X,w,\mathcal{P})$ as a* **PWS** *space with $A \subseteq X$.*
(i) *In the case where $A \subseteq \mathcal{P}$, it follows that $A^{\diamond}_w = \emptyset$.*

(ii) *When $\mathcal{P} = \{\emptyset\}$, we observe that $A^{\diamond}_{w} = \emptyset$.*
(iii) *Given that $\mathcal{P} = 2^{X} \setminus \{X\}$, the result is that $A^{\diamond}_{w} = X$.*
Let $X = \mathbb{N}$, the set of natural numbers, and consider a weak structure $w = \{\emptyset\} \cup \{U \subseteq \mathbb{N} : U$ contains all but finitely many elements of $\mathbb{N}\}$. Define a primal collection $\mathcal{P} = \{A \subseteq \mathbb{N} : 1 \in A\}$. Let $A = \{2, 3, 4, \dots\} \subseteq \mathbb{N}$. In the classical weak structure w, the operator $A^{\diamond}_{w}$ gives $A^{\diamond}_{w} = \mathbb{N}$, since every weak neighborhood of 1 intersects A. However, for the primal weak structure space $(\mathbb{N}, w, \mathcal{P})$, the primal condition restricts inclusion: $A^{\diamond}_{w} = A = \{2, 3, 4, \dots\}$, because $A^{c} \cup U^{c} \notin \mathcal{P}$ if it does not contain 1.
Note that (iv) illustrates that a property holding in a **PWS** *may fail in the classical weak structure.*

Theorem 3. *Consider $(X, w, \mathcal{P})$ as a* **PWS** *space. The following holds for two subsets A, B of X.*

(i) *$A^{\diamond}_{w}$ is $(w, \mathcal{P})$-closed, which means $c_{w}(A^{\diamond}_{w}) = A^{\diamond}_{w}$.*
(ii) *$A^{\diamond}_{w} \subseteq B^{\diamond}_{w}$, whenever, $A \subseteq B$.*
(iii) *$(A \cap B)^{\diamond}_{w} \subseteq A^{\diamond}_{w} \cap B^{\diamond}_{w}$.*
(iv) *$A^{\diamond}_{w} \cup B^{\diamond}_{w} = (A \cup B)^{\diamond}_{w}$.*

Proof. (i) Clearly, $A^{\diamond}_{w} \subseteq c_{w}(A^{\diamond}_{w})$. To show the reverse inclusion, let $x \in c_{w}(A^{\diamond}_{w})$. By definition of the c_{w}-closure, for every $U \in \psi_{w}(x)$, we have $A^{\diamond}_{w} \cap U \neq \emptyset$. Thus, there exists $y \in A^{\diamond}_{w} \cap U$. By the definition of $A^{\diamond}_{w}$, we have $A^{c} \cup U^{c} \in \mathcal{P}$ for every $U \in \psi_{w}(y)$. Since $x \in U$, and $\mathcal{P}$ is closed under supersets, it follows that x also satisfies the condition defining $A^{\diamond}_{w}$. Hence, $x \in A^{\diamond}_{w}$, which shows that $c_{w}(A^{\diamond}_{w}) \subseteq A^{\diamond}_{w}$. Therefore, $A^{\diamond}_{w}$ is $(w, \mathcal{P})$-closed.

(ii) Suppose that $A \subseteq B$. Let $x \notin B^{\diamond}_{w}$. Then, there exists some $U \in \psi_{w}(x)$ satisfying $B^{c} \cup U^{c} \notin \mathcal{P}$.

However, since $A \cap U \subseteq B \cap U$, it follows from Corollary 1 that $A^{c} \cup U^{c} \notin \mathcal{P}$. Therefore, $x \notin A^{\diamond}_{w}$.

(iii) Suppose that x belongs to $(A \cap B)^{\diamond}_{w}$. Thus, for all $U \in \psi_{w}(x)$,

$$(A \cap B)^{c} \cup U^{c} \in \mathcal{P}.$$

Since $(A \cap B)^{c} = A^{c} \cup B^{c}$, it follows that

$$(A^{c} \cup B^{c}) \cup U^{c} \in \mathcal{P}.$$

By the primal definition, this implies

$$A^{c} \cup U^{c} \in \mathcal{P} \quad \text{and} \quad B^{c} \cup U^{c} \in \mathcal{P}, \quad \forall U \in \psi_{w}(x).$$

Therefore, $x \in A^{\diamond}_{w}$ and $x \in B^{\diamond}_{w}$, which implies $x \in A^{\diamond}_{w} \cap B^{\diamond}_{w}$.

(iv) Since

$$A^{\diamond}_{w} \subseteq (A \cup B)^{\diamond}_{w} \quad \text{and} \quad B^{\diamond}_{w} \subseteq (A \cup B)^{\diamond}_{w},$$

it follows that

$$A^{\diamond}_{w} \cup B^{\diamond}_{w} \subseteq (A \cup B)^{\diamond}_{w}.$$

Conversely, suppose that $x \notin A^{\diamond}_{w} \cup B^{\diamond}_{w}$. This implies that $x \notin A^{\diamond}_{w}$ and $x \notin B^{\diamond}_{w}$. Hence, there exist $U_1, U_2 \in \psi_{w}(x)$ such that

$$A^{c} \cup U_1^{c} \notin \mathcal{P} \quad \text{and} \quad B^{c} \cup U_2^{c} \notin \mathcal{P}.$$

Let $U = U_1 \cap U_2$, then $U \in \psi_{w}(x)$. Therefore,

$$A^{c} \cup U^{c} \notin \mathcal{P} \quad \text{and} \quad B^{c} \cup U^{c} \notin \mathcal{P}.$$

By the primal definition,

$$(A \cup B)^c \cup U^c = (A^c \cap B^c) \cup U^c = (A^c \cup U^c) \cap (B^c \cup U^c) \notin \mathcal{P}.$$

Consequently, $x \notin (A \cup B)^{\diamond}_{w}$.
□

Remark 2. *The converse inclusions stated in parts (i) and (ii) of Theorem 3 are not generally valid. The subsequent example serves to demonstrate this limitation explicitly.*

Example 7. *Let*

$$X = \mathbb{N}, \quad w = \{\emptyset\} \cup \{\{n\} : n \in \mathbb{N}\} \cup \{\{1, 3, 4, 5, \dots\}\}, \quad \mathcal{P} = 2^X \setminus \{X\}.$$

Then, $(X, w, \mathcal{P})$ forms a **PWS** *space. Consider the infinite sets*

$$A = \{2n : n \in \mathbb{N}\}, \quad B = \{1\} \cup \{2n + 1 : n \geq 1\}.$$

We have $A^{\diamond}_{w} = A \cup \{1\}$. Thus,

(i) $A^{\diamond}_{w} \subseteq B^{\diamond}_{w}$*; however,* $A \nsubseteq B$*.*
(ii) $A^{\diamond}_{w} \cap B^{\diamond}_{w} = A \cup \{1\} \nsubseteq (A \cap B)^{\diamond}_{w} = \emptyset$*.*

This infinite example shows explicitly that the converse inclusions of Theorem 3 are generally invalid.

Proposition 2. *Given a structure $(X, w, \mathcal{P})$ that defines a* **PWS** *space, assume that the complement A^c is $(w, \mathcal{P})$-open. Under this condition, the inclusion $A^{\diamond}_{w} \subseteq A$ holds. In addition, it follows that $(A^{\diamond}_{w})^{\diamond}_{w} \subseteq A^{\diamond}_{w}$.*

Proof. Assume that A is $(w, \mathcal{P})$-open, and let $x \in A^{\diamond}_{w}$. Suppose, for contradiction, that $x \notin A$, which means $x \in A^c$. By the definition of $A^{\diamond}_{w}$, for every $U \in \psi_w(x)$, the union $A^c \cup U^c$ belongs to $\mathcal{P}$. Since A^c is $(w, \mathcal{P})$-open and contains x, we have $A^c \in \psi_w(x)$. Then, $A^c \cup (A^c)^c = X \in \mathcal{P}$, which contradicts the assumption that $x \in A^{\diamond}_{w}$ while $x \notin A$. Hence, $x \in A$, showing that $A^{\diamond}_{w} \subseteq A$. Moreover, since $A^{\diamond}_{w}$ is $(w, \mathcal{P})$-closed, Theorem 3 (i) guarantees that its complement is $(w, \mathcal{P})$-open, and applying Theorem 3 (ii) yields $(A^{\diamond}_{w})^{\diamond}_{w} \subseteq A^{\diamond}_{w}$. □

Remark 3. *The converse inclusion of Proposition 2 does not hold; the next example illustrates that.*

Example 8. *Consider the set $X = \mathbb{N}$ equipped with*

$$w = \{\emptyset\} \cup \{\{n\} : n \in \mathbb{N}\}, \quad \mathcal{P} = \{\emptyset\} \cup \{\{2n : n \in \mathbb{N}\}\}.$$

Then, $(X, w, \mathcal{P})$ forms a **PWS** *space.*

Now, take the set $A = \{2n + 1 : n \in \mathbb{N}\}$. In this case, it follows that $A^{\diamond}_{w} = \emptyset \subseteq A$. However, the complement $A^c = \{2n : n \in \mathbb{N}\}$ is not $(w, \mathcal{P})$-open, showing that the condition in Proposition 2 is not necessary for the inclusion $A^{\diamond}_{w} \subseteq A$ to hold.

We investigate the conditions or additional assumptions under which the converse of the inclusion stated in Theorem 3 (iii) holds true. In particular, we analyze how these conditions affect the structural properties of the space $(X, w, \mathcal{P})$ and the behavior of the operator $c^{\diamond}_{w}$, providing insight into the interplay between the operator and the underlying primal weak structure.

Proposition 3. *Consider a structure* $(X, w, \mathcal{P})$ *that forms a* **PWS** *space. For any subsets* $A, B \subseteq X$*, whenever* A *is* $(w, \mathcal{P})$*-open, the following inclusion holds:* $A \cap B_w^\diamond \subseteq (A \cap B)_w^\diamond$*.*

Proof. Consider an element x belongs to the intersection $A \cap B_w^\diamond$. This membership implies that $x \in A \wedge x \in B_w^\diamond$. By the definition of $B_w^\diamond$, for every neighborhood U in $\psi_w(x)$, the set $B^c \cup U^c$ lies in $\mathcal{P}$. Simultaneously, since A is $(w, \mathcal{P})$-open, it belongs to $\psi_w(x)$. Combining these facts, the union

$$(A \cap B)^c \cup U^c = B^c \cup (A \cap U)^c$$

also belongs to $\mathcal{P}$ for all $U \in \psi_w(x)$. Consequently, $x \in (A \cap B)_w^\diamond$. □

Definition 9. *Consider a structure* $(X, w, \mathcal{P})$ *forming a* **PWS** *space. For any subset* $A \subseteq X$*, an operator* $c_w^\diamond : 2^X \to 2^X$ *is introduced by the relation*

$$c_w^\diamond(A) = A \cup A_w^\diamond.$$

Additionally, the collection

$$w^\diamond = \{c_w^\diamond(A^c) = A^c \mid A \subseteq X\}$$

constitutes a primal weak structure induced by the operator $c_w^\diamond$*; here,* w *denotes the weak structure on* X*.*

Proposition 4. *Consider* $(X, w, \mathcal{P})$ *as a* **PWS** *space. Thus, the operator that is defined in Definition 9 satisfies Kuratowski's closure operator axioms, which means the following:*

(i) $c_w^\diamond(\emptyset) = \emptyset$.
(ii) $A \subseteq c_w^\diamond(A)$.
(iii) $c_w^\diamond(A \cup B) = c_w^\diamond(A) \cup c_w^\diamond(B)$.
(iv) $c_w^\diamond(c_w^\diamond(A)) = c_w^\diamond(A)$.

Proof. We first note that, by Theorem 3 (i), $A_w^\diamond$ is $(w, \mathcal{P})$-closed, which will be used in verifying the closure axioms.

(i) By definition, the closure of the empty set contains no points, so $c_w^\diamond(\emptyset) = \emptyset$.
(ii) By the definition of $c_w^\diamond$ and Theorem 3 (i), every point of A is contained in some weak neighborhood intersecting A, hence $A \subseteq c_w^\diamond(A)$.
(iii) For any $x \in c_w^\diamond(A \cup B)$, every weak neighborhood of x intersects $A \cup B$. Thus, it intersects A or B, which implies $x \in c_w^\diamond(A) \cup c_w^\diamond(B)$, proving equality.
(iv) Let $x \in c_w^\diamond(c_w^\diamond(A))$. Then, every weak neighborhood of x intersects $c_w^\diamond(A)$, and by Theorem 3 (i) this ensures $x \in c_w^\diamond(A)$. Hence, $c_w^\diamond(c_w^\diamond(A)) = c_w^\diamond(A)$.

□

Remark 4. *The subsequent theorem provides a comprehensive resolution to the inquiry posed in* **Question 1,** *elucidating the underlying structural nuances and establishing the foundational framework essential for further developments.*

Theorem 4. *Consider a structure* $(X, w, \mathcal{P})$ *forming a* **PWS** *space. The following assertions hold:*

(i) *The equality* $w^\diamond = 2^X$ *arises in the case where* $\mathcal{P} = \{\emptyset\}$*.*
(ii) *The equality* $w^\diamond = w$ *is satisfied when* $\mathcal{P} = 2^X \setminus \{X\}$*.*

Proof. (i) By definition, the inclusion $w^\diamond \subseteq 2^X$ holds trivially. To establish the reverse inclusion, consider any subset $A \subseteq X$ under the assumption that $\mathcal{P} = \emptyset$. In this setting, the $(w,\mathcal{P})$-derived set $A_w^\diamond$ reduces to the empty set for all $A \subseteq X$, as no nontrivial condition is satisfied by any element. Consequently, the operator satisfies

$$c_w^\diamond(A^c) = A^c \cup (A^c)_w^\diamond = A^c \cup \emptyset = A^c,$$

implying that each $A \in 2^X$ belongs to $w^\diamond$. This yields the inclusion $2^X \subseteq w^\diamond$. Therefore, the equality $w^\diamond = 2^X$ is obtained.

(ii) Assume that A is $(w,\mathcal{P})$-open. By definition, this implies that the complement A^c is $(w,\mathcal{P})$-closed. Consequently, the inclusion $(A^c)_w^\diamond \subseteq A^c$ holds. Applying the operator yields

$$c_w^\diamond(A^c) = A^c \cup (A^c)_w^\diamond \subseteq A^c.$$

At the same time, the containment $A^c \subseteq c_w^\diamond(A^c)$ is always valid by construction of the operator. Therefore, equality follows,

$$c_w^\diamond(A^c) = A^c,$$

which implies that $A \in w^\diamond$, establishing the inclusion $w \subseteq w^\diamond$. For the reverse inclusion, consider any $A \in w^\diamond$. By definition, this means

$$c_w^\diamond(A^c) = A^c \cup (A^c)_w^\diamond = A^c,$$

and hence $(A^c)_w^\diamond \subseteq A^c$. The goal is to verify that A^c is $(w,\mathcal{P})$-closed. Given an element $x \notin (A^c)_w^\diamond$, the definition of the derived set guarantees the existence of a neighborhood $U \in \psi_w(x)$ such that the union $U^c \cup A$ fails to belong to $\mathcal{P}$. Under the assumption $\mathcal{P} = 2^X \setminus \{X\}$, this failure occurs if and only if $U^c \cup A = X$. This implies that $U \cap A^c = \emptyset$, so x does not belong to $c_w(A^c)$. Hence, the inclusion

$$c_w(A^c) \subseteq (A^c)_w^\diamond \subseteq A^c$$

is established, confirming that A^c is $(w,\mathcal{P})$-closed. As a result, the subset A must belong to w, and therefore $w^\diamond \subseteq w$. Combining both directions yields the equality $w^\diamond = w$. □

4. New Structures Form a GPT Space

This part introduces novel mathematical constructs based on the operators i_w, c_w, and $c_w^\diamond$. These constructs extend weak structures into the more sophisticated framework of the **GPT** space [26], providing a resolution to **Question 2** as follows.

Definition 10. *Consider $(X,w,\mathcal{P})$ as a* **PWS** *space with $A \subseteq X$. Define the following subsets of X:*

(i) $\sigma(w,\mathcal{P}) = \{A \subseteq X : A \subseteq c_w^\diamond(i_w(A))\}$.
(ii) $\pi(w,\mathcal{P}) = \{A \subseteq X : A \subseteq i_w(c_w^\diamond(A))\}$.
(iii) $\beta(w,\mathcal{P}) = \{A \subseteq X : A \subseteq c_w(i_w(c_w^\diamond(A)))\}$.
(iv) $\alpha(w,\mathcal{P}) = \{A \subseteq X : A \subseteq i_w(c_w^\diamond(i_w(A)))\}$.

Theorem 5. *Assume $(X,w,\mathcal{P})$ is a* **PWS** *space and $A \subseteq X$. Each of the structures $\sigma(w,\mathcal{P})$, $\pi(w,\mathcal{P})$, $\beta(w,\mathcal{P})$, and $\alpha(w,\mathcal{P})$ constitutes a* **GPT** *space.*

Proof. Clearly, $\emptyset \in \sigma(w,\mathcal{P})$ since $\emptyset \subseteq c_w^\diamond(i_w(\emptyset)) = \emptyset$. Consider an arbitrary family $\{A_\beta : \beta \in \Delta\} \subseteq \sigma(w,\mathcal{P})$. By definition, for each index β,

$$A_\beta \subseteq c_w^\diamond(i_w(A_\beta)).$$

It follows that

$$\bigcup_{\beta\in\Delta} A_\beta \subseteq \bigcup_{\beta\in\Delta} c_w^\diamond(i_w(A_\beta)) \subseteq c_w^\diamond\left(i_w\left(\bigcup_{\beta\in\Delta} A_\beta\right)\right).$$

Consequently, the union $\bigcup_{\beta\in\Delta} A_\beta$ belongs to $\sigma(w,\mathcal{P})$. An analogous argument establishes the corresponding closure properties for the remaining structures. □

Proposition 5. *Within the* **PWS** *space* $(X,w,\mathcal{P})$ *and for any subset* $A\subseteq X$*, the collection* $w^\diamond$ *constitutes a* **GPT** *space on* X*.*

Proof. Clearly $\emptyset\in w^\diamond$. Consider a family $\{A_\beta : \beta\in\Delta\}\subseteq w^\diamond$. By definition, for each index β, the equality

$$c_w^\diamond\big((A_\beta)^c\big) = (A_\beta)^c$$

holds. Taking the union over β yields

$$\bigcup_{\beta\in\Delta}(A_\beta)^c = \bigcup_{\beta\in\Delta} c_w^\diamond\big((A_\beta)^c\big) = c_w^\diamond\left(\bigcup_{\beta\in\Delta}(A_\beta)^c\right).$$

The union $\bigcup_{\beta\in\Delta} A_\beta$ belongs to $w^\diamond$. Therefore, $w^\diamond$ forms a generalized primal topology on X. □

Theorem 6. *Consider* $(X,w,\mathcal{P})$ *as a* **PWS** *space. Thus, the following holds:*

(i) $w\subseteq\alpha(w,\mathcal{P})\subseteq\sigma(w,\mathcal{P})\subseteq\beta(w,\mathcal{P})$.
(ii) $w\subseteq\alpha(w,\mathcal{P})\subseteq\pi(w,\mathcal{P})\subseteq\beta(w,\mathcal{P})$.

Proof. (i) For any set $A\in w$, the equality $A=i_w(A)$ holds. From this, the inclusion $A\subseteq i_w(A)$ implies

$$c_w^\diamond(A)\subseteq c_w^\diamond\big(i_w(A)\big).$$

Since $A\subseteq c_w^\diamond(A)$, it follows that

$$A\subseteq c_w^\diamond\big(i_w(A)\big).$$

Applying the operator i_w yields

$$i_w(A) = A\subseteq i_w\big(c_w^\diamond(i_w(A))\big),$$

which establishes membership of A in $\alpha(w,\mathcal{P})$. Moreover, the inclusion

$$A\subseteq i_w\big(c_w^\diamond(i_w(A))\big)\subseteq c_w^\diamond(i_w(A))$$

shows that A belongs to $\sigma(w,\mathcal{P})$. Consequently,

$$A\subseteq c_w^\diamond(i_w(A))\subseteq c_w\big(i_w(A)\big).$$

Since $A\subseteq c_w^\diamond(A)$, one obtains

$$A\subseteq c_w\big(i_w(c_w^\diamond(A))\big),$$

which confirms that A is an element of $\beta(w,\mathcal{P})$.

(ii) The assertion follows by an analogous argument to that in part (i). □

https://doi.org/10.3390/math14050864

5. Separation Axioms on a PWS Space

We introduce the separation axioms in the context of primal weak structure **PWS** spaces and examine their interrelations. These axioms are crucial for distinguishing points and sets, forming a core part of the structural analysis of topological spaces. Their study in PWS spaces extends classical results and reveals new topological behaviors.

Definition 11. *Let $(X, w, \mathcal{P})$ be a* **PWS** *space. A subset $A \subseteq X$ is called a $D_{(w,\mathcal{P})}$-set if $A = U \setminus V$, where $U \neq X$ and U, V are $(w, \mathcal{P})$-open sets.*

Remark 5. *Every $(w, \mathcal{P})$-open set $U \subseteq X$ is a $D_{(w,\mathcal{P})}$-set by taking $V = \emptyset$.*

Definition 12. *Let $(X, w, \mathcal{P})$ be a* **PWS** *space. The space $(X, w, \mathcal{P})$ is said to satisfy the following separation axioms:*

(i) *$(w, \mathcal{P})$-D_0 (resp., T_0) if for any two distinct points $x \neq y \in X$, there exists a $D_{(w,\mathcal{P})}$-set (resp., open set) containing exactly one of x or y.*
(ii) *$(w, \mathcal{P})$-D_1 (resp., T_1) if for any two distinct points $x \neq y \in X$, there exist sets U, V as above such that $x \in U$, $y \notin U$, $x \notin V$, and $y \in V$.*
(iii) *$(w, \mathcal{P})$-D_2 (resp., T_2) if the separating sets U and V in (ii) are disjoint.*
(iv) *$(\mathfrak{g}, \mathcal{P})$-$T_0$ if for any two distinct points $x \neq y \in X$, there exists a $(w, \mathcal{P})$-open set containing precisely one of x or y.*
(v) *$(w, \mathcal{P})$-T_1 if for any two distinct points $x \neq y \in X$, there exist $(w, \mathcal{P})$-open sets U, V such that $x \in U$, $y \notin U$, $y \in V$, and $x \notin V$.*
(vi) *$(w, \mathcal{P})$-T_2 (Hausdorff) if for any two distinct points $x \neq y \in X$, there exist disjoint $(w, \mathcal{P})$-open sets U, V with $x \in U$ and $y \in V$.*

Theorem 7. *A* **PWS** *space attains the $(w, \mathcal{P})$-D_0 property exactly when it possesses the $(w, \mathcal{P})$-T_0 property.*

Proof. Assume that the **PWS** space $(X, w, \mathcal{P})$ has the $(w, \mathcal{P})$-D_0 property. For any distinct points $x, y \in X$, there exists a $D_{(w,\mathcal{P})}$-set U such that $x \in U$ and $y \notin U$.

Express U as $V \setminus W$, where $V \neq X$ and V, W are $(w, \mathcal{P})$-open. Then, either $x \in V$ and $y \notin V$, or $y \in W$ and $x \notin W$. In both situations, there exists a $(w, \mathcal{P})$-open set containing exactly one of x or y. Therefore, $(X, w, \mathcal{P})$ satisfies the $(w, \mathcal{P})$-T_0 condition.

The converse follows directly from Remark 5, since every $(w, \mathcal{P})$-open set is a $D_{(w,\mathcal{P})}$-set. □

Corollary 2. *Every* **PWS** *space satisfying the $(w, \mathcal{P})$-D_2 condition satisfies the $(w, \mathcal{P})$-T_0 condition as well.*

Remark 6. *The following diagram shows the relationship among all the previous notions.*

The converse of the relationship in Figure 1 is not always true; the next examples explain that.

$$\begin{array}{ccccc} (w,\mathcal{P})-T_2 & \longrightarrow & (w,\mathcal{P})-T_1 & \longrightarrow & (w,\mathcal{P})-T_0 \\ \downarrow & & \downarrow & & \updownarrow \\ (w,\mathcal{P})-D_2 & \longrightarrow & (w,\mathcal{P})-D_1 & \longrightarrow & (w,\mathcal{P})-D_0 \end{array}$$

Figure 1. The relationships between all separation axioms on a **PWS** space.

https://doi.org/10.3390/math14050864

Example 9. *Let $X = \mathbb{N}$, the set of natural numbers, and consider the primal collection*

$$\mathcal{P} = \{A \subseteq X : A \text{ is finite}\}.$$

Define the **PWS** *structure w in three distinct ways:*

(i) $w_1 = \{\emptyset\} \cup \{\{1\}, \{3\}, \{n, n+1, \dots\} : n \geq 2\}$. *Then, $(X, w_1, \mathcal{P})$ is $(w_1, \mathcal{P})$-D_0 and hence $(w_1, \mathcal{P})$-T_0, but it is not $(w_1, \mathcal{P})$-T_1.*

(ii) $w_2 = \{\emptyset\} \cup \{\{1,2\}, \{2,3\}, \{n, n+1\} : n \geq 1\}$. *Then, $(X, w_2, \mathcal{P})$ is $(w_2, \mathcal{P})$-T_1 and $(w_2, \mathcal{P})$-D_2, but it is not $(w_2, \mathcal{P})$-T_2.*

(iii) $w_3 = \{\emptyset\} \cup \{\{1,3\}, \{2,3\}, \{n, n+2, n+3, \dots\} : n \geq 1\}$. *Then, $(X, w_3, \mathcal{P})$ is $(w_3, \mathcal{P})$-D_1 (since there exist $D_{(w_3, \mathcal{P})}$-sets corresponding to singletons in $\mathcal{P}$), but it is not $(w_3, \mathcal{P})$-T_1.*

This construction generalizes the finite case to an infinite **PWS** *space, illustrating how different choices of w affect the separation axioms.*

Effect of the Primal on Separation Axioms

In this subsection, we investigate how the primal collection $\mathcal{P}$ influences classical separation axioms in a **PWS** space. By focusing on special classes of sets defined via the primal, we uncover new structural behaviors and distinctions between $(w, \mathcal{P})$-Hausdorff and $\mathcal{P}$-Hausdorff spaces.

In particular, this subsection addresses **Question 3**, examining the conditions under which the primal weak structure can be further developed to generate a generalized primal topological space and the resulting effects on separation properties.

Definition 13. *Let $(X, w, \mathcal{P})$ be a* **PWS** *space. A subset $A \subseteq X$ is called* $\mathcal{P}^{\diamond}$-open *if*

$$A \subseteq i_w(A)^{\diamond}_w.$$

Definition 14. *A* **PWS** *space $(X, w, \mathcal{P})$ is* $\mathcal{P}^{\diamond}$-Hausdorff *if for every pair of distinct points $x, y \in X$, there exist disjoint $\mathcal{P}^{\diamond}$-open sets U and V such that $x \in U$ and $y \in V$.*

Remark 7. *A $(w, \mathcal{P})$-Hausdorff space need not be $\mathcal{P}^{\diamond}$-Hausdorff, and vice versa. This shows that the standard separation axioms do not automatically transfer when restricting to $\mathcal{P}^{\diamond}$-open sets.*

Example 10. *Let $X = \mathbb{N}$, the set of natural numbers, and define $w = \{\emptyset, \{1\}, \{2\}, \dots, X\}$, with primal collection*

$$\mathcal{P} = \{A \subseteq \mathbb{N} : 1 \notin A \text{ or } 2 \notin A\}.$$

Then, for any $A \subseteq \mathcal{P}$, we have $A^{\diamond}_w = \emptyset$, so no set other than the empty set is $\mathcal{P}^{\diamond}$-open. Therefore, this infinite **PWS** *space is $(w, \mathcal{P})$-Hausdorff but not $\mathcal{P}^{\diamond}$-Hausdorff.*

Example 11. *Consider $X = \mathbb{Z}$, the set of all integers, with $w = \{\emptyset, \{0\}, X\}$ and primal collection $\mathcal{P} = \{\emptyset, \{n\} : n \in \mathbb{Z}\}$. In this infinite setting, every $(w, \mathcal{P})$-Hausdorff space is automatically $\mathcal{P}^{\diamond}$-Hausdorff, since for each pair of distinct points x, y, the sets $U = \{x\}$ and $V = \{y\}$ satisfy $U \cap V \in \mathcal{P}$.*

Definition 15. *A* **PWS** *space $(X, w, \mathcal{P})$ is called* $\mathcal{P}$-Hausdorff *if for every pair of distinct points $x, y \in X$, there exist $(w, \mathcal{P})$-open sets U_1, U_2 such that*

$$x \in U_1, \quad y \in U_2, \quad \text{and} \quad U_1 \cap U_2 \in \mathcal{P}.$$

Remark 8. *From Definition 15, every $(w, \mathcal{P})$-Hausdorff space is $\mathcal{P}$-Hausdorff, independent of the choice of the primal $\mathcal{P}$. Conversely, a $\mathcal{P}$-Hausdorff space may not be $(w, \mathcal{P})$-Hausdorff unless additional conditions are satisfied on the primal.*

Example 12. *Let $X = \mathbb{N}$ with $w = \{\emptyset, \{1\}, \{n\}_{n \geq 2}, X\}$ and $\mathcal{P} = \{\emptyset\} \cup \{\{n\} : n \geq 2\}$. Then, for any $x, y \geq 2$, the singleton sets $\{x\}$ and $\{y\}$ are $\mathcal{P}$-open and disjoint, so the space is $\mathcal{P}$-Hausdorff. However, $(w, \mathcal{P})$-Hausdorff fails since 1 cannot be separated from 2 with $(w, \mathcal{P})$-open sets.*

Theorem 8. *Every $(w, \mathcal{P})$-Hausdorff space is $\mathcal{P}$-Hausdorff. There exist infinite $\mathcal{P}$-Hausdorff spaces that are not $(w, \mathcal{P})$-Hausdorff, demonstrating that the primal strongly affects the separation properties.*

Proof. Let $(X, w, \mathcal{P})$ be $(w, \mathcal{P})$-Hausdorff. For each distinct $x, y \in X$, there exist disjoint $(w, \mathcal{P})$-open sets U_1, U_2 such that $x \in U_1$ and $y \in U_2$. Then, $U_1 \cap U_2 = \emptyset \in \mathcal{P}$, so $(X, w, \mathcal{P})$ is $\mathcal{P}$-Hausdorff.

To see that the converse fails in general, consider Example 4 with $X = \mathbb{N}$. For $x = 1$ and $y = 2$, any $(w, \mathcal{P})$-open sets containing them intersect outside $\mathcal{P}$, showing $(w, \mathcal{P})$-Hausdorff does not hold. □

Theorem 9. *Let $(X, w, \mathcal{P})$ be a $\mathcal{P}$-Hausdorff* **PWS** *space. Suppose that for every countable family $\{U_n\}_{n \in \mathbb{N}}$ of $(w, \mathcal{P})$-open sets, the intersection*

$$\bigcap_{n \in \mathbb{N}} U_n \in \mathcal{P}.$$

Then, $(X, w, \mathcal{P})$ is $(w, \mathcal{P})$-Hausdorff.

Proof. Consider any distinct points $x, y \in X$. Since $(X, w, \mathcal{P})$ is $\mathcal{P}$-Hausdorff, there exist $(w, \mathcal{P})$-open sets U_1 and U_2 such that $x \in U_1$, $y \in U_2$, and $U_1 \cap U_2 \in \mathcal{P}$.

Construct the countable family $\{U_n\}_{n \in \mathbb{N}}$ by iteratively refining neighborhoods of x and y:

$$U_1 \supset U_3 \supset U_5 \supset \ldots, \quad U_2 \supset U_4 \supset U_6 \supset \ldots$$

where each $U_{2n-1} \cap U_{2n} \in \mathcal{P}$ for all $n \in \mathbb{N}$. By the infinite intersection property, we have

$$\bigcap_{n \in \mathbb{N}} (U_{2n-1} \cap U_{2n}) \in \mathcal{P}.$$

This intersection forms disjoint $(w, \mathcal{P})$-open sets around x and y, proving that $(X, w, \mathcal{P})$ satisfies the $(w, \mathcal{P})$-Hausdorff condition. □

Example 13. *Let $X = \mathbb{N}$, the set of natural numbers, and define a* **PWS** *structure as follows:*

$$w = \{\emptyset\} \cup \{\{n, n+1, n+2, \ldots\} : n \in \mathbb{N}\}, \quad \mathcal{P} = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}.$$

For any distinct $x, y \in \mathbb{N}$, choose $U_x = \{x, x+1, x+2, \ldots\}$ and $U_y = \{y, y+1, y+2, \ldots\}$. Then,

$$U_x \cap U_y = \{\max(x, y), \max(x, y) + 1, \ldots\} \in \mathcal{P},$$

demonstrating the $\mathcal{P}$-Hausdorff property.

Consider the countable family $\{U_n\}_{n \in \mathbb{N}}$, where $U_n = \{n, n+1, \ldots\}$. Then,

$$\bigcap_{n \in \mathbb{N}} U_n = \emptyset \in \mathcal{P}.$$

https://doi.org/10.3390/math14050864

This confirms the infinite intersection property of the theorem and ensures that this $\mathcal{P}$-Hausdorff space is also $(w, \mathcal{P})$-Hausdorff.

Thus, this construction provides a concrete infinite **PWS** *space illustrating the impact of the primal $\mathcal{P}$ on separation axioms.*

Theorem 10. *Let $(X, w, \mathcal{P})$ be a* **PWS** *space. If the space is $(w, \mathcal{P})$-T_0, then for any distinct $x, y \in X$ we have*

$$c_w(\{x\}) \neq c_w(\{y\}).$$

Proof. For distinct $x, y \in X$, there exists a $(w, \mathcal{P})$-open set U such that $x \in U$ and $y \notin U$. Since $X \setminus U$ is $(w, \mathcal{P})$-closed and may belong to the primal $\mathcal{P}$, it contains y but not x, giving $c_w(\{x\}) \neq c_w(\{y\})$. Hence, the primal collection ensures that the distinction of closures respects $\mathcal{P}$. □

Theorem 11. *If $c_w(\{x\})$ is $(w, \mathcal{P})$-closed for all $x \in X$ and $c_w(\{x\}) \neq c_w(\{y\})$ for all distinct x, y, then the space is $(w, \mathcal{P})$-T_0.*

Proof. For distinct $x, y \in X$, pick $z \in c_w(\{x\}) \setminus c_w(\{y\})$. Then, $X \setminus c_w(\{y\})$ is $(w, \mathcal{P})$-open, containing x but not y. The primal $\mathcal{P}$ ensures these open sets are compatible with $\mathcal{P}$, so $(w, \mathcal{P})$-T_0 is satisfied. □

Theorem 12. *If $\{x\}$ is $(w, \mathcal{P})$-closed for all $x \in X$, then the space is $(w, \mathcal{P})$-T_1.*

Proof. For distinct $x, y \in X$, the complements $X \setminus \{x\}$ and $X \setminus \{y\}$ are $(w, \mathcal{P})$-open and may belong to $\mathcal{P}$, separating x and y. Hence, the space is $(w, \mathcal{P})$-T_1 under the primal constraints. □

Corollary 3. *If the union of $(w, \mathcal{P})$-open sets is $(w, \mathcal{P})$-open, then the converse of Theorem 11 holds, ensuring a stronger $\mathcal{P}$-compatibility.*

Example 14. *Let $X = \mathbb{N}$, $\mathcal{P} = \{F \subseteq \mathbb{N} : F \text{ is finite}\}$, and*

$$w = \{\emptyset\} \cup \{\{n, n+1, n+2, \dots\} : n \in \mathbb{N}\}.$$

(i) Each singleton $\{n\}$ is $(w, \mathcal{P})$-closed, hence the space is $(w, \mathcal{P})$-T_1.
(ii) For $n < m$, the set $\{n, n+1, \dots\}$ contains n but not m, showing $(w, \mathcal{P})$-T_0 separation.
(iii) The primal collection $\mathcal{P}$ ensures that finite intersections of closures respect $\mathcal{P}$, highlighting new behaviors in infinite spaces.

Remark 9. *The inclusion of a primal collection $\mathcal{P}$ modifies the classical separation axioms:*

(i) T_0 and T_1 properties are now sensitive to $\mathcal{P}$-closed sets.
(ii) Infinite **PWS** *spaces can be studied via $\mathcal{P}$, giving rise to subspaces and hierarchies that preserve separation under primal constraints.*
(iii) This framework allows constructing infinite examples where separation holds in ways impossible in ordinary topological spaces without $\mathcal{P}$.

Example 15. *Let $X = \mathbb{N}$, the set of natural numbers, and consider a weak structure*

$$w = \{\emptyset\} \cup \{U \subseteq \mathbb{N} : U \text{ contains all but finitely many elements of } \mathbb{N}\}.$$

Define a primal collection

$$\mathcal{P} = \{A \subseteq \mathbb{N} : 1 \in A\}.$$

Let $A = \{2, 3, 4, \dots\} \subseteq \mathbb{N}$*. In the weak structure* w*, the operator* $A^{\diamond}_{w}$ *gives* $A^{\diamond}_{w} = \mathbb{N}$*, since every weak neighborhood of* 1 *intersects* A*. In the primal weak structure space* $(\mathbb{N}, w, \mathcal{P})$*, the primal condition restricts inclusion:*

$$A^{\diamond}_{w} = A = \{2, 3, 4, \dots\},$$

because $A^c \cup U^c \notin \mathcal{P}$ *if it does not contain* 1.

This example illustrates that a property holding in a PWS may fail in the classical weak structure. Specifically, the primal condition modifies the closure behavior, highlighting the effect of the primal collection $\mathcal{P}$.

6. Conclusions

This paper has introduced the concept of a primal weak structure, built upon weak structures defined via primal sets, and has provided a thorough analysis of the associated operators and their fundamental properties. The construction of primal weak structures was further extended to generalized primal topological spaces, and separation axioms within these structures were examined, with relationships between them illustrated and distinctions from classical topological spaces highlighted.

The study has addressed the key questions posed in the Introduction; specifically, the conditions under which a primal weak structure coincides with one induced by an operator satisfying Kuratowski's closure axioms, the potential for further development into generalized primal topological spaces, and the influence of the primal collection on separation axioms compared to classical weak structures.

As a direction for future research, it would be of interest to explore the interplay between primal weak structures and additional algebraic or topological constraints, such as compactness or connectedness, and to investigate how these conditions affect the behavior of the associated operators. Such investigations may lead to new classes of generalized topological spaces with richer structural and functional properties.

Author Contributions: Methodology, H.A.-S. and H.A.-M.; writing—original draft, H.A.-S. and H.A.-M.; writing—review and editing, H.A.-S. and H.A.-M.; supervision, H.A.-S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Acknowledgments: The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflicts of Interest: The authors declare no conflicts of interest.

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https://doi.org/10.3390/math14050864