

Article

Boundary Value Problems and Propagation of Singularities for Several Partial Differential Equations of Mathematical Physics

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Abstract

This paper deals with several equations of mathematical physics written in explicit form with their solutions. In Theorem 1, an oblique derivative problem for the string equation is studied. More precisely, the initial-boundary value problem for the string equation is investigated. The corresponding vector field on the boundary is non-vanishing and does not have a characteristic direction, but can be tangential to some part of the boundary, and it is allowed to change sign. A classical solution exists with suitable compatibility conditions at the corner points. The picture changes significantly in the case of the wave equation with several (say two: 2D) space variables in a circular cylinder. The initial-boundary value problem turns out to be underdetermined with an infinite-dimensional kernel if the boundary vector field is orthogonal to the time axis. By prescribing extra conditions on the generatrices of the cylinder where the vector field is tangential to the cylinder, we obtain a unique classical solution. In Theorem 2, we consider the Cauchy problem in the interior of the parabola of the Lorentzian-type eikonal equation and find its unique classical solution in $\{0 \leq x_2 \leq 1/2\} \cap \{x_2 \geq \frac{x_1^2}{2}\}$. Propagation of singularities for the D and $3D$ hyperbolic (Klein–Gordon) equations in $\mathbf{R}^4$, $\mathbf{R}^8$ is studied in Theorem 3. In the double characteristic points, the wave front propagates either along the surface of the characteristic cone, or in the solid cone starting from (t_0, x_0) .

Keywords: wave equation; string equation; Klein–Gordon equation; Lorentzian eikonal equation; initial-boundary value problem; oblique derivative problem; propagation of singularities; wave-front set; double characteristic points; fundamental solution

MSC: 35A18; 35A08; 35A21; 35A30; 35C05; 35F25; 35L05; 35L20; 49L25; 65M25; 81Q05; 83C57



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1. Introduction

This paper deals with boundary value problems and propagation of the wave-front of the solutions to several classical equations of mathematical physics as the string equation with oblique derivative data and Goursat–Darboux problem for it, the Cauchy problem for the Lorentzian eikonal equation, as well as with singularities of the solutions of degenerate hyperbolic equations with constant coefficients in $\mathbf{R}_{(t,x)}^2 \times \mathbf{R}_{(z,y)}^2$, $\mathbf{R}^3 \times \mathbf{R}^3$, $\mathbf{R}_{(t,x)}^4 \times \mathbf{R}_{(z,y)}^4$, $t, z \in \mathbf{R}^1$ (say, wave and Klein–Gordon equations in $\mathbf{R}_{t,x}^{n+1}$ with $n = 1, 2, 3$) and degenerate elliptic operators: $\Delta_n + c$ in $\mathbf{R}_x^{n+k}$, $n = 1, 2, 3$; $k \geq 1$ (see [1–3]).

We also consider pseudo-differential operators with multiple (double) involutive characteristics having a principal symbol $p_2^0 = u_1(x, \xi)v_1(x, \xi)$. If $\rho^0 \in T^*(\Omega) \setminus 0$ is such that

$p_2^0(\rho^0) \neq 0$, there are no singularities of the solutions of $P(x, D)u = f$. Simple characteristics (or principal-type pseudo-differential operator) of the real-valued second order in ξ symbol p_2^0 at ρ^0 exists if for the real-valued first-order symbol $u_1(\rho)v_1(\rho)$ we have: $p_2^0(\rho^0) = 0$, but $|\nabla_{x,\xi} p_2^0(\rho^0)| > 0$. Thus $\left| \begin{array}{l} u_1(\rho^0) = 0, \nabla u_1(\rho^0) \neq 0 \\ v_1(\rho^0) \neq 0 \end{array} \right|$ or $\left| \begin{array}{l} v_1(\rho^0) = 0, \nabla v_1(\rho^0) \neq 0 \\ u_1(\rho^0) \neq 0 \end{array} \right|$.

Then, the singularities starting from the characteristic point ρ^0 propagate along the zero-bicharacteristic of u_1 and v_1 , respectively. The zero bicharacteristic of the symbol u_1 , namely $\mathcal{L}_{u_1}$, is the nondegenerate integral curve of the Hamiltonian vector field H_{u_1} passing through

$$\rho^0 : \begin{cases} \dot{x} = \nabla_{\xi} u_1(x(0), \xi(0)) = \rho^0 \\ \dot{\xi} = -\nabla_x u_1, \rho(t) = (x(t), \xi(t)). \end{cases}$$

Then, $u_1(\rho) \equiv u_1(\rho^0) = 0$ for each $\rho(t) \in \mathcal{L}_{u_1}$. We denote by $CharP = \{\rho \in T^*(\Omega) \setminus 0 : p_2^0(\rho) = 0\}$ and by $CrP = \{\rho \in T^*(\Omega) \setminus 0 : p_2^0(\rho) = \nabla_{x,\xi} p_2^0(\rho^0) = 0\}$ —the critical points of p_2^0 . Assume that the characteristics of P with principal symbol $p_2^0 = u_1^0(\rho)v_1^0(\rho)$ are such that $u_1(\rho^0) = 0 = v_1(\rho^0)$, $\nabla_{\xi} u_1(\rho^0)$ and $\nabla_{\xi} v_1(\rho^0)$ are linearly independent. Then, CrP is a smooth manifold of codimension 2 near ρ^0 . P is called an operator with double involutive characteristics near ρ^0 if $\{u_1, v_1\} \equiv 0$ on CrP . As usual, $\{\}$ stands for the Poisson bracket of u_1, v_1 . The identity $[H_{u_1}, H_{v_1}] = H_{\{u_1, v_1\}}$ on CrP enables us to conclude that the manifold CrP foliates at two dimensional leaves—integral manifolds M_{ρ} of the vector fields H_{u_1}, H_{v_1} (Frobenius theorem), $H_{u_1} = \sum_1^n (\frac{\partial u_1}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial u_1}{\partial x_j} \frac{\partial}{\partial \xi_j})$, $H_1(\rho^0) \neq 0$, $H_{u_1}(\rho^0), H_{v_1}(\rho^0)$ being linearly independent and commuting on $CharP$.

The subprincipal symbol of P , $p_1'(\rho) = p_1(\rho) + \frac{i}{2} \sum \frac{\partial^2 p_2^0}{\partial x_j \partial \xi_j}(\rho)$ is invariant with respect to the homogeneous canonical transformations at CrP (not at $CharP$). We recall that the homogeneous canonical transformations from $T^*(\Omega) \setminus 0 \rightarrow T^*(\tilde{\Omega}) \setminus 0$ conserve the Poisson bracket, i.e., the classical symplectic form on the cotangential bundle $T^*(\Omega) \setminus 0$ [3]. The operator P satisfies the Levy condition if $p_1' \equiv 0$ on CrP .

There exists for the involutive characteristics near ρ^0 such a canonical transformation that $u_1 \rightarrow \xi_1, v_1 \rightarrow \xi_2; \rho^0 = (\tilde{x}_0; \xi_1 = 0, \xi_2 = 0, (\tilde{x}'')^0 \neq 0)$. Certainly, $p_1' \rightarrow 0$ on CrP , i.e., $p_1 \rightarrow A\xi_1 + B\xi_2$, A, B being 0 order pseudo-differential operators [3,4].

Very often, near ρ^0 , one considers operators Q with principal symbol $p_2^0 = u_1^2 + v_1^2$; i.e., on CrP , the principal symbol becomes $\xi_1^2 + \xi_2^2$. So, p_2^0 has a zero of sharp order 2 on the involutive manifold CrP and is positive outside it. The first-order terms are again $A\xi_1 + B\xi_2$ under the Levy condition. It was proved in [5] that the singularities of the latter degenerate elliptic operator propagate along M_{ρ^0} .

The initial oblique derivative problem for the string equation in half space $\{t > 0, 0 < x < l\}$ includes the case when the boundary vector field $\frac{\partial}{\partial t}$ is tangential to the boundary, and conserves or changes sign. The only restriction is that $\frac{\partial}{\partial t}$ does not have the characteristic directions $\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$ and does not vanish. We are looking for classical solutions of this mixed problem. Certainly, then, compatibility conditions must be imposed at the corner points $(0, 0), (l, 0)$. Moreover, the construction of classical solution requires other compatibility conditions at the points $(0, k\frac{l}{2}), (l, k\frac{l}{2}), k \in \mathbf{N}$. In the special

case when the boundary data are of the type $\begin{cases} A(t)\frac{\partial}{\partial t} + B(t)\frac{\partial}{\partial x}, \\ (A, B) \not\parallel (1, \pm 1) \\ A^2 + B^2 > 0 \end{cases}$, the uniqueness result

holds. The solvability in the rectangle $\{0 \leq x \leq l, 0 \leq t \leq \frac{l}{2}\}$ is reduced to the classical Cauchy problem for the string equation and to two Goursat–Darboux boundary value problems (see Theorem 1(a)), i.e., boundary value problems with data on one characteristic and one non-characteristic straight lines. The described procedure is repeated several times in rectangles of the type $\{0 \leq x \leq l, (k-1)\frac{l}{2} \leq t \leq k\frac{l}{2}, k \geq 2\}$. One can easily

see that the existence of points on the boundary lines $x = 0, x = l$, where $\frac{\partial}{\partial t}$ is tangential to $\vec{Ot}$, does not affect the existence results. Non-existence results, including a blow-up of the solution, could appear only if $\frac{\partial}{\partial t} \parallel \partial_t \pm \partial_x$ at at least one point ($x = 0, t_0 \geq 0$), ($x = l, t_0 \geq 0$). The situation changes substantially if the space variables are $n \geq 2$. To explain this, we consider the domain-bounded cylinder $K \times \{t \geq 0\}$, $K = \{x^2 + y^2 \leq L^2\}$. Then, the vector field $\frac{\partial}{\partial x}$ is tangential to the lateral sides of the cylinder at the generatrices: $\{t \geq 0\} \times \{(0, \pm L)\}$. This fact leads to the appearance of an undetermined mixed problem having an infinite-dimensional kernel, i.e., the mixed problem with zero data has infinitely many solutions.

The prescription of extra conditions on the above-mentioned two half lines leads to the existence of a unique classical solution of the corresponding mixed problem. Certainly, appropriate compatibility conditions should be imposed too. We shall only mention that in the multidimensional case, the mixed problem is rather complicated as the hyperbolic, diffractive and glancing points appear on $T^*(\partial K \times t \geq 0)$ [3].

Our second equation is a Lorentzian-type eikonal equation $p^2 - q^2 = -1$ with constant initial data on the parabola $\gamma : x_2 = \frac{x_1^2}{2}$. In the classical distance, the eikonal equation $p^2 + q^2 = 1$ possesses a local (near each point of the parabola curve) classical solution (see [6] for new results). If we consider the solution $u = u(x, y)$, $u \equiv 1$ on γ as an integral surface S inside γ , the latter develops singularities of the swallow tail type. For viscosity solutions, see [2,7]. For the Lorentzian eikonal equation, the situation is completely different, in contrast with the distance-type eikonal equation, where the characteristic straight lines are crossing inside γ and form an envelope there. Here, they form an envelope outside γ and the characteristics are not crossing in the domain $\{\frac{1}{2} \geq x_2 \geq x_1^2/2, |x_1| \leq 1\} = D$. Therefore, a classical solution can be found in D . To get this result, we apply the Cauchy method. Evidently, the data $u \equiv 1$ is given on the whole parabola $x_2 = \frac{x_1^2}{2}$, but u exists in D , $x_2 \leq 1/2$. Global viscosity solutions to the Lorentzian eikonal equation were found recently in [7,8].

Concerning the propagation of singularities, we shall mention that we investigate the propagation of singularities starting from the double characteristic points of the real-valued principal symbol. Then, the WF set propagates (for the wave, Klein–Gordon equation) in the (solid) characteristic cone $t^2 = |x|^2$, starting from the point (t_0, x_0) , i.e., (in the interior of) the cone $(t - t_0)^2 = |x - x_0|^2$. In the case of degenerate elliptic equations (say $\Delta_n \in \mathbf{R}^{2n}$), the singularities propagate along the $\mathbf{R}^n$ space with initial point at x_0 . Certainly, if $P(D_x)$ is given in $\mathbf{R}_x^n$ and the considerations are in $\mathbf{R}_x^n \times \mathbf{R}_y^n$, then $P(D_x)u = f(x, y)$, $y \in \mathbf{R}_y^n$, $u = u(x, y)$, $(x, y) \in \mathbf{R}^{2n}$.

The pseudo-differential operator with symbol $\xi_1 \xi_2 + A \xi_1 + B \xi_2 + \dots$ in $\mathbf{R}^4$ and near $0 = \xi_1^0 = \xi_2^0, (\xi_3^0)^2 + (\xi_4^0)^2 > 0$ was recently studied in [9] when investigating the waves across the event horizon of an extremal rotating Kerr black hole. Unlike point-like particles, fields can spread and interact non-causally across the horizons, but the theory of real principal-type operators (i.e., with simple characteristics) does not apply on the horizon. It was shown in [9] that the solutions near the horizon admit two more channels for the propagation of singularities. In the case of operators with constant coefficients, it turns out that these channels are two opposite solid cones (of the future and the past).

Consider operator satisfying Levy condition at ρ^0 :

$$P \sim \xi_1 \xi_2 + A \xi_1 + B \xi_2 + \dots, 0 = \xi_1^0 = \xi_2^0, (\xi_3^0)^2 + (\xi_4^0)^2 > 0.$$

As we know, through $Cr(P) = \{\xi_1 = \xi_2 = 0, \xi_3^0, \xi_4^0, (\xi_3^0)^2 + (\xi_4^0)^2 > 0\}$ is passing the 2-dimensional leave $M_{\rho^0}(\partial_{x_1}, \partial_{x_2})$, such that $H_{u_1}(\rho^0), H_{v_1}(\rho^0)$ are tangential to M_{ρ^0} at ρ^0 . Denote by L_{u_1}, L_{v_1} the integral curves of H_{u_1}, H_{v_1} , located at M_{ρ^0} . They are non-tangential

and crossing at the point $\rho^0 \in M_{\rho^0}$. Consider the opposite solid angles with vertex at ρ^0 and with sides L_{u_1}, L_{v_1} (they are not bicharacteristics). Then, the singularities propagate only along them. They are the channels for the propagation of singularities across the event horizon.

In Section 5, Discussion, we shall use the following remark.

Remark 1. In a similar way to the tangential oblique derivative problem, one can consider the Goursat–Darboux boundary value problem for the string equation: $u_{xy} + cu = 0$. The investigation is in a solid angle of the type $\{x = 0, y = \alpha x, 0 < \alpha\}$. Moreover, $u|_{x=0} = \varphi_0(y)$,

$$A(x)\frac{\partial u}{\partial y} + B(x)\frac{\partial u}{\partial x} + C(x)u = \varphi_1(x), (A, B) \nparallel (1, 0), (0, 1), A^2 + B^2 > 0,$$

on $y = \alpha x$. As the general solution of the PDE with $c = \pm 1$ and in the first quadrant is:

(a) for $c = -1$,

$$u = \int_0^x f(t)I_0(2\sqrt{y(x-t)})dt + \int_0^y g(s)I_0(2\sqrt{x(y-s)})ds + \tilde{c}I_0(2\sqrt{xy})$$

for arbitrary smooth f, g , I_0 —the Bessel function with purely imaginary argument [1] and

(b) for $c = 1$

$$u = \int_0^x f(t)J_0(2\sqrt{y(x-t)})dt + \int_0^y g(s)J_0(2\sqrt{x(y-s)})ds + \tilde{c}J_0(2\sqrt{xy}),$$

J_0 —being the standard Bessel function [1], while f, g are arbitrary smooth functions, $\tilde{c} = \text{const}$, the boundary value problem is reduced to the solvability of a system of Volterra-type integral equations of second type for f, g .

Notations for the Bessel functions are standard in the handbooks of special functions.

The non-standard Cauchy–Goursat boundary value problem for $u_{tt} = u_{xx}$ with applications in physics is studied in [10].

In this paper, we concentrate on PDE with constant coefficients, as the proofs are short and simple. Moreover, the string, the wave, the Klein–Gordon- and Schrödinger-type equations often appear in the applications, and the qualitative picture for them is clear and easy to explain and describe. We shall compare the classical solution of the Lorentzian eikonal equation, proposed here, with the global viscosity solution on the 2-dimensional Minkowski space time $(\mathbf{R}^2, g = dy^2 - dx^2)$: $g(\nabla f, \nabla f) = -1$ [7]. Then, $f(x, y) = |x|$ is a viscosity solution, but f is not locally semiconcave in any neighbourhood of $0 = x$. Another example is $f(x, y) = \sqrt{2}|x| - |y|$, which is a Lipschitz function. It is also a globally defined viscosity solution of the Lorentzian eikonal equation. The sets $\nabla^- f(0, 0)$ and $\nabla^+ f(0, 0)$ are empty. Thus, the graph of f is partially linear (consists of several planes). We point out that we shall not study viscosity solutions in this paper.

This paper is organized as follows. Theorems 1–3 are formulated and proved in Sections 2–4. This paper ends with a list of references. When the operators are with constant coefficients, they are locally solvable. The situation is completely different after perturbation of the lower-order terms with smooth, complex-valued, non-constant coefficients. For operators with double involute characteristics, microhypoellipticity could appear (i.e., no propagation of singularities arises) and the solution loses one unit regularly in the Sobolev spaces [4]. The microlocal Cauchy problem for a subclass of operators with involutive characteristics was studied in [11]. On the other hand, there exist such perturbations of the first (time) derivative of the Schrödinger operator that the new operator becomes (micro)-locally nonsolvable at the critical point ρ^0 and there exist solutions with stationary

singularity, i.e., $Pu = f \in C^\infty$, $WF(u) = \{\rho^0\}$ (see [12,13]). Applications of microlocal analysis in general relativity could be found in [14].

2. Mixed Problem for 1D and 2D Wave Equations with Oblique Derivative Data

Consider the following initial-boundary value problem with smooth coefficients for the string equation in a half strip, looking for a classical solution u .

$$\begin{cases} u_{tt} = u_{xx}, 0 \leq t, 0 \leq x \leq l, l = \text{const} > 0 \\ u|_{t=0} = \varphi_0(x), 0 \leq x \leq l \\ u_t|_{t=0} = \varphi_1(x), 0 \leq x \leq l \\ l_1(u) = \alpha_1(t)u_t + \beta_1(t)u_x + \gamma_1(t)u = h_1(t) \text{ on } x = 0, t \geq 0, \alpha_1^2 + \beta_1^2 > 0 \\ l_2(u) = \alpha_2(t)u_t + \beta_2(t)u_x + \gamma_2(t)u = h_2(t) \text{ on } x = l, t \geq 0, \alpha_2^2 + \beta_2^2 > 0. \end{cases} \quad (1)$$

We assume, moreover, that $\alpha_1(t) \neq \beta_1(t)$, $\alpha_2(t) \neq -\beta_2(t)$, $\forall t \geq 0$, i.e., the vector fields $l_1 \nparallel (1, +1)$, $l_2 \nparallel (1, -1)$ are non-characteristic. The data are smooth too.

The compatibility conditions for the existence of a classical solution u at the corner points of first and second order take the form:

$$\begin{cases} \alpha_1(0)\varphi_1(0) + \beta_1(0)\varphi_0'(0) + \gamma_1(0)\varphi_0(0) = h_1(0) \\ \alpha_2(0)\varphi_1(l) + \beta_2(0)\varphi_0'(l) + \gamma_2(0)\varphi_0(l) = h_2(0) \end{cases} \quad (2)$$

$$\begin{cases} \alpha_1'(0)\varphi_1(0) + \beta_1'(0)\varphi_0'(0) + \gamma_1'(0)\varphi_0(0) + \alpha_1(0)\varphi_0''(0) + \beta_1(0)\varphi_1'(0) + \\ \gamma_1(0)\varphi_1(0) = h_1'(0) \\ \alpha_2'(0)\varphi_1(l) + \beta_2'(0)\varphi_0'(l) + \gamma_2'(0)\varphi_0(l) + \alpha_2(0)\varphi_0''(l) + \beta_2(0)\varphi_1'(l) + \\ \gamma_2(0)\varphi_1(l) = h_2'(0) \end{cases} \quad (3)$$

etc.

Theorem 1. (a) The initial boundary value problem (1) (mixed problem) possesses a classical solution in $[0, \frac{l}{2}] \times [0, l]$ under several compatibility conditions imposed on the coefficients at the corner points $(t = 0, x = 0)$, $(t = 0, x = l)$ (see (14)–(16)). It is unique if $\gamma_1 \equiv \gamma_2 \equiv 0$. The global solution in $t \geq 0$ can be constructed inductively in the rectangle $[0, k\frac{l}{2}] \times [0, l]$, $k \geq 2$ under additional compatibility conditions imposed at $(t = (k-1)\frac{l}{2}, x = 0)$ and $(t = (k-1)\frac{l}{2}, x = l)$.

(b) Consider the initial-boundary value problem

$$\begin{cases} u_{tt} = \Delta_2 u \text{ in } M = \{t \geq 0\} \times K, K = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq L^2\} \\ u_x|_{\partial K \times [0, T)} = f(x, y, t), (x, y) \in \partial K, t \geq 0 \\ u|_{t=0} = \varphi_0(x, y), u_t|_{t=0} = \varphi_1(x, y), (x, y) \in K \end{cases} \quad (4)$$

equipped with the compatibility conditions $f(x, y, 0) = \frac{\partial \varphi_0}{\partial x}$ on $\partial K \times \{t = 0\}$, $f_t(x, y, 0) = \frac{\partial \varphi_1}{\partial x}$.

The boundary value problem (4) has infinitely many solutions.

Under the additional conditions $u(0, \pm L, t) = g_\pm(t)$, the initial boundary value problem (4), $u(0, \pm L, t) = g_\pm(t)$, $t \geq 0$ possesses a unique solution, $g_\pm(0) = \varphi_0(0, \pm L)$.

Therefore, (4) with $f = \varphi_0 = \varphi_1 = g_\pm = 0$ implies that the classical solution $u = 0$.

Below (Figure 1), we propose a geometric visualization of the results proposed in Theorem 1(b).

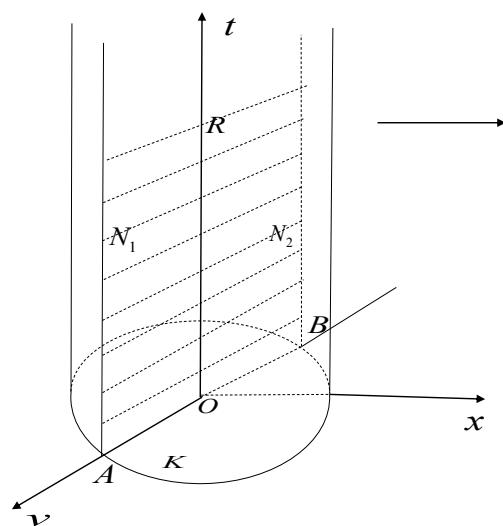


Figure 1. The vector field $\frac{\partial}{\partial x}$ is tangential to the generatrices AN_1 and BN_2 of the cylinder M . Extra data $g_{\pm}(t)$ are imposed on them.

Proof of Theorem 1. We shall first prove Theorem 1. To do this, we shall give several notations. The characteristics OA and AB , respectively, in the rectangle $Q = \{0 \leq x \leq l, 0 \leq t \leq \frac{l}{2}\}$ have the equations $x = t$ and $x + t = l$, respectively, while $A(\frac{l}{2}, \frac{l}{2})$, $B(l, 0)$. We introduce the triangles $\Delta_1 = \{0 \leq x \leq \frac{l}{2}, \frac{l}{2} \geq t \geq x\}$, $\Delta_2 = \{\frac{l}{2} \geq t \geq l - x, \frac{l}{2} \leq x \leq l\}$, $\Delta_3 = \{0 \leq t \leq x, 0 \leq x \leq \frac{l}{2}\} \cup \{\frac{l}{2} \leq x \leq l, 0 \leq t \leq l - x\}$, $\Delta_1 \cup \Delta_2 \cup \Delta_3 = Q$. The solution $u|_{\Delta_j} = u_j$, $j = 1, 2, 3$.

According to the D'Alembert formula

$$u_3(t, x) = \frac{\varphi_0(x+t) + \varphi_0(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \varphi_1(\lambda) d\lambda.$$

We write $u_3 = F_3(x-t) + G_3(x+t)$, where $F_3(z) = \frac{\varphi_0(z)}{2} + \frac{1}{2} \int_z^0 \varphi_1(\lambda) d\lambda$, $G_3(z) = \frac{\varphi_0(z)}{2} + \frac{1}{2} \int_0^z \varphi_1(\lambda) d\lambda$. Then, we shall find u_1 and u_2 . Evidently,

$$u_1|_{x=t} = u_3|_{x=t}, u_1|_{x+t=l} = u_3|_{x+t=l}. \quad (5)$$

The construction of u_1 in Δ_1 will be given in detail, but the considerations for u_2 will be omitted, as they are very similar.

As we know, u_1 has the form

$$u_1(t, x) = F_1(x-t) + G_1(x+t) \quad (6)$$

and $x-t \leq 0$ in Δ_1 with F_1, G_1 , the arbitrary smooth functions.

Applying (5), we conclude that

$$F_3(0) + G_3(2t) = \frac{\varphi_0(0)}{2} + G_3(2t) = F_1(0) + G_1(2t).$$

Put $z = 2t \in [0, l]$. Therefore,

$$G_1(z) = C + \frac{1}{2}(\varphi_0(z) + \int_0^z \varphi_1(\lambda) d\lambda),$$

where $C = \frac{\varphi_0(0)}{2} - F_1(0)$ and $u_1 = F_1(x-t) + G_3(x+t) + C$. Thus,

$$u_1|_{x=0} = F_1(-t) + G_3(t) + C, 0 \leq t \leq \frac{l}{2},$$

$$\frac{\partial u_1}{\partial x}|_{x=0} = F_1'(-t) + G_3'(t); \frac{\partial u_1}{\partial t}|_{x=0} = -F_1'(-t) + G_3'(t), \quad (7)$$

i.e.,

$$l_1(u_1)|_{x=0} = \alpha_1(t)(-F_1'(-t) + G_3'(t)) + \beta_1(t)(F_1'(-t) + G_3'(t)) + \gamma_1(t) \times \\ [C + F_1(-t) + G_3(t)] = h_1(t), 0 \leq t \leq \frac{l}{2}.$$

Consequently,

$$F_1'(-t)(\beta_1 - \alpha_1) + \gamma_1 F_1(-t) = h_1(t) - G_3'(\alpha_1 + \beta_1) - \gamma_1 G_3 - C\gamma_1. \quad (8)$$

Put $l_4(G_3) = (\alpha_1 + \beta_1)G_3' + \gamma_1 G_3$ and denote $w_1(t) = F_1(-t)$, $0 \leq t \leq \frac{l}{2} \Rightarrow w_1'(t) = -F_1'(t)$, $w_1''(t) = F_1''(-t)$. This way, we obtain a linear first-order ODE to the function $w_1(t)$, $0 \leq t \leq \frac{l}{2}$:

$$w_1'(\alpha_1 - \beta_1) + \gamma_1 w_1 = h_1 - l_4(G_3) - C\gamma_1, w_1|_{t=0} = F_1(0); \alpha_1 - \beta_1 \neq 0. \quad (9)$$

$F_1(0)$ will be a parameter in our study. So

$$w_1(t) = e^{-\int_0^t \frac{\gamma_1(s)ds}{\alpha_1(s) - \beta_1(s)}} [F_1(0) + \int_0^t e^{\int_0^s \frac{\gamma_1(\mu)d\mu}{\alpha_1(\mu) - \beta_1(\mu)}} \frac{h_1 - l_4(G_3) - C\gamma_1}{\alpha_1 - \beta_1} ds], 0 \leq t \leq \frac{l}{2}. \quad (10)$$

We conclude that in Δ_1 ,

$$u_1(t, x) = e^{-\int_0^{t-x} \frac{\gamma_1(s)ds}{\alpha_1(s) - \beta_1(s)}} [F_1(0) + \int_0^{t-x} e^{\int_0^s \frac{\gamma_1(\mu)d\mu}{\alpha_1(\mu) - \beta_1(\mu)}} \frac{h_1(s) - l_4(G_3)(s) - C\gamma_1(s)ds}{\alpha_1(s) - \beta_1(s)}] + \quad (11)$$

$$G_3(x+t) + \frac{\varphi_0(0)}{2} - F_1(0). \quad \square$$

Remark 2. In the special case when $\gamma_1 \equiv 0$, i.e., l_1 is a vector field in the plane

$$u_1 = \int_0^{t-x} \frac{h_1(s) - (\alpha_1 + \beta_1)G_3'}{\alpha_1(s) - \beta_1(s)} ds + \frac{\varphi_0(x+t) + \int_0^{x+t} \varphi_1(\lambda)d\lambda}{2} + \frac{\varphi_0(0)}{2}. \quad (12)$$

Thus, u_1 is uniquely determined in Δ_1 . From the construction of u_1 , we know that $u \in C^0(\Delta_1 \cup \Delta_3)$. We are interested when $u \in C^2(\Delta_1 \cup \Delta_3)$, i.e., when $\nabla u_1|_{x=t} = \nabla u_3|_{x=t}$ and $(\partial_t^2, \partial_x \partial_t, \partial_x^2)u_1|_{x=t} = (\partial_t^2, \partial_x \partial_t, \partial_x^2)u_3|_{x=t}$.

Having in mind that

$$\partial_x u_1|_{x=t} = F_1'(0) + G_3'(2t), \partial_t u_1|_{x=t} = -F_1'(0) + G_3'(2t),$$

$$\partial_x u_3|_{x=t} = F_3'(0) + G_3'(2t), \partial_t u_3|_{x=t} = -F_3'(0) + G_3'(2t),$$

we conclude that $u \in C^1$, near the characteristic line $x = t$ iff $F_1'(0) = F_3'(0) = \frac{\varphi_0'(0) - \varphi_1(0)}{2}$. As we know, $w_1(0) = F_1(0)$, $w_1'(0) = -F_1'(0)$. According to (9),

$$w_1'(0)(\alpha_1(0) - \beta_1(0)) + \gamma_1(0)w_1(0) = h_1(0) - l_4(G_3)(0) - \gamma_1(0)(\frac{\varphi_0(0)}{2} - F_1(0)),$$

i.e.,

$$w_1'(0)(\alpha_1(0) - \beta_1(0)) = h_1(0) - l_4(G_3)(0) + \frac{\gamma_1(0)\varphi_0(0)}{2}. \quad (13)$$

Then, the condition guaranteeing $u \in C^1$ near $x = t$ takes the form:

$$\frac{\varphi_0'(0) - \varphi_1(0)}{2} = \frac{h_1(0) - l_4(G_3)(0) + \frac{1}{2}\gamma_1(0)\varphi_0(0)}{\beta_1(0) - \alpha_1(0)}, \quad (14)$$

where

$$l_4(G_3)(0) = \frac{1}{2}(\alpha_1(0) + \beta_1(0))(\varphi_0'(0) + \varphi_1(0)) + \gamma_1(0)\varphi_0(0).$$

Concerning the coincidence of the second derivatives of u_1, u_3 on the characteristic $x = t$, we have the following condition:

$$F_1''(0) = F_3''(0) = w_1''(0) = \frac{1}{2}(\varphi_0''(0) - \varphi_1'(0)). \quad (15)$$

From (9), we get that

$$w_1''(0)(\alpha_1(0) - \beta_1(0)) + w_1'(0)(\alpha_1'(0) - \beta_1'(0)) + \gamma_1'(0)F_1(0) + \gamma_1(0)w_1'(0) = \frac{\partial}{\partial t}(h_1 - l_4(G_3))|_{t=0} - \gamma_1'(0)\left(\frac{\varphi_0(0)}{2} - F_1(0)\right),$$

i.e.,

$$w_1''(0)(\alpha_1(0) - \beta_1(0)) + w_1'(0)(\alpha_1'(0) - \beta_1'(0) + \gamma_1(0)) = h_1'(0) - \frac{\partial}{\partial t}\left[\frac{1}{2}(\alpha_1(t) + \beta_1(t))(\varphi_0'(t) + \varphi_1(t)) + \gamma_1(t)G_3(t)\right]|_{t=0} - \frac{1}{2}\gamma_1'(0)\frac{\varphi_0(0)}{2}, \quad (16)$$

$w_1'(0)$ given by (13), $G_3(t) = \frac{1}{2}\varphi_0(t) + \frac{1}{2}\int_0^t \varphi_1(\lambda)d\lambda$, $G_3' = \frac{1}{2}(\varphi_0' + \varphi_1)(t)$.

Conditions (14) and (15), with $w_1''(0), w_1'(0)$, taken from (13) and (16), imply that

$$u = \begin{cases} u_1 \in \Delta_1 \\ u_3 \in \Delta_3 \end{cases}$$

is C^2 smooth in $\Delta_1 \cup \Delta_3$.

The construction of u_2 in Δ_2 is the same as the construction of u_1 in Δ_1 . Conditions similar to (14) and (15) guarantee the existence of a smooth solution of (1) in $\{0 \leq x \leq l\} \times \{0 \leq t \leq \frac{l}{2}\}$. We repeat the above considerations in the rectangle $\{0 \leq x \leq l\} \times \{\frac{l}{2} \leq t \leq l\}$ and find (under corresponding regularity conditions of the type (14) and (15) imposed at the corner points $A_1(\frac{l}{2}, 0), B_1(\frac{l}{2}, l)$) the smooth solution u in this rectangle.

We shall illustrate Theorem 1 by Example 1.

Example 1. We shall find the unique classical solution of (1) in $\{0 \leq t \leq \frac{l}{2}\} \times \{0 \leq x \leq l\}$ for $\varphi_0 \equiv \varphi_1 \equiv 0$ and $\gamma_1 \equiv \gamma_2 \equiv 0$. Then, $u_3 \equiv 0$, while $u_1(t, x) = \int_0^{t-x} \frac{h_1(s)ds}{\alpha_1(s) - \beta_1(s)}$ in Δ_1 , $u_2 = \int_0^{x+t-l} \frac{h_2(s)ds}{\alpha_2(s) + \beta_2(s)}$ in Δ_2 ($x + t \geq l$ in Δ_2). Evidently, $u_1|_{t=x} = 0$, $u_2|_{t+x=l} = 0$. One can easily guess that $u_1 \in C^2(\Delta_1)$ if $h_1(0) = h_1'(0) = 0$ and $u_2 \in C^2(\Delta_2)$ if $h_2(0) = h_2'(0) = 0$. In fact,

$$\partial_{t,x}u_2 = \frac{h_2(t+x-l)}{\alpha_2(t+x-l) + \beta_2(t+x-l)} \Rightarrow \partial_{t,x}u_2|_{x+t=l} = \frac{h_2(0)}{\alpha_2(0) + \beta_2(0)}, \alpha_2(0) + \beta_2(0) \neq 0,$$

etc.

Remark 3. It is interesting to study (1) when $\alpha_1(t) \frac{\partial}{\partial t} + \beta_1 \frac{\partial}{\partial x} (\alpha_2(t) \frac{\partial}{\partial t} + \beta_2 \frac{\partial}{\partial x})$ has a characteristic direction at some point $t_0 \geq 0$, i.e., $(\alpha_1(t_0), \beta_1(t_0)) \parallel (1, 1)$ $((\alpha_2(t_0), \beta_2(t_0)) \parallel (1, -1))$. Further on, we may suppose that $t_0 = 0$, while $\alpha_1(t), \beta_1(t) \not\parallel (1, 1)$ for each $t > 0$.

Example 2. Let $\gamma_1 \equiv 0, \alpha_1(t) = a = \text{const} > 0, \beta_1 = t^k + a, k \in \mathbf{N}$. Then, (9) takes the form

$$-t^k w_1'(t) = h_1(t) - (2a + t^k) G_3'(t) \equiv H_1(t), w_1(0) = F_1(0), G_3'(t) = \frac{1}{2}(\varphi_0'(t) + \varphi_1(t)). \quad (17)$$

Equation (17) possesses a smooth solution for $t \geq 0$ if $H_1(t)$ vanishes of order $r \geq k$ at $t = 0$. Thus,

$$w_1(t) = F_1(0) - \int_0^t \frac{h_1(s) - G_3'(s)(2a + s^k)}{s^k} ds.$$

$w_1(t)$ is smooth if $h_1(s) - 2aG_3'(s) = O(s^k), s \rightarrow +0 \Rightarrow$

$$w_1(t) = F_1(0) - \int_0^t \frac{h_1(s) - 2aG_3'(s)}{s^k} ds + G_3(t) - \frac{1}{2}\varphi_0(0).$$

Example 3. We shall now analyze the case $\gamma_1 \equiv A = \text{const} \neq 0, \alpha_1(t) = a > 0, \beta_1(t) = a + t^k$. Evidently, from (9), we have

$$-t^k w_1'(t) + Aw_1(t) = h_1(t) - (2a + t^k) G_3'(t) - A(G_3 + C), w_1(0) = F_1(0).$$

The change $\tilde{w}_1(t) = w_1(t) - F_1(0)$ in the previous equation leads to

$$t^k \tilde{w}_1'(t) - A\tilde{w}_1 = -h_1(t) + (2a + t^k) G_3'(t) + A(G_3(t) + \frac{\varphi_0(0)}{2}) \equiv g_1(t), \tilde{w}_1(0) = 0; t \geq 0. \quad (18)$$

Thus, if $\tilde{w}_1 \in C^1(t \geq 0) \Rightarrow g_1(0) = 0$.

If we are looking for C^∞ or an analytic solution of (18), we can apply the theory of Briot–Bouquet. The simplest case is when $k = 1$. Equation (18) is written as

$$\begin{cases} ty'(t) = Ay + f(t) \\ y(0) = 0 \end{cases} \quad (19)$$

where the series $f(t) = \sum_1^\infty a_n t^n$ is convergent near 0. One can easily see that if $A > 0$ and $A \notin \mathbf{N}$, there exists a unique analytic solution of (19): $y(t) = \sum_{n=1}^\infty a_n \frac{t^n}{n-A}$. If $A = n_0 \in \mathbf{N}$, we assume that $a_{n_0} = 0$. Then, (19) has infinitely many analytic solutions near 0, namely: $y = Dt^{n_0} + \sum_{1 \leq n \neq n_0} \frac{a_n t^n}{n-n_0}, D = \text{const}$.

Consider now the initial boundary value problem (4). As the vector field $\frac{\partial}{\partial x}$ is tangential to M along the generatrices of the cylinder $M : N_1 = \{(0, L) = A\} \times \{t \geq 0\}$, and $N_2 = \{(0, -L) = B\} \times \{t \geq 0\}$, we expect some difficulties to appear.

Put $v(t, x, y) = \frac{\partial u}{\partial x}(t, x, y)$ for $t > 0$ and differentiate the wave Equation (4). The function v satisfies the Dirichlet-type initial boundary value problem

$$\begin{cases} \square_2 v = 0 \text{ in } M \\ v|_{\partial K \times [0, T)} = f(x, y, t), (x, y) \in \partial K, t \geq 0 \\ v|_{t=0} = \frac{\partial \varphi_0}{\partial x}(x, y), v_t|_{t=0} = \frac{\partial \varphi_1}{\partial x}(x, y) \text{ in } K. \end{cases} \quad (20)$$

Under several compatibility conditions, (20) has a unique solution v . Consequently, $u = \int_0^x v(t, s, y) ds + w(t, y)$. Below, we obtain a mixed boundary value problem for $w(t, y)$ in the half strip $R = \{0 \leq t\} \times \{|y| \leq L\} \subset \mathbf{R}_{t,y}^2$. There are no difficulties in verifying that

$$\begin{cases} \square_1 w = v_x(t, 0, y) \text{ in } R \\ w|_{t=0} = \varphi_0(0, y), w_t|_{t=0} = \varphi_1(0, y), |y| \leq L \end{cases} \quad (21)$$

$$u(0, 0, \pm L) = w(0, \pm L).$$

Equation (21) is evidently an undetermined boundary value problem, as we do not have any information about $w|_{y=\pm L, t \geq 0}$. That is why we prescribe extra conditions $u|_{N_1} = g_+(t)$, $u|_{N_2} = g_-(t) \Rightarrow w(t, \pm L) = u|_{x=0}(t, \pm L) = g_{\pm}$. Therefore,

$$\left\{ \begin{array}{l} \square_1 w = w_{1tt} - w_{1yy} = v_x(t, 0, y) \\ w|_{N_1} = g_+(t), w|_{N_2} = g_-(t) \\ w|_{t=0} = \varphi_0(0, y), w_t|_{t=0} = \varphi_1(0, y), |y| \leq L. \end{array} \right. \quad (22)$$

Under standard compatibility conditions imposed in the corner points $(0, L)$, $(0, -L)$, (22) possesses a unique classical solution $w(t, y)$. This way, we conclude that the initial boundary value problem (4) has an infinite-dimensional kernel. By putting two extra conditions on the generatrices of the cylinder M where the boundary vector field is tangential to ∂M , we obtain a unique classical solution of (4). Certainly, (20) and (22) must be equipped with appropriate compatibility conditions and then $u(t, x, y) = \int_0^x v(t, s, y) ds + w(t, y)$.

3. Cauchy Problem for the Lorentzian Eikonal Equation

In this section, we will find a classical solution to the following Cauchy problem for the Lorentzian eikonal equation:

$$\left\{ \begin{array}{l} p^2 - q^2 = -1, u = u(x_1, x_2), p = \frac{\partial u}{\partial x_1}, q = \frac{\partial u}{\partial x_2} \\ u|_{\gamma} = 1, \gamma : x_1 = \alpha, x_2 = \frac{\alpha^2}{2}, \text{i.e., } x_2 = x_1^2/2. \end{array} \right. \quad (23)$$

We are looking for a solution inside the parabola $\gamma : x_2 \geq \frac{1}{2}x_1^2$.

Theorem 2. *There exists a unique classical solution of (23) in the domain $D = \{0 \leq x_2 \leq 1/2\} \cap \{x_2 \geq \frac{x_1^2}{2}\}$. u is constructed via the characteristics of the corresponding Hamiltonian system. They are straight lines in $0x_1x_2$ and form a cusp-type envelope P located outside γ but P is tangential to γ at two points: $(\pm 1, 1/2)$. The characteristics are not crossing in D .*

Proof of Theorem 2. Let us now prove Theorem 2. According to the classical method of characteristics (Cauchy) $u = u(x_1(t, \alpha), x_2(t, \alpha))$, the bicharacteristics $(x_1(t, \alpha), x_2(t, \alpha); p(t, \alpha), q(t, \alpha))$ satisfy the Hamiltonian system of ODE in $\mathbf{R}^4$:

$$\left\{ \begin{array}{l} \dot{x}_1 = 2p, x_1(0) \equiv x_1(0, \alpha) = \alpha \\ \dot{x}_2 = -2q, x_2(0) \equiv x_2(0, \alpha) = \frac{\alpha^2}{2} \\ \dot{p} = 0, p^0(\alpha) \equiv p(0) \Rightarrow p(t, \alpha) \equiv p^0(\alpha) \\ \dot{q} = 0, q^0(\alpha) \equiv q(0) \Rightarrow q(t, \alpha) \equiv q^0(\alpha) \end{array} \right. \quad (24)$$

$$u = u(x_1(t, \alpha), x_2(t, \alpha)) \Rightarrow \dot{u} = 2(p^2 - q^2) = 2((p^0)^2 - (q^0)^2), u(0) = u(\alpha, \frac{\alpha^2}{2}) = 1 \\ \Rightarrow \frac{\partial}{\partial \alpha} u(0) = p^0(\alpha) + \alpha q^0(\alpha) = 0.$$

The characteristic (x_1, x_2) is not collinear to $\dot{\gamma} = (1, \alpha)$ at $t = 0$ if $(\dot{x}_1(0), \dot{x}_2(0)) \nparallel (1, \alpha)$, i.e., if $q^0 \neq 0$, $(p^0, -q^0) \nparallel (1, \alpha) \iff \alpha^2 - 1 \neq 0$. So, we will assume that $|\alpha| < 1$; $\alpha = 0 \Rightarrow$, and the characteristic curve is orthogonal to the parabola γ only at 0, while on $\gamma, 0 < |\alpha| < 1 \Rightarrow |x_1| < 1, 0 < x_1 < 1/2$.

Solving the system

$$\left\{ \begin{array}{l} (p^0)^2 - (q^0)^2 = -1 \\ p^0 + \alpha q^0 = 0 \end{array} \right.$$

we find

$$\begin{cases} q^0(\alpha) = \frac{\rho}{\sqrt{1-\alpha^2}}t, \rho^2 = 1, |\alpha| < 1, q^0 \neq 0 \\ p^0(\alpha) = -\frac{\rho\alpha}{\sqrt{1-\alpha^2}} \neq 0 \text{ for } \alpha \neq 0. \end{cases} \quad (25)$$

Equation (25) implies that the characteristics Γ_α are straight lines,

$$\Gamma_\alpha : \begin{cases} x_1(t, \alpha) = 2p^0t + \alpha = \alpha(1 - \frac{2\rho t}{\sqrt{1-\alpha^2}}) \\ x_2(t, \alpha) = -2q^0t + \frac{\alpha^2}{2} = -\frac{2\rho t}{\sqrt{1-\alpha^2}} + \frac{\alpha^2}{2}, \end{cases} \quad (26)$$

$$\dot{u} = 2((p^0)^2 - (q^0)^2) = -2, u(0) = 1 \Rightarrow$$

$$u = -2t + 1. \quad (27)$$

Γ_0 is parallel to $\overrightarrow{0x_2}$.

We eliminate t in (26) for $\alpha \neq 0, 0 < |\alpha| < 1$. Thus, in the plane Ox_1x_2

$$\Gamma_\alpha : x_2 = \frac{x_1}{\alpha} + \frac{\alpha^2}{2} - 1 \text{ for } 0 < |\alpha| < 1, |x_1| < 1; \quad (28)$$

$$\alpha = 0 \Rightarrow \Gamma_0 : \begin{cases} x_1 = 0 \\ x_2 = -2\rho t. \end{cases}$$

Evidently, Γ_α is monotonically increasing for $1 > \alpha > 0$ and monotonically decreasing for $\alpha < 0, x_1 = 0 \Rightarrow \frac{\alpha^2}{2} - 1 = x_2 \in [-1, -1/2)$. If $tg\Theta = \frac{1}{\alpha}, 1 > \alpha > 0$ (slope angle), then $\frac{\pi}{4} < \Theta < \frac{\pi}{2}$. On the other hand, from (28), it follows that $x_2 = 0 \Rightarrow x_1 = \alpha - \frac{\alpha^3}{2}$ for $0 < \alpha < 1$. Consequently, $x_1 \in (0, (\frac{2}{3})^{3/2}) \subset (0, 1)$. The curve $\Gamma_\alpha, \alpha > 0$ is located symmetrically to $\Gamma_{-\alpha}$ with respect to $\overrightarrow{0x_2}$. Our next step is to find the envelope P of the family Γ_α . Therefore,

$$P : \begin{cases} x_2 = \frac{x_1}{\alpha} + \frac{\alpha^2}{2} - 1, \alpha \neq 0 \\ 0 = -\frac{x_1}{\alpha} + \alpha. \end{cases}$$

$$\text{Evidently, } P : \begin{cases} x_1 = \alpha^3 \\ x_2 = \frac{3}{2}\alpha^2 - 1, \end{cases} \text{ i.e.,}$$

$$P : x_2 = -1 + \frac{3}{2}x_1^{2/3}; x_2 = 0 \Rightarrow x_1 = \pm(\frac{2}{3})^{3/2}, x_1 = \pm 1 \rightarrow x_2 = \frac{1}{2}. \quad (29)$$

To fix the ideas ($x_1^{2/3}$ is an even function), we shall assume that $\alpha \in (0, 1); x_2'(1) = 1, \gamma : x_2 = \frac{x_1^2}{2} \Rightarrow x_2'(1) = 1$.

We conclude that the envelope P is tangential to γ at $(1, 1/2)$ and is strictly below γ for $0 \leq x_1 < 1$. In fact, $\frac{x_1^2}{2} > -1 + \frac{3}{2}x_1^{2/3}$ in $[0, 1)$ (after the change $\mu = x_1^{2/3} \Rightarrow \mu^3 - 3\mu + 2 > 0 \iff (\mu - 1)^2(\mu + 2) > 0$). P is located outside D . As Γ_α is not crossing Γ_0 in D and $\Gamma_{\alpha_1}, \alpha_1 > 0$ is not crossing Γ_{α_2} , for $\alpha_2 < 0$, we are interested whether $\Gamma_{\alpha_1} \cap \Gamma_{\alpha_2} \cap D \neq \emptyset$ for some $\alpha_1 \neq \alpha_2, \alpha_1 > 0, \alpha_2 > 0$.

Take for $\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_1 \neq \alpha_2, \alpha_1 > 0, \alpha_2 > 0$ the corresponding characteristics $\Gamma_{\alpha_1}, \Gamma_{\alpha_2}$:

$$\begin{cases} x_1 = \frac{x_1}{\alpha_1} + \frac{\alpha_1^2}{2} - 1, 1 > \alpha_1 > 0 \\ x_2 = \frac{x_1}{\alpha_2} + \frac{\alpha_2^2}{2} - 1, 1 > \alpha_2 > 0, \alpha_1 \neq \alpha_2. \end{cases} \quad (30)$$

The system (30) has the solutions

$$\begin{cases} x_1 = \frac{1}{2}\alpha_1\alpha_2(\alpha_1 + \alpha_2) \in (0, 1) \\ x_2 = \frac{1}{2}(\alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2 - 2) < 1. \end{cases} \quad (31)$$

The point $(x_1, x_2) \in \text{int}D$ (interior of D) iff

$$x_2 > \frac{x_1^2}{2} \iff \alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2 - 2 > \frac{1}{4}\alpha_1^2\alpha_2^2(\alpha_1 + \alpha_2)^2, \quad (32)$$

$0 < \alpha_1 \neq \alpha_2 < 1$. To study inequality (32), we denote

$$\begin{cases} \alpha_1 + \alpha_2 = -p, p \in (-2, 0) \\ \alpha_1\alpha_2 = q \in (0, 1) \end{cases} \quad (33)$$

and make the change (33) in (31). Therefore,

$$4(p^2 - q) > 8 + q^2p^2, p \in (-2, 0), q \in (0, 1). \quad (34)$$

Equation (34) can be rewritten as $p^2 > \frac{4}{2-q}, q \in (0, 1)$, i.e.,

$$-2 < p < \frac{-2}{\sqrt{2-q}}, 0 < q < 1. \quad (35)$$

Exercise. Let α_1, α_2 be the roots of the quadratic equation $z^2 + pz + q = 0$. The latter possesses simple roots $\alpha_1, \alpha_2 \in (0, 1)$ iff $p^2 - 4q > 0, -2 < p < 0, 0 < q < 1, 1 + p + q > 0$. Sufficiency: $p^2 - 4q > 0 \Rightarrow \alpha_1 \neq \alpha_2$ are simple real roots and $\alpha_1\alpha_2 = q > 0 \Rightarrow \alpha_1, \alpha_2$ have the same sign. $\alpha_1 < 0, \alpha_2 < 0 \Rightarrow p > 0$ (absurd), $0 < \alpha_1 < 1 < \alpha_2 \Rightarrow p + q + 1 < 0$, $0 < \alpha_1 < \alpha_2 = 1 \Rightarrow p + q + 1 = 0$, $\alpha_1 > \alpha_2 \geq 1 \Rightarrow q > 1$. Therefore, $0 < \alpha_1 < \alpha_2 < 1 \Rightarrow p + q + 1 > 0$. More precisely, we draw the graphs of the functions $p = -\frac{2}{\sqrt{2-q}}, p = -q - 1, p = -2\sqrt{q}$ for $0 \leq q \leq 1, -2 \leq p \leq 0$ and verify that they cross only at $(-2, 1)$. Then, from the exercise and the latter fact, we conclude that the algebraic inequality (32) does not hold for $\alpha_1 \neq \alpha_2 \in (0, 1)$, i.e., Γ_{α_1} and Γ_{α_2} are not crossing inside D and so the solution u is single-valued there. In the special case, $q = 1, p = -2 \Rightarrow \alpha_1 = \alpha_2 = 1$ and (32) become equal. If $\alpha_1 = \alpha_2 \in (0, 1) \Rightarrow 0 > \alpha^6 - 3\alpha^2 + 2 \iff 0 > (1 - \alpha^2)^2(\alpha^2 + 2)$ —absurd. The proof is completed. $\square$

4. Singularities of the Solutions of Degenerate Hyperbolic Equations with Constant Coefficients in $\mathbf{R}^4, \mathbf{R}^8$

We shall now study the propagation of singularities, and we remind the definition and several properties of the wave front set of Schwartz distributions $u \in D'(\Omega)$ (see [3,15,16]). Denote by $T^*(\Omega) \setminus 0$ the cotangent bundle of the domain $\Omega \subset \mathbf{R}^n, (x, \xi) \in T^*(\Omega) \setminus 0 \iff x \in \mathbf{R}^n, \xi \neq 0, \xi \in \mathbf{R}^n$.

Definition 1. $(x_0, \xi^0) \notin WF(u)$ if there exist $\varphi \in C_0^\infty(\mathbf{R}^n), \varphi \equiv 1$ near x_0 and $\varepsilon > 0$ such that $\varphi u(\xi) = O(|\xi|^{-N}), \forall N \in \mathbf{N}$. Here, $|\frac{\xi}{|\xi|} - \frac{\xi^0}{|\xi^0|}| < \varepsilon$. Evidently, WF is a conical set with respect to ξ . As usual, $\hat{u}$ stands for the Fourier transform of u .

Put $D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}$ and $D_x \leftrightarrow \xi$ as dual variables to x . (1) The projection of $WF(u)$ with respect to x coincides with $\text{singsupp}u$; $\text{singsupp}u$ is the smallest closed set outside of which $u \in C^\infty$, (2) $a \in C_0^\infty \Rightarrow WF(au) \subset WF(u)$, (3) $WF(u + v) \subset WF(u) \cup WF(v), u, v \in D',$

(4) $WF(P(x, D)u) \subset WF(u) \subset WF(Pu) \cup CharP$, where $P(x, D)$ is arbitrary differential or pseudodifferential operator with C^∞ coefficients of order m with symbol p ,

$$p \sim p_m^0(x, \xi) + p_{m-1}(x, \xi) + \dots + p_{m-j}(x, \xi) + \dots,$$

where $p_{m-j}(x, t\xi) = t^{m-j}p_{m-j}(x, \xi), \forall t > 0$ (see [3]). P is micro-hypoelliptic if $WF(Pu) = WF(u)$. Each elliptic operator at $\rho^0 = (x_0, \xi^0 \neq 0)$, i.e., $p_m^0(\rho^0) \neq 0$ is micro-hypoelliptic. p_m^0 is invariant with respect to the homogeneous canonical transformations $\kappa : T^*(\Omega) \setminus 0 \rightarrow T^*(\tilde{\Omega}) \setminus 0$, i.e., non-degenerate transformations which conserve the Poisson bracket $\{f, g\}(x, \xi) = \sum_1^n (\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j})$.

Let $p_m^0(\rho)$ be real-valued and $p_m^0(\rho^0) = 0, \nabla_{x, \xi} p_m^0(\rho^0) \neq 0$, i.e., p_m^0 possesses simple characteristics at ρ^0 . p_m^0 is called a symbol of real principal type. Let $m = 1$ and $a \equiv p_1^0$. Consider the Hamiltonian integral curve of a through ρ^0 (the so-called 0 bicharacteristic):

$$\mathcal{L}_a \left| \begin{array}{l} \dot{x} = \nabla_{\xi} a, a(x(0), \xi(0)) = a(\rho^0) = 0 \\ \dot{\xi} = -\nabla_x a. \end{array} \right.$$

The manifold $CharP = \{p_1^0(\rho) = 0\}$, $codim CharP = 1$, foliates near ρ^0 to non-crossing 0 bicharacteristics $\mathcal{L}_a$ of a .

It is well known that $WF(u) \setminus WF(Pu) \subset CharP$ is invariant under the hamiltonian flow. Roughly speaking, singularities propagate along the bicharacteristic curves (rays).

Other useful results from the microlocal analysis are the following ones: (5) for $a \in D'(X), b \in D'(Y)$

$$WF(a \otimes b) \subset (WF(a) + WF(b)) \cup ((supp a \times \{0\}) \times WF(b)) \cup (WF(a) \times (supp b \times \{0\})) \quad (36)$$

$$WF(a * b) \subset \{(x + y, \xi) : (x, \xi) \in WF(a), (y, \xi) \in WF(b)\} \quad (37)$$

(see [3] and specially Chapter VIII).

Certainly, it is assumed that the convolution $a * b$ exists (say, a or b is compactly supported).

Evidently, $WF(\delta(y)) = T_0^*(\mathbf{R}_y^n \setminus 0)$. Let $\mathcal{E} \in \mathcal{J}'(\mathbf{R}^n)$ be the fundamental solution of $P(D_x)\mathcal{E} = \delta(x)$. Then,

$$P(D_x)u = f \in \mathcal{J}'(\mathbf{R}^n) \Rightarrow u = \mathcal{E} * f + KerP.$$

We point out that if $\square_n = u_{tt} - \Delta_n u$ is the wave operator and Δ_n is the Laplace operator, then the study of $\square_n + \sum_{j=1}^n a_j \frac{\partial}{\partial x_j} + b_1 \frac{\partial}{\partial t} + c$ and $\Delta_n + \sum_1^n a_j \frac{\partial}{\partial x_j} + c, a_j, b_1, c$, respectively—constants after the change—and $u = ve^{<A, x> + b_1 t}$ and $u = ve^{<A, x>}$, respectively, is reduced to the investigation of $\square_n + \tilde{c}_1, \Delta_n + \tilde{c}_2, \tilde{c}_{1,2}$ —constants. The change $y_1 = x_1 + x_2, y_2 = x_1 - x_2$ in $\mathbf{R}_x^2 : \frac{\partial^2}{\partial x_1 \partial x_2} \rightarrow \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}$.

To simplify the formulation, we shall deal only with $n = 3$ case —3 space variables. Then, $\square_3 : u_{tt} = \Delta_3 u$ has the fundamental solution

$$\mathcal{E}_3 = \frac{\Theta(t)}{4\pi t} \delta_{S_t}, \quad (38)$$

where δ_{S_t} is a delta function on the sphere: $S_t : t = r = \sqrt{x^2 + y^2 + z^2}$, $singsupp \mathcal{E}_3 = \{t = r\} = supp \mathcal{E}_3$, $\Theta(t)$ is the Heaviside function.

Concerning the Klein–Gordon operator $\square_3 - k^2, k > 0$ the fundamental solution

$$\mathcal{E}_{3,-} = \frac{1}{4\pi} \left(\frac{\Theta(t)}{t} \delta_{S_t} - \Theta(t-r) \frac{k I_1(k\sqrt{t^2 - r^2})}{\sqrt{t^2 - r^2}} \right), \quad (39)$$

where the Bessel function I_1 of purely imaginary argument (see [1]) is odd: $I_1(z) = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \left(\frac{z}{2}\right)^{2n+1}$, $I_1(0) = 0$. Therefore, $\frac{I_1(k\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} \in C^\infty$,

$$\text{singsupp}\mathcal{E}_{3,-} = \{t = r\}, \text{supp}\mathcal{E}_{3,-} = \{t \geq r\},$$

see [17].

The operator $\square_3 + k^2$, $k > 0$ possesses the fundamental solution

$$\mathcal{E}_{3,+} = \frac{1}{4\pi} \left(\frac{\Theta(t)}{t} \delta_{S_t} - k \frac{J_1(k\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} \Theta(t-r) \right), \quad (40)$$

Ref. [17], where the odd Bessel function $J_1(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{2n+1}}{n!(n+1)!} (-1)^n$ (see [1]). Thus, $\text{singsupp}\mathcal{E}_{3,+} = \{t = r\}$, $\text{supp}\mathcal{E}_{3,+} = \{t \geq r\}$. For the Laplace operator, things are simpler in its fundamental solution: $\mathcal{E}_n : WF(\mathcal{E}_n) = \{(0, \mathbf{R}_\xi^n \setminus 0)\}$. In fact, Δ_n is elliptic and therefore a hypoelliptic operator.

We shall study the behavior of our operators in $(\mathbf{R}_t^1 \times \mathbf{R}_x^3) \times \mathbf{R}_{(\tilde{t},y)}^4$. Then, the corresponding fundamental solutions are: $\tilde{\mathcal{E}} = \mathcal{E} \otimes \delta(\tilde{t}, y)$, $y \in \mathbf{R}^3$, $\tilde{t} \in \mathbf{R}^1$, $x \in \mathbf{R}^3$, $\tilde{\tau} \in \mathbf{R}^1$. We shall study the singularities on the critical set

$$\text{CrP} = \{(\tau = \xi = 0; (\tilde{\tau}, \eta) \neq 0)\}, \quad (41)$$

i.e., $Pu = f$, $\rho^0 \in WF(f)$, $\rho^0 = (t^0, x^0, \tilde{t}^0, y^0; \tau = \xi = 0, (\tilde{\tau}, \eta) \neq 0)$. The principal symbol of the wave equation is $\tau^2 - |\xi|^2$.

Theorem 3. (a) Consider the operator $\square_3 + k^2$, $k > 0$ in $\mathbf{R}^8$ near the point ρ^0 (41), $Pu = f$. Then, $WF(u) \subset \{(t, x) : |t - t_0|^2 \geq |x - x_0|^2, \tilde{t} = \tilde{t}^0, y = y^0, \tau = \xi = 0, (\tilde{\tau}^0, \eta^0) \neq 0\}$, i.e., singularities propagate in the solid cone $(t - t_0)^2 \geq |x - x_0|^2$ with vertex at (t_0, x_0) . In the special case $k = 0$ (wave equation), the singularities propagate along the conical surface with vertex at (t_0, x_0) .

(b) In the elliptic case, the singularities of u propagate along the space $\{x + x_0, x \in \mathbf{R}^n\}$, $y = y_0$, $\tilde{\zeta}^0 = 0$, $\eta = \eta^0 \neq 0$ if $\rho^0 = (x_0, y_0; \xi = 0, \eta^0 \neq 0)$.

Remark 4. Assume that p_2^0 has a zero of sharp order 2 at the involutive manifold $\Sigma = \{u_1(\rho) = v_1(\rho) = 0, \nabla u_1(\rho^0), \nabla v_1(\rho^0) \text{ linearly independent}, u_1(\rho^0) = v_1(\rho^0) = 0, \{u_1, u_2\}(\rho) = 0\}$. Thus, $p_2^0 = a_{11}u_1^2 + 2a_{12}u_1v_1 + a_{22}v_1^2$ near ρ^0 , $a_{12}^2 - a_{11}a_{22} < 0$, $a_{11} > 0$, $p_2|_\Sigma = 0$, while p_2^0 is > 0 outside Σ . According to the Frobenius theorem, Σ near ρ^0 foliates at two-dimensional integral manifold M_ρ of the linearly independent Hamiltonian fields, $[H_{u_1}, H_{v_1}] = 0$ on Σ .

If the condition of Levi is satisfied, i.e., the first-order term p_1 in the full symbol development is $p_1 = Au_1 + Bv_1$, then the singularities starting from ρ^0 propagate along the leave M_ρ . In the case when the Levi condition is not fulfilled, everything depends on the subcritical symbol $p_1'|_\Sigma$ ($p_1'|_\Sigma$ is the canonical invariant) (see [4]).

There are two different possibilities: 1) $p_1'|_\Sigma$ is away from the axis, $\{x < 0\} \Rightarrow P$ is hypoelliptic with a loss of 1 derivative near ρ^0 ; 2) p_1' takes negative value for some ρ^0 . Then, B. de Monvel found some f such that $WFf = \{\rho^0\}$, $Pf \in H_{loc}^\varepsilon$ for some $\varepsilon > 0$ but $f \notin H_{loc}^1$ (P micro-hypoelliptic with loss of 1 derivative would imply that $f \in H_{loc}^{1+\varepsilon}$) [4].

Example 4. The Schrödinger operator in $\mathbf{R}^{n+1}$: $-P_2 = iu_t + \Delta_n u$ with symbol $|\xi|^2 + \tau$, $\xi \in \mathbf{R}^n$, $\tau \in \mathbf{R}^1$ is (a) microelliptic at $\rho^0 = (t_0, x_0; \xi^0 \neq 0, \tau^0)$ and therefore it does not propagate singularities; (b) $\xi^0 = 0 \Rightarrow \tau^0 > 0$, P is micro-hypoelliptic ($|\xi^0| + \tau^0 > 0$), and again there is no propagation of singularities; (c) if $\tau^0 < 0$, we use the fact that the fundamental solution $\mathcal{E}_n(t, x)$

of P_2 is such that $\text{singsupp}\mathcal{E}_n = \{(t = 0, x), x \in \mathbf{R}^n\}$, and therefore, $WF(\mathcal{E}_n) = \{(0, x; \xi, \tau), x \in \mathbf{R}^n, (\xi, \tau) \in \mathbf{R}^{n+1} \setminus \{0\}\}$. Take $f \in \mathcal{E}'$, such that $\rho^0 = (t_0, x_0; \xi^0 = 0, \tau^0 > 0) \in WF(f)$. One can easily see that $WF(\mathcal{E}_n * f)$ propagates along the set $\{(t = t_0, x + x_0, \forall x \in \mathbf{R}^n; \xi^0 = 0, \tau^0 < 0)\}$. Certainly, $\mathcal{E}_n(t, x) = -i \frac{\Theta(t)}{(2\sqrt{\pi t})^n} e^{i(\frac{|x|^2}{4t} - \frac{n\pi}{4})}$ [17].

Proof of Theorem 3. The proof of Theorem 3 is based on the inclusions (36) and (37). In fact, take the fundamental solution $\mathcal{E}_{3,\pm}$ of the Klein–Gordon equation in $\mathbf{R}_t^1 \times \mathbf{R}_x^3$, $x = (x_1, x_2, x_3)$, $k \neq 0$, and $(\tau, \xi) \in \mathbf{R}^4 \setminus 0$ are the dual variables. Let $(\tilde{t}, y) \in \mathbf{R}^1 \times \mathbf{R}^3$ and $(\tilde{\tau}, \eta)$ be the dual variables to $(\tilde{t}, y)$. In $\mathbf{R}^4 \times \mathbf{R}^4$ $(\square_3 \pm k^2 \mathcal{E}_{3,\pm}) \otimes \delta(\tilde{t}, y) = \delta(t) \otimes \delta(x) \otimes \delta(\tilde{t}) \otimes \delta(y)$, i.e., $\tilde{\mathcal{E}}_{3,\pm} = \mathcal{E}_3(t, x) \otimes \delta(\tilde{t}, y)$ is the fundamental solution of $\square_3 \pm k^2$ in $\mathbf{R}^8$. In $\mathbf{R}^4 : WF(\mathcal{E}_{3,\pm}) = \{((t, x) : t = |x|; (\tau, \xi) : \tau^2 = |\xi|^2 > 0)\}$.

According to (36),

$$WF(\tilde{\mathcal{E}}_{3,\pm}) \subset \{((t, x, \tilde{t} = 0, y = 0) : t = |x|; (\tau, \xi, \tilde{\tau}, \eta) : \tau^2 = |\xi|^2 > 0)\} \cup$$

$$\{(t = 0, x = 0, \tilde{t} = 0, y = 0); \mathbf{R}_{(\tilde{\tau}, \eta)}^4 \times \mathbf{R}_{(\tau, \xi)}^4 \setminus \{0\}\} \cup$$

$$\{(t, x, \tilde{t} = 0, y = 0) : t \geq |x|; (\tau = 0, \xi = 0, \mathbf{R}_{(\tilde{\tau}, \eta)}^4 \setminus \{0\})\} \cup$$

$$\{(t, x, \tilde{t} = 0, y = 0) : t = |x|; ((\tau, \xi) : \tau^2 = |\xi|^2 > 0, \tilde{\tau} = 0, \eta = 0)\}.$$

If $\rho^0 = (t_0, x_0; \tilde{t}_0, y_0; \tau^0 = \xi^0 = 0, (\tilde{\tau}^0, \eta^0), (\tilde{\tau}^0)^2 + |\eta^0|^2 > 0)$ belongs to $WF(f)$, then (37) leads to:

$$WF(\tilde{\mathcal{E}}_{3,\pm} * f) \subset \{(t + t_0, x + x_0, \tilde{t}_0, y_0) : t \geq |x|; \tau^0 = 0, \xi^0 = 0, \tilde{\tau}^0, \eta^0\}$$

i.e., singularities propagate along the solid cone $K : (t - t_0)^2 \geq |x - x_0|^2$ with vertex at (t_0, x_0) , while $\tau^0 = \xi^0 = 0, (\tilde{\tau}^0, \eta^0) \in \mathbf{R}^4 \setminus 0, \tilde{t} = \tilde{t}_0, y = y_0$.

In the elliptic case, the fundamental solution of P in $\mathbf{R}^{2n}$ is

$$\tilde{\mathcal{E}} = \mathcal{E} \otimes \delta(y) \Rightarrow WF(\mathcal{E}) \subset \{(0, 0; \mathbf{R}_\xi^n \times \mathbf{R}_\eta^n \setminus \{0\})\} \cup \{(\mathbf{R}_x^n, y = 0); (0, \mathbf{R}_\eta^n \setminus \{0\})\} \cup$$

$$\{(0, 0; \mathbf{R}_\xi^n \setminus \{0, 0\})\}, CharP = \{(\xi, \eta) : \xi = 0, \eta \neq 0\}, CrP = \{\xi = 0, \eta \neq 0\}.$$

Assume that $\rho^0 = (x_0, y_0; \xi^0 = 0, \eta^0 \neq 0) \in WF(f)$, i.e., $\rho^0 \in CharP = CrP$. Then, $WF(\tilde{\mathcal{E}}_n * f) \subset \{(x + x_0, y_0 : \xi^0 = 0, \eta^0 \neq 0)\}$, i.e., singularities of f propagate along the $\mathbf{R}_x^n = \{x + x_0, \forall x \in \mathbf{R}^n, x_0 \in \mathbf{R}^n\}$, $y = y_0, \xi = \xi^0 = 0, \eta^0 = \eta \neq 0$. If $\xi^0 \neq 0$, no singularities propagate. Thus, everything is proved. $\square$

5. Discussion

1. We can consider Dirac's system of PDEs in $\mathbf{R}^4$, $\tilde{x} \in \mathbf{R}^4$, $\tilde{x} = (x_0, x) \in \mathbf{R}^4$, $x = (x_1, x_2, x_3)$ describing the relativistic particles: $\sum_{\kappa=0}^3 (\gamma^\kappa \frac{\partial}{\partial x_\kappa} - mI_4)\psi = 0$, I_4 being the unit matrix in $\mathbf{R}^4$ and the 4×4 matrices γ^κ are Dirac's constant ones. We can find the fundamental solution of the system in $\mathbf{R}^4$ as the action of the first-order linear partial differential operator at the fundamental solution of the scalar Klein–Gordon equation $(\square_3 - m^2)\eta = 0$ in $\mathbf{R}^4$. Thus, we can study the propagation of singularities (wave-front sets) in $\mathbf{R}^8 = \mathbf{R}_x^4 \times \mathbf{R}_y^4$ by applying Theorem 3. For massless particles (neutrino, photon) singularities in the double characteristic case propagate along the surface of the double characteristic cone of the type $x_0^2 = |x|^2$, while for particles with mass $m > 0$ (proton, neutron, electron) they propagate along the interior of the solid cone $x_0^2 = |x|^2$.

2. In Theorem 1, boundary data are given on two parallel lines, which are not characteristic. It is also possible to investigate in the first quadrant $u_{xy} \pm cu = 0$ with data $u|_{x=0} = f_1(y)$ and $l(u)|_{y=\alpha x} = a(x)u_x + b(x)u_y + d(x)u = f_2(y)$, $a^2 + b^2 > 0$, $\alpha > 0$,

i.e., the Goursat–Darboux boundary value problem. Evidently, the data are given on the characteristic $x = 0$ and on the non-characteristic line $y = \alpha x$. The only restriction is that $(a(x), b(x)) \not\parallel (0, 1)$. This problem was shortly discussed in Remark 1.

3. Another interesting Darboux problem is $(\square_2 \pm k^2)u = 0$, u being prescribed at three characteristic hyperplanes α_j passing through the origin, i.e., $\alpha_j \ni 0$, $j = 1, 2, 3$ and α_j are tangential to the characteristic cone $x_0^2 = |x|^2$, $x = (x_1, x_2)$, $x_0 \geq 0$. The multidimensional generalizations are evident. What about the construction of solutions into explicit form and the singularities of the solutions developed on the conical surface and on the rays $\alpha_i \cap \alpha_j$, $x_0 > 0$? Several Darboux and Goursat boundary value problems can be found in [18]. It is also interesting to consider the case of nonlinear hyperbolic equations and the blow-up of their solutions, as we did in Remark 3 and the examples below it. In this direction, the following papers could be useful [19,20].

6. Conclusions

In this paper, we study several equations of mathematical physics, and we find their solutions in explicit form. First, we consider the initial-boundary value oblique derivative problem for the string equation. The classical solution exists under some compatibility conditions in the corner points. The second problem is the case of the wave equation with several space variables in a circular cylinder. We add extra conditions on the generatrices of the cylinder where the vector field is tangential to the cylinder, and we obtain a unique classical solution. We study the Cauchy problem in the interior of the parabola of the Lorentzian-type eikonal equation, and we find its unique classical solution. It appears that in the double characteristic points, the wave front propagates either along the surface of the characteristic cone or in the solid cone. In the discussion, we propose some open problems.

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