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Critical Fractional Problems with Weights: Existence, Minimization, and Pohozaev Obstructions

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Abstract

Recently, a great amount of attention has been focused on the study of fractional and nonlocal operators of the elliptic type both for pure mathematical research and in view of concrete real-world applications. We are interested in proving the existence and nonexistence of solutions of a minimizing problem involving a fractional Laplacian with weight. We consider the nonlocal minimizing problem on $\mathbb{H}_0^s(\Omega) \subset L^{q_s}(\Omega)$, with $q_s := \frac{2n}{n-2s}$, $s \in (0, 1)$, and $n \geq 3$ $\inf_{u \in \mathbb{H}_0^s(\Omega)} \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx$, where Ω is a bounded domain in $\mathbb{R}^n$, $p : \mathbb{R}^n \rightarrow \mathbb{R}$ is a given positive weight presenting a global positive minimum $p_0 > 0$ at $a \in \Omega$, and λ is a real constant. The objective of this paper is to show that minimizers do not exist for some k, s, λ , and n . After that, we show some nonexistence results thanks to a fractional Pohozaev identity and fractional Hardy inequality.

Keywords: critical Sobolev exponent; nonlocal operator; fractional Laplacian; minimizing problem

MSC: 35J20; 35J25; 35H30; 35J60

1. Introduction, Notation, and Statement of the Result

1.1. The Fractional Nonlinear Problem with Weight and Its Relation to the Ordinary Nonlinear Problem

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Define the space $\mathbb{H}_0^s(\Omega)$ by

$$\mathbb{H}_0^s(\Omega) := \left\{ u \in L^2(\Omega), \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{2} + s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n \setminus \Omega^c \times \Omega^c), u(x) = 0, \forall x \in \mathbb{R}^n \setminus \Omega \right\}. \quad (1)$$

Note that it is well known that $\mathbb{H}_0^s(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for any $1 \leq r \leq \frac{2n}{n-2s}$. Moreover, this embedding is compact for $1 \leq q < \frac{2n}{n-2s}$ (see ([1], Theorem 1.4). We assume that $p : \mathbb{R}^n \rightarrow \mathbb{R}$ is a positive weight in $C(\mathbb{R}^n)$ and for every $x \in \mathbb{R}^n$

$$p(x) = p_0 + \kappa e^{-|x-a|^2} |x-a|^k, \quad (2)$$

where $\kappa > 0$, $a \in \Omega$. It is obvious that p represents a global minimum p_0 . We are interested in studying the following problem:



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$$\begin{cases} (-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} u \right) = u^{q_s-1} + \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (3)$$

with $u \in \mathbb{H}_0^s(\Omega)$. Here, $s \in (0, 1)$, Ω is a bounded $C^{1,1}$ and star-shaped domain, and p is as defined in (2).

In this paper, we extend the result which has been already presented for the ordinary Laplacian in [2] to the case of the fractional Laplacian.

Let p be defined as previously in (2), $s \in (0, 1)$, and $n \geq 3$; we define the infimum $\tilde{S}_{s,\lambda}(p)$ by

$$\tilde{S}_{s,\lambda}(p) := \inf_{\substack{u \in \mathbb{H}_0^s(\Omega) \\ \|u\|_{L^{q_s}(\mathbb{R}^n)}=1}} \left\{ \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \right\}. \quad (4)$$

The study of the infimum (4) shows that the existence of minimizers depends not only on the behavior of p near its global minimum but also on its behavior at $+\infty$.

The well-known fractional Sobolev inequalities were first considered in a remarkable paper by Lieb; see [3–6].

Let us compare this work to what is proved in the traditional minimization problem of [2]:

$$S_{\lambda}(p) = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^q(\Omega)}=1}} \int_{\Omega} p(x) |\nabla u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx, \quad (5)$$

where $q = \frac{2n}{n-2}$ is the critical exponent for the Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$. In [7], the authors treated the case where the problem is related to the ordinary Laplacian ($s = 1$). They proved, in particular, the existence of minimizers of $S_{\lambda}(p)$ for $0 < \lambda < \lambda_1^{div}$ if $n \geq 4$ and $k > 2$, and for $\lambda^* < \lambda < \lambda_1^{div}$ if $n \geq 3$ and $0 < k < 2$, and in other cases that are well detailed in Theorem 1.1 of [2] with k being a positive constant that appears in the expression of the weight p , λ_1^{div} the first eigenvalue of $-div(p(x)\nabla \cdot)$ on Ω with zero Dirichlet boundary condition, and λ^* a positive constant. The method used for the proof of this result is first to show that $S_{\lambda}(p) < p_0 S$ with S being the best Sobolev constant defined by

$$S := \inf_{\substack{u \in H_1(\mathbb{R}^n) \\ \|u\|_{L^q(\mathbb{R}^n)}=1}} \int_{\Omega} |\nabla u(x)|^2 dx, \quad (6)$$

then, they prove that the infimum $S_{\lambda}(p)$ is achieved.

Another similar result in the fractional case is given in [1,5], but where the weight p is a constant, they proved that

$$\tilde{S}_{s,\lambda} := \inf_{\substack{u \in \mathbb{H}_0^s(\Omega) \\ \|u\|_{L^{q_s}(\Omega)}=1}} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \quad (7)$$

is achieved for $s \in (0, 1)$, $n \geq 4s$, and $\lambda \in]0, \lambda_{1,s}[$, where $\lambda_{1,s}$ denotes the first eigenvalue of the nonlocal operator $(-\Delta)^s$ with a homogeneous Dirichlet boundary datum.

In [8], we dealt with the minimization problem

$$S_{s,\lambda}(p) := \inf_{\substack{u \in \mathbb{H}_0^s(\Omega) \\ \|u\|_{L^{q_s}(\mathbb{R}^n)}=1}} \left\{ \int_{\mathbb{R}^n} p(x) \left(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy \right) dx - \lambda \int_{\Omega} |u(x)|^2 dx \right\}, \quad (8)$$

and we proved that the infimum $S_{s,\lambda}(p)$ is achieved for some suitable values of n, s, k , and λ . Thanks to the identity

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy = 2c(n, s)^{-1} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}^2 \quad (9)$$

(see [1], (1.50)), we deduce that if the weight p is constant, $S_{s,\lambda}(p)$ and $\tilde{S}_{s,\lambda}(p)$ are equivalent. However, it remains unclear whether this equivalence persists when p is not constant. In this paper, we consider a significant example of a weight function p , defined in (2), which satisfies properties (3) and (4) as stated in [8]. This choice allows us to avoid the constraints on κ imposed in Theorem 1.2 of [8] and to obtain the result for all $\kappa > 0$.

There are extensive works that study properties of fractional Laplacian operators as nonlocal operators together with applications to free-boundary value problems and nonlocal minimal surfaces by Caffarelli and Silvestre in [9] and many others (see for example [10]).

Frank, Lieb and Seiringer treated in [4] weighted cases, namely $\frac{dx dy}{|x|^\alpha |y|^\beta}$, in order to prove Hardy–Lieb–Thirring inequalities.

Chang and González [11], as well as González Noguera and Qing [12], formulated a fractional s -Yamabe problem that includes the boundary Yamabe problem studied by Escobar [13]; see also [14]. They highlight a Hopf-type maximum principle together with interplay between the analysis of weighted trace Sobolev inequalities and the conformal structure of the underlying manifolds, and they obtained some properties for the fractional case that are analogous to the original Yamabe problem.

In [15], we studied the polyharmonic operator, namely $(-\Delta)^r$ where $r \in \mathbb{N}$, $r \geq 2$ with a critical Sobolev exponent.

Servadei and Valdinoci dealt in [16] with non-ground state solutions in the fractional Laplacian case without weight. The solutions are constructed with a variational method by a min–max procedure on the associated energy functional.

The problem involving the ordinary Laplacian in the case where p is a constant was originally studied by Aubin, Brezis–Nirenberg in [17–19].

We refer to [7,20] for the study of the existence and multiplicity of solutions of the above problem in the case where $s = 1$ with the presence of a smooth and positive weight and with homogeneous Dirichlet boundary conditions. Nevertheless, in [7], it is shown that if p has discontinuities, then a solution of $S_0(p, 0)$ still exists.

For more general weights, depending on x and on u , in [2], Hadiji and Vigneron showed that in the case of homogeneous Dirichlet boundary condition and in the presence of a linear perturbation, the corresponding minimizing problem possesses a solution. The model of the weight is $p(x, u) = \alpha + |x|^\beta |u|^k$ with positive parameters α , β and k . Note that in this case, natural scalings appear, and the answer depends on the ratio $\frac{\beta}{k}$. We refer to problems that resemble (3), (4), as well as have an extensive literature, and we refer to papers [17,18,21–23] and references therein and [24] for a complete history of the problem and geometrical motivations.

1.2. Some Definitions

One of the aims of this paper is to study nonlocal problems driven by $(-\Delta)^s$ (or its generalization) and with Dirichlet boundary data via variational methods. For this purpose, we need to work in a suitable fractional Sobolev space: for this, we consider a functional analytical setting that is inspired by (but not equivalent to) the fractional Sobolev spaces in order to correctly encode the Dirichlet boundary datum in the variational formulation. This section is devoted to the definition of this space as well as to its properties. Therefore, before

setting the main result, we start by defining the fractional Laplacian and the fractional Sobolev spaces.

Let $s \in (0, 1)$ and let $S(\mathbb{R}^n)$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$. We define the nonlocal operator $(-\Delta)^s : S(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by

$$\begin{aligned} (-\Delta)^s u(x) &:= c(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \\ &= c(n, s) \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B(x, \varepsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \end{aligned} \quad (10)$$

Here, $P.V.$ is a commonly used abbreviation for “in the principal value sense” (as defined by the latter equation), $x \in \mathbb{R}^n$, $B(x, \varepsilon)$ is the ball centered in $x \in \mathbb{R}^n$ of radius ε and $c(n, s)$ is the positive normalization constant, which is defined by

$$c(n, s) := \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \right)^{-1}, \quad (11)$$

with $\zeta := (\zeta_1, \zeta')$, $\zeta_1 \in \mathbb{R}$ and $\zeta' \in \mathbb{R}^{n-1}$. In the sequel, we omit to write “P.V.” in the integral $P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy$.

We recall that this operator is nonlocal since, for example, if we take $u \in S(\mathbb{R}^n)$, such that $u > 0$ in $\mathbb{R}^n \setminus B(0, 2)$ and $u(x) = 0$ in $B(0, 1)$, then $(-\Delta)^s u(0) > 0$, which depends on the value of u in the hole $\mathbb{R}^n$.

We define the Sobolev space $H^s(\mathbb{R}^n)$ as the set of functions u such that they are square integrable and their fractional Laplacian $(-\Delta)^{\frac{s}{2}} u$ is also square integrable

$$H^s(\mathbb{R}^n) := \{u : \mathbb{R}^n \rightarrow \mathbb{R}, u \in L^2(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy < +\infty\}. \quad (12)$$

The norm on $H^s(\mathbb{R}^n)$ can be defined by, for all $u \in H^s(\mathbb{R}^n)$,

$$\|u\|_{H^s(\mathbb{R}^n)} := \|u\|_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{\frac{1}{2}}.$$

We know that $H^s(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{H^s(\mathbb{R}^n)}$ is a Hilbert space. For more definitions, we refer to [1].

Thus, $\mathbb{H}_0^s(\Omega)$ is a subspace of $H^s(\mathbb{R}^n)$ and can be defined simply by

$$\mathbb{H}_0^s(\Omega) := \{u \in H^s(\mathbb{R}^n); u = 0 \text{ a.e. in } \Omega^c\},$$

where $\Omega^c := \mathbb{R}^n \setminus \Omega$ denotes the complement of Ω in $\mathbb{R}^n$, and “a.e. in Ω^c ” means almost everywhere with respect to the Lebesgue measure in $\mathbb{R}^n \setminus \Omega$.

The norm of $\mathbb{H}_0^s(\Omega)$ can be defined by

$$\mathcal{N}(u) := \|u\|_{L^2(\Omega)} + \left(\frac{c(n, s)}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{\frac{1}{2}}.$$

This norm is equivalent to

$$\|u\|_{\mathbb{H}_0^s(\Omega)} = \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{\frac{1}{2}},$$

(see ([1], Lemma 1.28) and ([4], Lemma 3.1)).

We notice that the following identity $\|u\|_{\mathbb{H}_0^s(\Omega)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}$, gives the relation between the fractional Laplacian operator $(-\Delta)^s$ and the fractional Sobolev space $\mathbb{H}_0^s(\Omega)$. See ([1], relation (4.35)). In this paper, $\mathbb{H}_0^s(\Omega)$ will be the functional analytic setting because the classical fractional Sobolev space approach is not sufficient for studying the problem; see [1,16,25].

We start by recalling some notations and some remarks which will be useful. First, we give some important definitions afterwards. Let us denote by $\tilde{S}_s(p) := \tilde{S}_{s,0}(p)$ and $S_s := \tilde{S}_{s,0}(1)$ the weightless case. Let us denote by $u_{\varepsilon,s,a}$ an extremal function for the weightless Sobolev inequality for the fractional Laplacian operator. Let us fix $\eta > 0$ such that

$$B(a, 4\eta) \subset \Omega \quad (13)$$

For $x \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$U_{\varepsilon,s,a}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x - a|^2} \right)^{\frac{n-2s}{2}}, \quad (14)$$

then we set

$$u_{\varepsilon,s,a}(x) = U_{\varepsilon,s,a}(x)\Psi(x), \quad (15)$$

where $\Psi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \Psi \leq 1$, $\Psi = 1$ in $B(a, \eta)$ and $\Psi = 0$ in $B(a, 2\eta)^c$, where η is a positive real.

Using [1,19], we have

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u_{\varepsilon,s,a}(x) - u_{\varepsilon,s,a}(y)|^2}{|x - y|^{n+2s}} dy dx = K_s + \mathcal{O}(\varepsilon^{n-2s}), \quad (16)$$

for $n \geq 4$ and $s \in (0, 1)$, or $n = 3$ and $s < \frac{3}{4}$, the square integration of $u_{\varepsilon,s,a}$ goes as follows for any $n > 4s$

$$\|u_{\varepsilon,s,a}\|_{L^2(\mathbb{R}^n)}^2 = K_{2,s}\varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}), \quad (17)$$

the q_s norm is given by

$$\|u_{\varepsilon,s,a}\|_{L^{q_s}(\mathbb{R}^n)}^{q_s} = K_{q_s} + \mathcal{O}(\varepsilon^n), \quad (18)$$

with $K_{q_s} = \int_{\mathbb{R}^n} \frac{dy}{(1 + |y|^2)^n}$ and $\frac{K_s}{K_{q_s}^{\frac{2}{q_s}}} = S_s$.

In order to present our main results, we will need to introduce the first eigenvalue $\tilde{\lambda}_{1,p,s}$ of $(-\Delta)^{\frac{s}{2}}(p(x)(-\Delta)^{\frac{s}{2}} \cdot)$ on Ω with zero Dirichlet boundary condition

$$\tilde{\lambda}_{1,p,s} := \min_{\substack{u \in \mathbb{H}_0^s(\Omega) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx}{\int_{\mathbb{R}^n} |u(x)|^2 dx}. \quad (19)$$

1.3. Natural Scaling(s) of the Problem

Let us define the energy

$$\tilde{E}_\lambda(u) := \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \lambda \int_{\Omega} u^2 dx. \quad (20)$$

We observe that the critical points of (20) are solutions to the Euler–Lagrange problem (3).

As in [8], it is useful to evaluate the problem with some natural scaling. We analyze a blow-up around a minimum of p which we suppose to be 0, and we are led to study $\tilde{E}_\lambda(v)$. For the convenience of the reader, we give the details. More precisely, let us define v_ε by $v(x) = \varepsilon^{-n/q_s} v_\varepsilon(\frac{x}{\varepsilon})$ and $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Note that if $\|v\|_{L^{q_s}(\mathbb{R}^n)} = 1$, then $\|v_\varepsilon\|_{L^{q_s}(\mathbb{R}^n)} = 1$. We have

$$\begin{aligned} \tilde{E}_\lambda(v) &= p_0 \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v(x)|^2 dx + \kappa \varepsilon^k \int_{B(0, \frac{\eta}{\varepsilon})} e^{-|x-a|^2} |x-a|^k \left(\int_{\mathbb{R}^n} \frac{v_\varepsilon(x) - v_\varepsilon(y)}{|x-y|^{n+s}} dy \right)^2 dx \\ &+ \int_{B(0, \eta)^c} p(x) \left(\int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x-y|^{n+s}} dy \right)^2 dx - \lambda \varepsilon^{2s} \int_{\Omega_\varepsilon} v_\varepsilon^2 dx. \end{aligned} \quad (21)$$

Unlike, for example, [2], we have a positive term in the previous expression which is the following $\int_{B(0, \eta)^c} p(x) \left(\int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x-y|^{n+s}} dy \right)^2 dx$, which can explain the conditions on k announced later in our result.

When $k > 2s$, the term related to ε^{2s} dominates the one related to ε^k provided that the quantities

$$\int_{B(0, \frac{\eta}{\varepsilon})} e^{-|x-a|^2} |x-a|^k \left(\int_{\mathbb{R}^n} \frac{v_\varepsilon(x) - v_\varepsilon(y)}{|x-y|^{n+s}} dy \right)^2 dx \quad (22)$$

is finite, so we find that the energy becomes strictly below the critical value, and we expect that $\tilde{E}_\lambda(v) < p_0 S_s$; then, $\tilde{S}_{s, \lambda}(p)$ is achieved.

It is important to mention that since the problem is not local, at infinity, the term $e^{-|x-a|^2} |x-a|^k$ has an impact on the integral (22), which is not finite in general for $u \in \mathbb{H}_0^s(\Omega)$, so we will restrict ourselves to specific values of k .

We expect a competition between the local character of the weight and the nonlocal operator. It turns out that the local nature of the weight wins for a class of weights. The presence of weight pushes the problem to be local. In other words, when k is smaller than $n - 4s$, we find that the local character dominates; on the other hand, when k is very large, we expect that the problem does not admit solutions.

If we take $\lambda = 0$, we obtain the following nonexistence result.

Proposition 1. *Let p be defined as in (2). For $n = 3$ and $s \in]0, \frac{1}{4}]$ or $n = 4$ and $s \in]0, \frac{1}{2}]$ or $n = 5$ and $s \in]0, \frac{3}{4}]$ or $n \geq 6$ and $s \in (0, 1)$, and for all $k \in [2, n - 4s[$, we have $\tilde{S}_{s,0}(p) = p_0 \tilde{S}_{s,0}(1)$ and $\tilde{S}_{s,0}(p)$ is never achieved, so there is no minimizing solution.*

Proof. Let $s \in (0, 1)$ and $2 \leq k < n - 4s$. We recall the notations $\tilde{S}_s(p) := \tilde{S}_{s,0}(p)$ and $S_s := \tilde{S}_{s,0}(1)$.

By Theorem 3, please see Section 2, we write

$$p_0 S_s \leq \tilde{S}_s(p) \leq \frac{1}{\|u_{\varepsilon,s,a}\|_{L^{qs}(\mathbb{R}^n)}^2} \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u_{\varepsilon,s,a}(x)|^2 dx \leq p_0 S_s + o(1), \quad (23)$$

as ε tends to zero. Then, we obtain $\tilde{S}_s(p) = p_0 S_s$.

We suppose that $\tilde{S}_s(p)$ is achieved by some function $u \in \mathbb{H}_0^s(\Omega)$ such that $\|u\|_{L^{qs}(\mathbb{R}^n)} = 1$; then,

$$p_0 \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx \leq \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx = p_0 S_s. \quad (24)$$

Therefore, $S_s = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx$, $\|u\|_{L^{qs}(\mathbb{R}^n)} = 1$, which means that S_s is achieved by a function $u \in \mathbb{H}_0^s(\Omega)$ and this is absurd. See [3–6]. $\square$

In what follows, we will concentrate on the case where $\lambda > 0$.

1.4. Statement of the Main Results

We now state the main statements of this paper.

Theorem 1 (Existence). Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with continuous boundary, p as in (2), $\lambda \in]0, \tilde{\lambda}_{1,p,s}[$, and $2 \leq k < n - 4s$. Assume one of the following conditions holds:

1. $n \geq 6$ and $s \in (0, 1)$;
2. $n \in \{3, 4, 5\}$ and $s \in \left]0, \frac{n-2}{4}\right[$.

Then, for every $\kappa > 0$, $\tilde{S}_{s,\lambda}(p)$ is achieved.

In order to state our main second theorem, we need to define the following constant $\alpha(k, s)$:

$$\alpha(k, s) := \frac{k}{2s} \inf_{\substack{u \in \mathbb{H}_0^s(\Omega) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\int_{\Omega} |u(x)|^2 dx}. \quad (25)$$

Theorem 2 (Nonexistence). Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $n \geq 3$, p as in (2), and $s \in (0, 1)$. The following hold:

1. If $\lambda \geq \tilde{\lambda}_{1,p,s}$, then $\tilde{S}_{s,\lambda}(p)$ is not achieved.
If Ω is star-shaped,
2. If $n \in \{3, 4, 5\}$ and $s \in \left]0, \frac{n-2}{4}\right[$, or $n \geq 6$ and $s \in (0, 1)$, and $2 \leq k < n - 4s$, then $\alpha(k, s) = 0$ and $\tilde{S}_{s,\lambda}(p)$ is not achieved for every $\lambda \leq 0$.
3. If $k < 2s$, then there exists a constant $\alpha_0 > 0$ such that $\alpha(k, s) \geq \alpha_0 > 0$, and $\tilde{S}_{s,\lambda}(p)$ is not achieved for every $\lambda < \alpha(k, s)$.

In order to achieve this result (Theorem 2), we needed to find a formula adapted to our case similar to the Pohozaev's identity written in [26]. We were inspired by [26] in order to find a fractional weighted Pohozaev identity and apply it in order to show that minimizers do not exist in some cases of weight. For a star-shaped domain Ω and for a weight p defined as in (2) and for $u \in \mathbb{H}^s(\mathbb{R}^n)$, we found an identity that was the key to conclude that the infimum is not achieved if $\lambda < \alpha(k, s)$ with $\alpha(k, s)$ is defined in (25).

1.5. Structure of the Paper

This paper is structured as follows. The next Section 2 proves the a priori estimate $\tilde{S}_{s,\lambda}(p) < p_0 S_s$. Theorem 1, which is the first main result of this paper, is proved by a mathematically adequate technique using the previous section in order to obtain the existence of minimizing solutions to $\tilde{S}_{s,\lambda}(p)$. In Section 3, we carry out a Pohozaev identity that spans a good part of this section and a sort of Hardy fractional inequality for which have a crucial role to prove nonexistence results of minimizing solutions to $\tilde{S}_{s,\lambda}(p)$.

2. Existence of Minimizers

First of all, let us prove that the infimum $\tilde{S}_{s,\lambda}(p)$ does exist for every $\lambda \in \mathbb{R}$. In fact, thanks to Sobolev inequalities, we have

$$\int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \geq C \|u\|_{L^{qs}(\mathbb{R}^n)} - \lambda C_1 \|u\|_{L^{qs}(\mathbb{R}^n)}, \quad (26)$$

where C and C_1 are positive constants. Therefore, we obtain

$$\tilde{S}_{s,\lambda}(p) \geq C - \lambda C_1 \quad (27)$$

So that the infimum $\tilde{S}_{s,\lambda}(p)$ exists.

Here, we used the fact that $N_1(u) := \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx$ is a norm on $\mathbb{H}_0^s(\Omega)$.

The following theorem plays a crucial role to prove the existence of solutions; it is an adaptation of an original argument due to [19] in the context of Yamabe's conjecture.

Proposition 2. *Let $s \in (0, 1)$ and let a weight p satisfy (2). If $\tilde{S}_{s,\lambda}(p) < p_0 S_s$, then $\tilde{S}_{s,\lambda}(p)$ is achieved.*

Proof. Let us recall the expression of $\tilde{S}_{s,\lambda}(p)$:

$$\tilde{S}_{s,\lambda} := \inf_{\substack{u \in \mathbb{H}_0^s(\Omega) \\ \|u\|_{L^{qs}(\Omega)}=1}} \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \quad (28)$$

Let (u_j) be a minimizing sequence of $\tilde{S}_{s,\lambda}(p)$; then, $\|u_j\|_{L^{qs}(\mathbb{R}^n)} = 1$ and

$$\int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{u_j(x) - u_j(y)}{|x - y|^{n+s}} dy \right|^2 dx - \lambda \|u_j\|_2^2 = \tilde{S}_{s,\lambda}(p) + o(1), \quad (29)$$

as j tends to $+\infty$.

We define a scalar product on $\mathbb{H}_0^s(\Omega)$ by

$$\langle u, v \rangle_p = \int_{\mathbb{R}^n} p(x) \left(\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} dy \right) \left(\int_{\mathbb{R}^n} \frac{v(x) - v(z)}{|x - z|^{n+s}} dz \right) dx, \quad (30)$$

for all $u, v \in \mathbb{H}_0^s(\Omega)$. It is obvious that the norm associated with this scalar product is equivalent to the ordinary norm over $\mathbb{H}_0^s(\Omega)$.

Since (u_j) is bounded in $\mathbb{H}_0^s(\Omega)$, we extract a subsequence still denoted by (u_j) such that (u_j) tends weakly to u in $\mathbb{H}_0^s(\Omega)$ (since $\mathbb{H}_0^s(\Omega)$ is reflexive space). Then, (u_j) tends strongly to u in $L^2(\Omega)$, and (u_j) tends to u almost everywhere in Ω , with

$$\|u\|_{L^{qs}(\mathbb{R}^n)} \leq 1. \quad (31)$$

By the definition of S_s we have

$$\int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{u_j(x) - u_j(y)}{|x - y|^{n+s}} dy \right|^2 dx \geq p_0 S_s, \quad (32)$$

then

$$\int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{u_j(x) - u_j(y)}{|x - y|^{n+s}} dy \right|^2 dx - \tilde{S}_{s,\lambda}(p) + o(1) \geq p_0 S_s - \tilde{S}_{s,\lambda}(p) + o(1), \quad (33)$$

as j tends to $+\infty$, therefore

$$\lambda \|u\|_{L^2(\mathbb{R}^n)}^2 \geq p_0 S_s - \tilde{S}_{s,\lambda}(p) > 0.$$

Therefore, $u \neq 0$.

Let us take $v_j := u_j - u$; then, (v_j) tends weakly to 0 in $\mathbb{H}_0^s(\Omega)$ and (v_j) tends to 0 almost everywhere in Ω . Using the definition of $\tilde{S}_{s,\lambda}(p)$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} dy \right|^2 dx + \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{v_j(x) - v_j(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ & - \lambda \|u\|_{L^2(\mathbb{R}^n)}^2 = \tilde{S}_{s,\lambda}(p) + o(1), \end{aligned} \quad (34)$$

as j tends to $+\infty$.

On the other hand, since (v_j) is bounded in $L^{q_s}(\Omega)$ and (v_j) tends to 0 almost everywhere in Ω , we deduce from a result of Brezis-Lieb, see [27], that

$$1 = \|u + v_j\|_{L^{q_s}(\mathbb{R}^n)}^{q_s} = \|u\|_{L^{q_s}(\mathbb{R}^n)}^{q_s} + \|v_j\|_{L^{q_s}(\mathbb{R}^n)}^{q_s} + o(1), \quad (35)$$

as j tends to $+\infty$. Thus,

$$1 \leq \|u\|_{L^{q_s}(\mathbb{R}^n)}^2 + \|v_j\|_{L^{q_s}(\mathbb{R}^n)}^2 + o(1),$$

as j tends to $+\infty$. Denoting by $\tilde{v}_j := \frac{v_j}{\|v_j\|_{L^{q_s}(\mathbb{R}^n)}}$. Since we have

$$p_0 S_s \leq \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{\tilde{v}_j(x) - \tilde{v}_j(y)}{|x - y|^{n+s}} dy \right|^2 dx, \quad (36)$$

we obtain

$$1 \leq \|u\|_{L^{q_s}(\mathbb{R}^n)}^2 + \frac{1}{p_0 S_s} \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{v_j(x) - v_j(y)}{|x - y|^{n+s}} dy \right|^2 dx + o(1), \quad (37)$$

as j tends to $+\infty$. Since we have already proved that $\tilde{S}_{s,\lambda}(p) \geq 0$, we deduce from (37) that

$$\tilde{S}_{s,\lambda}(p) \leq \tilde{S}_{s,\lambda}(p) \|u\|_{L^{q_s}(\mathbb{R}^n)}^2 + \frac{\tilde{S}_{s,\lambda}(p)}{p_0 S_s} \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{v_j(x) - v_j(y)}{|x - y|^{n+s}} dy \right|^2 dx + o(1), \quad (38)$$

as j tends to $+\infty$. Combining (34) and (38), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} dy \right|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx + \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{v_j(x) - v_j(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ & \leq \tilde{S}_{s,\lambda}(p) \|u\|_{L^{q_s}(\mathbb{R}^n)}^2 + \frac{\tilde{S}_{s,\lambda}(p)}{p_0 S_s} \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{v_j(x) - v_j(y)}{|x - y|^{n+s}} dy \right|^2 dx + o(1), \end{aligned} \quad (39)$$

as j tends to $+\infty$. Thus,

$$\begin{aligned} & \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} dy \right|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \\ & \leq \tilde{S}_{s,\lambda}(p) \|u\|_{L^{q_s}(\mathbb{R}^n)}^2 + \left[\frac{\tilde{S}_{s,\lambda}(p)}{p_0 S_s} - 1 \right] \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{v_j(x) - v_j(y)}{|x - y|^{n+s}} dy \right|^2 dx + o(1), \end{aligned} \quad (40)$$

as j tends to $+\infty$. Since $\tilde{S}_{s,\lambda}(p) < p_0 S_s$, we deduce

$$\int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} dy \right|^2 dx - \lambda \int_{\Omega} |u(x)|^2 dx \leq \tilde{S}_{s,\lambda}(p) \|u\|_{L^{q_s}(\mathbb{R}^n)}^2. \quad (41)$$

This means that u is a minimum of $\tilde{S}_{s,\lambda}(p)$. $\square$

Remark 1. 1. In the case where the weight p and λ are such that $-\infty < \tilde{S}_{s,\lambda}(p) \leq 0$, we prove that the infimum $\tilde{S}_{s,\lambda}(p)$ is achieved. In fact, as in (31), we have $\|u\|_{L^{q_s}(\mathbb{R}^n)} \leq 1$, then

$$\tilde{S}_{s,\lambda}(p) \leq \tilde{S}_{s,\lambda}(p) \|u\|_{L^{q_s}(\mathbb{R}^n)}^2. \quad (42)$$

Again, we deduce (41) from (34).

2. Since the norm associated with the scalar product defined in (30) is equivalent to the standard norm on $\mathbb{H}_0^s(\Omega)$, and it is established that the norm induced by the scalar product given in ([8], (23)) is also equivalent to the standard norm on $\mathbb{H}_0^s(\Omega)$, it follows that these two weighted norms are equivalent. That is, there exist two positive constants C_1 and C_2 such that

$$\begin{aligned} C_1 \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx &\leq \int_{\mathbb{R}^n} p(x) \left(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy \right) dx \\ &\leq C_2 \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx. \end{aligned} \quad (43)$$

Remark 2. Thanks to the following inequality $||u_j(x)| - |u_j(y)|| \leq |u_j(x) - u_j(y)|$, if (u_j) is a minimizing sequence of $\tilde{S}_{s,\lambda}(p)$, $(|u_j|)$ is also a minimizing sequence. Therefore, we can take a positive minimizing sequence of $\tilde{S}_{s,\lambda}(p)$.

Clearly, if $\tilde{S}_{s,\lambda}(p)$ is achieved by some function u , then it is also achieved by $|u|$. Therefore, there exists a positive solution for the infimum $\tilde{S}_{s,\lambda}(p)$.

A Priori Estimate on $\tilde{S}_{s,\lambda}(p)$

We need to prove the following proposition and the techniques that it uses, since it constitutes an official key in order to apply Proposition 2.

Theorem 3. Let n, s such that $n = 3$ and $s \in]0, \frac{1}{4}[$ or $n = 4$ and $s \in]0, \frac{1}{2}[$ or $n = 5$ and $s \in]0, \frac{3}{4}[$ or $n \geq 6$ and $s \in (0, 1)$. Let p be as defined in (2). For any $k \in [2, n - 4s]$, the following estimate holds true

$$\int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u_{\varepsilon,s,a}(x)|^2 dx \leq p_0 K_s + \kappa C \varepsilon^2 + \mathcal{O}(\varepsilon^{n-2s}) + \mathcal{O}(\varepsilon^{k+2s}), \quad (44)$$

as ε tends to zero and C is a positive constant depending on k, s and the dimension n .

Proof. First of all and without loss of generality, we assume that $a = 0$, and we note $U_{\varepsilon,s,0} = U_{\varepsilon,s}$ $u_{\varepsilon,s,0} = u_{\varepsilon,s}$.

Let A be a subset domain of $\mathbb{R}^n$; then, we define the function 1_A by $\forall x \in \mathbb{R}^n$, $1_A(x) = 0$ if $x \notin A$ and $1_A(x) = 1$ if $x \in A$.

In the sequel, we will assume that n and s are such that

$$n = 3 \text{ and } s \in]0, \frac{1}{4}[\text{ or } n = 4 \text{ and } s \in]0, \frac{1}{2}[\text{ or } n = 5 \text{ and } s \in]0, \frac{3}{4}[\text{ or } n \geq 6 \text{ and } s \in (0, 1). \quad (45)$$

The proof makes use of the following estimates and is a bit more complicated than the one for similar results in the case of the fractional Laplacian without weight, as it is in the [1].

We easily see that for $\rho > 0$ and $x \in B(0, \rho)^c$, then

$$u_{\varepsilon,s}(x) \leq U_{\varepsilon,s}(x) \leq C \varepsilon^{\frac{n-2s}{2}}, \quad (46)$$

for any $\varepsilon > 0$ and for some positive constant C , possibly depending on η, ρ, s and the dimension n .

We mention that the following assertions hold true

1. For any $x \in \mathbb{R}^n$ and $y \in B(0, \eta)^c$, with $|x - y| \leq \frac{\eta}{2}$,

$$|u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)| \leq C \varepsilon^{\frac{n-2s}{2}} |x - y|. \quad (47)$$

2. For any $x, y \in B(0, \eta)^c$,

$$|u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)| \leq C\varepsilon^{\frac{n-2s}{2}} \min\{1, |x - y|\}, \quad (48)$$

for any $\varepsilon > 0$ and for some positive constant C , possibly depending on η, ρ, s and n .

By (2) and (16), we have

$$\begin{aligned} \int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u_{\varepsilon,s}(x)|^2 dx &= \int_{\mathbb{R}^n} p(x) \left| \int_{\mathbb{R}^n} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ &\leq p_0 K_s + \int_{\mathbb{R}^n} (p(x) - p_0) \left| \int_{\mathbb{R}^n} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx. \end{aligned} \quad (49)$$

Here, without loss of generality, we retain the same notation K_s here, although, by using (9), it actually stands for $K_s \cdot \frac{C(n,s)}{2}$, which is still a constant depending on s .

In the following of the proof, we will estimate the second term of the right-hand side of (49). Since, for all $x \in \mathbb{R}^n$, $p(x) - p_0 = \kappa e^{-|x|^2} |x|^k$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} (p(x) - p_0) \left| \int_{\mathbb{R}^n} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ &= \int_{B(0,\eta)} (p(x) - p_0) \left| \int_{\mathbb{R}^n} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ &\quad + \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{\mathbb{R}^n} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ &= \int_{B(0,\eta)} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ &\quad + \int_{B(0,\eta)} (p(x) - p_0) \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ &\quad + \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ &\quad + \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ &\leq \kappa \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \\ &\quad + \kappa \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})}(x - y) dy \right|^2 dx \\ &\quad + \kappa \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx \\ &\quad + \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})}(x - y) dy \right|^2 dx \\ &\quad + \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx \\ &\quad + \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \end{aligned} \quad (50)$$

We begin by estimating the first term of the last sum, which is

$$\tilde{A}_{s,k,\varepsilon} := \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \quad (51)$$

Proposition 3. For $s \in (0, 1)$, $2 \leq k < n - 4s$ and $\varepsilon > 0$ being very small, we have

$$\tilde{A}_{s,k,\varepsilon} \leq C\varepsilon^2, \quad (52)$$

where C is a positive constant depending on s, k and the dimension n .

Proof. Let $s \in (0, 1)$, $2 \leq k < n - 4s$. After performing a change of variable in (51), we obtain

$$\begin{aligned} \tilde{A}_{s,k,\varepsilon} &= \varepsilon^k \int_{|x| \leq \frac{\eta}{\varepsilon}} e^{-|x|^2} |x|^k \left| \int_{|y| \leq \frac{\eta}{\varepsilon}} \left(\frac{1}{(1+|x|^2)^{\frac{n-2s}{2}}} - \frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \frac{dy}{|x-y|^{n+s}} \right|^2 dx \\ &\leq \varepsilon^k \int_{|x| \leq \frac{\eta}{\varepsilon}} \left| \int_{|y| \leq \frac{\eta}{\varepsilon}} \left(\frac{|x|^{\frac{k}{2}}}{(1+|x|^2)^{\frac{n-2s}{2}}} - \frac{|x|^{\frac{k}{2}}}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \frac{dy}{|x-y|^{n+s}} \right|^2 dx. \end{aligned} \quad (53)$$

Let $f_k(t) := \frac{|t|^{\frac{k}{2}}}{(1+|t|^2)^{\frac{n-2s}{2}}}$, $t \in \mathbb{R}^n$; thus,

$$\begin{aligned} \tilde{A}_{s,k,\varepsilon} &\leq \varepsilon^k \int_{|x| \leq \frac{\eta}{\varepsilon}} \left| \int_{|y| \leq \frac{\eta}{\varepsilon}} \left(|f_k(x) - f_k(y)| + \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \frac{dy}{|x-y|^{n+s}} \right|^2 dx \\ &\leq 2\varepsilon^k \int_{|x| \leq \frac{\eta}{\varepsilon}} \left(\int_{|y| \leq \frac{\eta}{\varepsilon}} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ &\quad + 2\varepsilon^k \int_{|x| \leq \frac{\eta}{\varepsilon}} \left(\int_{|y| \leq \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \end{aligned} \quad (54)$$

As a consequence, in order to obtain the inequality (52), we will prove the two following assertions

$$\int_{|x| \leq \frac{\eta}{\varepsilon}} \left(\int_{|y| \leq \frac{\eta}{\varepsilon}} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \leq \bar{C} < +\infty, \quad (55)$$

where $\bar{C}$ is a constant and

$$\int_{|x| \leq \frac{\eta}{\varepsilon}} \left(\int_{|y| \leq \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \leq C \quad (56)$$

where C is a positive constant depending on s, k and n .

We start proving (55). Let $\gamma > 0$.

$$\begin{aligned} &\int_{|x| \leq \frac{\eta}{\varepsilon}} \left(\int_{|y| \leq \frac{\eta}{\varepsilon}} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ &= \int_{\mathbb{R}^n} \left(\int_{|x-y| \geq \gamma} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ &\quad + \int_{\mathbb{R}^n} \left(\int_{|x-y| < \gamma} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx. \end{aligned} \quad (57)$$

For the first time, we take the case where $|x - y| \geq \gamma$.

$$\begin{aligned}
 & \int_{\mathbb{R}^n} \left(\int_{|x-y| \geq \gamma} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\
 &= \int_{\mathbb{R}^n} \left(\int_{|x-y| \geq \gamma} \left| \frac{|x|^{\frac{k}{2}}}{(1+|x|^2)^{\frac{n-2s}{2}}} - \frac{|y|^{\frac{k}{2}}}{(1+|y|^2)^{\frac{n-2s}{2}}} \right| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\
 &\leq \int_{\mathbb{R}^n} \left(\int_{|x-y| \geq \gamma} \frac{|x|^{\frac{k}{2}}}{(1+|x|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} + \int_{|x-y| \geq \gamma} \frac{|y|^{\frac{k}{2}}}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\
 &\leq 2 \left[\int_{\mathbb{R}^n} \left(\int_{|x-y| \geq \gamma} \frac{|x|^{\frac{k}{2}}}{(1+|x|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \right. \\
 &\quad \left. + \int_{\mathbb{R}^n} \left(\int_{|x-y| \geq \gamma} \frac{|y|^{\frac{k}{2}}}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \right] \\
 &= 2 \int_{\mathbb{R}^n} \frac{|x|^k}{(1+|x|^2)^{n-2s}} \left(\int_{|x-y| \geq \gamma} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\
 &\quad + 2 \int_{\mathbb{R}^n} |f_k * \tilde{h}(x)|^2 dx \\
 &= 2 \int_{\mathbb{R}^n} \frac{|x|^k}{(1+|x|^2)^{n-2s}} dx \left(\int_{|z| \geq \gamma} \frac{dz}{|z|^{n+s}} \right)^2 + 2 \int_{\mathbb{R}^n} |f_k * \tilde{h}(x)|^2 dx
 \end{aligned} \tag{58}$$

where f_k and $\tilde{h}$ are functions defined by, for all $t \in \mathbb{R}^n$,

$$f_k(t) = \frac{|t|^{\frac{k}{2}}}{(1+|t|^2)^{\frac{n-2s}{2}}}, \text{ and } \tilde{h}(t) := \frac{1_{B(0,\gamma)^c}(t)}{|t|^{n+s}}.$$

Provided that $k < n - 4s$, we have $f_k \in L^2(\mathbb{R}^n)$. Since $s > 0, \tilde{h} \in L^1(\mathbb{R}^n)$. This implies $f_k * \tilde{h} \in L^2(\mathbb{R}^n)$, and we have

$$\|f_k * \tilde{h}\|_{L^2(\mathbb{R}^n)} \leq \|f_k\|_{L^2(\mathbb{R}^n)} \|\tilde{h}\|_{L^1(\mathbb{R}^n)} < +\infty, \tag{59}$$

Furthermore, since $k < n - 4s$, $\int_{\mathbb{R}^n} \frac{|x|^k}{(1+|x|^2)^{n-2s}} dx < +\infty$ and since $s > 0$, $\int_{|z| \geq \gamma} \frac{dz}{|z|^{n+s}} < +\infty$; thus,

$$\int_{\mathbb{R}^n} \left(\int_{|x-y| \geq \gamma} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx < +\infty. \tag{60}$$

Now, let us take the case where $|x - y| < \gamma$. We have

$$f_k(x) - f_k(y) = \int_0^1 Df_k(y + t(x - y))(x - y) dt \tag{61}$$

with $f_k(z) := \frac{|z|^{\frac{k}{2}}}{(1+|z|^2)^{\frac{n-2s}{2}}}$, $z \in \mathbb{R}^n$, and

$$Df_k(z) = \frac{|z|^{\frac{k}{2}-2}z}{(1+|z|^2)^{\frac{n-2s}{2}}} \left(\frac{k}{2} - (n-2s) \frac{|z|^2}{(1+|z|^2)} \right)$$

Let $R > 0$ be such that $R \geq 2\gamma$ and let $|y| \geq R$.

For the sake of clarity, we note that since we have $|x - y| < \gamma$ and $|y| \geq R$, we can suppose also that $|x| \geq R$.

Since $t \in [0, 1]$ and $|x - y| < \gamma$, then

$$\begin{aligned}
|y + t(x - y)| &\geq |y| - t|x - y| \\
&\geq |y| - \gamma \\
&\geq |y|(1 - \frac{\gamma}{|y|}) \\
&\geq \frac{1}{2}|y|
\end{aligned} \tag{62}$$

So, we have

$$\begin{aligned}
|Df_k(y + t(x - y))(x - y)| &\leq C \frac{|y + t(x - y)|^{\frac{k}{2}-1} |x - y|}{(1 + |y + t(x - y)|^2)^{\frac{n-2s}{2}}} \\
&\leq C \frac{|x - y|}{|y + t(x - y)|^{n-2s-\frac{k}{2}+1}} \\
&\leq C \frac{|x - y|}{|y|^{n-2s-\frac{k}{2}+1}},
\end{aligned} \tag{63}$$

where C is a constant.

Thus, by (63) and (61),

$$|f_k(x) - f_k(y)| \leq |Df_k(y + t(x - y))(x - y)| \leq C \frac{|x - y|}{|y|^{n-2s-\frac{k}{2}+1}}, \tag{64}$$

Since $k < n - 4s$, then $k < n - 4s + 2$, and by using (64), we have

$$\begin{aligned}
&\int_{|x| \geq R} \left(\int_{|x-y| < \gamma, |y| \geq R} |f_k(x) - f_k(y)| \frac{dy}{|x - y|^{n+s}} \right)^2 dx \\
&\leq C \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{1_{B(0,R)^c}(y)}{|y|^{n-2s-\frac{k}{2}+1}} \frac{1_{B(0,\gamma)}(x - y)}{|x - y|^{n+s-1}} dy \right|^2 dx \\
&= \|\tilde{f} * \tilde{g}\|_{L^2(\mathbb{R}^n)}^2 \\
&\leq \|\tilde{f}\|_{L^2(\mathbb{R}^n)}^2 \|\tilde{g}\|_{L^1(\mathbb{R}^n)}^2 \\
&< +\infty.
\end{aligned} \tag{65}$$

where for all $y \in \mathbb{R}^n$, $\tilde{f}(y) := \frac{1_{B(0,R)^c}(y)}{|y|^{n-2s-\frac{k}{2}+1}}$ and $\tilde{g}(y) := \frac{1_{B(0,\gamma)}(y)}{|y|^{n+s-1}}$, and we have $\tilde{f} \in L^2(\mathbb{R}^n)$ and $\tilde{g} \in L^1(\mathbb{R}^n)$.

Now, let $|y| < R$, and so we take $|x| < R$ since $|x - y| < \gamma$.

Since for all $t \in [0, 1]$, $|y + t(x - y)| < 2R$ and $\frac{1}{(1 + |y + t(x - y)|^2)^{\frac{n-2s}{2}}} \leq 1$, we have

$$\begin{aligned}
|Df_k(y + t(x - y))(x - y)| &\leq C \frac{|y + t(x - y)|^{\frac{k}{2}-1} |x - y|}{(1 + |y + t(x - y)|^2)^{\frac{n-2s}{2}}} \\
&\leq C' |x - y|,
\end{aligned} \tag{66}$$

where C' is a constant depending on R .

Since $s < 1$ and by (66), we have

$$\begin{aligned}
&\int_{|x| < R} \left(\int_{|x-y| < \gamma, |y| < R} |f_k(x) - f_k(y)| \frac{dy}{|x - y|^{n+s}} \right)^2 dx \\
&\leq \tilde{C}' \int_{|x| < R} \left(\int_{|x-y| < \gamma, |y| < R} \frac{dy}{|x - y|^{n+s-1}} \right)^2 dx \\
&\leq \tilde{C}' \int_{|z| < \gamma} \frac{dz}{|z|^{n+s-1}} \left(\int_{|y| < R} dy \right)^2 \\
&\leq C'' \\
&< +\infty
\end{aligned} \tag{67}$$

where C'' is a positive constant depending on R, γ, s and n .

Therefore, by (65) and (67), for $2 \leq k < n - 4s$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\int_{|x-y|<\gamma} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ & \leq \int_{|x|<R} \left(\int_{|x-y|<\gamma, |y|<R} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ & \quad + \int_{|x|\geq R} \left(\int_{|x-y|<\gamma, |y|\geq R} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ & < +\infty. \end{aligned} \quad (68)$$

and so by using (60) and (68) together with (57), for $2 \leq k < n - 4s$, we obtain

$$\int_{|x|\leq \frac{\eta}{\varepsilon}} \left(\int_{|y|\leq \frac{\eta}{\varepsilon}} |f_k(x) - f_k(y)| \frac{dy}{|x-y|^{n+s}} \right)^2 dx \leq \tilde{C} < +\infty, \quad (69)$$

where $\tilde{C}$ is a positive constant depending on s, k and n . Thus, (55) holds true.

Now, to prove (56), we need to apply Lemma 2.6 in [8]. To guarantee the comfort of the reader, we recall the statement of this lemma.

Lemma 1. Let $\gamma, R \in \mathbb{R}_+^*$ and $s \in (0, 1)$. Let $k \geq 2$ and let the set $\mathcal{A}$ defined by $\mathcal{A} := \{x, y \in \mathbb{R}^n, |x - y| < \gamma, |x| \leq R, |y| \leq R\}$. Then, there exists $\delta := \delta(R, k, s) > 0$, such that for every $x, y \in \mathcal{A}$, we have

$$||x|^{\frac{k}{2}} - |y|^{\frac{k}{2}}|^2 \leq \delta |x - y|^2. \quad (70)$$

(In fact, $\delta = 2^{k-4}k^2R^{k-2}$.)

So now, we apply this lemma for $R = \frac{\eta}{\varepsilon}$ to prove (56), and we take $|x - y| < \gamma < 1$. Since $s < 1$, we have

$$\begin{aligned} & \int_{|x|\leq \frac{\eta}{\varepsilon}} \left(\int_{|y|\leq \frac{\eta}{\varepsilon}, |x-y|<\gamma} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ & \leq \delta_2 \int_{|x|\leq \frac{\eta}{\varepsilon}} \left(\int_{|y|\leq \frac{\eta}{\varepsilon}} \frac{1}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)}(x-y)}{|x-y|^{n+s-1}} dy \right)^2 dx \\ & = 2^{k-4}k^2\eta^{k-2}\varepsilon^{-k+2} \int_{\mathbb{R}^n} |\tilde{h}_1 * \tilde{g}_1(z)|^2 dz \end{aligned} \quad (71)$$

where $\tilde{h}_1$ and $\tilde{g}_1$ are defined by

$$\tilde{h}_1(z) := \frac{1}{(1 + |z|^2)^{\frac{n-2s}{2}}}, \text{ and } \tilde{g}_1(z) := \frac{1_{B(0,\gamma)}(z)}{|z|^{n+s-1}}. \quad (72)$$

For $n > 4s$, we have $\tilde{h}_1 \in L^2(\mathbb{R}^n)$, in other words, provided that

$$\begin{aligned} & \text{if } n = 3 \text{ and } s < \frac{3}{4} \\ & \text{or } n \geq 4 \text{ and } 0 < s < 1, \end{aligned} \quad (73)$$

which is satisfied since n and s satisfy (45).

Since $s < 1$, then $\tilde{g}_1 \in L^1(\mathbb{R}^n)$. Therefore, $\tilde{h}_1 * \tilde{g}_1 \in L^2(\mathbb{R}^n)$.

Finally, we obtain

$$\int_{|x| \leq \frac{\eta}{\varepsilon}} \left(\int_{|y| \leq \frac{\eta}{\varepsilon}, |x-y| < \gamma} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \leq \tilde{C}'_0 \varepsilon^{-k+2}, \quad (74)$$

where $\tilde{C}'_0$ is a positive constant depending on s, k, n and η . Now, to finish the proof of (56), it remains to study the integral in (56) over the set $\Gamma_\varepsilon := \{x, y \in \mathbb{R}^n \text{ such that } |x| < \frac{\eta}{\varepsilon}, |y| < \frac{\eta}{\varepsilon} \text{ and } |x-y| > \gamma\}$. Let

$$\tilde{X}_\varepsilon := \int_{|x| < \frac{\eta}{\varepsilon}} \left(\int_{|y| < \frac{\eta}{\varepsilon}, |x-y| > \gamma} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx. \quad (75)$$

In the following, we prove that

$$\tilde{X}_\varepsilon \leq C \quad (76)$$

with C being a positive constant depending on k, s and n .

We have

$$\begin{aligned} \tilde{X}_\varepsilon &= \int_{0 \leq |x| \leq 1} \left(\int_{|y| < \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y) dy}{|x-y|^{n+s}} \right)^2 dx \\ &\quad + \int_{1 < |x| < \frac{\eta}{\varepsilon}} \left(\int_{|y| < \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y) dy}{|x-y|^{n+s}} \right)^2 dx. \end{aligned} \quad (77)$$

We start by treating the first term on the right-hand side of (77).

Let $x, y \in \mathbb{R}^n$ such that $|x| \leq 1$ and $|x-y| \geq \gamma$; then, we have

$$|y| \leq |x-y| + |x| \leq |x-y| + 1. \quad (78)$$

Then, by computing the minimum of the function $f : t \mapsto \frac{t}{2+t}, t \in [\gamma, +\infty[$ which is achieved for $t = \gamma$, we have

$$\frac{|x-y|}{1+|y|} \geq \frac{|x-y|}{2+|x-y|} \geq \frac{\gamma}{2+\gamma} := C_0 > 0, \quad (79)$$

where C_0 is a positive constant.

Since $k < n - 4s$, then $k < 2n - 2s$, and by (79), we have

$$\begin{aligned} &\int_{0 \leq |x| \leq 1} \left(\int_{|y| < \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y) dy}{|x-y|^{n+s}} \right)^2 dx \\ &\leq C \int_{0 \leq |x| \leq 1} dx \left(\int_{|y| < \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} + 1|}{(1+|y|^2)^{\frac{n-2s}{2}} (1+|y|)^{n+s}} dy \right)^2 \\ &\leq C', \end{aligned} \quad (80)$$

where C' is a positive constant depending on n, s, k and γ .

We treat the second term on the right-hand side of (77).

$$\begin{aligned} &\int_{1 < |x| < \frac{\eta}{\varepsilon}} \left(\int_{|y| < \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y) dy}{|x-y|^{n+s}} \right)^2 dx \\ &= \int_{1 < |x| < \frac{\eta}{\varepsilon}} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y) dy}{|x-y|^{n+s}} \right)^2 dx \\ &\quad + \int_{1 < |x| < \frac{\eta}{\varepsilon}} \left(\int_{1 \leq |y| < \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1+|y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y) dy}{|x-y|^{n+s}} \right)^2 dx \end{aligned} \quad (81)$$

For the first term of the previous sum in (81), since for $x, y \in \mathbb{R}^n$ such that $|y| \leq 1$, we have $|x - y|^{n+s} \geq (|x| - |y|)^{n+s} \geq (|x| - 1)^{n+s}$, thus we have

$$\begin{aligned} & \int_{1 < |x| < \frac{\eta}{\epsilon}} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+s}} dy \right)^2 dx \\ &= \int_{1 < |x| \leq 5} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+s}} dy \right)^2 dx \\ &+ \int_{5 < |x| < \frac{\eta}{\epsilon}} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+s}} dy \right)^2 dx \\ &\leq \frac{1}{\gamma^{2n+2s}} \int_{1 < |x| \leq 5} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy \right)^2 dx \\ &+ \int_{5 < |x| < \frac{\eta}{\epsilon}} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy \right)^2 (|x| - 1)^{n+s} dx. \end{aligned} \quad (82)$$

We estimate the first term of the last sum (82), and we have

$$\begin{aligned} \int_{1 < |x| \leq 5} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy \right)^2 dx &\leq \int_{1 < |x| \leq 5} \left(\int_{0 \leq |y| \leq 1} \frac{|y|^{\frac{k}{2}} + |x|^{\frac{k}{2}}}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy \right)^2 dx \\ &\leq 2 \int_{1 < |x| \leq 5} \left(\int_{0 \leq |y| \leq 1} \frac{|x|^{\frac{k}{2}}}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy \right)^2 dx \\ &+ 2 \int_{1 < |x| \leq 5} \left(\int_{0 \leq |y| \leq 1} \frac{|y|^{\frac{k}{2}}}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy \right)^2 dx \\ &= 2 \int_{1 < |x| \leq 5} |x|^k dx \left(\int_{0 \leq |y| \leq 1} \frac{dy}{(1 + |y|^2)^{\frac{n-2s}{2}}} \right)^2 dx \\ &+ 2 \int_{1 < |x| \leq 5} dx \left(\int_{0 \leq |y| \leq 1} \frac{|y|^{\frac{k}{2}}}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy \right)^2, \end{aligned} \quad (83)$$

Every integral here represents an integral of a continuous function over a compact set; then, each one is convergent (and is bounded independently of ϵ), and we obtain

$$\int_{1 < |x| \leq 5} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy \right)^2 dx \leq C, \quad (84)$$

where C is a positive constant depending on n, s and k .

Now, for the second term of the last sum in (82), we write

$$\begin{aligned} & \int_{5 < |x| < \frac{\eta}{\epsilon}} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} (|x| - 1)^{n+s} dy \right)^2 dx \\ &\leq 2 \int_{5 < |x| < \frac{\eta}{\epsilon}} \left(\int_{0 \leq |y| \leq 1} \frac{|x|^{\frac{k}{2}}}{(1 + |y|^2)^{\frac{n-2s}{2}}} (|x| - 1)^{n+s} dy \right)^2 dx \\ &+ 2 \int_{5 < |x| < \frac{\eta}{\epsilon}} \left(\int_{0 \leq |y| \leq 1} \frac{|y|^{\frac{k}{2}}}{(1 + |y|^2)^{\frac{n-2s}{2}}} (|x| - 1)^{n+s} dy \right)^2 dx \\ &\leq 2 \int_{5 < |x| \leq +\infty} \frac{|x|^k}{(|x| - 1)^{2n+2s}} dx \left(\int_{0 \leq |y| \leq 1} \frac{dy}{(1 + |y|^2)^{\frac{n-2s}{2}}} \right)^2 \\ &+ 2 \int_{5 < |x| \leq +\infty} \frac{dx}{(|x| - 1)^{2n+2s}} \left(\int_{0 \leq |y| \leq 1} \frac{|y|^{\frac{k}{2}}}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy \right)^2 dx \end{aligned} \quad (85)$$

Clearly, the integrals $\int_{0 \leq |y| \leq 1} \frac{dy}{(1 + |y|^2)^{\frac{n-2s}{2}}}$ and $\int_{0 \leq |y| \leq 1} \frac{|y|^{\frac{k}{2}}}{(1 + |y|^2)^{\frac{n-2s}{2}}} dy$ are convergent as integrals of continuous functions over a compact set. Since $k < n - 4s$, then $k < n + 2s$, and we have

$$\int_{5 < |x| \leq +\infty} \frac{|x|^k}{(|x| - 1)^{2n+2s}} dx \leq C_0,$$

where C_0 is a positive constant possibly depending on n, s and k . Also, since $n + 2s > 0$, we have

$$\int_{5 < |x| \leq +\infty} \frac{dx}{(|x| - 1)^{2n+2s}} \leq C_1,$$

where C_1 is a positive constant possibly depending on n, s and k . Therefore, we obtain

$$\int_{5 < |x| < \frac{\eta}{\varepsilon}} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}} (|x| - 1)^{n+s}} dy \right)^2 dx \leq C, \quad (86)$$

where C is a positive constant possibly depending on n, s, η and k .

Thus, thanks to (84), (86) together with (82), we obtain

$$\int_{1 < |x| < \frac{\eta}{\varepsilon}} \left(\int_{0 \leq |y| \leq 1} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+s}} dy \right)^2 dx \leq C, \quad (87)$$

where C is a positive constant possibly depending on n, s, η and k .

Let us consider the second term of the sum (81). Here, we refer the reader to [28]. For $x, y \in \mathbb{R}^n$, let $x = rx'$, $y = \rho y'$ and $|x'| = |y'| = 1$, and perform the following change of variables $\tau = \frac{\rho}{r}$; then, we have

$$\begin{aligned} & \int_{1 < |x| < \frac{\eta}{\varepsilon}} \left(\int_{1 \leq |y| < \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+s}} dy \right)^2 dx \\ & \leq \int_1^{\frac{\eta}{\varepsilon}} \left(\int_1^{\frac{\eta}{\varepsilon}} \int_{|y'|=1} \int_{|x'|=1} \frac{|\rho^{\frac{k}{2}} - r^{\frac{k}{2}}| d\sigma(x') d\sigma(y')}{(1 + \rho^2)^{\frac{n-2s}{2}} |rx' - \rho y'|^{n+s}} \rho^{n-1} d\rho \right)^2 r^{n-1} dr \\ & \leq \int_1^{\frac{\eta}{\varepsilon}} \left(\int_1^{\frac{\eta}{\varepsilon}} \int_{|y'|=1} \int_{|x'|=1} \frac{r^{\frac{k}{2}} \left| \left(\frac{\rho}{r} \right)^{\frac{k}{2}} - 1 \right| \rho^{n-1} d\sigma(x') d\sigma(y')}{\rho^{n-2s} r^{n+s} |x' - \frac{\rho}{r} y'|^{n+s}} d\rho \right)^2 r^{n-1} dr \\ & = \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{r}}^{\frac{\eta}{r}} \int_{|y'|=1} \int_{|x'|=1} \frac{|\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\sigma(y')}{|x' - \tau y'|^{n+s}} d\tau \right)^2 dr \\ & = \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{r}}^{\frac{1}{2}} \int_{|y'|=1} \int_{|x'|=1} \frac{|\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\sigma(y')}{|x' - \tau y'|^{n+s}} d\tau \right)^2 dr \\ & + \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{2}}^1 \int_{|y'|=1} \int_{|x'|=1} \frac{|\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\sigma(y')}{|x' - \tau y'|^{n+s}} d\tau \right)^2 dr \\ & + \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_1^{\frac{\eta}{r}} \int_{|y'|=1} \int_{|x'|=1} \frac{|\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\sigma(y')}{|x' - \tau y'|^{n+s}} d\tau \right)^2 dr. \end{aligned} \quad (88)$$

We will estimate the terms of the last sum in (88), and we begin with the first term. Since $|x - y| > \gamma$, then there exists $\bar{\delta} > 0$ such that $|x' - \tau y'| > \bar{\delta}$, and since $k < n - 4s$, then $k < n - 2s + 2$ and $k < n + 2s - 2$. When ε is small enough, we have

$$\begin{aligned} & \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{r}}^{\frac{1}{2}} \int_{|y'|=1} \int_{|x'|=1} \frac{|\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\sigma(y')}{|x' - \tau y'|^{n+s}} d\tau \right)^2 dr \\ & \leq C \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{r}}^{\frac{1}{2}} |\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\tau \right)^2 dr \\ & \leq C \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{r}}^{\frac{1}{2}} \tau^{\frac{k}{2}+2s-1} d\tau \right)^2 dr \\ & + C \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{2}}^1 \tau^{2s-1} d\tau \right)^2 dr \\ & = C \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \frac{1}{\left(\frac{k}{2} + 2s\right)^2} \left(\left(\frac{1}{2}\right)^{\frac{k}{2}+2s} - r^{-\frac{k}{2}-2s} \right)^2 dr \\ & + C \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \frac{1}{(2s)^2} \left(\frac{1}{2^{2s}} - r^{-2s} \right)^2 dr \\ & \leq C \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\frac{1}{2}\right)^{k+4s} dr + C \int_1^{\frac{\eta}{\varepsilon}} r^{-n-2s-3} dr \\ & + C \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \frac{1}{2^{4s}} dr + C \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k-2s-3} dr \\ & \leq C \left[\frac{r^{-n+k+2s-2}}{-n+k+2s-2} \right]_1^{\frac{\eta}{\varepsilon}} + C \left[\frac{r^{-n-2s-2}}{-n-2s-2} \right]_1^{\frac{\eta}{\varepsilon}} \\ & + C \left[\frac{r^{-n+k+2s-2}}{-n+k+2s-2} \right]_1^{\frac{\eta}{\varepsilon}} + C \left[\frac{r^{-n+k-2s-2}}{-n+k-2s-2} \right]_1^{\frac{\eta}{\varepsilon}} \\ & \leq C, \end{aligned} \quad (89)$$

where C is a positive constant possibly depending on η, s, k and n .

Notice that $K(\tau) := \int_{|y'|=1} \frac{d\sigma(y')}{|x' - \tau y'|^{n+2s}}$ is independent of $x' \in \{|y| = 1\}$ (see [28]), and we have $K(\tau) = \tau^{2-n}(\tau^2 - 1)^{-1-2s}H(\tau)$, $\tau \geq 1$, where $H(\tau)$ is a positive continuous function on $[1, +\infty[$ such that $H(\tau) \approx \tau^{2s}$ as τ tends to $+\infty$.

Therefore, for the second term of the sum in (88), performing a change of variables $\xi := \frac{1}{\tau}$, we have

$$\begin{aligned} & \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{2}}^1 \int_{|y'|=1} \int_{|x'|=1} \frac{|\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\sigma(y')}{|x' - \tau y'|^{n+2s}} d\tau \right)^2 dr \\ &= \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{2}}^1 \int_{|x'|=1} \left(\int_{|y'|=1} \frac{d\sigma(y')}{|x' - \tau y'|^{n+2s}} \right) |\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\tau \right)^2 dr \\ &= \sigma^2(S^{n-1}) \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_{\frac{1}{2}}^1 K(\tau) |\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\tau \right)^2 dr \\ &= \sigma^2(S^{n-1}) \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} dr \left(\int_1^2 |\xi^{-\frac{k}{2}} - 1| \xi^{-2s-1} K(\frac{1}{\xi}) d\xi \right)^2 \\ &= \sigma^2(S^{n-1}) \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} dr \left(\int_1^2 |\xi^{-\frac{k}{2}} - 1| \xi^{-2s-1} \xi^{n+2s} K(\xi) d\xi \right)^2 \\ &= \sigma^2(S^{n-1}) \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} dr \left(\int_1^2 |\xi^{-\frac{k}{2}} - 1| \xi^{n-1} \xi^{2-n} (\xi^2 - 1)^{-1-s} H(\xi) d\xi \right)^2 \\ &= \sigma^2(S^{n-1}) \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} dr \left(\int_1^2 |\xi^{-\frac{k}{2}} - 1| \xi (\xi^2 - 1)^{-1-s} H(\xi) d\xi \right)^2, \end{aligned} \quad (90)$$

where $\sigma(S^{n-1})$ is the Lebesgue measure of the unit sphere in $\mathbb{R}^n$. Since $k < n - 4s$, then $k < n - 2s + 2$, and we have

$$\int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} dr = \left[\frac{r^{-n+k+2s-2}}{-n+k+2s-2} \right]_1^{\frac{\eta}{\varepsilon}} \leq C, \quad (91)$$

where C is a positive constant depending on n, s and k .

The function $|1 - \xi^{-\frac{k}{2}}|^2 \xi^{n-2s+1} H(\xi) (\xi^2 - 1)^{-1-2s}$ is continuous on $]1, 2]$, and in the neighborhood of 1, this function is equivalent to $\frac{|1 - \xi^{-\frac{k}{2}}| H(1)}{(\xi^2 - 1)^{1+s}} = \xi^{-\frac{k}{2}} \frac{|\xi^{\frac{k}{2}} - 1| H(1)}{(\xi^2 - 1)^{1+s}}$, which in the neighborhood of $\xi = 1$ is equivalent to $\frac{|\xi^{\frac{k}{2}} - 1| H(1)}{(\xi^2 - 1)^{1+s}}$, and if we note $\xi' := \xi - 1$, we obtain after performing a limited development on $\xi' = 0$

$$\begin{aligned} \frac{|\xi^{\frac{k}{2}} - 1|}{(\xi^2 - 1)^{1+s}} &= \frac{|(\xi' + 1)^{\frac{k}{2}} - 1|}{((\xi' + 1)^2 - 1)^{1+s}} \\ &= \xi'^{-s} \left(\frac{(\frac{k}{2})}{2^{1+s}} + o(1) \right), \end{aligned} \quad (92)$$

which is integrable in the neighborhood of $\xi' = 0$, since $s < 1$.

Therefore, for $k < n - 4s$ and $s \in (0, 1)$, we obtain

$$\int_1^2 |\xi^{-\frac{k}{2}} - 1| \xi (\xi^2 - 1)^{-1-s} H(\xi) d\xi \leq C, \quad (93)$$

where C is a positive constant depending on s, k and n .

Thus, using (91) and (93) together with (90), for $2 \leq k < n - 4s$, we find

$$\int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_1^1 \int_{|y'|=1} \int_{|x'|=1} \frac{|\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\sigma(y')}{|x' - \tau y'|^{n+s}} d\tau \right)^2 dr \leq C, \quad (94)$$

where C is a positive constant depending on s, k and n .

We treat the third term of the sum in (88) and we proceed as in (90); we write

$$\begin{aligned} & \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_1^{\frac{\eta}{\varepsilon}} \int_{|y'|=1} \int_{|x'|=1} \frac{|\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\sigma(y')}{|x' - \tau y'|^{n+s}} d\tau \right)^2 dr \\ &= \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_1^{\frac{\eta}{\varepsilon}} \int_{|x'|=1} K(\tau) |\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\tau \right)^2 dr \\ &= \sigma^2(S^{n-1}) \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} dr \left(\int_1^{\frac{\eta}{\varepsilon}} \tau^{2-n} (\tau^2 - 1)^{-1-s} H(\tau) |\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\tau \right)^2 \\ &= \sigma^2(S^{n-1}) \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} dr \left(\int_1^{\frac{\eta}{\varepsilon}} \tau^{-n+2s+1} (\tau^2 - 1)^{-1-s} |\tau^{\frac{k}{2}} - 1| H(\tau) d\tau \right)^2 \\ &\leq \sigma^2(S^{n-1}) \int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} dr \left(\int_1^{+\infty} \tau^{-n+2s+1} (\tau^2 - 1)^{-1-s} |\tau^{\frac{k}{2}} - 1| H(\tau) d\tau \right)^2. \end{aligned} \quad (95)$$

Let us treat both of the integrals in the last expression of (95), for $k < n - 2s + 2$ (which is satisfied since $k < n - 4s$) by (91), we have

$$\int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} dr \leq C. \quad (96)$$

The second integral of (95), which is the following,

$$\int_1^{+\infty} \tau^{-n+2s+1} (\tau^2 - 1)^{-1-s} |\tau^{\frac{k}{2}} - 1| H(\tau) d\tau, \quad (97)$$

is the integral of a continuous function on $]1, +\infty[$. In the neighborhood of $+\infty$, since $H(\tau) \approx \tau^{2s}$ as τ tends to $+\infty$, the function inside the integral is equivalent to $\frac{1}{\tau^{n-2s+1-\frac{k}{2}}}$, which is integrable at $+\infty$, since $k < 2n - 4s$ (in fact, $k < n - 4s$).

In the neighborhood of 1, after performing the change of variables $\tau' = \tau - 1$, the function inside the integral (97) is equivalent to $\frac{|\tau^{\frac{k}{2}} - 1|}{(\tau^2 - 1)^{1+s}}$, we use the same computations as in (92), and since $s < 1$, we find that it is integrable in the neighborhood of $\tau = 1$. Thus, for $2 \leq k < n - 4s$, we obtain

$$\int_1^{+\infty} \tau^{-n+2s+1} (\tau^2 - 1)^{-1-s} |\tau^{\frac{k}{2}} - 1| H(\tau) d\tau \leq C, \quad (98)$$

where C is a positive constant depending on s, k and n .

Therefore, using (96) and (98) together with (95), for $2 \leq k < n - 4s$, we obtain

$$\int_1^{\frac{\eta}{\varepsilon}} r^{-n+k+2s-3} \left(\int_1^{\frac{\eta}{\varepsilon}} \int_{|y'|=1} \int_{|x'|=1} \frac{|\tau^{\frac{k}{2}} - 1| \tau^{2s-1} d\sigma(x') d\sigma(y')}{|x' - \tau y'|^{n+s}} d\tau \right)^2 dr \leq C, \quad (99)$$

where C is a positive constant depending on s, k and n .

Thereby, using (89), (94) and (99) together with (88), for $2 \leq k < n - 4s$, we obtain

$$\int_{1 < |x| < \frac{\eta}{\varepsilon}} \left(\int_{1 \leq |y| < \frac{\eta}{\varepsilon}} \frac{||y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+s}} dy \right)^2 dx \leq C, \quad (100)$$

where C is a positive constant depending on s, k, η and n .

Then, by (87) and (100) together with (81), we obtain

$$\int_{1 < |x| < \frac{\eta}{\varepsilon}} \left(\int_{|y| < \frac{\eta}{\varepsilon}} \frac{|y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{1_{B(0,\gamma)^c}(x-y)}{|x-y|^{n+s}} dy \right)^2 dx \leq C, \quad (101)$$

where C is a positive constant depending on s, k, η and n .

Thereby, thanks to (80) and (101) together with (77), we obtain

$$\begin{aligned} \tilde{X}_\varepsilon &= \int_{|x| \leq \frac{\eta}{\varepsilon}} \left(\int_{|y| < \frac{\eta}{\varepsilon}, |x-y| > \gamma} \frac{|y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ &\leq C, \end{aligned} \quad (102)$$

where C is a positive constant depending on s, k, η and n . Then, (76) yields.

Consequently, by (74), (75) and (76), for $2 \leq k < n - 4s$, we have

$$\begin{aligned} &\int_{|x| \leq \frac{\eta}{\varepsilon}} \left(\int_{|y| \leq \frac{\eta}{\varepsilon}} \frac{|y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_{|x-y| < \gamma} \frac{|y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ &\quad + \int_{|x| < \frac{\eta}{\varepsilon}} \left(\int_{|y| < \frac{\eta}{\varepsilon}, |x-y| > \gamma} \frac{|y|^{\frac{k}{2}} - |x|^{\frac{k}{2}}|}{(1 + |y|^2)^{\frac{n-2s}{2}}} \frac{dy}{|x-y|^{n+s}} \right)^2 dx \\ &\leq C + \tilde{C}_0' \varepsilon^{-k+2}, \end{aligned} \quad (103)$$

where C is a positive constant possibly depending on s, k, η and n . Thus, (56) yields.

Now, since (55) and (56) both hold true, in (54), for $2 \leq k < n - 4s$, we obtain

$$\begin{aligned} \tilde{A}_{s,k,\varepsilon} &= \int_{B(0,\eta)} |x|^k \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x-y|^{n+s}} dy \right|^2 dx \\ &\leq 2\varepsilon^k (\tilde{C} + C + \tilde{C}_0' \varepsilon^{-k+2}) \\ &= C\varepsilon^2, \end{aligned} \quad (104)$$

where C is a positive constant depending on s, k and the dimension n . Thus, we finish the proof of Proposition 3. $\square$

Now, we estimate the last term on the right-hand side of (50). Using (48), we have

$$\begin{aligned} &\int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x-y|^{n+s}} dy \right|^2 dx \\ &\leq C\varepsilon^{n-2s} \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)^c} \frac{\min\{1, |x-y|\}}{|x-y|^{n+s}} dy \right|^2 dx \\ &= C\varepsilon^{n-2s} \left[\int_{|x| > \eta} (p(x) - p_0) \left| \int_{|y| > \eta} \frac{1_{B(0,1)}(x-y)}{|x-y|^{n+s-1}} dy \right|^2 dx \right. \\ &\quad \left. + \int_{|x| > \eta} (p(x) - p_0) \left| \int_{|y| > \eta} \frac{1_{B(0,1)^c}(x-y)}{|x-y|^{n+s}} dy \right|^2 dx \right] \\ &= C\varepsilon^{n-2s} \left[\int_{|x| > \eta} (p(x) - p_0) \left| \int_{|x+z| > \eta} \frac{1_{B(0,1)}(z)}{|z|^{n+s-1}} dz \right|^2 dx \right. \\ &\quad \left. + \int_{|x| > \eta} (p(x) - p_0) \left| \int_{|x+z| > \eta} \frac{1_{B(0,1)^c}(z)}{|z|^{n+s}} dz \right|^2 dx \right] \\ &\leq C\varepsilon^{n-2s} \left[\int_{|x| > \eta} (p(x) - p_0) dx \left| \int_{\mathbb{R}^n} \frac{1_{B(0,1)}(z)}{|z|^{n+s-1}} dz \right|^2 \right. \\ &\quad \left. + \int_{|x| > \eta} (p(x) - p_0) dx \left| \int_{\mathbb{R}^n} \frac{1_{B(0,1)^c}(z)}{|z|^{n+s}} dz \right|^2 \right], \end{aligned} \quad (105)$$

Since $s \in (0, 1)$, $\int_{\mathbb{R}^n} \frac{1_{B(0,1)}(z)}{|z|^{n+s-1}} dz < +\infty$ and $\int_{\mathbb{R}^n} \frac{1_{B(0,1)^c}(z)}{|z|^{n+s}} dz < +\infty$. Furthermore, by (2), $p - p_0 \in L^1(\mathbb{R}^n)$, then $\int_{|x|>\eta} (p(x) - p_0) dx < \infty$.

Thus,

$$\int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx = \mathcal{O}(\varepsilon^{n-2s}), \quad (106)$$

as ε tends to zero.

We move now to estimate the second term of the sum in (50). Using the assertion (47) and the fact that $s < 1$, we have

$$\begin{aligned} & \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx \\ & \leq C\varepsilon^{n-2s} \int_{|x|<\eta} \eta^k \left| \int_{|y|\geq\eta} \frac{1_{B(0,\frac{\eta}{2})}(x - y)}{|x - y|^{n+s-1}} dy \right|^2 dx \\ & \leq C\varepsilon^{n-2s} \eta^k \int_{|x|<\eta} \left| \int_{|z+x|\geq\eta} \frac{1_{B(0,\frac{\eta}{2})}(z)}{|z|^{n+s-1}} dz \right|^2 dx \\ & \leq C\varepsilon^{n-2s} \eta^k \int_{|x|<\eta} dx \left| \int_{\mathbb{R}^n} \frac{1_{B(0,\frac{\eta}{2})}(z)}{|z|^{n+s-1}} dz \right|^2 \\ & = \mathcal{O}(\varepsilon^{n-2s}), \end{aligned} \quad (107)$$

as ε tends to zero.

We treat now the third term of the sum (50), and we will prove the following estimation

$$\int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx = \mathcal{O}(\varepsilon^{n-2s}) + \mathcal{O}(\varepsilon^{k+2s}), \quad (108)$$

as ε tends to zero.

Let us recall that $u_{\varepsilon,s}(x) = U_{\varepsilon,s}(x)$ for all $x \in B(0, \eta)$. We use (46) for $\eta = \rho$, and since $s > 0$, we have

$$\begin{aligned} & \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{|u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)|}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx \\ & \leq 2 \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(x)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx \\ & + 2 \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx \end{aligned} \quad (109)$$

We treat the second term of the sum in (109)

$$\begin{aligned} & \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{u_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx \\ & \leq \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{U_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx \\ & \leq C\varepsilon^{n-2s} \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{1_{B(0,\frac{\eta}{2})^c}(x - y)}{|x - y|^{n+s}} dy \right|^2 dx \\ & \leq C\varepsilon^{n-2s} \int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{\mathbb{R}^n} \frac{1_{B(0,\frac{\eta}{2})^c}(x - y)}{|x - y|^{n+s}} dy \right|^2 dx \\ & = C\varepsilon^{n-2s} \int_{B(0,\eta)} e^{-|x|^2} |x|^k dx \left| \int_{\mathbb{R}^n} \frac{1_{B(0,\frac{\eta}{2})^c}(z)}{|z|^{n+s}} dz \right|^2 \\ & \leq C\eta^k \varepsilon^{n-2s} \int_{B(0,\eta)} dx \left| \int_{|z|>\frac{\eta}{2}} \frac{1}{|z|^{n+s}} dz \right|^2 \\ & = \mathcal{O}(\varepsilon^{n-2s}), \end{aligned} \quad (110)$$

as ε tends to zero.

Now, we move to the first term of the right-hand side of (109), and we will prove that

$$\int_{B(0,\eta)} e^{-|x|^2} |x|^k \left| \int_{B(0,\eta)^c} \frac{U_{\varepsilon,s}(x)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x-y) dy \right|^2 dx = \mathcal{O}(\varepsilon^{k+2s}), \quad (111)$$

as ε tends to zero.

Let $\tilde{g}_{1,k,\varepsilon}$ and h be two functions such that for all $x \in \mathbb{R}^n$, $\tilde{g}_{1,k,\varepsilon} := e^{-\frac{|x|^2}{2}} |x|^{\frac{k}{2}} U_{\varepsilon,s}(x)$ and $h(x) := \frac{1_{B(0,\frac{\eta}{2})^c}(x)}{|x|^{n+s}}$. Since $s > 0$, $h \in L^1(\mathbb{R}^n)$. We have

$$\begin{aligned} \|\tilde{g}_{1,k,\varepsilon}\|_{L^2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |U_{\varepsilon,s}(x)|^2 dx \\ &= \varepsilon^{n-2s} \int_{\mathbb{R}^n} \frac{e^{-|x|^2} |x|^k}{(\varepsilon^2 + |x|^2)^{n-2s}} dx \\ &= \varepsilon^{-n+2s} \int_{\mathbb{R}^n} \frac{e^{-|x|^2} |x|^k}{(1 + \frac{|x|^2}{\varepsilon^2})^{n-2s}} dx \\ &= \varepsilon^{k+2s} \left(\int_{\mathbb{R}^n} \frac{|z|^k}{(1 + |z|^2)^{n-2s}} dz + o(1) \right). \end{aligned} \quad (112)$$

Since $2 \leq k < n - 4s$ (for n and s as in (45)), the integral $\int_{\mathbb{R}^n} \frac{|z|^k}{(1 + |z|^2)^{n-2s}} dz$ is finite. Therefore, $\|\tilde{g}_{1,k,\varepsilon}\|_{L^2(\mathbb{R}^n)}^2 = \mathcal{O}(\varepsilon^{k+2s})$ as ε tends to zero. Then, we have

$$\begin{aligned} &\int_{B(0,\eta)} |x|^k \left| \int_{B(0,\eta)^c} \frac{U_{\varepsilon,s}(x)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x-y) dy \right|^2 dx \\ &\leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} |x|^{\frac{k}{2}} U_{\varepsilon,s}(x) \frac{1_{B(0,\frac{\eta}{2})^c}(x-y)}{|x-y|^{n+s}} dy \right|^2 dx \\ &= \|\tilde{g}_{1,k,\varepsilon} * h\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \|\tilde{g}_{1,k,\varepsilon}\|_{L^2(\mathbb{R}^n)}^2 \|h\|_{L^1(\mathbb{R}^n)}^2 \\ &= \mathcal{O}(\varepsilon^{k+2s}), \end{aligned} \quad (113)$$

as ε tends to zero. Thus, (111) yields.

Using (110) and (111) together with (109), we deduce that for n and s as described in (45) and for $2 \leq k < n - 4s$, (108) holds true.

Now, let us move to treat the fourth term of the sum (50). Thanks to (2) and (47) and since $s < 1$, we have

$$\begin{aligned} &\int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})}(x-y) dy \right|^2 dx \\ &\leq \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} C \varepsilon^{\frac{n-2s}{2}} \frac{1_{B(0,\frac{\eta}{2})}(x-y)}{|x-y|^{n+s-1}} dy \right|^2 dx \\ &\leq C \varepsilon^{n-2s} \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{\mathbb{R}^n} \frac{1_{B(0,\frac{\eta}{2})}(x-y)}{|x-y|^{n+s-1}} dy \right|^2 dx \\ &= C \varepsilon^{n-2s} \int_{B(0,\eta)^c} (p(x) - p_0) dx \left| \int_{\mathbb{R}^n} \frac{1_{B(0,\frac{\eta}{2})}(z)}{|z|^{n+s-1}} dz \right|^2 \\ &= \mathcal{O}(\varepsilon^{n-2s}), \end{aligned} \quad (114)$$

where C and C' are positive constants possibly depending on η, s and n .

Now, we move to the fifth term of the sum (50), and we will prove the following

$$\begin{aligned} & \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x-y) dy \right|^2 dx \\ & \leq 2 \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x-y) dy \right|^2 dx \\ & + 2 \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(y)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x-y) dy \right|^2 dx \end{aligned} \quad (115)$$

We start by the estimation of the first term of the previous sum; we use (46) for $\eta = \rho$

$$\begin{aligned} & \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x-y) dy \right|^2 dx \\ & \leq \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{U_{\varepsilon,s}(x)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x-y) dy \right|^2 dx \\ & \leq C\varepsilon^{n-2s} \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{1_{B(0,\frac{\eta}{2})^c}(x-y)}{|x-y|^{n+s}} dy \right|^2 dx \\ & \leq C\varepsilon^{n-2s} \int_{B(0,\eta)^c} (p(x) - p_0) dx \left| \int_{\mathbb{R}^n} \frac{1_{B(0,\frac{\eta}{2})^c}(z)}{|z|^{n+s}} dz \right|^2. \end{aligned} \quad (116)$$

Since we have $p - p_0 \in L^1(\mathbb{R}^n)$ (by (2)) and since $s > 0$, $\int_{B(0,\eta)} \frac{1_{B(0,\frac{\eta}{2})^c}(z)}{|z|^{n+s}} dz$ is bounded. Therefore,

$$\int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x-y) dy \right|^2 dx = \mathcal{O}(\varepsilon^{n-2s}), \quad (117)$$

as ε tends to zero.

For the the second term of the sum in (115), using (2) and (14), we have

$$\begin{aligned} & \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(y)}{|x-y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x-y) dy \right|^2 dx \\ & \leq C' \int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} U_{\varepsilon,s}(y) dy \right|^2 dx \\ & \leq C \left| \int_{B(0,\eta)} U_{\varepsilon,s}(y) dy \right|^2 \\ & = C\varepsilon^{n-2s} \left| \int_{|y|<\eta} \frac{dy}{(\varepsilon^2 + |y|^2)^{\frac{n-2s}{2}}} \right|^2 \\ & = C\varepsilon^{n+2s} \left| \int_{|z|<\frac{\eta}{\varepsilon}} \frac{dz}{(1 + |z|^2)^{\frac{n-2s}{2}}} \right|^2 \\ & = C\varepsilon^{n+2s} \left| \int_0^{\frac{\eta}{\varepsilon}} \frac{r^{n-1} dr}{(1 + r^2)^{\frac{n-2s}{2}}} \right|^2 \\ & = \mathcal{O}(\varepsilon^{n-2s}), \end{aligned} \quad (118)$$

since $\varepsilon^{2s} \int_0^{\frac{\eta}{\varepsilon}} \frac{r^{n-1} dr}{(1 + r^2)^{\frac{n-2s}{2}}}$ tends to a positive constant independent of ε as ε tends to zero.

Thus, using (117), (118) together with (115), we obtain

$$\int_{B(0,\eta)^c} (p(x) - p_0) \left| \int_{B(0,\eta)} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} 1_{B(0,\frac{\eta}{2})^c}(x - y) dy \right|^2 dx = \mathcal{O}(\varepsilon^{n-2s}) \quad (119)$$

Therefore, by using Proposition 3, (107), (108), (114), (119), and (106), together with (50), for $2 \leq k < n - 4s$, we obtain

$$\int_{\mathbb{R}^n} (p(x) - p_0) \left| \int_{\mathbb{R}^n} \frac{u_{\varepsilon,s}(x) - u_{\varepsilon,s}(y)}{|x - y|^{n+s}} dy \right|^2 dx \leq \kappa C \varepsilon^2 + \mathcal{O}(\varepsilon^{n-2s}) + \mathcal{O}(\varepsilon^{k+2s}). \quad (120)$$

Then, by this and (49), for n and s as in (45) (here we have already taken into account the conditions in (73)) and for $2 \leq k < n - 4s$, we write

$$\int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} u_{\varepsilon,s,a}(x)|^2 dx \leq p_0 K_s + \kappa C \varepsilon^2 + \mathcal{O}(\varepsilon^{n-2s}) + \mathcal{O}(\varepsilon^{k+2s}), \quad (121)$$

which completes the proof of Theorem 3.

Remark 3. Theorem 3 remains valid under the weaker assumption that the weight function p satisfies $p(x) = p_0 + \kappa|x|^k$ in a neighborhood $B(0, \eta)$ with $p - p_0 \in L^1(\mathbb{R}^n)$. As shown in the proof above, the term ε^2 appearing in inequality (121) does not arise from the exponential weight $e^{-|x|^2}$ and hence has no impact on the argument (see [8]). Nevertheless, this term will play a significant role in the next sections, where it will be essential in deriving nonexistence results.

Remark 4. Note that if we take $v_{\varepsilon,s,a} := \frac{u_{\varepsilon,s,a}}{\|u_{\varepsilon,s,a}\|_{L^{q_s}(\mathbb{R}^n)}}$, inequality (44) becomes

$$\int_{\mathbb{R}^n} p(x) |(-\Delta)^{\frac{s}{2}} v_{\varepsilon,s,a}(x)|^2 dx \leq p_0 S_s + \kappa C \varepsilon^2 + \mathcal{O}(\varepsilon^{n-2s}) + \mathcal{O}(\varepsilon^{k+2s}). \quad (122)$$

Let us conclude the proof of Theorem 1. Thanks to Theorem 3, (18) and (17), we write the inequality satisfied by the energy

$$\begin{aligned} \tilde{S}_{s,\lambda}(p) &\leq \tilde{E}_\lambda(v_{\varepsilon,s,a}) \\ &= \tilde{E}_\lambda\left(\frac{u_{\varepsilon,s,a}}{\|u_{\varepsilon,s,a}\|_{L^{q_s}(\mathbb{R}^n)}}\right) \\ &\leq p_0 S_s + \kappa C \varepsilon^2 - \lambda \frac{K_{2s}}{K_{q_s}^{\frac{2}{q_s}}} \varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s}) + \mathcal{O}(\varepsilon^{k+2s}) \\ &= p_0 S_s + \varepsilon^{2s} \left(-\lambda \frac{K_{2s}}{K_{q_s}^{\frac{2}{q_s}}} + \kappa C \varepsilon^{2-2s} + \mathcal{O}(\varepsilon^{n-4s}) + \mathcal{O}(\varepsilon^k) \right). \end{aligned} \quad (123)$$

We deduce that if $n = 3$ and $s \in]0, \frac{1}{4}]$ or $n = 4$ and $s \in]0, \frac{1}{2}]$ or $n = 5$ and $s \in]0, \frac{3}{4}]$ or if $n \geq 6$ and $s \in (0, 1)$, for $k \in [2, n - 4s[$, and when ε is small enough, we have $\tilde{S}_{s,\lambda}(p) \leq \tilde{E}_\lambda(v_{\varepsilon,s,a}) < p_0 S_s$. In all the cases above, $\tilde{S}_{s,\lambda}(p)$ is achieved for $0 < \lambda < \tilde{\lambda}_{1,p,s}$ and for every $\kappa > 0$; thus, the Theorem 1 follows at once. $\square$

Remark 5. In this Theorem 1, we proved the result for a weight $p(x) = p_0 + |x|^k e^{-|x|^2}$, $x \in \mathbb{R}^n$ since in the neighborhood of zero, $p_0 + |x|^k e^{-|x|^2}$ is equivalent to $p_0 + |x|^k$. Then, we obtain the same result if we take $p(x) = p_0 + |x|^k$ in a neighborhood of zero, and we suppose that $p - p_0 \in L^1(\mathbb{R}^n)$ as we supposed in [8].

Remark 6. For $p = 1$, Theorem 1 coincides with Theorem 1.2 of [8], since we have $S_{s,\lambda}(1) = \tilde{S}_{s,\lambda}(1)$.

In the following section, we prove Pohozaev's identity in the case of the fractional Laplacian with weight, which plays a role in proving the nonexistence of minimizers.

3. Pohozaev's Identity with Weight

The classical Pohozaev identity was introduced by Pohozaev [29] in 1965 for local operators as a fundamental tool to establish nonexistence results for elliptic equations in star-shaped domains. In [26], Ros-Oton and Serra established a generalization of the Pohozaev identity to the nonlocal setting, namely for the fractional Laplacian $(-\Delta)^s$, and used it to derive analogous nonexistence results in the fractional case. In this paper, we follow the method of [26] to establish a weighted Pohozaev identity adapted to our operator $(-\Delta)^{s/2}(p(-\Delta)^{s/2}\cdot)$, which is the key tool in the proof of the nonexistence results of Theorem 2.

Our aim is to prove the Pohozaev identity for the semilinear Dirichlet problem with weight

$$\begin{cases} (-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} u \right) = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (124)$$

with $u \in \mathbb{H}_0^s(\Omega)$. Here, $s \in (0, 1)$, Ω is a bounded $C^{1,1}$ and star-shaped domain and p is as defined in (2)

The weak formulation of this problem is given by, for all $v \in \mathbb{H}_0^s(\Omega)$,

$$\int_{\mathbb{R}^n} p(x)(-\Delta)^{\frac{s}{2}} u(x)(-\Delta)^{\frac{s}{2}} v(x) dx = \int_{\Omega} f(x, u)v(x) dx. \quad (125)$$

In the sequel and without any loss of generality, we assume that $a = 0$.

Theorem 4. Let $u \in \mathbb{H}_0^s(\Omega)$, where Ω a star-shaped domain, $s \in (0, 1)$ and a weight p is defined as in (2); then, we have

$$\begin{aligned} \int_{\Omega} (x \cdot \nabla u(x)) \left[(-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} u \right)(x) \right] dx &= \int_{\mathbb{R}^n} \left(\frac{2s-n}{2} p(x) - \frac{\kappa k}{2} e^{-|x|^2} |x|^k \right) \left| (-\Delta)^{\frac{s}{2}} u(x) \right|^2 dx \\ &\quad - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} p(x)(x \cdot \nu) \left(\frac{u}{\delta^s}(x) \right)^2 d\sigma(x). \end{aligned} \quad (126)$$

where c is a constant called the mean constant, ν is the unit outward normal to $\partial\Omega$ at x , and Γ is the Gamma function.

Proof. Let $u \in \mathbb{H}_0^s(\Omega)$ and $s \in (0, 1)$. We write the left-hand side of the identity (126) as

$$\int_{\Omega} (x \cdot \nabla u(x)) \left[(-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} u \right)(x) \right] dx = \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\Omega} u_{\lambda}(x)(-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} u \right)(x) dx,$$

where $u_{\lambda}(x) = u(\lambda x)$. Note that $u_{\lambda} \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, since Ω is star-shaped, and we take $\lambda > 1$ in the above derivative. By replacing p by its expression, we obtain

$$\begin{aligned} \int_{\Omega} (x \cdot \nabla u(x)) \left[(-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} u \right)(x) \right] dx &= p_0 \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\Omega} u_{\lambda}(x)(-\Delta)^s u(x) dx \\ &\quad + \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\Omega} u_{\lambda}(x)(-\Delta)^{\frac{s}{2}} \left(\kappa e^{-|x|^2} |x|^k (-\Delta)^{\frac{s}{2}} u \right)(x) dx. \end{aligned} \quad (127)$$

Using the classical Pohozaev formula (see [26]), we obtain

$$\begin{aligned}
 & \int_{\Omega} (x \cdot \nabla u(x)) [(-\Delta)^{\frac{s}{2}} (p(-\Delta)^{\frac{s}{2}} u)(x)] dx \\
 &= p_0 \left(\frac{2s-n}{2} \int_{\Omega} u(x) (-\Delta)^s u(x) dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}(x) \right)^2 (x \cdot \nu) d\sigma \right) \\
 &+ \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\mathbb{R}^n} ((-\Delta)^{\frac{s}{2}} u_{\lambda}(x)) \kappa e^{-|x|^2} |x|^k (-\Delta)^{\frac{s}{2}} u(x) dx \\
 &= p_0 \left(\frac{2s-n}{2} \int_{\Omega} u(x) (-\Delta)^s u(x) dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}(x) \right)^2 (x \cdot \nu) d\sigma \right) \\
 &+ \frac{d}{d\lambda} \Big|_{\lambda=1} \left(\lambda^{s-\frac{k}{2}} \kappa \int_{\mathbb{R}^n} w_{1,\lambda}(x) w_1(x) dx \right),
 \end{aligned} \tag{128}$$

where $w_1(x) = e^{-\frac{|x|^2}{2}} |x|^{\frac{k}{2}} (-\Delta)^{\frac{s}{2}} u(x)$ and $w_{1,\lambda}(x) = w_1(\lambda x)$, see [30] for the function Γ . We recall that if u and v are $\mathbb{H}_0^s(\Omega)$, then

$$\int_{\Omega} v(-\Delta)^s u dx = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx. \tag{129}$$

As a consequence, we may use the previous identity (129) with $v = u_{\lambda}$, and we make the change of variables $y = \sqrt{\lambda}x$ to obtain

$$\begin{aligned}
 & \int_{\Omega} (x \cdot \nabla u(x)) [(-\Delta)^{\frac{s}{2}} (p(-\Delta)^{\frac{s}{2}} u)(x)] dx \\
 &= p_0 \left(\frac{2s-n}{2} \int_{\Omega} u(x) (-\Delta)^s u(x) dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}(x) \right)^2 (x \cdot \nu) d\sigma \right) \\
 &+ \frac{d}{d\lambda} \Big|_{\lambda=1} \left(\lambda^{\frac{2s-k-n}{2}} \kappa \int_{\mathbb{R}^n} w_{1,\sqrt{\lambda}}(x) w_{1,\frac{1}{\sqrt{\lambda}}}(x) dx \right) \\
 &= p_0 \left(\frac{2s-n}{2} \int_{\Omega} u(x) (-\Delta)^s u(x) dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}(x) \right)^2 (x \cdot \nu) d\sigma \right) \\
 &+ \frac{2s-k-n}{2} \kappa \int_{\mathbb{R}^n} e^{-|x|^2} |x|^k \left| (-\Delta)^{\frac{s}{2}} u(x) \right|^2 dx + \frac{\kappa}{2} \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\mathbb{R}^n} w_{1,\lambda}(x) w_{1,\frac{1}{\lambda}}(x) dx.
 \end{aligned} \tag{130}$$

Using again [26], we have

$$\frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\mathbb{R}^n} w_{1,\lambda}(x) w_{1,\frac{1}{\lambda}}(x) dx = \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\partial\Omega} e^{-|x|^2} |x|^k (x \cdot \nu) \left(\int_0^{\infty} t^{n-1} w(\lambda t x) w\left(\frac{tx}{\lambda}\right) dt \right) d\sigma(x), \tag{131}$$

where $w(x) = (-\Delta)^{\frac{s}{2}} u(x)$.

We write

$$\int_0^{\infty} t^{n-1} w(\lambda t x) w\left(\frac{tx}{\lambda}\right) dt = \int_0^{\infty} \phi(t\lambda) \phi\left(\frac{t}{\lambda}\right) dt,$$

with

$$\phi(t) := t^{\frac{n-1}{2}} w(tx) = t^{\frac{n-1}{2}} (-\Delta)^{\frac{s}{2}} u(tx).$$

From the Pohozaev identity for the classical fractional laplacian (see [26]), we have

$$\frac{d}{d\lambda} \Big|_{\lambda=1} \int_0^{\infty} t^{n-1} w(\lambda t x) w\left(\frac{tx}{\lambda}\right) dt = -\Gamma(1+s)^2 \left(\frac{u}{\delta^s}(x) \right)^2.$$

Therefore,

$$\frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\mathbb{R}^n} w_{1,\lambda}(x) w_{1,\frac{1}{\lambda}}(x) dx = -\Gamma(1+s)^2 \int_{\partial\Omega} e^{-|x|^2} |x|^k (x \cdot \nu) \left(\frac{u}{\delta^s}(x) \right)^2 d\sigma(x). \tag{132}$$

Finally, we obtain

$$\begin{aligned}
& \int_{\Omega} (x \cdot \nabla u(x)) \left[(-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} \right) \right] dx \\
&= p_0 \left(\frac{2s-n}{2} \int_{\Omega} u(x) (-\Delta)^s u(x) dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}(x) \right)^2 (x \cdot \nu) d\sigma \right) \\
&+ \frac{2s-k-n}{2} \int_{\mathbb{R}^n} \kappa e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \kappa e^{-|x|^2} |x|^k (x \cdot \nu) \left(\frac{u}{\delta^s}(x) \right)^2 d\sigma(x).
\end{aligned} \tag{133}$$

Then, we write

$$\begin{aligned}
\int_{\Omega} (x \cdot \nabla u(x)) \left[(-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} \right) \right] dx &= \int_{\mathbb{R}^n} \left(\frac{2s-n}{2} p(x) - \frac{\kappa k}{2} e^{-|x|^2} |x|^k \right) \left| (-\Delta)^{\frac{s}{2}} u(x) \right|^2 dx \\
&- \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} p(x) (x \cdot \nu) \left(\frac{u}{\delta^s}(x) \right)^2 d\sigma(x).
\end{aligned} \tag{134}$$

□

We find a similar formula if we gain a weight of the type $p(x) = p_0 + p_{k_1}|x|^{k_1} + p_{k_2}|x|^{k_2}$. We define it in the following proposition.

Proposition 4. Let $u \in \mathbb{H}_0^s(\Omega)$ and suppose that $p(x) = p_0 + p_{k_1}|x|^{k_1} + p_{k_2}|x|^{k_2}$; then, we have the following similar formula of Pohozaev's identity

$$\begin{aligned}
\int_{\Omega} (x \cdot \nabla u(x)) \left[(-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} \right) \right] dx &= \int_{\mathbb{R}^n} \left(\frac{2s-n}{2} p(x) - \sum_{i=1}^2 \frac{p_{k_i} k_i}{2} |x|^{k_i} \right) \left| (-\Delta)^{\frac{s}{2}} u(x) \right|^2 dx \\
&- \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} p(x) (x \cdot \nu) \left(\frac{u}{\delta^s}(x) \right)^2 d\sigma(x).
\end{aligned} \tag{135}$$

Now, we collected all the tools to prove Theorem 2.

Proof of Theorem 2

Proof. We start by proving assertion (1). Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. (To prove this assertion we don't need to suppose Ω to be starshaped domain.) Let u be a function which achieves the infimum $\tilde{S}_{s,\lambda}(p)$. The minimizer u satisfies Euler's equation given by

$$\begin{aligned}
& \int_{\mathbb{R}^n} p(x) \left(\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} dy \right) \left(\int_{\mathbb{R}^n} \frac{\zeta(x) - \zeta(z)}{|x - z|^{n+s}} dz \right) dx - \lambda \int_{\Omega} u(x) \zeta(x) dx \\
&= \Lambda_{s,p} \int_{\Omega} |u(x)|^{q_s-2} u(x) \zeta(x) dx,
\end{aligned} \tag{136}$$

for all $\zeta \in \mathbb{H}_0^s(\Omega)$ where $\Lambda_{s,p}$ is a Lagrange multiplier.

Let $\Phi_{1,p,s}$ be an eigenvector associated to $\tilde{\lambda}_{1,p,s}$. We take $\zeta = \phi_{1,p,s}$, and we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n} p(x) \left(\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} dy \right) \left(\int_{\mathbb{R}^n} \frac{\phi_{1,p,s}(x) - \phi_{1,p,s}(z)}{|x - z|^{n+s}} dz \right) dx - \lambda \int_{\Omega} u(x) \phi_{1,p,s}(x) dx \\
&= \Lambda_{s,p} \int_{\Omega} |u(x)|^{q_s-2} u(x) \phi_{1,p,s}(x) dx,
\end{aligned} \tag{137}$$

and

$$(\tilde{\lambda}_{1,p,s} - \lambda) \int_{\Omega} u(x) \Phi_{1,p,s}(x) dx = \Lambda_{s,p} \int_{\Omega} |u(x)|^{q_s-2} u(x) \phi_{1,p,s}(x) dx. \tag{138}$$

Identically as in ([1], Chapter 3), we can prove that $\phi_{1,p,s} > 0$. Therefore, we deduce

$$\int_{\Omega} |u|^{q_s-2} \phi_{1,p,s} dx > 0. \tag{139}$$

Then, we obtain $(\tilde{\lambda}_{1,p,s} - \lambda) \int_{\mathbb{R}^n} u(x) \phi_{1,p,s}(x) dx > 0$ and so $\tilde{\lambda}_{1,p,s} > \lambda$.

Finally, if $\lambda \geq \tilde{\lambda}_{1,p,s}$, there exists no solution achieving the infimum (4).

We move to detail the proof of assertion (2). Let Ω be a star-shaped domain and let n, s such that $n = 3$ and $s \in]0, \frac{1}{4}[$ or $n = 4$ and $s \in]0, \frac{1}{2}[$ or $n = 5$ and $s \in]0, \frac{3}{4}[$ or $n \geq 6$ and $s \in (0, 1)$. We will establish that $\alpha(k, s) = 0$ and then for every $\lambda \leq 0$, there exists no solution to (4).

By the Pohozaev identity in the fractional Laplacian with weight, we have

$$\begin{aligned} \int_{\Omega} (x \cdot \nabla u) \left[(-\Delta)^{\frac{s}{2}} \left(p(-\Delta)^{\frac{s}{2}} u \right) \right] dx &= \int_{\mathbb{R}^n} \left(\frac{2s-n}{2} p(x) - \frac{k}{2} e^{-|x|^2} |x|^k \right) |(-\Delta)^{\frac{s}{2}} u|^2 dx \\ &\quad - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} p(x) (x \cdot \nu) \left(\frac{u}{\delta^s} \right)^2 d\sigma(x). \end{aligned} \quad (140)$$

The left-hand side is

$$\begin{aligned} \int_{\Omega} (x \cdot \nabla u) (u^{q_s-1} + \lambda u) &= \int_{\Omega} \left(\frac{x \cdot \nabla u^{q_s}}{q_s} + \frac{\lambda x \cdot \nabla u^2}{2} \right) dx \\ &= \frac{-n}{q_s} \int_{\Omega} u^{q_s} dx - \frac{n\lambda}{2} \int_{\Omega} u^2 dx \\ &= -\frac{n-2s}{2} - \frac{n\lambda}{2} \int_{\Omega} u^2 dx. \end{aligned} \quad (141)$$

We study the right-hand side by taking

$$A := \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} p(x) (x \cdot \nu) \left(\frac{u}{\delta^s} \right)^2 d\sigma(x),$$

using the fact that $(-\Delta)^{\frac{s}{2}} (p(-\Delta)^{\frac{s}{2}} u) = u^{q_s-1} + \lambda u$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\frac{2s-n}{2} p(x) - \frac{k}{2} |x|^k \right) |(-\Delta)^{\frac{s}{2}} u|^2 dx - A \\ &= \int_{\mathbb{R}^n} \frac{2s-n}{2} p(x) |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{k}{2} \int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u|^2 dx - A \\ &= \frac{2s-n}{2} \left(\int_{\mathbb{R}^n} u^{q_s} dx + \lambda \int_{\mathbb{R}^n} u^2 dx \right) - \frac{k}{2} \int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - A. \end{aligned} \quad (142)$$

Therefore, by combining (141) and (142) and taking into account $\|u\|_{q_s} = 1$, we obtain

$$-\frac{n\lambda}{2} \int_{\Omega} u^2 dx = \frac{2s-n}{2} \lambda \int_{\Omega} u^2 dx - \frac{k}{2} \int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - A, \quad (143)$$

so

$$-\lambda s \int_{\Omega} u^2 dx = -\frac{k}{2} \int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - A. \quad (144)$$

Therefore, we obtain

$$\lambda \geq \frac{k}{2s} \frac{\int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx}{\int_{\Omega} |u(x)|^2 dx} > 0. \quad (145)$$

An immediate result that we can announce here is that if $\lambda = 0$, the infimum (4) cannot be achieved.

For n, s such that $n = 3$ and $s \in]0, \frac{1}{4}[$ or $n = 4$ and $s \in]0, \frac{1}{2}[$ or $n = 5$ and $s \in]0, \frac{3}{4}[$ or $n \geq 6$ and $s \in (0, 1)$, by Theorem 3, for every $2 \leq k < n - 4s$, we have

$$\begin{aligned}
0 \leq \alpha(k, s) &= \inf_{\substack{u \in \mathbb{H}_0^s(\Omega) \\ u \neq 0}} \frac{k \int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u|^2 dx}{2s \int_{\Omega} |u(x)|^2 dx} \\
&\leq \frac{k \int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u_{\varepsilon, s}|^2 dx}{2s \int_{\Omega} |u_{\varepsilon, s}(x)|^2 dx} \\
&\leq \frac{k (C\varepsilon^2 + \mathcal{O}(\varepsilon^{n-2s}))}{2s K_{1,s} \varepsilon^{2s} + \mathcal{O}(\varepsilon^{n-2s})} \\
&\leq \varepsilon^{2-2s} (C + \mathcal{O}(\varepsilon^{n-2})),
\end{aligned} \tag{146}$$

where C is a positive constant.

Since $n > 2, s \in (0, 1)$, $\alpha(k, s) = 0$, then we obtain $\lambda \geq \alpha(k, s) = 0$. Therefore, if $\lambda < 0$, the infimum (4) is not achieved, which finishes the proof of assertion (2).

We end the proof of Theorem 2 by proving the assertion (3). For the convenience of the reader, we recall Hardy's formula, and for further details, please see [10,31].

Proposition 5. Let $k \in \mathbb{R}_+$ such that $k < 2s$, then we have $\forall u \in \mathbb{H}_0^s(\Omega)$

$$\int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx \geq \frac{C_{n,s,k}^2}{4} \int_{\Omega} |x|^{k-2s} |u(x)|^2 dx + \frac{C_{n,s,k}}{2} \int_{\mathbb{R}^n} \frac{I_{\frac{s}{2}}(u, u)}{|x|^{s-k}} dx, \tag{147}$$

where

$$I_{\frac{s}{2}}(u, u) = c(n, \frac{s}{2}) \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+s}} dy, \tag{148}$$

and $C_{n,s,k}$ is a positive constant depending on s, k and the dimension n .

Proof. $\forall t \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} \left| e^{-\frac{|x|^2}{2}} |x|^{\frac{k}{2}} |(-\Delta)^{\frac{s}{2}} u(x)| + t e^{-\frac{(\text{diam}(\Omega))^2}{2}} \frac{e^{\frac{|x|^2}{2}}}{|x|^{s-\frac{k}{2}}} u(x) \right|^2 dx \geq 0, \tag{149}$$

where $\text{diam}(\Omega)$ is the diameter of Ω . We apply the product formula of the fractional Laplacian $u(-\Delta)^{\frac{s}{2}} u = \frac{1}{2} ((-\Delta)^{\frac{s}{2}} |u|^2 + I_{\frac{s}{2}}(u, u))$, and we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx \\
&+ t e^{-\frac{(\text{diam}(\Omega))^2}{2}} \int_{\mathbb{R}^n} |x|^{k-s} \left((-\Delta)^{\frac{s}{2}} |u(x)|^2 + I_{\frac{s}{2}}(u, u) \right) dx \\
&+ t^2 e^{-(\text{diam}(\Omega))^2} \int_{\Omega} \frac{e^{|x|^2} |u(x)|^2}{|x|^{2s-k}} dx \geq 0.
\end{aligned} \tag{150}$$

with $I_s(u, u)$ defined in (148). Since $k < 2s$, we have

$$\int_0^{+\infty} \frac{r^{k-s}}{(1+r)^{n+s}} r^{n-1} dr < +\infty, \tag{151}$$

Then, we apply the result shown in [28], Theorem 1.1, and with $r = |x|$, for every $x \in \mathbb{R}^n \setminus \{0\}$, we obtain

$$(-\Delta)^{\frac{s}{2}} |x|^{k-s} = C_{n,s} \frac{1}{|x|^{2s-k}}, \tag{152}$$

where $C_{n,s}$ is a positive constant depending on n, s . After application of the Green formula, there exists a positive constant $C_{n,s,k}$ such that

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx + t e^{-\frac{(\text{diam}(\Omega))^2}{2}} C_{n,s,k} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^{2s-k}} dx \\ & + t e^{-\frac{(\text{diam}(\Omega))^2}{2}} \int_{\mathbb{R}^n} \frac{I_{\frac{s}{2}}(u, u)}{|x|^{s-k}} dx + t^2 e^{-(\text{diam}(\Omega))^2} \int_{\Omega} \frac{e^{|x|^2} |u(x)|^2}{|x|^{2s-k}} dx \geq 0. \end{aligned} \quad (153)$$

since $e^{-\frac{(\text{diam}(\Omega))^2}{2}} \leq 1$ and for $x \in \Omega$, $e^{|x|^2} \leq e^{(\text{diam}(\Omega))^2}$; then, we obtain

$$\int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx + t C_{n,s,k} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^{2s-k}} dx + t \int_{\mathbb{R}^n} \frac{I_{\frac{s}{2}}(u, u)}{|x|^{s-k}} dx + t^2 \int_{\Omega} \frac{|u(x)|^2}{|x|^{2s-k}} dx \geq 0. \quad (154)$$

For $t = -\frac{C_{n,s,k}}{2}$, we obtain

$$\int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx - \frac{C_{n,s,k}}{2} \int_{\mathbb{R}^n} \frac{I_{\frac{s}{2}}(u, u)}{|x|^{s-k}} dx \geq \frac{C_{n,s,k}^2}{4} \int_{\Omega} \frac{|u(x)|^2}{|x|^{2s-k}} dx, \quad (155)$$

then, we obtain the desired result.

□

Now, we can easily prove (3) of Theorem 2.

Let Ω be a star-shaped domain and let u be a solution of (4), so we have $u = 0$ in $\mathbb{R}^n \setminus \Omega$. By (145) and using Proposition 5, for Ω star-shaped domain and $k < 2s$, we have

$$\begin{aligned} \lambda & \geq \frac{k}{2s} \frac{\int_{\mathbb{R}^n} e^{-|x|^2} |x|^k |(-\Delta)^{\frac{s}{2}} u(x)|^2 dx}{\int_{\Omega} |u(x)|^2 dx} \\ & \geq \alpha(k, s) \\ & \geq C_{(n,s,k)} \frac{\int_{\Omega} |x|^{k-2s} |u(x)|^2 dx}{\int_{\Omega} |u(x)|^2 dx} \\ & \geq C_{(n,s,k)} (\text{diam}(\Omega))^{k-2s} \\ & := \alpha_0 > 0. \end{aligned} \quad (156)$$

Then, we conclude that for $\lambda < \alpha(k, s)$ the infimum (4) is not achieved.

We conclude that for $k < 2s$, we have $\lambda \geq \alpha(k, s) \geq \alpha_0 > 0$, so that if $\lambda < \alpha(k, s)$, $\tilde{S}_{s,\lambda}(p)$ is not achieved. In other words, when λ is small, there is no solution, and to have a solution λ must be greater than a value $\alpha_0 > 0$. □

We can, finally, end this section by the following remark where we give a short comparison between the local and fractional cases.

Remark 7. In the local case, everything happens in the neighborhood of the minimum, which is 0, and when x is close to 0, $|x|^k$ is very small. In contrast to what happens in the nonlocal case, the vectors x which are far from 0 intervene, which means that $|x|^k$ will be large. Therefore, we can exceed S_s , and consequently this is not reached.

4. Conclusions

In this paper, we have studied the fractional minimization problem

$$\tilde{S}_{s,\lambda}(p) := \inf_{\substack{u \in H_0^s(\Omega) \\ \|u\|_{L^q} = 1}} \int_{\mathbb{R}^n} p(x) |(-\Delta)^{s/2} u|^2 dx - \lambda \int_{\Omega} |u|^2 dx,$$

where p is a positive weight admitting a global minimum $p_0 > 0$, and $s \in (0, 1)$, $n \geq 3$. Our main contributions are the following.

Existence. We proved in Theorem 1 that for $k \in [2, n - 4s[$, $\lambda \in]0, \tilde{\lambda}_{1,p,s}[$ and under the dimensional conditions of Theorem 1, the infimum $\tilde{S}_{s,\lambda}(p)$ is achieved for every $\kappa > 0$. The key step is the a priori estimate $\tilde{S}_{s,\lambda}(p) < p_0 S_s$, established in Theorem 3, which allows us to recover compactness despite the critical Sobolev exponent. Compared to [8], where a restriction on κ was imposed, our result holds for all $\kappa > 0$ thanks to the specific structure of the weight p defined in (2).

Nonexistence. We proved in Theorem 2 three nonexistence results. First, if $\lambda \geq \tilde{\lambda}_{1,p,s}$, no minimizer exists. Second, for $k \in [2, n - 4s[$, there is no minimizer when $\lambda \leq 0$. Third, for $k < 2s$, there exists $\alpha_0 > 0$ such that no minimizer exists when $\lambda < \alpha(k, s)$. The proofs rely on two key tools that we developed: a fractional weighted Pohozaev identity (Theorem 4), adapted to the operator $(-\Delta)^{s/2}(p(-\Delta)^{s/2} \cdot)$, and a fractional Hardy-type inequality with weight (Proposition 5).

Comparison with related results. Our results extend those of [7] (local case $s = 1$) and [26] (unweighted fractional case $p \equiv 1$) to the weighted fractional setting. The condition $k \in [2, n - 4s[$ for existence is a natural fractional analogue of the condition $k > 2$ of [7], recovering the latter as $s \rightarrow 1$. The nonlocal character of the operator introduces new difficulties that are not present in the local case, in particular the contribution of the weight at infinity, which is responsible for the additional condition on k discussed in Remark 7.

Perspectives. Several natural extensions of this paper remain open. First, it would be interesting to consider more general weights p , for instance weights with multiple local minima or with a different behavior at infinity. Second, the case of systems of equations involving the fractional Laplacian with weight deserves investigation. It is also interesting to consider a system with n components in the case of a local operator; see, for instance [32]. Finally, it is important to extend these results to the polyharmonic operator case, $(-\Delta)^{s+n}$ with $s \in (0, 1)$ and $n \in \mathbb{N}$; see [15,22].

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References

1. Bisci, G.M.; Rădulescu, V.D.; Servadei, R. *Variational Methods for Nonlocal Fractional Problems*; Cambridge University Press: Cambridge, UK, 2016; Volume 162.
2. Hadji, R.; Vigneron, F. Existence of solutions of a nonlinear eigenvalue problem with a variable weight. *J. Differ. Equ.* **2019**, *266*, 1488–1513.
3. Cotioli, A.; Tavoularis, N.K. Best constants for Sobolev inequalities for higher order fractional derivatives. *J. Math. Anal. Appl.* **2004**, *295*, 225–236.
4. Frank, R.; Lieb, E.; Seiringer, R. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators. *J. Am. Math. Soc.* **2008**, *21*, 925–950.
5. Lieb, E. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. Math.* **1983**, *118*, 349–374.
6. Di Nezza, E.; Palatucci, G.; Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Mathématiques* **2012**, *136*, 521–573.

7. Hadiji, R.; Yazidi, H. Problem with critical Sobolev exponent and with weight. *Chin. Ann. Math. Ser.* **2007**, *28*, 327–352.
8. Benhafsia, S.; Hadiji, R. Existence results for problems involving nonlocal operator with an asymmetric weight and with a critical nonlinearity. *arXiv* **2024**, arXiv:2404.07531.
9. Caffarelli, L.; Silvestre, L. An extension problem related to the fractional Laplacian. *Commun. Partial. Differ. Equ.* **2007**, *32*, 1245–1260.
10. Nguyen, H.; Squassina, M. Fractional Caffarelli-Kohn-Nirenberg inequalities. *J. Funct. Anal.* **2018**, *274*, 2661–2672.
11. Chang, S.-Y.A.; González, M.d.M. Fractional Laplacian in conformal geometry. *Adv. Math.* **2011**, *226*, 1410–1432.
12. González Noguera, M.; Qing, J. Fractional conformal Laplacians and fractional Yamabe problems. *Anal. PDE* **2013**, *6*, 1535–1576.
13. Escobar, J.F. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. Math.* **1992**, *136*, 1–50.
14. Servadei, R. The Yamabe equation in a nonlocal setting. *Adv. Nonlinear Anal.* **2013**, *3*, 235–270.
15. Benhamida, A.; Hadiji, R.; Yazidi, H. Existence results for elliptic equation involving polyharmonic operator and a critical growth. *J. Elliptic Parabol. Equ.* **2025**, *11*, 19–43.
16. Servadei, R.; Valdinoci, E. Mountain pass solutions for nonlocal elliptic operators. *J. Math. Anal. Appl.* **2012**, *389*, 887–898.
17. Aubin, T. Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl.* **1976**, *55*, 269–296.
18. Brezis, H. Some variational problems with lack of compactness. *Proc. Symp. Pure Math.* **1986**, *45*, 165–201.
19. Brezis, H.; Nirenberg, L. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Commun. Pure Appl. Math.* **1983**, *36*, 437–477.
20. Hadiji, R.; Molle, R.; Passaseo, D.; Yazidi, H. Localization of solutions for nonlinear elliptic problems with critical growth. *Comptes Rendus. Mathématiques* **2006**, *343*, 725–730.
21. Ambrosetti, A.; Rabinowitz, P. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **1973**, *14*, 349–381.
22. Beaulieu, A.; Hadiji, R. Remarks on solutions of a fourth-order problem. *Appl. Math. Lett.* **2006**, *19*, 661–666.
23. Hadiji, R. A nonlinear problem with a weight and a non-vanishing boundary datum. *Pure Appl. Funct. Anal.* **2020**, *5*, 965–980.
24. Lee, J.M.; Parker, T.H. The Yamabe problem. *Bull. Amer. Math. Soc.* **1987**, *17*, 37–91.
25. Appolloni, L.; Bisci, G.M.; Secchi, S. On critical Kirchhoff problems driven by the fractional Laplacian. *Calc. Var. Partial. Differ. Equ.* **2021**, *60*, 209.
26. Ros-Oton, X.; Serra, J. The Pohozaev identity for the fractional Laplacian. *Arch. Ration. Mech. Anal.* **2014**, *213*, 587–628.
27. Brezis, H.; Lieb, E. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.* **1983**, *88*, 486–490.
28. Ferrari, F.; Verbitsky, I. Radial fractional Laplace operators and Hessian inequalities. *J. Differ. Equ.* **2012**, *253*, 244–272.
29. Pohozaev, S. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Soviet Math. Doklady* **1965**, *6*, 1408–1411.
30. Magnus, W.; Oberhettinger, F.; Soni, R.P.; Wigner, E. Formulas and theorems for the special functions of mathematical physics. *Phys. Today* **1967**, *20*, 81–83.
31. Abdellaoui, B.; Bentifour, R. Caffarelli-Kohn-Nirenberg type inequalities of fractional order with applications. *J. Funct. Anal.* **2017**, *272*, 3998–4029.
32. Benhamida, A.; Hadiji, R. A System with weights and with critical Sobolev exponent. *Eur. J. Math.* **2023**, *9*, 54.

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