

Article

Wick Products Under Bogoliubov Transformations on Guichardet–Fock Space

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Abstract

We construct Wick products by creation and annihilation operators on Guichardet–Fock spaces $\mathcal{F}$ and obtain commutation relations and s —adapted characters about the Wick products. Meanwhile, we introduce a concept of singular Bogoliubov transformation on Guichardet–Fock space, and prove that the Wick product under the singular Bogoliubov transformation still satisfies the commutative relationships.

Keywords: Wick product; s —adapted operator; creation operator; annihilation operator; Guichardet–Fock spaces

MSC: 81Q112; 70S20; 81S20; 35Q35

1. Introduction

Hida and Ikeda [1] first rigorously defined the Wick renormalization on functionals of Brownian motion and called it the Wick product instead. Meyer and Yan [2] extended the Wick product to cover generalized functionals of Gaussian white noises (also known as Hida distributions). Now, the Wick product is widely used as a tool in stochastic differential equations, stochastic partial differential equations, stochastic quantization [3], and many other fields. Attal [4] discussed and extended quantum stochastic calculus by means of the Skorohod integral of anticipative processes and the related gradient operator on Guichardet–Fock spaces. Usually, Fock spaces, as the models of Particle Systems, are widely used in quantum physics.

The study of Wick products on Fock spaces is closely related to the theory of quantum stochastic calculus and has deep connections with quantum physics. Guichardet–Fock space provides a natural framework for formulating discrete stochastic processes and has attracted considerable attention in recent years. In particular, the extension of Wick products to settings involving Bogoliubov transformations is important for understanding the structure of quantum fields on non-vacuum states.

Wang, Lu, and Chai [5] developed an alternative approach to Privault’s discrete-time chaotic calculus on the Bernoulli–Fock space, which provides a combinatorial counterpart to the continuous-time theory developed here. Jorgensen and Werner [6] studied coherent states of the q -canonical commutation relations, a deformation of the standard CCR that motivates the general algebraic framework underlying our results. It should be noted that the recent paper by Jiang et al. [7] on localized radial basis function collocation methods for phase-field models and the paper by Hu et al. [8] on giant elastic-wave asymmetry operate in an applied PDE and wave-physics setting that is mathematically quite different from the



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operator-theoretic Fock-space setting of the present paper; they are included to indicate the breadth of physical contexts in which normal-ordering and operator-algebraic techniques are relevant.

The algebraic structure of Wick products on CCR and CAR algebras was studied by Bermudez, Gonzalez, and Negrin [9] and by Wang and Zhang [10] for Bernoulli noise functionals; a recursion relation for Wick products of the CAR algebra was established in [11]. The treatment of singular Bogoliubov transformations and inequivalent representations of CCR follows Asao [12], whose results on unbounded Bogoliubov pairs provide the foundation for Section 5 of the present paper.

The main contributions of the present paper are as follows: (1) We establish a rigorous construction of Wick products on Guichardet–Fock space using corrected creation and annihilation operator definitions with the Lebesgue measure on the configuration space Γ . (2) We prove commutation relations for Wick products, extending the results of Bermudez–Gonzalez–Negrin [9] and Wang–Zhang [10] to the Guichardet–Fock setting. (3) We introduce and study s -adaptedness of Wick products, providing a precise domain framework and conditional expectation factorization. (4) We introduce the concept of singular Bogoliubov transformation induced by a pair (U, V) of unbounded operators and prove that Wick products under such transformations retain their commutation-relation structure.

In this paper, we construct Wick product operators on Guichardet–Fock space and develop a Wick-type calculus on them. Meanwhile, we introduce a concept of Bogoliubov transformation which is induced by a pair (U, V) of unbounded linear operators obeying some consistency conditions on the Guichardet–Fock space.

The outline of the present paper is as follows: In Section 2, we fix some necessary notation and recall main notions and facts about Guichardet–Fock space for our later use. we define Wick products on $\mathcal{F}$ by annihilation and creation operators, meanwhile showing recursion and commutation relations for the Wick products. In Section 3, we discuss the commuting, projective and differential definitions of adaptedness of Wick products on Guichardet–Fock spaces $\mathcal{F}$ and prove that $\mathcal{E}(S) \in \mathcal{F}$ forms a commutative algebra with the Wick product. In the Section 4, we introduce a singular Bogoliubov transformation and prove that the Wick product under the singular Bogoliubov transformation still satisfies the commutative relationships. Section 5 provides an example.

2. Preliminaries

Let the one-particle space be $\eta = L^2(\mathbb{R}_+)$, where $\mathbb{R}_+ = [0, +\infty)$. We denote by Γ the collection of all finite subsets of $\mathbb{R}_+$:

$$\Gamma := \{\sigma | \sigma \subset \mathbb{R}_+, \#\sigma < \infty\},$$

with $\Gamma^{(n)}$ denoting the collection of n -element subsets. For $n \geq 1$, the Lebesgue measure induces a measure on $\Gamma^{(n)}$ through the bijection $\mathbf{s} \rightarrow \{s_1, \dots, s_n\}$ from $\{s \in \mathbb{R}_+^n : s_1 < \dots < s_n\}$ to $\Gamma^{(n)}$. By letting $\emptyset \in \Gamma^{(0)}$ be an atom of measure 1, we arrive at a σ -finite measure on $\bigcup_{n \geq 0} \Gamma^{(n)} = \Gamma$ called the symmetric measure of the Lebesgue measure on $\mathbb{R}_+$.

Fixing a complex separable Hilbert space η , Guichardet–Fock space is the tensor product $\eta \otimes L^2(\Gamma)$, which we identify with the space of square-integrable functions $L^2(\Gamma; \eta)$, and is denoted by $\mathcal{F}$. Elements of Γ will always be denoted by lowercase Greek letters such as $\alpha, \beta, \sigma, \tau$ and ω , and these will be used exclusively for this purpose. With this convention, we write simply $\int f(\sigma) d\sigma$ to denote the integral of a Hilbert space-valued function f over Γ with respect to the symmetric measure of the Lebesgue measure on $\mathbb{R}_+$. Similarly, $\int \varphi(s) ds$ will always denote the integral of a function φ over $\mathbb{R}_+$ with respect to the Lebesgue measure. The following elementary identity is fundamental; a proof may be found in [13].

Lemma 1 (Integral–sum lemma). *Let g be a nonnegative measurable function $\Gamma \times \Gamma \rightarrow \mathbb{R}_+$ (or a Bochner-integrable function $\Gamma \times \Gamma \rightarrow \eta$) and let G be the function defined by $G(\sigma) = \sum_{\alpha \subset \sigma} g(\alpha, \sigma \setminus \alpha)$. Then G is nonnegative and measurable (resp. integrable) and*

$$\int G(\sigma) d\sigma = \int \int g(\alpha, \beta) d\alpha d\beta.$$

The exponential vector $\varepsilon(\varphi)$ of the test function φ in Guichardet–Fock space is defined by

$$\varepsilon(\varphi)(\sigma) = \prod_{s \in \sigma} \varphi(s).$$

We define $\mathcal{E}(S)$ to be the linear span of $\{\varepsilon(\varphi) : \varphi \in S\}$, i.e., $\mathcal{E}(S) := \text{Lin}\{\varepsilon(\varphi) : \varphi \in S\}$, where S is a subset of $L^2(\mathbb{R}_+)$. Obviously, $\mathcal{E}(S)$ is a subspace of Guichardet–Fock space.

Guichardet–Fock space enjoys a continuous tensor product structure; for each $s \geq 0$,

$$(f \otimes g)(\omega) = f(\omega_s)g(\omega_{(s)}) = f(\omega \cap [0, s))g(\omega \cap [s, \infty)).$$

This structure is elegantly carried by the exponential vectors, being determined by the following restriction of $\mathcal{F}_s \otimes \mathcal{F}^s \rightarrow \mathcal{F}$:

$$v\varepsilon(\varphi_{[0,s)}) \otimes \varepsilon(\varphi_{[s,\infty)}) \mapsto v\varepsilon(\varphi),$$

where $\mathcal{F}_s$ is first viewed as a subspace of $\mathcal{F}$, $\mathcal{F}^s = L^2(\Gamma^s)$, $\Gamma^s = \{\omega \in \Gamma : \omega \subset [s, \infty)\}$, $\Gamma_s = \{\omega \in \Gamma : \omega \subset [0, s)\}$. The notion here is $\varphi_{[a,b)} = \varphi \mathbf{1}_{[a,b)}$, and $\mathbf{1}$ denotes the indicator function.

An important consequence of the integral–sum lemma is the following identity:

$$\int \int f(\sigma \cup t) = \int \sharp \omega f(\omega) d\omega.$$

which is valid for nonnegative measurable functions $\Gamma \rightarrow \mathbb{R}_+$ and for measurable functions $\Gamma \rightarrow \eta$ for which either side is/both sides are defined.

3. The Wick Products

We define a Wick product operator on $\mathcal{E}(S)$ which is a subspace of $\mathcal{F}$ by annihilation and creation operators on $\mathcal{E}(S)$ and show recursion and commutation relations for the Wick products.

Definition 1. *If $f \in \eta$, then annihilation operator $a(f)$ and creation operator $a^*(f)$ are defined on exponential vectors as follows:*

$$\begin{aligned} a(f)\varepsilon(\varphi)(\sigma) &= \langle f, \varphi \rangle \varepsilon(\varphi)(\sigma), \\ a^*(f)\varepsilon(\varphi)(\sigma) &= \sum_{s \in \sigma} f(s)\varepsilon(\varphi)(\sigma \setminus s). \end{aligned}$$

We extend $a(f)$ to the full exponential domain $\mathcal{E}(S)$ by linearity and density. For a general $h \in \mathcal{E}(S)$, the annihilation operator acts via the integral formula

$$(a(f)h)(\sigma) = \int_{\mathbb{R}_+} f(t)h(\sigma \cup \{t\})dt,$$

which coincides with the formula above on exponential vectors and is well defined on the invariant core $\mathcal{E}(S)$.

Let $\mathcal{A}(\mathcal{F})$ be the canonical commutation-relation algebra over $\mathcal{F}$, i.e., the algebra generated by the set $\{a(f), a^*(f) : f \in \eta\}$ together with a unit I ; those elements satisfy the canonical commutation relations.

Proposition 1. For all $f, g \in \eta$, and $\varepsilon(\phi) \in \mathcal{E}(S)$, we have

- (a) $a(f)a^*(g)\varepsilon(\phi)(\sigma) - a^*(g)a(f)\varepsilon(\phi)(\sigma) = \langle f, g \rangle \varepsilon(\phi)(\sigma).$
- (b) $a^*(f)a^*(g) - a^*(g)a^*(f) = 0.$
- (c) $a(f)a(g) - a(g)a(f) = 0.$

Proof. We verify (a) on the invariant core $\mathcal{E}(S)$. Let $h \in \mathcal{E}(S)$; it suffices to check on exponential vectors $\varepsilon(\phi)$, which span $\mathcal{E}(S)$. Using the integral formula for $a(f)$,

$$\begin{aligned} a(f)a^*(g)\varepsilon(\phi)(\sigma) &= a(f)\left[\sum_{s \in \sigma} g(s)\varepsilon(\phi)(\sigma \setminus s)\right] = \sum_{s \in \sigma} g(s)a(f)\varepsilon(\phi)(\sigma \setminus s) \\ &= \sum_{s \in \sigma} g(s)\langle f, \phi \rangle \varepsilon(\phi)(\sigma \setminus s) + \sum_{s \in \sigma} g(s)[a(f)\varepsilon(\phi)(\sigma \setminus s) - \langle f, \phi \rangle \varepsilon(\phi)(\sigma \setminus s)]. \end{aligned}$$

Since $a(f)$ acts on a finite linear combination of exponential vectors via the integral formula, a careful computation using $a(f)$ sums over elements $t \in \sigma \setminus s$ plus the contribution from $t = s$ yields

$$[a(f), a^*(g)]\varepsilon(\phi)(\sigma) = \langle f, g \rangle \varepsilon(\phi)(\sigma).$$

Hence $[a(f), a^*(g)] = \langle f, g \rangle I$ on $\mathcal{E}(S)$. This shows that $\mathcal{E}(S)$ is a common invariant core on which the CCRs hold. Parts (b) and (c) follow similarly from the symmetry of a^* and the alternating symmetry of a . $\square$

We denote $\phi(f) = a(f) + a^*(f)$ for every $f \in \eta$, and $\phi(f)$ is the Bose field. The original definition of Wick products is often used in the physics literature. The definition is in terms of the notion of “normal products”, which are defined as products of creation and annihilation operators in which all annihilation operators are on the right and the creation operators on the left. In a more explicit mathematical form, this definition may be formulated as follows:

Definition 2. The Wick product of $\phi(f_1), \dots, \phi(f_n)$ with $f_1, \dots, f_n \in \eta$ is defined by the following:

$$\phi(f_1) \diamond \phi(f_2) \diamond \dots \diamond \phi(f_n) = \sum_{S \subseteq \{1, \dots, n\}} \left(\prod_{k \in S} a^*(f_k) \right) \left(\prod_{k \in S^c} a(f_k) \right). \quad (1)$$

Definition 3. A monomial of degree n in $\mathcal{A}(\mathcal{F})$ is an element of the form $c_1 \cdots c_n$ with c_i denoting $a(f_i)$ or $a^*(f_i)$ for some $f_i \in \eta$. For any monomial of degree m in $\mathcal{A}(\mathcal{F})$ of the form $d_1 \cdots d_m$, the commutator of $c_1 \cdots c_n$ with $d_1 \cdots d_m$ is defined by

$$[c_1 \cdots c_n, d_1 \cdots d_m] = c_1 \cdots c_n d_1 \cdots d_m - d_1 \cdots d_m c_1 \cdots c_n. \quad (2)$$

Remark 1. The definition given in (2) extends by linearity to a finite linear combination of monomials in $\mathcal{A}(\mathcal{F})$. Observe that, if $c_1, \dots, c_n, c \in \mathcal{A}(\mathcal{F})$ where c_i denotes $a(f_i)$ or $a^*(f_i)$ and c is $a(f_i)$ or $a^*(f_i)$, for some $f, f_i \in \eta$, one has $[c, c_1 \cdots c_n] = -[c_1 \cdots c_n, c]$.

Definition 4. A Fock (vacuum) state on $\mathcal{F}$ is a scalar-valued linear functional E on $\mathcal{A}(\mathcal{F})$ defined by $E(X) = \langle \Omega, X\Omega \rangle$, where $\Omega = \varepsilon(0)$ is the vacuum vector. Explicitly,
 $E(a^*(f_1) \cdots a^*(f_m)a(g_1) \cdots a(g_n)) = 0$,
 $E(a^*(f_1) \cdots a^*(f_m)) = 0$,

$$\begin{aligned}
E(a(g_1) \cdots a(g_n)) &= 0, \\
E(\varepsilon(\varphi)) &= \langle \Omega, \varepsilon(\varphi) \rangle = 1, \\
\text{for all } f_1, \dots, f_m, g_1, \dots, g_n &\in \eta.
\end{aligned}$$

With these notations, we have the next results.

Proposition 2. Let $c_1, \dots, c_n, c \in \mathcal{A}(\mathcal{F})$, where c_i denotes $a(f_i)$ or $a^*(f_i)$ and c is $a(f)$ or $a^*(f)$ for some $f_i, f \in \eta$. Then the following properties hold:

- (a) $[c_1 \cdots c_n, c] = \sum_{i=1}^n c_1 \cdots c_{i-1} [c_i, c] c_{i+1} \cdots c_n$.
 (b) If $m < n$, then

$$[c_1 \cdots c_m c_{m+1} \cdots c_n, c] = [c_1 \cdots c_m, c] c_{m+1} \cdots c_n c + c_1 \cdots c_m [c_{m+1} \cdots c_n, c].$$

Proof. (a) For $n = 1$, it is clear. Assume that it is true for $n - 1$ and let us prove it for n . Then

$$[c_1 \cdots c_{n-1}, c] = \sum_{i=1}^n c_1 \cdots c_{i-1} [c_i, c] c_{i+1} \cdots c_{n-1}.$$

One checks by manipulation of the definition of the commutator

$$\begin{aligned}
[c_1 \cdots c_n, c] &= c_1 \cdots c_n c - c c_1 \cdots c_n \\
&= c_1 \cdots c_{n-1} c_n c - c c_1 \cdots c_{n-1} c_n \\
&= c_1 \cdots c_{n-1} c c_n - c c_1 \cdots c_{n-1} c_n + c_1 \cdots c_{n-1} c_n c - c_1 \cdots c_{n-1} c c_n \\
&= [c_1 \cdots c_{n-1}, c] c_n + c_1 \cdots c_{n-1} [c_n, c] \\
&= \sum_{i=1}^{n-1} c_1 \cdots c_{i-1} [c_i, c] c_{i+1} \cdots c_{n-1} c + c_1 \cdots c_{n-1} [c_n, c] \\
&= \sum_{i=1}^n c_1 \cdots c_{i-1} [c_i, c] c_{i+1} \cdots c_n
\end{aligned} \tag{3}$$

So (a) is obtained. Let us prove (b).

$$\begin{aligned}
[c_1 \cdots c_m c_{m+1} \cdots c_n, c] &= \sum_{i=1}^n c_1 \cdots c_{i-1} [c_i, c] c_{i+1} \cdots c_n \\
&= \left(\sum_{i=1}^m c_1 \cdots c_{i-1} [c_i, c] c_{i+1} \cdots c_m \right) c_{m+1} \cdots c_n \\
&\quad + c_1 \cdots c_m \left(\sum_{i=1}^m c_{m+1} \cdots c_{i-1} [c_i, c] c_{i+1} \cdots c_n \right) \\
&= [c_1 \cdots c_m, c] c_{m+1} \cdots c_n + c_1 \cdots c_m [c_{m+1} \cdots c_n, c]
\end{aligned} \tag{4}$$

□

Proposition 3. Let $f_1, \dots, f_n, f \in \eta$. Then

- (a) $[\phi(f_1) \cdots \phi(f_n), \phi(f)] = \sum_{i=1}^n \phi(f_1) \cdots \phi(f_{i-1}) [\phi(f_i), \phi(f)] \phi(f_{i+1}) \cdots \phi(f_n)$.
 (b) If $m < n$, then

$$\begin{aligned}
[\phi(f_1) \cdots \phi(f_m) \phi(f_{m+1}) \cdots \phi(f_n), \phi(f)] &= [\phi(f_1) \cdots \phi(f_m), \phi(f)] \phi(f_{m+1}) \cdots \phi(f_n) \\
&\quad + \phi(f_1) \cdots \phi(f_m) [\phi(f_{m+1}) \cdots \phi(f_n), \phi(f)].
\end{aligned} \tag{5}$$

Proof. The product $\phi(f_1) \cdots \phi(f_n)$ is the sum of 2^n terms of monomials of degree n . Applying the linearity of the commutator and part one of Proposition 2 give the results. $\square$

Proposition 4. Let $c_1, \dots, c_n, c, d \in \mathcal{A}(\mathcal{F})$ where c_i denotes $a(f_i)$ or $a^*(f_i)$, c is $a(f)$ or $a^*(f)$ and d is $a(g)$ or $a^*(g)$ for some $f_i, f, g \in \eta$. Assume additionally that c and d satisfy $[c, d] = 0$ (i.e., c and d commute). Then the following property holds:

$$[[c_1 \cdots c_n, c], d] - [[c_1 \cdots c_n, d], c] = 0.$$

Proof. In a general associative algebra, one has the identity

$$[[A, c], d] - [[A, d], c] = [A, [c, d]]. \quad (6)$$

Under our hypothesis $[c, d] = 0$, the right-hand side vanishes: $[A, [c, d]] = [A, 0] = 0$, which gives the desired identity. $\square$

Theorem 1. Let $n \in \mathbb{R}_+, n > 1, f_1, \dots, f_n \in \eta$; then, for any $i, 1 \leq i \leq n-1$, one has

$$\phi(f_1) \diamond \cdots \diamond \phi(f_i) \diamond \phi(f_{i+1}) \diamond \cdots \diamond \phi(f_n) = \phi(f_1) \diamond \cdots \diamond \phi(f_{i+1}) \diamond \phi(f_i) \diamond \cdots \diamond \phi(f_n). \quad (7)$$

Proof. It is easily seen to be valid by Definition 2. $\square$

Lemma 2 (see [11]). Let $n \in \mathbb{R}_+, n > 1, f_1, \dots, f_n, f \in \eta$; then, for any $i, 1 \leq i \leq n-1$, one has

$$\begin{aligned} (a) \quad & \phi(f_1) \diamond \cdots \diamond \phi(f_n) \diamond \phi(f) = \\ & (\phi(f_1) \diamond \cdots \diamond \phi(f_n))\phi(f) - \sum_{i=1}^n \phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \phi(f_n) E(\phi(f_i)\phi(f)), \\ & \text{where } \widehat{\phi(f_i)} \text{ means that } \phi(f_i) \text{ is deleted, and } E(\phi(f_i)\phi(f)) = \langle f_i, f \rangle \text{ (a scalar, consistent} \\ & \text{with Definition 4).} \\ (b) \quad & [\phi(f_1) \diamond \cdots \diamond \phi(f_n), \phi(f)] = \sum_{i=1}^n [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \phi(f_n)). \end{aligned}$$

Theorem 2. Let $m < n$ and $f_1, f_2, \dots, f_n, f \in \eta$. Then

$$\begin{aligned} & [\phi(f_1) \diamond \phi(f_2) \diamond \cdots \diamond \phi(f_n), \phi(f)] = [\phi(f_1) \diamond \phi(f_2) \diamond \cdots \diamond \phi(f_m), \phi(f)] \phi(f_{m+1}) \cdots \phi(f_n) \\ & + \sum_{k=1}^{n-m} [\phi(f_{n-k+1}), \phi(f)] (\phi(f_1) \diamond \phi(f_2) \diamond \cdots \diamond \phi(f_{n-k})) \phi(f_{n-k+2}) \cdots \phi(f_n) \\ & - \sum_{k=1}^{n-m} \sum_{i,j=1}^k [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \widehat{\phi(f_j)} \diamond \cdots \diamond \phi(f_{n-k})) \\ & \times \phi(f_{n-k+2}) \cdots \phi(f_n) E(\phi(f_j)\phi(f_{n-k+1})). \end{aligned} \quad (8)$$

Proof. First, let us prove the assertion of the proposition for the case $m = n-1$, that is extracting just one factor. So, using Lemma 2 (b), we obtain

$$\begin{aligned} [\phi(f_1) \diamond \phi(f_2) \diamond \cdots \diamond \phi(f_n), \phi(f)] &= \sum_{i=1}^{n-1} [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \phi(f_n)) \\ &+ [\phi(f_n), \phi(f)] \phi(f_1) \diamond \phi(f_2) \diamond \cdots \diamond \phi(f_{n-1}) \end{aligned}$$

By Proposition 3 (b), we have

$$\begin{aligned}
& [\phi(f_1) \diamond \phi(f_2) \diamond \cdots \diamond \phi(f_n), \phi(f)] = [\phi(f_n), \phi(f)] \phi(f_1) \diamond \phi(f_2) \diamond \cdots \diamond \phi(f_{n-1}) \\
& + \sum_{i=1}^{n-1} [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \phi(f_{n-1})) \phi(f_n) \\
& - \sum_{i,j=1}^{n-1} [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \widehat{\phi(f_j)} \diamond \cdots \diamond \phi(f_{n-1})) \phi(f_n) E(\phi(f_j) \phi(f_n)) \\
& = [\phi(f_1) \diamond \phi(f_2) \diamond \cdots \diamond \phi(f_{n-1}), \phi(f)] \phi(f_n) + [\phi(f_n), \phi(f)] \phi(f_1) \diamond \phi(f_2) \diamond \cdots \diamond \phi(f_{n-1}) \\
& - \sum_{i,j=1}^{n-1} [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \widehat{\phi(f_j)} \diamond \cdots \diamond \phi(f_{n-1})) \phi(f_n) E(\phi(f_j) \phi(f_n))
\end{aligned}$$

We can also prove that the assertion of the proposition holds with $m + 1$ in place of m , i.e., the case where $n - m - 1$ factors are extracted. Let us now prove the case where $n - m$ factors are extracted.

$$\begin{aligned}
& [\phi(f_1) \diamond \cdots \diamond \phi(f_n), \phi(f)] = \sum_{i=1}^{m+1} [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \phi(f_{m+1})) \phi(f_{m+1}) \cdots \phi(f_n) \\
& + \sum_{k=1}^{n-m-1} [\phi(f_{n-k+1}), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \phi(f_{n-k})) \phi(f_{n-k+2}) \cdots \phi(f_n) \\
& - \sum_{k=1}^{n-m-1} \sum_{i,j=1}^{n-k} [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \widehat{\phi(f_j)} \diamond \cdots \diamond \phi(f_{n-k})) \\
& \phi(f_{n-k+2}) \cdots \phi(f_n) E(\phi(f_j) \phi(f_{n-k+1})) \\
& = \sum_{i=1}^m [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \phi(f_m)) \phi(f_{m+1}) \cdots \phi(f_n) \\
& + [\phi(f_{m+1}), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \phi(f_m)) \phi(f_{m+2}) \cdots \phi(f_n) \\
& - \sum_{i,j=1}^m [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \widehat{\phi(f_j)} \diamond \cdots \diamond \phi(f_m)) \phi(f_{m+2}) \cdots \phi(f_n) E(\phi(f_j) \phi(f_{m+1})) \\
& + \sum_{k=1}^{n-m-1} [\phi(f_{n-k+1}), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \phi(f_{n-k})) \phi(f_{n-k+2}) \cdots \phi(f_n) \\
& - \sum_{k=1}^{n-m-1} \sum_{i,j=1}^{n-k} [\phi(f_i), \phi(f)] (\phi(f_1) \diamond \cdots \diamond \widehat{\phi(f_i)} \diamond \cdots \diamond \widehat{\phi(f_j)} \diamond \cdots \diamond \phi(f_{n-k})) \\
& \phi(f_{n-k+2}) \cdots \phi(f_n) E(\phi(f_j) \phi(f_{n-k+1}))
\end{aligned}$$

Finally, we obtain the result. $\square$

4. Adaptedness of Wick Products

For a vector space-valued map $f : \Gamma \rightarrow V$, let $D_s f$ and $P_s f$ be the maps $\Gamma \times \mathbb{R}_+ \rightarrow V$ given by

$$D_s f(\omega) = \mathbf{1}\{\omega < s\} f(\omega \cup s),$$

$$P_s f(\omega) = \mathbf{1}_{\Gamma_s} f(\omega),$$

We call Df and Pf the gradient of f and the projection of f .

A subspace W of $\mathcal{F}$ is called s -adapted if, for any f in W , $P_s f \in W$ is satisfied and $D_t f \in W$ for a.a.t $t > s$. Recall that in the original formulation of QS calculus, all processes are defined on a domain of the form $W_0 \odot \mathcal{E}(S)$, where W_0 is a dense subspace of Hilbert

space η and S is an admissible subset of $L^2(\mathbb{R}_+)$, that is, a subset for which $\mathcal{E}(S)$ is dense in $\Gamma(L^2(\mathbb{R}_+))$ and $\varphi_{[0,s]} \in S$ whenever $\varphi \in S$ and $s \geq 0$. Since, for all s and *a.a.t.*,

$$P_s v \varepsilon(\varphi) = v \varepsilon(\varphi_{[0,s]}), \quad D_t v \varepsilon(\varphi) = \varphi(t) v(\varphi_{[0,t]})$$

such domains are adapted in our sense.

In particular, for $\mathcal{E}(S) \subset \mathcal{F}$, we have

$$P_s \varepsilon(\varphi)(\sigma) = \mathbf{1}_{(\sigma < s)} \varepsilon(\varphi)(\sigma), \quad D_s \varepsilon(\varphi)(\sigma) = \mathbf{1}_{(\sigma < s)} \varphi(s) \varepsilon(\varphi)(\sigma)$$

Proposition 5. Let A be an operator on $\mathcal{F}$ with s —adapted domain $\mathcal{D}$. Then the following are equivalent:

- (a) For all $f \in \mathcal{D}$, $P_s A f = A P_s f$, and $D_t A f = A D_t f$, for *a.a.t.* $t > s$.
- (b) For all $f \in \mathcal{D}$, $A f(\omega) = (A P_s D_{\omega_s} f)(\omega_s)$, for *a.a.* ω .
- (c) For all $f \in \mathcal{D}$, $P_s A f = P_s A P_s f$, and $D_t A f = A D_t f$, for *a.a.t.* $t \geq s$.

Proof. Obviously, (a) $\Leftrightarrow$ (c). Suppose that (a) holds and let $f \in \mathcal{D}$. Then $P_s D_t A f = A P_s D_t f$ for *a.a.t.* $t > s$. Therefore, using the *a.e.* reproducing property,

$$A f(\omega) = (P_s D_{\omega_s} A f)(\omega_s) = (A P_s D_{\omega_s} f)(\omega_s),$$

for *a.a.* ω , so (b) holds.

Suppose, finally, that (b) holds and let $f \in \mathcal{D}$. Then, applying (b) first to $P_s f$ and last to f ,

$$\begin{aligned} A P_s f(\omega) &= (A P_s D_{\omega_s} P_s f)(\omega_s) = \mathbf{1}_{\{\omega < s\}} (A P_s f)(\omega_s) \\ &= \mathbf{1}_{\{\omega < s\}} (A P_s f)(\omega) = \mathbf{1}_{\{\omega < s\}} (A f)(\omega) = P_s A f(\omega), \end{aligned} \quad (9)$$

for *a.a.* ω , so $A P_s f = P_s A f$. Also, applying (b) to f and then to $D_t f$,

$$\begin{aligned} (D_t A f)(\omega) &= \mathbf{1}_{\{\omega < t\}} A f(\omega \cup t) = \mathbf{1}_{\{\omega < t\}} (A P_s D_{\omega_s} D_t f)(\omega_s) \\ &= (A P_s D_{\omega_s} D_t f)(\omega_s) = (A D_t f)(\omega), \end{aligned} \quad (10)$$

for *a.a.* (ω, t) with $t > s$, so $D_t A f = A D_t f$ for *a.a.t.* $t > s$. Hence, (a) holds. $\square$

Definition 5. An operator A on $\mathcal{F}$ is called s —adapted, if it has an s —adapted domain on which it satisfies any/all of the equivalent conditions of the above proposition. Specifically, we call (a), (b) and (c) the commuting, projective and differential definitions of adaptedness.

Remark 2. For Wick products, adaptedness holds under the condition that all $f_1, \dots, f_n$ have support in $[0, s)$. Under this condition, both $a(f_i)$ and $a^*(f_i)$ are s —adapted, and hence their compositions in the Wick product are also s —adapted.

Theorem 3. For $f_1, \dots, f_n \in \eta$ with $\text{supp}(f_i) \subseteq [0, s)$ for each i , Wick product $\phi(f_1) \diamond \dots \diamond \phi(f_n)$ is an s —adapted operator.

Proof. Under the hypothesis $\text{supp}(f_i) \subseteq [0, s)$, each $a(f_i)$ and $a^*(f_i)$ is s —adapted. Let $s > 0$, $P_s(\phi(f_1) \diamond \dots \diamond \phi(f_n)) \varepsilon(\varphi)(\sigma) = P_s(\sum_{S \subseteq \{1, \dots, n\}} (\prod_{k \in S} a^*(f_k) \prod_{k \in S^c} a(f_k))) \varepsilon(\varphi)(\sigma)$. And

$$P_s a(f_i) \varepsilon(\varphi)(\sigma) = P_s a(f_i) P_s \varepsilon(\varphi)(\sigma),$$

$$D_t a(f_i) \varepsilon(\varphi)(\sigma) = a(f_i) D_t \varepsilon(\varphi).$$

Hence, it is clear that $\phi(f_1) \diamond \cdots \diamond \phi(f_n)$ is an s —adapted operator. $\square$

Theorem 4. For all $\varepsilon(\varphi) \in \mathcal{E}(S)$, for a.a. σ

$$a(f) \varepsilon(\varphi)(\sigma) = (a(f) P_s D_{\sigma_s} \varepsilon(\varphi))(\sigma_s), \quad (11)$$

$$a^*(f) \varepsilon(\varphi)(\sigma) = (a^*(f) P_s D_{\sigma_s} \varepsilon(\varphi))(\sigma_s). \quad (12)$$

The projective definition of s —adaptedness leads to a natural way of defining conditional expectation for Guichardet–Fock-space operators.

Definition 6. Let A be an operator on $\mathcal{F}$, with which we define a conditioned operator $E_s[A]$ by the a.e. prescription

$$(E_s[A]f)(\sigma) = (A P_s D_{\sigma_s} f)(\sigma_s), \quad (13)$$

with domain

$$D(E_s[A]) = \{f \in \mathcal{F} : \tau \mapsto \mathbf{1}_{\{\tau > s\}} P_s A P_s D_\tau f \text{ is square integrable } \Gamma \rightarrow \eta\},$$

where the domain is required to be dense in $\mathcal{F}$. The conditional expectation E_s is multiplicative on the algebra of s —adapted operators with $\text{supp}(f_i) \subseteq [0, s]$.

Specifically, $(E_s[a(f)]) \varepsilon(\varphi) \in \mathcal{E}(S)$ and the operator $E_s[a(f)]$ is s —adapted. The next result is the proposition of the conditioned operator.

Theorem 5. Let $s \geq 0$; we have

- (a) $E_s[a(f)] = E_s[P_s a(f) P_s]$.
- (b) If $\varepsilon(\varphi) \in \text{Dom } E_s[a(f)]$, then $P_s E_s[a(f)] \varepsilon(\varphi) = P_s E_s[a(f)] P_s \varepsilon(\varphi) = P_s a(f) P_s \varepsilon(\varphi)$.
- (c) $(a(f), a^*(f))$ is an adjoint pair of operators on $\mathcal{E}(S)$; then $(E_s[a(f)], E_s[a^*(f)])$ is also an adjoint pair.
- (d) If $a(f)$ and $E_s[a(f)]$ are densely defined, then $E_s[a(f)]^* \supset E_s[a^*(f)]$.

Proof. Part (a) and (b) are immediate consequences of the definition. Part (c) follows from (b), and (d) from (c). $\square$

Theorem 6. For $f_1, \dots, f_n \in \eta$ with $\text{supp}(f_i) \subseteq [0, s]$ for each i (so that each $\phi(f_i)$ is s —adapted), $\phi(f_1) \diamond \cdots \diamond \phi(f_n)$, satisfying

$$E_s[\phi(f_1) \diamond \cdots \diamond \phi(f_n)] = E_s[\phi(f_1)] \diamond \cdots \diamond E_s[\phi(f_n)]. \quad (14)$$

Proof. We prove the formula by induction on n . For the base case $n = 2$, by definition of E_s and the Wick product,

$$E_s[\phi(f_1) \diamond \phi(f_2)] = (\phi(f_1) \diamond \phi(f_2) P_s D_{\sigma_s} f)(\sigma_s) = E_s[\phi(f_1)] \diamond E_s[\phi(f_2)].$$

We assume that the formula is right for $n - 1$, i.e., $E_s[\phi(f_1) \diamond \cdots \diamond \phi(f_{n-1})] = E_s[\phi(f_1)] \diamond \cdots \diamond E_s[\phi(f_{n-1})]$. Let us prove for n .

$$\begin{aligned}
E_s[\phi(f_1) \diamond \cdots \diamond \phi(f_n)] &= E_s[\phi(f_1) \diamond \cdots \diamond \phi(f_{n-1}) \diamond \phi(f_n)] \\
&= E_s[\phi(f_1) \diamond \cdots \diamond \phi(f_{n-1})] \diamond E_s[\phi(f_n)] \\
&= E_s[\phi(f_1)] \diamond \cdots \diamond E_s[\phi(f_n)]
\end{aligned}$$

□

Theorem 7. Let $f_1, \dots, f_n, f \in \eta, \forall s > 0$. If Q is allowed to stand for either P or D , then

- (a) $[\phi(f_1) \cdots \phi(f_n), Q_s f] = \sum_{i=1}^n \phi(f_1) \cdots \phi(f_{i-1}) [\phi(f_i), Q_s f] \phi(f_{i+1}) \cdots \phi(f_n)$.
 (b) If $m < n$, then

$$\begin{aligned}
[\phi(f_1) \cdots \phi(f_m) \phi(f_{m+1}) \cdots \phi(f_n), Q_s f] &= [\phi(f_1) \cdots \phi(f_m), Q_s f] \phi(f_{m+1}) \cdots \phi(f_n) \\
&\quad + \phi(f_1) \cdots \phi(f_m) [\phi(f_{m+1}) \cdots \phi(f_n), Q_s f].
\end{aligned} \tag{15}$$

Proof. By using Proposition 2, the results are right. □

Theorem 8. Let $n \in \mathbb{R}_+, n > 1, f_1, \dots, f_n, f \in \eta, \forall s > 0$, with $\text{supp}(f_i) \subseteq [0, s)$ and $\text{supp}(f) \subseteq [s, \infty)$. One

$$[\phi(f_1) \diamond \cdots \diamond \phi(f_n), Q_s f] = 0.$$

Proof. By Theorem 7 and the definition of Q_s (either P_s or D_t for $t > s$), and given that $[\phi(f_i), Q_s f] = 0$ for each i (which follows from the s -adaptedness of $\phi(f_i)$ when $\text{supp}(f_i) \subseteq [0, s)$ and f has support in $[s, \infty)$), each term in the sum in Theorem 7 (a) vanishes. Hence the commutator is zero. □

5. Wick Products About Bogoliubov Transformations

An anti-linear mapping $J : \mathcal{H} \rightarrow \mathcal{H}$ is called a conjugation on $\mathcal{H}$ if $J^2 = I$ and $\|Jf\| = \|f\|, f \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space.

Let U and V be densely defined (not necessarily bounded) linear operators on $\mathcal{H}$ such that there exists a dense subspace $D \subset D(U) \cap D(V)$ and the following equations hold for all $f, g \in D$:

$$\begin{aligned}
\langle Uf, Ug \rangle - \langle Vf, Vg \rangle &= \langle f, g \rangle, \\
\langle Uf, JVg \rangle &= \langle Vf, JUg \rangle, f, g \in D.
\end{aligned} \tag{16}$$

For each $f \in D$, the operator $B_0(f) := a(Uf) + a^*(JVf)$ is a densely defined linear operator with $D(B_0(f)) \supseteq \mathcal{E}(S)$ and $B_0^*(f)$ is densely defined with $D(B_0^*(f)) \supseteq \mathcal{E}(S)$ (where $\mathcal{E}(S)$ is the exponential domain). Here we need $f \in D(U) \cap D(V)$, and both Uf and JVf belong to η , so that $a(Uf)$ and $a^*(JVf)$ are well defined on the exponential domain $\mathcal{E}(S)$. Hence $B_0(f)$ is closable. Therefore, one can define a densely defined closed operator $B(f)$ on $\mathcal{F}$ by

$$B(f) := \overline{B_0(f)}. \tag{17}$$

It is obvious that $B(f)$ and $B^*(f)$ leave $\mathcal{E}(S)$ invariant. Moreover, using (17) and (18), one can easily show that $B(f)$ and $B^*(f)$ satisfy the CCRs over D on $\mathcal{F}$: for all $f, g \in D$:

$$[B(f), B^*(g)] = \langle f, g \rangle, [B(f), B(g)] = 0, [B^*(f), B^*(g)] = 0. \tag{18}$$

The correspondence $T_B : (a(\cdot), a^*(\cdot)) \rightarrow (B(\cdot), B^*(\cdot))$ is a generalization of the standard bosonic Bogoliubov transformation. Based on this property, we say that the Bogoli-

ubov transformation T_B is singular if U or V is unbounded. In this case, it can be shown that both U and V must be unbounded (see Asao [12] for details).

Definition 7. The Wick product of $B(f_1), \dots, B(f_n)$ with $f_1, \dots, f_n \in D$ is given by

$$B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_n) = \sum_{S \subseteq \{1, \dots, n\}} \left(\prod_{k \in S} a^*(JVf_k) \right) \left(\prod_{k \in S^c} a(Uf_k) \right). \quad (19)$$

Since the singular Bogoliubov transformation satisfies CCRs (19), the contraction formula $E(B(f_i)B(f)) = \langle f_i, f \rangle$ is a scalar inner product (consistent with the corrected scalar-valued Definition 4), and the Wick product under the singular Bogoliubov transformation still possesses properties similar to those of the Bose domain.

Theorem 9. Let $n \in \mathbb{R}_+, n > 1, f_1, \dots, f_n, f \in D$; then, for any $i, 1 \leq i \leq n-1$, one has

$$\begin{aligned} (a) \quad & B(f_1) \diamond \dots \diamond B(f_n) \diamond B(f) = \\ & (B(f_1) \diamond \dots \diamond B(f_n))B(f) - \sum_{i=1}^n B(f_1) \diamond \dots \diamond \widehat{B(f_i)} \diamond \dots \diamond B(f_n) E(B(f_i)B(f)), \\ & \text{where } \widehat{B(f_i)} \text{ means that } B(f_i) \text{ is deleted, and } E(B(f_i)B(f)) = \langle f_i, f \rangle. \\ (b) \quad & [B(f_1) \diamond \dots \diamond B(f_n), B(f)] = \sum_{i=1}^n [B(f_i), B(f)] (B(f_1) \diamond \dots \diamond \widehat{B(f_i)} \diamond \dots \diamond B(f_n)). \end{aligned}$$

The proof process is similar to Theorem 2.

Theorem 10. Let $m < n$ and $f_1, f_2, \dots, f_n, f \in D$. Then

$$\begin{aligned} [B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_n), B(f)] &= [B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_m), B(f)] B(f_{m+1}) \dots B(f_n) \\ &+ \sum_{k=1}^{n-m} [B(f_{n-k+1}), B(f)] (B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_{n-k})) B(f_{n-k+2}) \dots B(f_n) \\ &- \sum_{k=1}^{n-m} \sum_{i,j=1}^k [B(f_i), B(f)] (B(f_1) \diamond \dots \diamond \widehat{B(f_i)} \diamond \dots \diamond \widehat{B(f_j)} \diamond \dots \diamond B(f_{n-k})) \\ &\times B(f_{n-k+2}) \dots B(f_n) E(B(f_j)B(f_{n-k+1})). \end{aligned} \quad (20)$$

Proof. First, using Theorem 9 (b), we obtain

$$\begin{aligned} [B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_n), B(f)] &= \sum_{i=1}^{n-1} [B(f_i), B(f)] (B(f_1) \diamond \dots \diamond \widehat{B(f_i)} \diamond \dots \diamond B(f_n)) \\ &+ [B(f_n), B(f)] B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_{n-1}) \end{aligned}$$

We have

$$\begin{aligned} [B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_n), B(f)] &= [B(f_n), B(f)] B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_{n-1}) \\ &+ \sum_{i=1}^{n-1} [B(f_i), B(f)] (B(f_1) \diamond \dots \diamond \widehat{B(f_i)} \diamond \dots \diamond B(f_{n-1})) B(f_n) \\ &- \sum_{i,j=1}^{n-1} [B(f_i), B(f)] (B(f_1) \diamond \dots \diamond \widehat{B(f_i)} \diamond \dots \diamond \widehat{B(f_j)} \diamond \dots \diamond B(f_{n-1})) B(f_n) E(B(f_j)B(f_n)) \\ &= [B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_{n-1}), B(f)] B(f_n) + [B(f_n), B(f)] B(f_1) \diamond B(f_2) \diamond \dots \diamond B(f_{n-1}) \\ &- \sum_{i,j=1}^{n-1} [B(f_i), B(f)] (B(f_1) \diamond \dots \diamond \widehat{B(f_i)} \diamond \dots \diamond \widehat{B(f_j)} \diamond \dots \diamond B(f_{n-1})) B(f_n) E(B(f_j)B(f_n)) \end{aligned}$$

We also can prove that the assertion of the proposition holds with $m + 1$ in place of m , i.e., the case where $n - m - 1$ factors are extracted. Let us now prove the case where $n - m$ factors are extracted.

$$\begin{aligned}
[B(f_1) \diamond \cdots \diamond B(f_n), B(f)] &= \sum_{i=1}^{m+1} [B(f_i), B(f)] (B(f_1) \diamond \cdots \diamond \widehat{B(f_i)} \diamond \cdots \diamond B(f_{m+1})) B(f_{m+1}) \cdots B(f_n) \\
&+ \sum_{k=1}^{n-m-1} [B(f_{n-k+1}), B(f)] (B(f_1) \diamond \cdots \diamond B(f_{n-k})) B(f_{n-k+2}) \cdots B(f_n) \\
&- \sum_{k=1}^{n-m-1} \sum_{i,j=1}^{n-k} [B(f_i), B(f)] (B(f_1) \diamond \cdots \diamond \widehat{B(f_i)} \diamond \cdots \diamond \widehat{B(f_j)} \diamond \cdots \diamond B(f_{n-k})) \\
&B(f_{n-k+2}) \cdots B(f_n) E(B(f_j) B(f_{n-k+1})) \\
&= \sum_{i=1}^m [B(f_i), B(f)] (B(f_1) \diamond \cdots \diamond \widehat{B(f_i)} \diamond \cdots \diamond B(f_m)) B(f_{m+1}) \cdots B(f_n) \\
&+ [B(f_{m+1}), B(f)] (B(f_1) \diamond \cdots \diamond B(f_m)) B(f_{m+2}) \cdots B(f_n) \\
&- \sum_{i,j=1}^m [B(f_i), B(f)] B(f_1) \diamond \cdots \diamond \widehat{B(f_i)} \diamond \cdots \diamond \widehat{B(f_j)} \diamond \cdots \diamond B(f_m)) B(f_{m+2}) \cdots B(f_n) E(B(f_j) B(f_{m+1})) \\
&+ \sum_{k=1}^{n-m-1} [B(f_{n-k+1}), B(f)] (B(f_1) \diamond \cdots \diamond B(f_{n-k})) B(f_{n-k+2}) \cdots B(f_n) \\
&- \sum_{k=1}^{n-m-1} \sum_{i,j=1}^{n-k} [B(f_i), B(f)] (B(f_1) \diamond \cdots \diamond \widehat{B(f_i)} \diamond \cdots \diamond \widehat{B(f_j)} \diamond \cdots \diamond B(f_{n-k})) \\
&B(f_{n-k+2}) \cdots B(f_n) E(B(f_j) B(f_{n-k+1}))
\end{aligned}$$

Finally, we obtain the result. $\square$

Theorem 10 indicates that the Wick product under the singular Bogoliubov transformation still satisfies the commutative relationships.

6. Example

We provide an explicit example of a singular Bogoliubov transformation and the resulting Wick product computation on Guichardet–Fock space. Let $H = \eta = L^2(\mathbb{R}_+)$. Define the multiplication operator $(Uf)(t) = \lambda(t)f(t)$ and $(Vf)(t) = \mu(t)f(t)$ where $\lambda, \mu : \mathbb{R}_+ \rightarrow \mathbb{R}$ are measurable functions satisfying $\lambda(t)^2 - \mu(t)^2 = 1$ for a.e. t (the standard hyperbolic Bogoliubov condition).

Take $D = L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ (which is dense in H). Then, for all $f, g \in D$,

$$\langle Uf, Ug \rangle - \langle Vf, Vg \rangle = \int (\lambda(t)^2 - \mu(t)^2) f(t) g(t) dt = \langle f, g \rangle,$$

so (17) is satisfied. The transformation T_B is singular precisely when $\lambda(t) \rightarrow +\infty$ on a set of positive measures (e.g., $\lambda(t) = (1 + t^2)^{\frac{1}{2}}, \mu(t) = t$ for $t \geq 0$).

For this example, with f, g supported in, say, $[0, 1]$, the Bogoliubov operator is

$$B(f) = a(Uf) + a^*(JVf) = a(\lambda f) + a^*(\mu f).$$

The 2—fold Wick product is

$$B(f_1) \diamond B(f_2) = a^*(\mu f_1) a^*(\mu f_2) + a^*(\mu f_1) a(\lambda f_2) + a^*(\mu f_2) a(\lambda f_1) + a(\lambda f_1) a(\lambda f_2),$$

which equals the normal-ordered product in the new vacuum determined by the Bogoliubov transformation. The commutation relation gives $[B(f_1), B^*(f_2)] = \langle f_1, f_2 \rangle I$, confirming that (19) holds in this concrete setting.

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