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On a Fractional p -Laplacian Problem in the Whole Space and with Singular Reaction

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Abstract

The existence of solutions that are positive, pointwise decaying at infinity, and weak to a fractional p -Laplacian problem in the whole space and exhibit a singular reaction is established. Truncation arguments, variational methods, as well as suitable a priori estimates are exploited.

Keywords: fractional p -Laplacian problem; singular reaction; positive solution in the whole space; decay estimates

MSC: 35J60; 35J75; 35D30; 35B08

1. Introduction

Nonlocal differential operators, like the fractional p -Laplacian of order s , usually denoted with $(-\Delta_p)^s$, have been introduced in recent years to model a variety of phenomena, including anomalous diffusion and processes characterized by long-range interactions. Due to its broad applicability, they have attracted considerable attention, also in connection with problems arising from game theory, finance, image processing, materials science, etc. This daily growing interest has led to a vast amount of studies that treat existence, uniqueness, or multiplicity of solutions for equations with such operators. See, e.g., refs. [1–7] with regard to existence or multiplicity and [8–11] for regularity. The survey in [12] provides a comprehensive account of underlying fractional Sobolev spaces.

Singular equations driven by nonlocal differential operators have recently been investigated. For instance, Dirichlet fractional p -Laplacian problems are studied in [13], while [14,15] deal with the fractional (p, q) -Laplacian. All these works are set in a bounded domain $\Omega \subseteq \mathbb{R}^N$. As well-pointed out by Carl, Perera, and Teherani [8], proofs involving nonlocal differential operators on the whole space are usually carried out within the framework of homogeneous fractional Sobolev spaces, where the natural decay of functions is only measure-theoretic. Here, we consider the problem

$$\begin{cases} (-\Delta_p)^s u = f(x, u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases} \quad (1)$$

with $N \geq 3$, $0 < s < 1$, $2 \leq p < \frac{N}{s}$, and $(-\Delta_p)^s$ formally defined by



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$$(-\Delta_p)^s u(x) := 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy, \quad x \in \mathbb{R}^N.$$

The reaction term $f : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}_0^+$ satisfies Carathéodory's conditions. Moreover,

$$a(x)t^{-\gamma} \leq f(x, t) \leq a(x)(t^{-\gamma} + t^r) \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}^+, \quad (2)$$

where $0 < \gamma < 1 < r < p - 1$ and $a : \mathbb{R}^N \rightarrow \mathbb{R}_0^+$ is a measurable function such that

$$0 \leq a(x) \leq c_a w(x), \quad \text{with} \quad w(x) := \frac{1}{1 + |x|^{N+\alpha}}, \quad x \in \mathbb{R}^N, \quad (3)$$

for some $c_a > 0$, $\alpha \in \left(\gamma \frac{N-sp}{p-1}, \gamma \frac{N-sp}{p-1} + sp\right)$.

Hypothesis (3) imposes a precise decay at infinity to the weight a . It allows comparison both with the fractional p -harmonic function and with the fractional p -Laplacian of power functions; see Lemma 6.

If $D^{s,p}(\mathbb{R}^N)$ denotes the completion of $C_c^\infty(\mathbb{R}^N)$ through Gagliardo's norm (cf. Section 2), then a function $u \in D^{s,p}(\mathbb{R}^N)$ is called a *weak solution* to (1) provided

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy = \int_{\mathbb{R}^N} f(\cdot, u) \varphi dx$$

for all $\varphi \in D^{s,p}(\mathbb{R}^N)$, $u > 0$ in $\mathbb{R}^N$, and $\lim_{|x| \rightarrow \infty} u(x) = 0$. Our main result, namely Theorem 6 below, ensures that, under (2) and (3), problem (1) admits a weak solution.

From a technical point of view, there are at least four noteworthy features:

- (a) The driving differential operator is nonlocal;
- (b) $f(x, \cdot)$ turns out to be singular at zero, which means $\lim_{t \rightarrow 0^+} f(x, t) = +\infty$;
- (c) One seeks solutions defined on the whole $\mathbb{R}^N$;
- (d) A pointwise decay of the solution at infinity is required.

Working in $\mathbb{R}^N$ causes a lack of compactness for fractional Sobolev embeddings. So, right compact embeddings, as well as comparison principles in exterior domains, are necessary. To achieve (d), we first derive suitable lower and upper bounds and then match them with appropriately chosen radial functions having prescribed decay at infinity. In doing this, the papers [8,16] play a significant role. Precisely, ref. [8] provides global L^∞ -bounds and decay estimates for solutions $u \in D^{s,p}(\mathbb{R}^N)$ of fractional p -Laplacian equations, while [16] contains a rigorous computation of the fractional p -Laplacian fundamental solution and establishes the sharp asymptotic behavior at infinity of extremal functions for the fractional critical Sobolev embedding.

This paper is organized as follows. Section 2 contains preliminary facts. In the next section, we focus on the pure singular problem and, via a shifting method, show that it admits a weak solution. Finally, Section 4 is devoted to the study of (1), which we solve by means of a truncation argument, combined with a variational approach and suitable a priori estimates.

2. Basic Definitions and Auxiliary Results

Given two topological spaces X and Y , the symbol $X \hookrightarrow Y$ means that X continuously embeds in Y .

Let X be a real Banach space with topological dual X^* and duality brackets $\langle \cdot, \cdot \rangle$. A function $A : X \rightarrow X^*$ is called *strictly monotone* when $\langle A(x) - A(z), x - z \rangle > 0$ for all $x, z \in X$ with $x \neq z$.

If $r > 0$ and $x_0 \in \mathbb{R}^N$, then $B_r(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < r\}$; $B_r := B_r(0)$; $\bar{B}_r$ denotes the closure of B_r ; $B_r^c := \mathbb{R}^N \setminus B_r$; $|E|$ indicates the N -dimensional Lebesgue measure of $E \subseteq \mathbb{R}^N$; χ_E stands for the characteristic function of E ;

$$t_{\pm} := \max\{\pm t, 0\}, \quad t \in \mathbb{R};$$

and C, C_1 , etc. are positive constants, which may change value from line to line, whose dependencies will be specified when necessary.

Let $X(\mathbb{R}^N)$ be a real-valued function space on $\mathbb{R}^N$, and let $u, v \in X(\mathbb{R}^N)$. We simply write $u \leq v$ when $u(x) \leq v(x)$ a.e. in $\mathbb{R}^N$, and analogously, for $u < v$, etc. To shorten the notation, define

$$\{u \leq v\} := \{x \in \mathbb{R}^N : u(x) \leq v(x)\}, \quad X(\mathbb{R}^N)_+ := \{w \in X(\mathbb{R}^N) : w \geq 0\}.$$

The meaning of $\{u < v\}$ and so on is similar. Henceforth, q' indicates the conjugate exponent of $q \geq 1$, while

$$\|v\|_q := \begin{cases} \left(\int_{\mathbb{R}^N} |v(x)|^q dx \right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty, \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^N} |v(x)| & \text{when } q = \infty. \end{cases}$$

Given $s \in (0, 1)$ and $p \in [1, +\infty)$, the Gagliardo semi-norm of a measurable function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is

$$[u]_{s,p} := \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

If $1 < p < \frac{N}{s}$, write (when no confusion can arise)

$$p^* := \frac{Np}{N - sp}$$

and denote by $D^{s,p}(\mathbb{R}^N)$ the homogeneous fractional Sobolev space

$$D^{s,p}(\mathbb{R}^N) := \left\{ u \in L^{p^*}(\mathbb{R}^N) : [u]_{s,p} < +\infty \right\},$$

endowed with the norm

$$\|u\|_{s,p} := [u]_{s,p}, \quad u \in D^{s,p}(\mathbb{R}^N).$$

It is known [17] that $(D^{s,p}(\mathbb{R}^N), \|\cdot\|_{s,p})$ turns out to be a uniformly convex (hence, reflexive) Banach space.

The operator $(-\Delta_p)^s : D^{s,p}(\mathbb{R}^N) \rightarrow (D^{s,p}(\mathbb{R}^N))^*$ defined by setting

$$\langle (-\Delta_p)^s u, v \rangle := \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+sp}} dx dy$$

for every $u, v \in D^{s,p}(\mathbb{R}^N)$ is called fractional p -Laplacian of order s . It enjoys the next useful property.

Lemma 1. Suppose that $2 \leq p < \frac{N}{s}$. Then, $(-\Delta_p)^s$ turns out to be strictly monotone.

Proof. Given any $a, b \in \mathbb{R}$, one has

$$\left(|a|^{p-2}a - |b|^{p-2}b\right)(a - b) \geq C_p|a - b|^p,$$

with appropriate $C_p > 0$; see, e.g., ref. [18], Example 25.5. If $u, v \in D^{s,p}(\mathbb{R}^N)$ then choosing $a := u(x) - u(y)$, $b := v(x) - v(y)$, dividing by $|x - y|^{N+sp}$, and integrating on $\mathbb{R}^N \times \mathbb{R}^N$ easily produces

$$\langle (-\Delta_p)^s u - (-\Delta_p)^s v, u - v \rangle \geq C_p \|u - v\|_{s,p}^p,$$

as desired. $\square$

Let $w : \mathbb{R}^N \rightarrow \mathbb{R}_0^+$ be as in (3), namely

$$w(x) := \frac{1}{1 + |x|^{N+\alpha}} \quad \forall x \in \mathbb{R}^N,$$

where $\alpha > 0$. Since $w \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the interpolation inequality ([19], p. 93) entails $w \in L^q(\mathbb{R}^N)$ for whatever $q \in [1, \infty]$. On the linear space

$$L^q(\mathbb{R}^N, w) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} w|u|^q dx < \infty \right\},$$

we will consider the norm

$$\|u\|_{q,w} = \left(\int_{\mathbb{R}^N} |u|^q w dx \right)^{\frac{1}{q}}.$$

If $1 < q < \infty$, then $(L^q(\mathbb{R}^N, w), \|\cdot\|_{q,w})$ turns out reflexive and separable. Moreover,

$$u \in D^{s,p}(\mathbb{R}^N) \implies \|u\|_{p^*,w} \leq \|w\|_\infty \|u\|_{p^*} \leq C(w)[u]_{s,p},$$

whence $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N, w)$. Actually, a nontrivial argument ([8], Lemma 3.3) ensures that

Lemma 2. Assume that $1 < p < \frac{N}{s}$. Then:

- (a) $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, w)$, for whatever $1 \leq q \leq p^*$.
- (b) The embedding in (a) is compact once $q < p^*$.

Given any open set $\Omega \subseteq \mathbb{R}^N$, we define

$$\begin{aligned} \tilde{D}^{s,p}(\Omega) := & \left\{ u \in L_{\text{loc}}^{p-1}(\mathbb{R}^N) \cap L^{p^*}(\Omega) : \exists E \supseteq \Omega \text{ with } E^c \text{ compact, } \text{dist}(E^c, \Omega) > 0, \right. \\ & \left. \text{and } [u]_{W^{s,p}(E)} < \infty \right\}. \end{aligned}$$

Here,

$$[u]_{W^{s,p}(E)} := \left(\int_{E \times E} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

The following results, established in Theorem 2.7, Lemma A.2, and Propositions 3.5–3.6 of [16], will play a key role in the proof of Theorem 6.

Theorem 1. (Comparison principle in general domains) Let $\Omega \subseteq \mathbb{R}^N$ be an open set. If $u, v \in \tilde{D}^{s,p}(\Omega)$ satisfies

$$u \leq v \text{ in } \Omega^c \quad \text{and} \quad (-\Delta_p)^s u \leq (-\Delta_p)^s v \text{ in } \Omega$$

then $u \leq v$ in Ω .

Theorem 2. Suppose that $0 < \frac{N-sp}{p} < \beta < \frac{N}{p-1}$. Then, for any $R > 0$, the function $v_\beta(x) := |x|^{-\beta}$ fulfills

$$(-\Delta_p)^s v = C(\beta)|x|^{-\beta(p-1)-sp} \text{ weakly in } \mathbb{R}^N \setminus \bar{B}_R,$$

where $C(\beta)$ is given by

$$C(\beta) := 2 \int_0^1 \rho^{sp-1} \left[1 - \rho^{N-sp-\beta(p-1)} \right] |1 - \rho^\beta|^{p-1} \Phi(\rho) \, d\rho$$

while

$$\Phi(\rho) := \mathcal{H}^{N-2}(S^{N-2}) \int_{-1}^1 \frac{(1-t^2)^{\frac{N-2}{2}}}{(1-2t\rho+\rho^2)^{\frac{N+sp}{2}}} \, dt.$$

Remark 1. If $\beta = \frac{N-sp}{p-1}$, then $C(\beta) = 0$. Accordingly, the function $v_\beta(x) := |x|^{-\frac{N-sp}{p-1}}$ belongs to $\tilde{D}^{s,p}(B_R^c)$ and solves $(-\Delta_p)^s v = 0$ in $\mathbb{R}^N \setminus \bar{B}_R$.

Proposition 1. Let $1 < p < \frac{N}{s}$. Then, for every $R > 0$, there exists a unique, radial, non-increasing function $u_R \in D^{s,p}(\mathbb{R}^N)$ such that

$$\begin{cases} (-\Delta_p)^s u_R = 0 & \text{in } \mathbb{R}^N \setminus \bar{B}_R, \\ u_R \equiv 1 & \text{in } \bar{B}_R. \end{cases}$$

Moreover, when $R = 1$, the following decay estimate holds:

$$\frac{1}{C} |x|^{-\frac{N-sp}{p-1}} \leq u_1(x) \leq p^{\frac{1}{p-1}} |x|^{-\frac{N-sp}{p-1}} \quad \forall x \in B_1^c, \quad (4)$$

where $C := C(N, p, s) > 1$.

Finally, we point out the next weak Harnack-type inequality ([9], Theorem 5.2) for non-negative super-solutions.

Let $\Omega \subseteq \mathbb{R}^N$ be bounded. We set

$$\tilde{W}^{s,p}(\Omega) := \left\{ u \in L_{\text{loc}}^p(\mathbb{R}^N) : \exists E \ni \Omega : \|u\|_{L^p(E)} + [u]_{W^{s,p}(E)} + \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}} \, dx < \infty \right\}.$$

Theorem 3. If $u \in \tilde{W}^{s,p}(B_{R/3})$ satisfies

$$\begin{cases} (-\Delta_p)^s u \geq K & \text{in } B_{R/3}, \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $K \geq 0$, then there exist $\sigma \in (0, 1)$, $C^* > 0$ such that

$$\inf_{B_{R/4}} u \geq \sigma \left(\int_{B_R \setminus B_{R/2}} u^{p-1} \, dx \right)^{\frac{1}{p-1}} - C^* (KR^{ps})^{\frac{1}{p-1}}.$$

3. A Pure Singular Problem

In this section, we focus on the pure singular problem

$$\begin{cases} (-\Delta_p)^s u = a(x)u^{-\gamma} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases} \quad (5)$$

and prove that it admits a weak solution $\bar{u} \in D^{s,p}(\mathbb{R}^N)$. With this aim, like [13], let us first study the regularized problem

$$\begin{cases} (-\Delta_p)^s u = a(x) \left(u_+ + \frac{1}{n}\right)^{-\gamma} & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (6)$$

where $n \in \mathbb{N}$. Define, for every $(x, t) \in \mathbb{R}^N \times \mathbb{R}$,

$$A(x, t) := \int_0^t a(x) \left(\tau_+ + \frac{1}{n}\right)^{-\gamma} d\tau.$$

One evidently has

$$A(x, t) \leq \int_0^{|t|} a(x) \left(\tau_+ + \frac{1}{n}\right)^{-\gamma} d\tau \leq \int_0^{|t|} a(x) \tau^{-\gamma} d\tau = a(x) \frac{|t|^{1-\gamma}}{1-\gamma}, \quad (7)$$

while the energy functional arising from (6) is

$$J_n(u) := \frac{1}{p} \|u\|_{s,p}^p - \int_{\mathbb{R}^N} A(\cdot, u) dx, \quad u \in D^{s,p}(\mathbb{R}^N).$$

Lemma 3. *If (3) holds, then J_n is of class C^1 in $D^{s,p}(\mathbb{R}^N)$, weakly sequentially lower semi-continuous, and coercive.*

Proof. From $a \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, it follows that $a \in L^r(\mathbb{R}^N)$, for whatever $r > 1$; see, e.g., ref. [20], Proposition A.6. Hence, thanks to Hölder's inequality,

$$u \in D^{s,p}(\mathbb{R}^N) \implies \int_{\mathbb{R}^N} a(x) |u(x)|^{1-\gamma} dx \leq C \|a\|_{\frac{p^*}{p^*-1+\gamma}} \|u\|_{p^*}^{1-\gamma} < \infty.$$

This means that J_n is well defined. Further, $J_n \in C^1(D^{s,p}(\mathbb{R}^N))$, as a standard argument shows. Now, let $u_m \rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$. The same reasoning made for proving Lemma 3.3 in [8] ensures here that to each $R > 0$, there corresponds a subsequence, again denoted by $\{u_m\}$, fulfilling $u_m(x) \rightarrow u(x)$ a.e. in B_R . Through a diagonalization procedure, we next arrive at $u_m(x) \rightarrow u(x)$ for almost all $x \in \mathbb{R}^N$. Due to Fatou's lemma and (7), one thus has

$$\limsup_{m \rightarrow +\infty} \int_{\mathbb{R}^N} A(x, u_m(x)) dx \leq \int_{\mathbb{R}^N} A(x, u(x)) dx.$$

On the other hand, Proposition 3.5 of [19] entails

$$\liminf_{m \rightarrow \infty} \|u_m\|_{s,p}^p \geq \|u\|_{s,p}^p.$$

Hence,

$$\begin{aligned}\liminf_{m \rightarrow \infty} J_n(u_m) &\geq \liminf_{m \rightarrow \infty} \frac{1}{p} \|u_m\|_{s,p}^p - \limsup_{m \rightarrow \infty} \int_{\mathbb{R}^N} A(x, u_m(x)) \, dx \\ &\geq \frac{1}{p} \|u\|_{s,p}^p - \int_{\mathbb{R}^N} A(x, u(x)) \, dx = J_n(u),\end{aligned}$$

i.e., J_n is weakly sequentially lower semi-continuous. Let us finally come to coercivity. Via Young's inequality with ε and (a) of Lemma 2, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} a(x) |u(x)|^{1-\gamma} \, dx &= \int_{\mathbb{R}^N} a(x)^{\frac{p-1+\gamma}{p}} a(x)^{\frac{1-\gamma}{p}} |u(x)|^{1-\gamma} \, dx \\ &\leq \varepsilon \int_{\mathbb{R}^N} a(x) |u(x)|^p \, dx + C_\varepsilon \|a\|_1 \\ &\leq C_\varepsilon \|u\|_{s,p}^p + C_\varepsilon \|a\|_1,\end{aligned}$$

whence

$$J_n(u) \geq \left(\frac{1}{p} - C_\varepsilon\right) \|u\|_{s,p}^p - (C_\varepsilon + C) \|a\|_1 \quad \forall u \in D^{s,p}(\mathbb{R}^N).$$

Choosing sufficiently small $\varepsilon > 0$ yields $\lim_{\|u\|_{s,p} \rightarrow \infty} J_n(u) = +\infty$, as desired. $\square$

Theorem 4. Under hypothesis (3), for every $n \in \mathbb{N}$, problem (6) possesses a solution $u_n \in D^{s,p}(\mathbb{R}^N)_+$. The sequence $\{u_n\}$ is bounded in $D^{s,p}(\mathbb{R}^N)$.

Proof. By Lemma 3, the Weierstrass–Tonelli theorem ([21], Theorem 1.2) can be applied to J_n . So, there exists $u_n \in D^{s,p}(\mathbb{R}^N)$ such that

$$J_n(u_n) = \inf_{u \in D^{s,p}(\mathbb{R}^N)} J_n(u).$$

In particular, u_n weakly solves the differential equation of (6), i.e.,

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy \\ = \int_{\mathbb{R}^N} a \left((u_n)_+ + \frac{1}{n} \right)^{-\gamma} v \, dx \quad \forall v \in D^{s,p}(\mathbb{R}^N).\end{aligned}\tag{8}$$

We will show that $u_n \geq 0$. Testing (8) with $v := -(u_n)_-$ and using the inequality

$$\pm |a - b|^{p-2} (a - b) (a_\pm - b_\pm) \geq |a_\pm - b_\pm|^p, \quad a, b \in \mathbb{R}$$

(cf. [22], p. 89) produces

$$\begin{aligned}\|(u_n)_-\|_{s,p}^p &\leq \int_{\mathbb{R}^{2N}} - \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) ((u_n)_-(x) - (u_n)_-(y))}{|x - y|^{N+sp}} \, dx \, dy \\ &= - \int_{\mathbb{R}^N} a \left((u_n)_+ + \frac{1}{n} \right)^{-\gamma} (u_n)_- \, dx = -n^\gamma \int_{\mathbb{R}^N} a (u_n)_- \, dx \leq 0,\end{aligned}$$

which obviously forces $u_n = (u_n)_+ \geq 0$. Finally, via (8) written for $v := u_n$, Hölder's inequality, besides the continuous embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, we obtain

$$\|u_n\|_{s,p}^p \leq \int_{\mathbb{R}^N} a |u_n|^{1-\gamma} \, dx \leq \|a\|_{\frac{p^*}{p^*-1+\gamma}} \|u_n\|_{p^*}^{1-\gamma} \leq C \|a\|_{\frac{p^*}{p^*-1+\gamma}} \|u_n\|_{s,p}^{1-\gamma} \quad \forall n \in \mathbb{N}.$$

The boundedness of $\{u_n\}$ now follows from $p > 1 - \gamma > 0$. $\square$

Since $D^{s,p}(\mathbb{R}^N)$ is reflexive, there exists $\bar{u} \in D^{s,p}(\mathbb{R}^N)$ such that, up to subsequences,

$$u_n \rightharpoonup \bar{u} \text{ in } D^{s,p}(\mathbb{R}^N), \quad u_n(x) \rightarrow \bar{u}(x) \text{ for almost every } x \in \mathbb{R}^N; \quad (9)$$

see the proof of Lemma 3. Moreover,

Lemma 4. *The sequence $\{u_n\}$ given by Theorem 4 turns out monotone non-decreasing.*

In fact, on the set $\{u_n > u_{n+1}\}$, one evidently has

$$(-\Delta_p)^s u_n = a \left(u_n + \frac{1}{n} \right)^{-\gamma} \leq a \left(u_{n+1} + \frac{1}{n+1} \right)^{-\gamma} = (-\Delta_p)^s u_{n+1}.$$

Consequently, like in [23] (Lemma 2.8), the conclusion is achieved once we show the following:

Lemma 5. *If $v, w \in D^{s,p}(\mathbb{R}^N)$ and $\langle (-\Delta_p)^s w, \varphi \rangle \leq \langle (-\Delta_p)^s v, \varphi \rangle$ for every $\varphi \in D^{s,p}(\mathbb{R}^N)_+$ fulfilling $\varphi \equiv 0$ on $\{w \leq v\}$, then $w \leq v$ in $\mathbb{R}^N$.*

Proof. To simplify the formulae, define

$$V(x, y) := \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{N+sp}}, \quad W(x, y) := \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))}{|x - y|^{N+sp}}.$$

and pick $\varphi := (w - v)_+$. Using the assumptions, we get

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^{2N}} V(x, y)(\varphi(x) - \varphi(y)) \, dx \, dy - \int_{\mathbb{R}^{2N}} W(x, y)(\varphi(x) - \varphi(y)) \, dx \, dy \\ &= \int_{\{w \geq v\}} dy \int_{\{w \geq v\}} (V(x, y) - W(x, y))(w(x) - v(x) - w(y) + v(y)) \, dx \\ &\quad + \int_{\{w < v\}} dy \int_{\{w > v\}} (V(x, y) - W(x, y))\varphi(x) \, dx \\ &\quad - \int_{\{w > v\}} dy \int_{\{w < v\}} (V(x, y) - W(x, y))\varphi(y) \, dx. \end{aligned}$$

Since $L(a, b) := |a - b|^{p-2}(a - b)$, $(a, b) \in \mathbb{R}^2$, is increasing with respect to a and decreasing in b , this entails

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^{2N}} (V(x, y) - W(x, y))(\varphi(x) - \varphi(y)) \, dx \, dy \\ &= - \int_{\{w \geq v\}} dy \int_{\{w \geq v\}} (V(x, y) - W(x, y))(v(x) - v(y) - w(x) + w(y)) \, dx \\ &\quad + \int_{\{w < v\}} dy \int_{\{w > v\}} (W(x, y) - W(x, y))\varphi(x) \, dx \\ &\quad - \int_{\{w > v\}} dy \int_{\{w < v\}} (W(x, y) - W(x, y))\varphi(y) \, dx \\ &= - \int_{\{w \geq v\} \times \{w \geq v\}} (V(x, y) - W(x, y))(v(x) - v(y) - w(x) + w(y)) \, dx \, dy \leq 0 \end{aligned}$$

by Lemma 1. Therefore,

$$\int_{\{w \geq v\} \times \{w \geq v\}} (V(x, y) - W(x, y))(\varphi(x) - \varphi(y)) \, dx \, dy = 0,$$

which, due to the choice of φ , implies $w \leq v$ on the whole $\mathbb{R}^N$. $\square$

Theorem 5. Let (3) be satisfied and let $\bar{u}$ be as in (9). Then, $\bar{u}$ weakly solves (5).

Proof. Theorem 4 ensures that (8) holds for every $n \in \mathbb{N}$, $v \in D^{s,p}(\mathbb{R}^n)$. Lemma 4 and the arguments adopted in the proof of Lemma 2.9 in [23] produce

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} a \left(u_n + \frac{1}{n} \right)^{-\gamma} v \, dx = \int_{\mathbb{R}^N} a \bar{u}^{-\gamma} v \, dx. \quad (10)$$

We next claim that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^{2N}} U_n(x, y) (v(x) - v(y)) \, dx \, dy = \int_{\mathbb{R}^{2N}} \bar{U}(x, y) (v(x) - v(y)) \, dx \, dy, \quad (11)$$

where, to shorten the formulae,

$$U_n(x, y) := \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y))}{|x - y|^{N+sp}},$$

$$\bar{U}(x, y) := \frac{|\bar{u}(x) - \bar{u}(y)|^{p-2} (\bar{u}(x) - \bar{u}(y))}{|x - y|^{N+sp}}.$$

In fact, by Theorem 4 again, the sequence $\left\{ |x - y|^{\frac{N+sp}{p}} U_n(x, y) \right\}$ turns out bounded in $L^{p'}(\mathbb{R}^{2N})$, while (9) entails

$$\lim_{n \rightarrow +\infty} |x - y|^{\frac{N+sp}{p}} U_n(x, y) = |x - y|^{\frac{N+sp}{p}} \bar{U}(x, y) \text{ for almost every } (x, y) \in \mathbb{R}^{2N}.$$

So, taking a subsequence if necessary,

$$|x - y|^{\frac{N+sp}{p}} U_n(x, y) \rightharpoonup |x - y|^{\frac{N+sp}{p}} \bar{U}(x, y) \text{ in } L^{p'}(\mathbb{R}^{2N}),$$

which yields (11) because the function $|x - y|^{-\frac{N+sp}{p}} (v(x) - v(y))$ belongs to $L^p(\mathbb{R}^{2N})$, as $v \in D^{s,p}(\mathbb{R}^N)$. Now, through (10)–(11), we see that $\bar{u}$ weakly solves the equation

$$(-\Delta_p)^s u = a(x) u^{-\gamma} \text{ in } \mathbb{R}^N. \quad (12)$$

Since one evidently has $\bar{u} \geq 0$, the conclusion will follow from the next lemma. $\square$

Lemma 6. If $\bar{u} \in D^{s,p}(\mathbb{R}^N)_+$ is a solution of (12), then

$$\tilde{C} |x|^{-\frac{N-sp}{p-1}} \leq \bar{u}(x) \leq D |x|^{-\frac{N-sp}{p-1}} \text{ for almost every } x \in B_1^c, \quad (13)$$

with $\tilde{C}, D > 0$ suitable constants. So, $\bar{u}(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

Proof. By Theorem 3, written for $K := 0$ and $R := 4$, one has

$$\inf_{B_1} \bar{u} \geq \sigma \left(\int_{B_4 \setminus B_2} \bar{u}^{p-1} \, dx \right)^{\frac{1}{p-1}} = \frac{\sigma}{[\omega_N(4^N - 2^N)]^{\frac{1}{p-1}}} \left(\int_{B_4 \setminus B_2} \bar{u}^{p-1} \, dx \right)^{\frac{1}{p-1}}.$$

Hence,

$$\inf_{B_1} \bar{u} \geq \sigma M, \quad (14)$$

where $\sigma \in (0, 1)$, $M := M(N, p, \|u\|_{L^p(B_4)}) > 0$. Let $u_1 \in D^{s,p}(\mathbb{R}^N)$ be as in Proposition 1, with $R := 1$, and let $0 < c < \sigma M$. The function $\tilde{u}_1 := cu_1$ enjoys the properties

$$(-\Delta_p)^s \tilde{u}_1 = 0 \text{ in } \mathbb{R}^N \setminus \bar{B}_1, \quad \tilde{u}_1 \equiv c \text{ on } \bar{B}_1.$$

Moreover, thanks to (14),

$$\tilde{u}_1 \leq \bar{u} \text{ in } \bar{B}_1, \quad (-\Delta_p)^s \tilde{u}_1 \leq (-\Delta_p)^s \bar{u} \text{ in } \mathbb{R}^N \setminus \bar{B}_1.$$

Because of Theorem 1 and (4), this implies

$$\tilde{C}|x|^{-\frac{N-sp}{p-1}} \leq \tilde{u}_1(x) \leq \bar{u}(x) \quad \forall x \in B_1^c, \quad (15)$$

where $\tilde{C} := \tilde{C}(N, p, s, C, c) > 0$. Via (3) and (15), we next obtain

$$a(x)\bar{u}(x)^{-\gamma} \leq \frac{c_a}{1+|x|^{N+\alpha}} \tilde{C}^{-\gamma} |x|^{\gamma \frac{N-sp}{p-1}} \leq \frac{\hat{C}}{|x|^{N+\alpha-\gamma \frac{N-sp}{p-1}}}, \quad x \in B_1^c, \quad (16)$$

for some $\hat{C} > 0$. Putting (3), (14), and (16) together gives $a\bar{u}^{-\gamma} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. In fact,

$$\begin{aligned} \int_{\mathbb{R}^N} a\bar{u}^{-\gamma} dx &\leq \|a\|_\infty \left(\inf_{B_1} \bar{u} \right)^{-\gamma} |B_1| + \hat{C} \int_{B_1^c} \frac{1}{|x|^{N+\alpha-\gamma \frac{N-sp}{p-1}}} dx < \infty, \\ \sup_{x \in \mathbb{R}^N} a(x)\bar{u}(x)^{-\gamma} &\leq \|a\|_\infty \left(\inf_{B_1} \bar{u} \right)^{-\gamma} + \hat{C} \sup_{x \in B_1^c} \frac{1}{|x|^{N+\alpha-\gamma \frac{N-sp}{p-1}}} < \infty. \end{aligned}$$

Consequently,

$$a\bar{u}^{-\gamma} \in L^\tau(\mathbb{R}^N) \text{ whatever } \tau \in [1, \infty], \quad (17)$$

which entails

$$\bar{u} \in L^\infty(\mathbb{R}^N); \quad (18)$$

see [8], Theorem 1.2. Let $\beta \in \mathbb{R}$ satisfy

$$\beta(p-1) + sp = N + \alpha - \gamma \frac{N-sp}{p-1}. \quad (19)$$

One has $\beta < \frac{N}{p-1}$ because $\alpha < sp + \gamma \frac{N-sp}{p-1}$. So, Theorem 2 and Proposition 1 ensure that $w(x) := |x|^{-\beta} \chi_{\mathbb{R}^N \setminus \bar{B}_1}(x) + u_1(x) \chi_{\bar{B}_1}(x)$ weakly solves the problem

$$\begin{cases} (-\Delta_p)^s w = C(\beta)|x|^{-\beta(p-1)-sp} & \text{in } \mathbb{R}^N \setminus \bar{B}_1, \\ w \equiv 1 & \text{on } \bar{B}_1. \end{cases}$$

If $\bar{w} := Dw$, where $D > 0$ complies with

$$D > \max \left\{ \left(\frac{\hat{C}}{C(\beta)} \right)^{\frac{1}{p-1}}, \sup_{B_1} \bar{u} \right\},$$

then, due to (16) and (19),

$$\bar{u}(x) \leq \bar{w}(x) \text{ in } \bar{B}_1, \quad (-\Delta_p)^s \bar{u} \leq (-\Delta_p)^s \bar{w} \text{ in } \mathbb{R}^N \setminus \bar{B}_1.$$

Now, Theorem 1 yields $\bar{u} \leq \bar{w}$ on the whole $\mathbb{R}^N$. Thus,

$$\bar{u}(x) \leq D|x|^{-\beta} = D|x|^{\frac{1}{p-1}(-N-\alpha+\gamma\frac{N-sp}{p-1}+sp)} \leq D|x|^{-\frac{N-sp}{p-1}} \quad \forall x \in B_1^c. \quad (20)$$

Estimate (13) follows from (15), (19), and (20). $\square$

Remark 2. Observe that (13) is consistent with its local counterparts for equations driven by the p -Laplacian or, more generally, the Φ -Laplacian; see [24], Theorems B–C, and [20], Section 4, respectively. Further, it agrees with related results known for fractional p -Laplacian non-singular problems, whose reaction terms are of the type $g(x, t) := a(x)(1 + t^{\eta-1})$, where a fulfills (3) and $1 \leq \eta < p^*$; cf. [8], Theorem 1.3.

Remark 3. Obviously, $\bar{u}$ weakly solves the equation $(-\Delta_p)^s u = a(x)u^{-\gamma}$ on every open ball $B_R(x)$. By [9], Theorem 5.4, there exist $\theta \in (0, 1)$ and $C > 0$, not dependent on x , such that

$$\text{osc}_{B_r(x)} \bar{u} \leq C \left[(\|a\bar{u}^{-\gamma}\|_{\infty} R^{ps})^{\frac{1}{p-1}} + \|\bar{u}\|_{L^{\infty}(B_R(x))} + \text{Tail}(\bar{u}; x, R) \right] \left(\frac{r}{R} \right)^{\theta} \quad (21)$$

for all $r \in (0, R)$, where

$$\text{Tail}(\bar{u}; x, R) := \left(R^{ps} \int_{\mathbb{R}^N \setminus B_R(x)} \frac{|\bar{u}(y)|^{p-1}}{|x-y|^{N+ps}} dy \right)^{\frac{1}{p-1}}.$$

Since (18) entails

$$\text{Tail}(\bar{u}; x, R) \leq C_1 \|\bar{u}\|_{\infty},$$

using (21), we achieve

$$\text{osc}_{B_r(x)} \bar{u} \leq C_2 \left[(\|a\bar{u}^{-\gamma}\|_{\infty} R^{ps})^{\frac{1}{p-1}} + \|\bar{u}\|_{\infty} \right] \left(\frac{r}{R} \right)^{\theta},$$

with $C_2 > 0$ independent of x . Thus, $\bar{u} \in C_{\text{loc}}^{0,\theta}(\mathbb{R}^N)$.

4. Existence of Solutions

Let $\bar{u}$ be the solution to (5) given by Theorem 5. Consider the truncated problem

$$\begin{cases} (-\Delta_p)^s u = \bar{f}(x, u) & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (22)$$

where

$$\bar{f}(x, t) := f(x, \max\{\bar{u}(x), t\}), \quad (x, t) \in \Omega \times \mathbb{R}.$$

Thanks to (2), one has

$$0 \leq \bar{f}(\cdot, t) \leq a(\max\{\bar{u}, t\})^{-\gamma} + a(\max\{\bar{u}, t\})^r \leq a\bar{u}^{-\gamma} + a(\bar{u}^r + |t|^r) \quad (23)$$

while the energy functional associated with (22) is

$$\bar{J}(u) := \frac{1}{p} \|u\|_{s,p}^p - \int_{\Omega} \left(\int_0^{u(x)} \bar{f}(x, t) dt \right) dx \quad \forall u \in D^{s,p}(\mathbb{R}^N).$$

Lemma 7. Suppose that (3) and (2) hold. Then, $\bar{J}$ admits a minimizer $\tilde{u} \in D^{s,p}(\mathbb{R}^N)_+$, which weakly solves (22).

Proof. Since $r < p - 1$, recalling (17)–(18), and using (23), we easily see that $\bar{J}$ turns out both well-defined and of class C^1 . Moreover, Hölder's inequality, Young's inequality, and Lemma 2, besides the continuous embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, produce

$$\begin{aligned}\bar{J}(u) &\geq \frac{1}{p} \|u\|_{s,p}^p - \int_{\mathbb{R}^N} a \bar{u}^{-\gamma} |u| \, dx - \int_{\mathbb{R}^N} a \bar{u}^r |u| \, dx - \frac{1}{r+1} \int_{\mathbb{R}^N} a |u|^{r+1} \, dx \\ &\geq \frac{1}{p} \|u\|_{s,p}^p - \|u\|_{p^*} \|a \bar{u}^{-\gamma}\|_{(p^*)'} - c_1 \int_{\mathbb{R}^N} a \bar{u}^p \, dx - c_2 \int_{\mathbb{R}^N} a |u|^{\frac{p}{p-r}} \, dx - c_3 \|u\|_{s,p}^{r+1} \\ &\geq \frac{1}{p} \|u\|_{s,p}^p - C \left(\|a \bar{u}^{-\gamma}\|_{(p^*)'} \|u\|_{s,p} - \|\bar{u}\|_{s,p}^p - \|u\|_{s,p}^{\frac{p}{p-r}} - \|u\|_{s,p}^{r+1} \right), \quad u \in D^{s,p}(\mathbb{R}^N),\end{aligned}$$

with appropriate $C > 0$. Therefore, $\bar{J}$ is coercive because $r + 1 < p$. Now, the Weierstrass–Tonelli theorem ensures that it attains its infimum, i.e., there exists $\tilde{u} \in D^{s,p}(\mathbb{R}^N)$ fulfilling

$$\bar{J}(\tilde{u}) = \inf_{u \in D^{s,p}(\mathbb{R}^N)} \bar{J}(u).$$

Finally, from $\bar{f}(x, t) \geq 0$ in $\mathbb{R}^N \times \mathbb{R}$, it easily follows that $\tilde{u} \geq 0$. $\square$

Theorem 6. Under conditions (3) and (2), problem (1) possesses a weak solution.

Proof. Let $\bar{u}$ be as in Theorem 5, let $\tilde{u}$ be given by Lemma 7, and let $\varphi := (\bar{u} - \tilde{u})^+$. Due to (2), testing (5) and (22) with φ yields

$$\langle (-\Delta_p)^s \bar{u}, \varphi \rangle = \int_{\mathbb{R}^N} a \bar{u}^{-\gamma} \varphi \, dx \leq \int_{\mathbb{R}^N} f(\cdot, \bar{u}) \varphi \, dx = \int_{\mathbb{R}^N} \bar{f}(\cdot, \tilde{u}) \varphi \, dx = \langle (-\Delta_p)^s \tilde{u}, \varphi \rangle,$$

namely

$$\langle (-\Delta_p)^s \bar{u} - (-\Delta_p)^s \tilde{u}, (\bar{u} - \tilde{u})^+ \rangle \leq 0.$$

By Lemma 1, this entails $\bar{u} \leq \tilde{u}$. From (15) and the properties of $\tilde{u}_1$, it next follows that

$$0 < \tilde{u}_1 \leq \bar{u} \leq \tilde{u}. \quad (24)$$

Since $\bar{u} \leq \tilde{u}$, we have $\bar{f}(x, \bar{u}(x)) = f(x, \tilde{u}(x))$ for all $x \in \mathbb{R}^N$, confirming that $\tilde{u}$ is indeed a positive solution to the equation in (1) and not merely of the truncated problem (22); see Lemma 7. Let us finally verify that $\lim_{|x| \rightarrow \infty} \tilde{u}(x) = 0$. Combining (2) with (24) produces

$$0 \leq f(\cdot, \tilde{u}) \leq a(\tilde{u}^{-\gamma} + \tilde{u}^r) \leq a(\bar{u}^{-\gamma} + \tilde{u}^r) \leq a(\bar{u}^{-\gamma} + 1) + a \tilde{u}^{p-1}.$$

Moreover, if

$$b_2(x) := a(x), \quad b_3(x) := a(x)(\bar{u}(x)^{-\gamma} + 1) \quad \forall x \in \mathbb{R}^N$$

then $b_2 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and, because of (17), $b_3 \in L^{\frac{p^*}{p-1}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Thus, through Theorem 1.2 in [8], we obtain

$$\tilde{u} \in L^\infty(\mathbb{R}^N). \quad (25)$$

Inequalities (15) and (24) yield

$$\tilde{C}|x|^{-\frac{N-sp}{p-1}} \leq \bar{u}(x) \leq \tilde{u}(x), \quad x \in B_1^c, \quad (26)$$

while, thanks to (16) and (25),

$$\begin{aligned} f(x, \tilde{u}(x)) &\leq a(x) (\tilde{u}(x)^{-\gamma} + \tilde{u}(x)^r) \leq a(x) (\tilde{u}(x)^{-\gamma} + \|\tilde{u}\|_{\infty}) \\ &\leq \frac{c_a}{1 + |x|^{N+\alpha}} \left(\tilde{C}^{-\gamma} |x|^{\gamma \frac{N-sp}{p-1}} + \|\tilde{u}\|_{\infty} \right) \leq C |x|^{-\beta} \quad \forall x \in B_1^c, \end{aligned}$$

where β comes from (19). Proceeding as already done for (20), with $D > 0$ bigger than $\|\tilde{u}\|_{\infty}$ too, we arrive at

$$\tilde{u}(x) \leq D |x|^{-\frac{N-sp}{p-1}}, \quad x \in B_1^c, \quad (27)$$

whence, because of (26), the conclusion follows. $\square$

Remark 4. Since $\tilde{u} \in L^{\infty}(\mathbb{R}^N)$, the same argument adopted in [8] (Corollary 3.8) ensures that (27) can be extended to a global pointwise upper bound of the form

$$\tilde{u}(x) \leq \frac{C}{1 + |x|^{\frac{N-sp}{p-1}}} \quad \forall x \in \mathbb{R}^N,$$

where $C := C(N, p, s, \|u\|_{L^{\infty}(B_1)}) > 0$.

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