

Article

The Unified Transform for Burgers' Equation: Application to Unsaturated Flow in Finite Interval

Konstantinos Kalimeris ¹, Leonidas Mindrinos ² and Athanasios Paraskevopoulos ^{1,2,*}

¹ Mathematics Research Center, Academy of Athens, Soranou Efessiou 4, 11527 Athens, Greece; kkalimeris@academyofathens.gr

² Department of Natural Resources Development and Agricultural Engineering, Agricultural University of Athens, Iera Odos 75, 11855 Athens, Greece; leonidas.mindrinos@aua.gr

* Correspondence: at.paraskevopoulos@aua.gr

Abstract

In this paper, we focus on one-dimensional vertical infiltration, assuming constant diffusivity and a quadratic relationship between hydraulic conductivity and water content. Under these assumptions, Richards' equation reduces to Burgers' equation, which we then linearize via the Hopf–Cole transformation. This turns the initial boundary value problem into a diffusion equation on a finite interval with mixed boundary conditions. To solve it, we use the Unified Transform Method (also known as the Fokas method). This approach gives an explicit integral representation of the solution, and when evaluated numerically, the results match classical Fourier series solutions exactly, but with better convergence and stability. Two examples from hydrological applications are examined.

Keywords: vertical infiltration; Burgers' equation; Fokas method; Unified Transform; analytical solution

MSC: 35A22; 35G16; 35G31; 35K05; 76S05

1. Introduction

The problem of water flow in unsaturated porous media has attracted much attention since the early years of soil physics research, as it presents significant theoretical and experimental challenges. From a mathematical perspective, great effort has been devoted to the derivation of both analytical and numerical solutions that can be compared with experimental observations to enhance understanding of the underlying physical processes.

In this work, we focus on the vertical infiltration problem (one-dimensional) in an unsaturated porous medium. This process is described by Richards' equation [1], which is a nonlinear partial differential equation (PDE) that combines diffusion and transport phenomena. Since its introduction in the early 20th century, Richards' equation has been widely used to model various physical conditions. Despite its strong physical foundation, the nonlinear nature of the equation significantly complicates the derivation of analytical solutions.

A common approach to simplify this model is to impose specific functional forms for the soil hydraulic parameters. Typically, the soil water diffusivity (D) is assumed constant, and the hydraulic conductivity (K) is a function of the water content (θ). Variations also arise depending on the imposed boundary conditions, which may represent flooding or rainfall scenarios and are commonly expressed as Dirichlet or Robin conditions, respectively. Additional restrictions are made if the soil domain is treated as bounded or semi-infinite.



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The following cases admit approximate analytical solutions for both flooding and rainfall conditions:

Case 1 $K(\theta) = \text{constant}$ [2,3].

Case 2 $K(\theta) = \text{linear function of } \theta$ [2,4].

Case 3 $K(\theta) = \text{quadratic function of } \theta$ [5,6].

The first case corresponds to neglecting gravitational effects and reduces Richards' equation to the linear diffusion equation. Case 2 simplifies the governing equation to the linear advection–dispersion equation, whereas in the third case, the PDE reduces to a form of Burgers' equation. The first two cases have been widely examined in the literature; here, we focus on the third case. Although the resulting equation remains nonlinear, it can be transformed into a linear diffusion equation through the Hopf–Cole transformation [7]. Consequently, the original problem can be reformulated as an initial boundary value problem (IBVP) for the heat equation, similarly to Case 1 [6,8].

A variety of analytical methods for the heat equation have been developed, among which separation of variables and Fourier series expansions are the most widely used. These classical techniques have been successfully applied to both infiltration and drainage problems in finite or semi-infinite domains under various boundary conditions [8–10]. Indeed, all previously reported analytical studies rely on such eigenfunction-based representations.

Despite their effectiveness, these approaches impose restrictive assumptions on problem formulation. In particular, they are typically limited to finite spatial intervals and require homogeneous or simple local boundary conditions. As a result, the approximate analytical solution often leads to lengthy derivations and series solutions that may converge slowly, especially for short times or near boundaries where compatibility conditions might not apply. These limitations, reflecting the lack of uniform convergence of the solution to their traces on the boundary of the IBVP, motivate the development of alternative analytical frameworks that can address such problems in a more direct and unified manner.

In this study, we apply the Unified Transform Method, which was introduced in the seminal work of Fokas in 1997 [11]; hence, the method is now known as the Fokas method. This approach yields an explicit integral representation of the solution relying on the naturally arising spectral transforms of the initial and boundary conditions. We refer to a book [12] and recent work [13] where the Fokas method is applied to various evolution PDEs, and the insight of the appearance of continuous spectrum is developed for finite domains, which is in contrast to the classically derived discrete Fourier spectrum. As a result, the analytical formulation is significantly simplified, and the resulting integral representation is often better suited for numerical evaluation, exhibiting improved stability and convergence properties when compared with truncated Fourier series solutions. In the present work, these analytical and numerical improvements are displayed by scrutinizing the solution representation (14). The analytical viewpoint: (i) Obviously, this is an integral representation (to be compared to the series in [8]); (ii) as it is proven in [14], this weak solution directly satisfies the associated boundary conditions due to uniform convergence, in contrast to the classical Fourier series, which fails this process, unless homogeneous conditions are imposed. The numerical viewpoint: (i) Apart from the two corner points $(x, t) = (0, 0)$ and $(x, t) = (L, 0)$, the integrand is exponentially decaying, allowing for low computation cost; (ii) the analytical uniform convergence yields reliable illustrations of the solution closer to the boundaries $x = 0$, $x = L$, and for short times, namely, for $t \rightarrow 0$, see Section 4.

Case 1 on the half-line was first studied in the original work [11], followed by its extension to finite intervals [15]. Further applications have been reported in later studies [16–20]. The linear evolution equation (see Case 2) has also been extensively examined on the half-line [21–23] and on the finite interval [14,24,25] under various boundary conditions. To our

knowledge, the application of the Fokas method to Burgers' equation (Case 3) has so far been considered only on the half-line [26]. The present work addresses this gap by extending the method to a finite interval with mixed boundary conditions. The possibility of investigating inverse problems in the vein of the above-mentioned literature, namely, the determination of boundary controllers associated with physically relevant problems, will be considered in future work.

The structure of the paper is as follows. In Section 2, we derive Burgers' equation for the specific physical problem and reduce it to a diffusion equation on a finite interval under mixed boundary conditions. The resulting IBVP is then solved analytically using the Unified Transform Method. The numerical implementation of the integral representation is presented in Section 3. In the final section, the analytical solution is compared with existing Fourier series solutions under physically relevant conditions, demonstrating agreement while highlighting the advantages of the proposed approach.

2. Problem Formulation

In soil physics, the vertical water flow in an unsaturated porous medium is modeled through the one-dimensional nonlinear Richards' equation [1]

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left(D(\theta) \frac{\partial \theta}{\partial x} \right) - \frac{\partial K(\theta)}{\partial x}, \quad 0 < x < L, \quad t > 0, \quad (1)$$

where θ is the water content, K is the hydraulic conductivity and D describes water diffusivity, which are all positive quantities.

There are several ways to describe how the material parameters D and K depend on θ . In this work, we focus on Case 3 of the Introduction by assuming constant diffusivity ($D > 0$) and that K is a quadratic function of θ [6],

$$K(\theta) = a(\theta + b)^2, \quad (2)$$

with $a, b \in \mathbb{R}$ constants. Then, Equation (1) takes the form

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2} - 2a(\theta + b) \frac{\partial \theta}{\partial x}, \quad (3)$$

which appears in the literature as the mathematical modeling for several physical processes without yet specifying the associated functional analysis; indeed, this is the well-known Burgers' equation [27], which is still nonlinear but can be linearized using the Hopf–Cole transformation. In what follows, we present the formulation of a physically relevant IBVP, which yields (11a–d); we discuss the rigorous mathematical background in that formulation at the end of this paragraph.

Following [8], one can show that (3) reduces to the linear diffusion equation (see Appendix A)

$$\frac{\partial w}{\partial t} = D \frac{\partial^2 w}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \quad (4)$$

for the new variable

$$w = e^{-\frac{a}{D}u}, \quad \text{where} \quad \frac{\partial u}{\partial x} = \theta + b. \quad (5)$$

Thus, once (4) is solved for w (subject to appropriate initial and boundary conditions), the water content is given by

$$\theta = -\frac{D}{a} \frac{\partial}{\partial x} (\ln w) - b = -\frac{D}{a} \frac{w_x}{w} - b. \quad (6)$$

We consider (3) in a finite interval $x \in [0, L]$ together with a constant initial condition of the form

$$\theta(x, 0) = \theta_0, \quad 0 < x < L. \quad (7)$$

At the surface $x = 0$, we assume a given time-dependent flux q , modeled through Darcy's law

$$K(\theta) - D(\theta) \frac{\partial \theta}{\partial x}(0, t) = q(t), \quad t > 0, \quad (8)$$

and at the bottom ($x = L$), we impose a Dirichlet boundary condition (given moisture)

$$\theta(L, t) = \theta_L, \quad t > 0, \quad (9)$$

with $\theta_0 \geq \theta_L > 0$.

Equation (8) using (2) takes the form of a nonlinear Robin-type boundary condition

$$a(\theta(0, t) + b)^2 - D \frac{\partial \theta}{\partial x}(0, t) = q(t), \quad t > 0. \quad (10)$$

The initial (7) and boundary conditions (9) and (10) have to be transformed using the new variables (5). This derivation, starting from the Burger's equation, yields the following IBVP and is a standard procedure, but for matters of completeness, we summarize it in Appendix A. Hence, in accordance with [8], the final form of the IBVP is given by

$$\frac{\partial w}{\partial t} = D \frac{\partial^2 w}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \quad (11a)$$

$$w(x, 0) = w_0(x), \quad 0 < x < L, \quad (11b)$$

$$w(0, t) = f(t), \quad t > 0, \quad (11c)$$

$$\frac{\partial w}{\partial x}(L, t) + C w(L, t) = 0, \quad t > 0, \quad (11d)$$

where

$$w_0(x) = e^{-\frac{a}{D}(\theta_0 + b)x}, \quad f(t) = e^{\frac{a}{D} \int_0^t q(s) ds}, \quad (12)$$

and $C = \frac{a}{D}(\theta_L + b)$.

Considering the rigorous treatment of this IBVP, we refer to [28] for a more general IBVP than (11a–d), which also improves the relevant results of [29]. A slight modification of Theorem 3.1 in [28] yields the global well-posedness (in the classical Hadamard sense in Sobolev spaces). Indeed, for any $T > 0$, $s > \frac{1}{2}$, with $(s - \frac{1}{2}) \notin \mathbb{N}$, $f \in H_t^{\frac{2s+1}{4}}(0, T)$, $w_0 \in H_x^s(0, L)$ and assuming the necessary compatibility of (w_0, f) at the space-time corner point $(x, t) = (0, 0)$, the IBVP (11a–d) has a unique solution $w \in C([0, T]; H_x^s(0, L))$. Moreover, the data to solution map $(w_0, f) \mapsto w$ is continuous from $H_x^s(0, L) \times H_t^{\frac{2s+1}{4}}(0, T)$ into $C([0, T]; H_x^s(0, L))$.

The above functional regime holds for the original IBVP, with a regularity shift. Of course, in this example, w_0 is already analytic. Considering the initial boundary condition q , the definition (12) allows for one extra unit of less regularity, namely, $q \in H_t^{\frac{2s-3}{4}}(0, T)$. This is also reflected in the solution of the original problem; indeed, under definition (6) and the comprehension that w is not vanishing (via the maximum principle), the function space of the solution θ becomes $C([0, T]; H_x^{s-1}(0, L))$.

Analytical Solution of the IBVP (11a–d)

We emphasize that the IBVP (11a–d) can now be treated as a purely mathematical structure. In this viewpoint, we use the solution of a more general IBVP that was considered

in [14], originated by mathematically modeling a different physical problem, namely, two of the authors derived the solution of the vertical infiltration problem considering linear homogeneous soil and various boundary conditions. However, the mathematical treatment remains invariant when the appropriate parameters of the IBVP are used.

We are interested in finding the analytical solution of (11a–d) using the Fokas method. This is an IBVP for the heat equation in a finite interval with mixed boundary conditions. The corresponding IBVP for $\alpha, \beta, K_0 \in \mathbb{R}$ and $D_0 > 0$ has the form

$$\frac{\partial \theta}{\partial t} + K_0 \frac{\partial \theta}{\partial x} = D_0 \frac{\partial^2 \theta}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \quad (13a)$$

$$\theta(x, 0) = \theta_0(x), \quad 0 < x < L, \quad (13b)$$

$$\theta(0, t) - \alpha \frac{\partial \theta}{\partial x}(0, t) = f(t), \quad t > 0, \quad (13c)$$

$$\theta(L, t) - \beta \frac{\partial \theta}{\partial x}(L, t) = g(t), \quad t > 0. \quad (13d)$$

To obtain the analytical solution of (11a–d), we set $K_0 = 0$, $D_0 = D$, in (13a), $\alpha = 0$ in (13c) and $\beta = -\frac{1}{C}$, $g(t) = 0$ in (13d). Thus, we get (see [14] Equation (16))

$$\begin{aligned} w(x, t) = & \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda x - D\lambda^2 t} \hat{w}_0(\lambda) d\lambda \\ & - \frac{1}{2\pi} \int_{\partial \mathcal{D}^+} \frac{e^{-D\lambda^2 t}}{\Delta(\lambda, -L)} \left[\sin(\lambda x) (i\lambda + C) e^{i\lambda L} \hat{w}_0(\lambda) \right. \\ & \left. + \Delta(\lambda, x - L) \hat{w}_0(-\lambda) + i2\lambda D \Delta(\lambda, x - L) \tilde{f}(D\lambda^2, t) \right] d\lambda, \end{aligned} \quad (14)$$

where

$$\Delta(\lambda, y) = \lambda \cos(\lambda y) - C \sin(\lambda y), \quad 0 \leq y \leq L \quad (15)$$

and the integral transforms appearing in (14) are defined by

$$\hat{w}_0(\lambda) = \int_0^L e^{-i\lambda x} w_0(x) dx, \quad \lambda \in \mathbb{C}, \quad \tilde{f}(k, t) = \int_0^t e^{ks} f(s) ds, \quad k \in \mathbb{C}, \quad (16)$$

namely, the Fourier transform of the initial condition and the t -transform of the boundary conditions, respectively.

Note that the roots of Δ lie outside of the domain of integration; consequently, their residue contributions may be exploited to obtain a series representation of the solution. For simple boundary conditions, the structure of Δ is also simplified [14]. Crucially, however, the strength of the approach lies in its unified framework, which remains valid, regardless of such modifications.

The contour $\partial \mathcal{D}^+$ is taken to be a positively oriented curve associated with the domain

$$\mathcal{D}^+ = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda^2) < 0, \operatorname{Im}(\lambda) > 0\}.$$

In practice, there is considerable freedom in the choice of $\partial \mathcal{D}^+$. As discussed in Remark 2.1 of [14], it may be chosen as any simply connected curve lying in $\mathbb{C}_+$, whose linear asymptotes remain in $\mathbb{C}_+ \setminus \mathcal{D}^+$ and are defined by $\operatorname{Im}(\lambda) = c \operatorname{Re}(\lambda)$, $c \in [-1, 1]$, for $\operatorname{Im}(\lambda) > 0$. In this work, for matters of clarity, we introduce a new curve, $\mathcal{C}^+$, which possesses these properties with the choice $c = \pm \tan \frac{\pi}{8} = \pm(\sqrt{2} - 1)$.

Different cases will be examined in the next section. Alternatively, one can compute (14) directly using numerical integration; however, this approach can be time-consuming and may require special treatment at early times or near the boundaries.

3. Numerical Implementation

In this section, we deal with the numerical evaluation of the integral representation of the solution (14).

Without being restrictive for the realization of the numerical scheme, and considering a physically relevant scenario, we assume constant flux $q(t) = q$, and the formulas in (12) are simplified as

$$w_0(x) = e^{-Ax}, \quad f(t) = e^{Bt},$$

with

$$A = \frac{a}{D}(\theta_0 + b), \quad B = \frac{a}{D}q.$$

Then, from the definitions (16), we obtain

$$\begin{aligned} \hat{w}_0(\lambda) &= \int_0^L e^{-Ax} e^{-i\lambda x} dx = \frac{1 - e^{-(A+i\lambda)L}}{A + i\lambda}, \\ \tilde{f}(D\lambda^2, t) &= \int_0^t e^{D\lambda^2 s} e^{Bs} ds = \frac{e^{(D\lambda^2+B)t} - 1}{D\lambda^2 + B}. \end{aligned} \quad (17)$$

and by substituting in (14), we get the form

$$\begin{aligned} w(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - D\lambda^2 t} \frac{1 - e^{-(A+i\lambda)L}}{A + i\lambda} d\lambda \\ &\quad - \frac{1}{2\pi} \int_{\partial\mathcal{D}^+} \frac{e^{-D\lambda^2 t}}{\Delta(\lambda, -L)} \left[e^{i\lambda L} (i\lambda + C) \sin(\lambda x) \frac{1 - e^{-(A+i\lambda)L}}{A + i\lambda} \right. \\ &\quad \left. + \Delta(\lambda, x - L) \frac{1 - e^{-(A-i\lambda)L}}{A - i\lambda} + 2i\lambda D \Delta(\lambda, x - L) \frac{e^{(D\lambda^2+B)t} - 1}{D\lambda^2 + B} \right] d\lambda, \end{aligned} \quad (18)$$

where $\Delta(\lambda, y)$ is defined in (15).

The analyticity and boundedness of the integrand of the first integral on the right-hand side of the above equation allow the deformation from the real line to the contour $\partial\mathcal{D}^+$. In more detail, we split the integral into two parts, whose boundedness allows for deformation to the upper and lower complex λ -plane, respectively. Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\lambda x - D\lambda^2 t} \frac{1 - e^{-(A+i\lambda)L}}{A + i\lambda} d\lambda &= \int_{\partial\mathcal{D}^+} e^{i\lambda x - D\lambda^2 t} \frac{1}{A + i\lambda} d\lambda \\ &\quad - \int_{\partial\mathcal{D}^-} e^{i\lambda x - D\lambda^2 t} \frac{e^{-(A+i\lambda)L}}{A + i\lambda} d\lambda \\ &= \int_{\partial\mathcal{D}^+} e^{-D\lambda^2 t} \left[\frac{e^{i\lambda x}}{A + i\lambda} - \frac{e^{-AL} e^{i\lambda(L-x)}}{A - i\lambda} \right] d\lambda, \end{aligned}$$

where, in the last step, we used the change of variables $\lambda \rightarrow -\lambda$, to map $\partial\mathcal{D}^-$ to $\partial\mathcal{D}^+$. Combining all the above, we obtain one integral over $\partial\mathcal{D}^+$; following the standard process described in detail in ([14] (Remark 2.1)), we deform $\partial\mathcal{D}^+$ to $\mathcal{C}^+$, where $\mathcal{C}^+$ is any simply connected contour with two asymptotes between the real axis and $\partial\mathcal{D}^+$. Then, we decompose the solution into the following two parts:

$$w(x, t) = w_1(x, t) + w_2(x, t), \quad (19)$$

where

$$\begin{aligned}
 w_1(x, t) &= \frac{1}{2\pi} \int_{C^+} \frac{e^{-D\lambda^2 t}}{\Delta(\lambda, -L)} \left[\frac{e^{i\lambda x}(A - i\lambda) - e^{-AL}e^{i\lambda(L-x)}(A + i\lambda)}{A^2 + \lambda^2} \Delta(\lambda, -L) \right. \\
 &\quad \left. - e^{i\lambda L}(i\lambda + C) \sin(\lambda x) \frac{1 - e^{-AL}e^{-i\lambda L}}{A + i\lambda} \right. \\
 &\quad \left. - \Delta(\lambda, x - L) \frac{1 - e^{-AL}e^{i\lambda L}}{A - i\lambda} + 2i\lambda D \Delta(\lambda, x - L) \frac{1}{D\lambda^2 + B} \right] d\lambda, \\
 w_2(x, t) &= -\frac{1}{2\pi} \int_{C^+} \frac{1}{\Delta(\lambda, -L)} \left[2i\lambda D \Delta(\lambda, x - L) \frac{e^{Bt}}{D\lambda^2 + B} \right] d\lambda.
 \end{aligned}$$

The rationale for this decomposition is described in [12,13]. Both integrands in w_1 and w_2 display exponential decay in the domain $(x, t) \in (0, L) \times (0, \infty)$, but for numerical purposes, when the points (x, t) are close to the boundaries of that domain, this decay becomes weaker. The presence of the term $e^{-D\lambda^2 t}$ in w_1 yields a strong exponential decay even if x is on the boundary, namely, $x = 0$ or $x = L$. This is not true for w_2 , which would make the numerical computation less efficient when x is on (or close to) the boundary; however, one can treat this term analytically, as shown in the two different treatments of the numerical examples in Section 4. In fact, a non-trivial choice of the contour of integration C^+ vanishes the contribution of w_2 .

After computing w using (19), we substitute it into (6) to obtain θ . The spatial derivative can be formally evaluated, resulting in

$$\begin{aligned}
 \frac{\partial w_1}{\partial x}(x, t) &= \frac{1}{2\pi} \int_{C^+} \frac{e^{-D\lambda^2 t}}{\Delta(\lambda, -L)} \left[i\lambda \frac{e^{i\lambda x}(A - i\lambda) + e^{-AL}e^{i\lambda(L-x)}(A + i\lambda)}{A^2 + \lambda^2} \Delta(\lambda, -L) \right. \\
 &\quad \left. - e^{i\lambda L}\lambda(i\lambda + C) \cos(\lambda x) \frac{1 - e^{-AL}e^{-i\lambda L}}{A + i\lambda} \right. \\
 &\quad \left. - F(\lambda, x - L) \frac{1 - e^{-AL}e^{i\lambda L}}{A - i\lambda} + 2i\lambda D F(\lambda, x - L) \frac{1}{D\lambda^2 + B} \right] d\lambda, \\
 \frac{\partial w_2}{\partial x}(x, t) &= -\frac{1}{2\pi} \int_{C^+} \frac{1}{\Delta(\lambda, -L)} \left[2i\lambda D F(\lambda, x - L) \frac{e^{Bt}}{D\lambda^2 + B} \right] d\lambda,
 \end{aligned}$$

where

$$F(\lambda, y) = \frac{\partial \Delta}{\partial y}(\lambda, y) = -\lambda^2 \sin(\lambda y) - C\lambda \cos(\lambda y).$$

The first integral in (19) can be computed efficiently (numerically) due to the exponential decay of the term $e^{-D\lambda^2 t}$. On the other hand, it is preferable to compute the term w_2 analytically using the Cauchy or Residue theorem, depending on the appearance of singularities in the domain of integration. Different cases will be examined in the next section.

Initially, we choose an infinite trapezoidal contour C^+ (positive oriented) with parametrization $z : \mathbb{R} \rightarrow \mathbb{C}$:

$$z(s) = \begin{cases} z_1(s) = (-\ell + \ell i) + (-s - \ell) e^{i\frac{5\pi}{6}}, & s \in (-\infty, -\ell), \\ z_2(s) = s + \ell i, & s \in [-\ell, \ell], \\ z_3(s) = (\ell + \ell i) + (s - \ell) e^{i\frac{\pi}{6}}, & s \in (\ell, +\infty), \end{cases} \quad (20)$$

as shown in Figure 1.

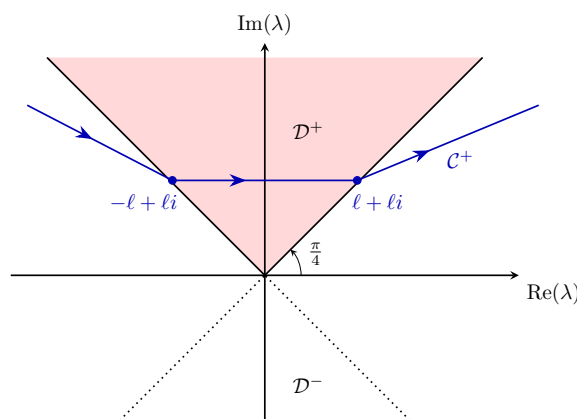


Figure 1. The boundary ∂D^+ (black curve) of the domain D^+ and the deformed contour C^+ (blue curve) given by (20).

Let $V_1(\lambda, x, t)$ denote the integrand of w_1 . Then, the term w_1 is expressed as

$$\begin{aligned} \int_{C^+} V_1(\lambda, x, t) d\lambda &= \int_{-\infty}^{-\ell} V_1(z_1(s), x, t) z_1'(s) ds + \int_{-\ell}^{\ell} V_1(z_2(s), x, t) z_2'(s) ds \\ &\quad + \int_{\ell}^{\infty} V_1(z_3(s), x, t) z_3'(s) ds \\ &= \int_{\ell}^{\infty} \left(V_1(z_3(s), x, t) e^{i\frac{\pi}{6}} - V_1(z_1(-s), x, t) e^{i\frac{5\pi}{6}} \right) ds + \int_{-\ell}^{\ell} V_1(z_2(s), x, t) ds. \end{aligned}$$

4. Numerical Examples

We verified the applicability of the analytical solution (14) by comparing it with the Fourier series solution presented in [5,8]. The comparison reproduces the same results; therefore, we kept the same soil properties, namely, $a = 9.88 \times 10^{-5}$ m/s, $b = -0.0065$ m³/m³, and $D = 3.51 \times 10^{-7}$ m²/s.

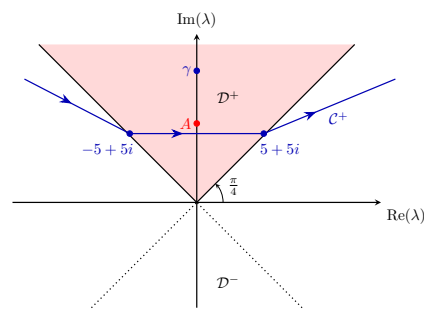
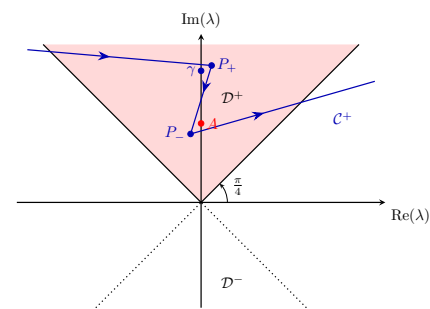
The construction of the contour C^+ , belonging entirely to the upper half λ -plane, ensures that the roots of $\Delta(\lambda, -L)$, which all lie on the real axis, are avoided. From (19), we observe that the denominators vanish at $\lambda = \pm Ai$ and at $\lambda = \pm \gamma i$, with $\gamma = \sqrt{\frac{B}{D}}$. The negative roots are outside the domain of integration, and $\lambda = Ai$ is a removable singularity since the numerator of the relevant ratio also vanishes (see the second integral in (18)).

In the first example, the finite column has length $L = 0.25$ m, and the constant flux on the surface is given by $q = 3.4 \times 10^{-6}$ m/s. The initial and boundary water moisture values are $\theta_0 = \theta_L = 0.03$.

Following the discussion about the decomposition $w = w_1 + w_2$ in (19), we treat this example in two equivalent ways for matters of illustration, with the difference lying in the treatment of the quantity w_2 . In earlier works [12], w_2 was computed analytically via its residue contribution, whereas in later works, a proper choice of contour integration yields $w_2 = 0$; both approaches induce negligible computational effort, but the second approach is also analytically convenient. In what follows, we treat the current example with these two (different but equivalent) approaches for matters of presentation to the interested reader.

The above-mentioned values for the physical parameters yield $A \approx 6.6$ and $\gamma \approx 52.2$; the latter is a simple root in the interior of the domain with boundary C^+ . In order to avoid computing potential residue contribution from the appearance of Ai , we consider the parametrization (20) with $\ell = 5 < A$, but any value $0 < \ell < A$ is acceptable; see the left image in Figure 2. Thus, we compute w_2 analytically using the residue theorem, namely,

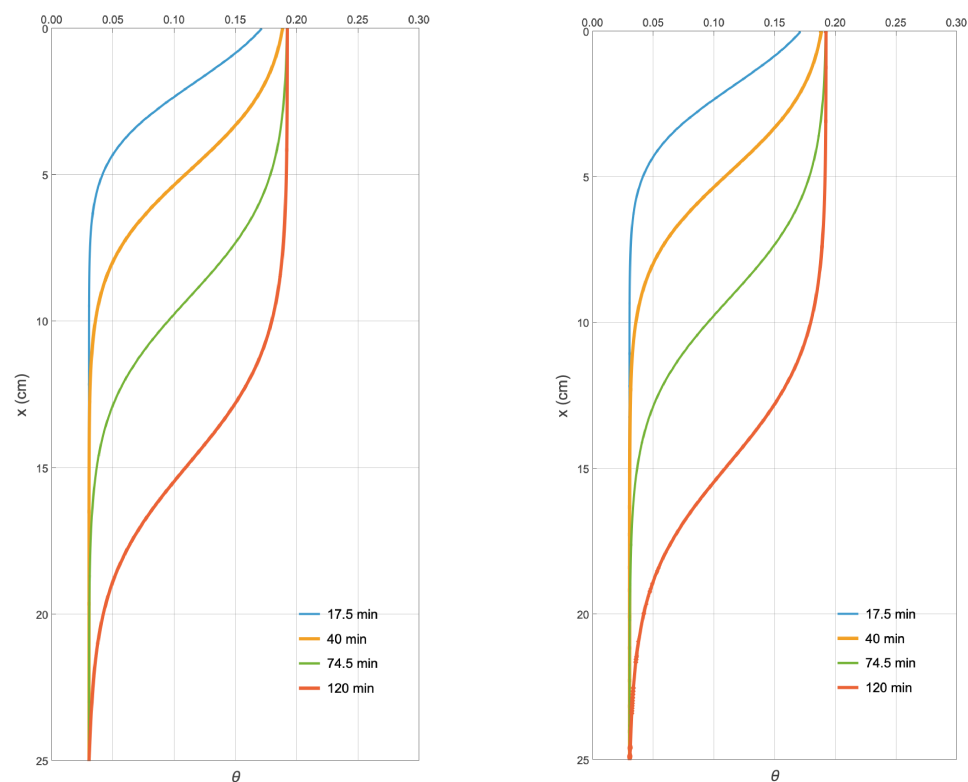
$$\begin{aligned}
 w_2(x, t) &= -\frac{1}{2\pi} \int_{\mathcal{C}^+} V_2(\lambda, x, t) d\lambda = -i \operatorname{Res}[V_2, \gamma i] \\
 &= e^{Bt} \frac{\sqrt{a\bar{q}} \cosh(\gamma(L-x)) + a(\theta_L + b) \sinh(\gamma(L-x))}{\sqrt{a\bar{q}} \cosh(\gamma L) + a(\theta_L + b) \sinh(\gamma L)}.
 \end{aligned}$$

(a) Equation (20) with $\ell = 5$.

(b) Equation (21).

Figure 2. The two different contour deformations $\mathcal{C}^+$ considered in the first example.

In Figure 3, we present the numerical evaluation of the analytical solution, and we compare it with the Fourier solution presented in [8]. We observe that the expected physically meaningful behavior is clearly observed, and the two solutions fit perfectly.

**Figure 3.** The water content θ (6) using the analytical solution (14) (left) and the Fourier series solution; see [8] (right), for the setup of the first example.

Note that the series had to be truncated at $N = 2000$ to obtain an illustration of the solution with a sufficiently small deviation from the actual solution. A visualization of the truncation error for different numbers of Fourier nodes is shown in Figure 4 for $t = 40$ min.

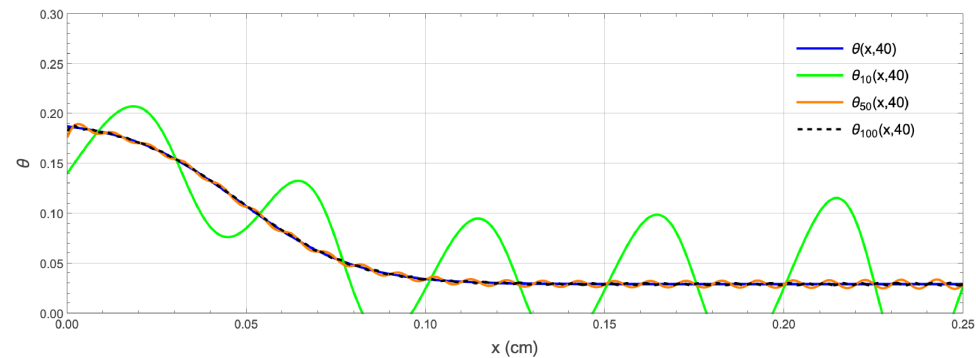


Figure 4. Convergence of the series solution to the analytical one (blue curve) for $t = 40$ min for the setup of the first example.

In the following, we do not consider (20). We present a different approach by introducing an alternative parametrization of $\mathcal{C}^+$ to avoid the computation of the singularity at $\lambda = \gamma i$, namely,

$$z(s) = \begin{cases} P_+ + s e^{i\pi/8}, & s \in (-\infty, 0), \\ (P_- - P_+)s + P_+, & s \in [0, 1], \\ P_- + (s - 1) e^{i7\pi/8}, & s \in (1, +\infty), \end{cases} \quad (21)$$

where $P_- = iA - (1 + i)$ and $P_+ = i\gamma + (1 + i)$; see the right image in Figure 2. This way, the root $\lambda = \gamma i$ lies in the exterior of the domain defined by the boundary $\mathcal{C}^+$. Furthermore, the roots of Δ also lie in the exterior of this domain, and the integrand is exponentially decaying for $\lambda \rightarrow \infty$ in this domain by definition; hence, the integrand is bounded and analytic, and using Cauchy's theorem, we get

$$w_2(x, t) = 0. \quad (22)$$

In Figure 5, we present the analytical solution $\theta(x, t)$ for $x \in [0, 25]$ cm and $t \in [0, 120]$ min.

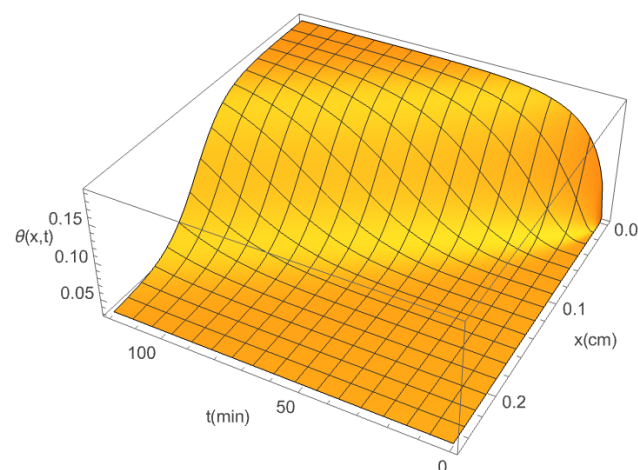


Figure 5. Example 1: Three-dimensional visualization of the analytical solution $\theta(x, t)$ for $x \in [0, 25]$ cm and $t \in [0, 120]$ min.

In the second example, the length is $L = 0.08$ m and no-flux (outflow) is assumed, meaning $q = 0$. In this case, $\theta_0 = 0.355$ and $\theta_L = 0.10$. Here, $\gamma = 0$, and the root is outside the region defined by (20), so (22) holds true by Cauchy's theorem.

A comparison between the Fokas method solution and the Fourier series solution for $N = 2000$ is presented in Figure 6. We emphasize that this choice of Fourier nodes is in

accordance with the illustrations in [8], which we reproduce here. On the other hand, this kind of truncation is not needed for the integral representation derived by the Fokas method, since this illustration is produced via the appropriate symbolic computation package in Mathematica; for more detail, we refer to the Supplementary Material, where the command `NIntegrate` is implemented in the interval $(0, \infty)$. Computational-wise, the Fokas method offers a clear computational advantage over the classical eigenfunction expansion. An extensive numerical comparison is out of the scope of this work, but, for example, on a laptop equipped with an Apple M3 processor and 16 GB of RAM, with the parameter values considered above, and for the same precision requirements, the Fokas method requires only 0.04 s per evaluation at a fixed (x, t) point in the domain $(0, L) \times (0, T)$, compared to 0.12 s for the Fourier series with $N = 2000$ terms. This scaling of approximately 3 times lower computational cost held in all the runs we performed for different points (x, t) . It is clear in Figure 6 that the two solutions are, again, in complete agreement. A three-dimensional visualization of the analytical solution $\theta(x, t)$ for $x \in [0, 8]$ cm and $t \in [0, 120]$ min is shown in Figure 7.

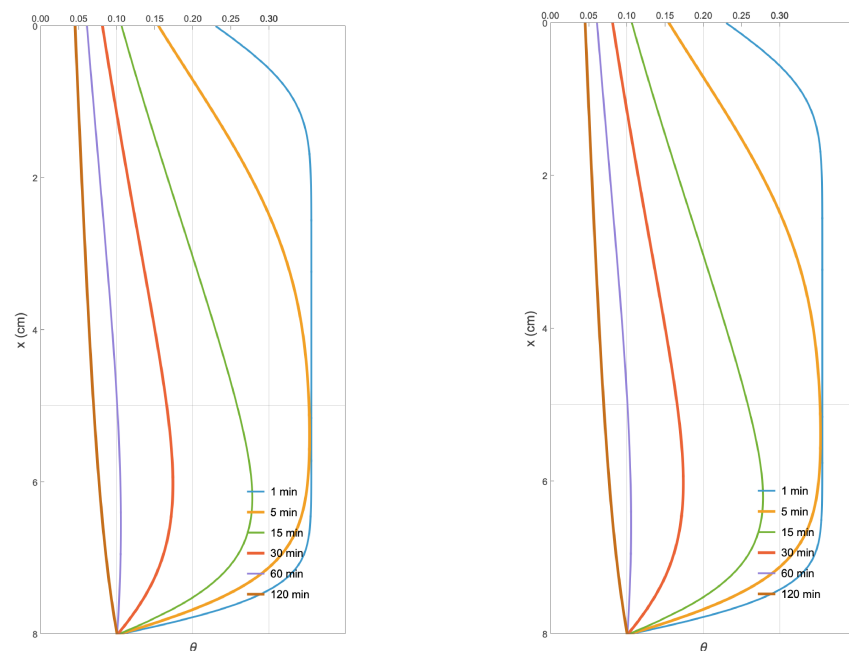


Figure 6. Water content θ (6) using the analytical solution (14) (left) and the Fourier series solution; see [8] (right) for the setup of the second example.

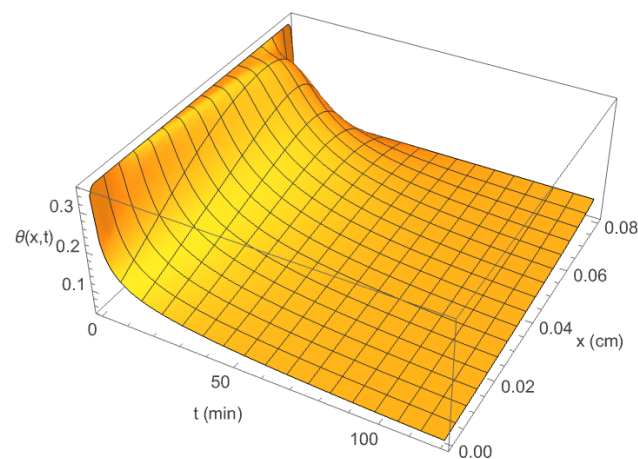


Figure 7. Example 2: The three-dimensional visualization of the analytical solution $\theta(x, t)$ for $x \in [0, 8]$ cm and $t \in [0, 120]$ min.

5. Conclusions

In this work, we examined the one-dimensional vertical infiltration problem in an unsaturated porous medium modeled through the nonlinear Richards' equation. Under suitable assumptions on the soil properties, in particular, constant diffusivity D and a quadratic constitutive relation for $K(\theta)$, the initial boundary value problem (IBVP) reduces to Burgers' equation. While these assumptions may appear restrictive for modeling infiltration in field conditions, they nevertheless represent a significant generalization compared to Cases 1 and 2 considered in previous works. This is still a nonlinear partial differential equation, but it can be easily linearized using the Hopf–Cole transformation. This resulted in a diffusion equation on a finite interval with Robin–Dirichlet boundary conditions.

Building on previous works by the two authors, we solved the simplified IBVP using the Fokas method. The key advantage is that we obtained an explicit integral representation of the solution without relying on eigenfunction expansions, with the limitations of slowly converging series.

We tested our solution on two numerical experiments modeling real-world applications of subsurface water flow under different ponding conditions. By exploiting tools from complex analysis, we derived efficient numerical schemes that are easy to implement. The results show perfect agreement with existing solutions while offering clear practical advantages, particularly in capturing short-time behavior and near-boundary dynamics.

As future work, it would be of interest to extend this approach to more general nonlinearities, multidimensional settings, and time-dependent boundary conditions that more accurately reflect realistic hydrological scenarios. Applications to layered soil structures also appear feasible.

Supplementary Materials: The following supporting information can be downloaded at: <https://www.mdpi.com/article/10.3390/math14132268/s1>.

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Appendix A

In this section, we briefly outline the steps required to derive the IBVP (11a–d) from the original nonlinear initial boundary value problem. For the full derivation, we refer to [8].

Appendix A.1. Derivation of (11a) from (3)

Let

$$\frac{\partial u}{\partial x} = \theta + b.$$

Then, Equation (3) becomes

$$\frac{\partial}{\partial x} \left[\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} + a \left(\frac{\partial u}{\partial x} \right)^2 \right] = 0,$$

which means that the term inside the brackets depends only on time t . Thus,

$$\frac{\partial u}{\partial t} - D \frac{\partial^2 u}{\partial x^2} + a \left(\frac{\partial u}{\partial x} \right)^2 = h(t).$$

Any arbitrary choice of $h(t)$ will not affect the solution of the original problem; see Appendix of [8] for more details. Hence, without loss of generality, we set $h(t) = 0$ and the above equation using the Hopf–Cole transformation

$$w = e^{-\frac{a}{D}u},$$

resulting in (11a).

Appendix A.2. Derivation of (11b) from (7)

Definition (5) together with (7) yields

$$u_x(x, 0) = \theta(x, 0) + b = \theta_0 + b.$$

Integrating both sides with respect to x , we obtain

$$u(x, 0) = (\theta_0 + b)x + C_0.$$

We can choose $C_0 = 0$ without loss of generality. Then, the initial condition takes the form

$$w(x, 0) = e^{-\frac{a}{D}(\theta_0 + b)x} = w_0(x),$$

which is (11b).

Appendix A.3. Derivation of (11c) from (10)

The boundary condition (10) using the new variable (5) reads

$$\begin{aligned} \alpha(u_x(0, t))^2 - Du_{xx}(0, t) &= f(t) \Leftrightarrow \\ \alpha(u_x)^2 - D \left[\frac{\alpha}{D}(u_x)^2 - \frac{D}{\alpha} \frac{w_{xx}}{w} \right] &= f(t) \Leftrightarrow \\ \frac{D^2}{\alpha} \frac{w_{xx}}{w} &= f(t) \end{aligned} \quad (\text{A1})$$

Using the fact that $w_{xx}(0, t) = \frac{1}{D}w_t(0, t)$, we get

$$\frac{D^2}{\alpha} \frac{1}{D} \frac{w_t(0, t)}{w(0, t)} = f(t) \Rightarrow \frac{w_t(0, t)}{w(0, t)} = \frac{\alpha}{D} f(t).$$

Solving this ODE for $w(0, \tau)$, by integrating both sides from 0 to t , yields (11c).

Appendix A.4. Derivation of (11d) from (9)

For the boundary condition at $x = L$, the new variable gives

$$u_x(L, t) = \theta_L + b.$$

Since both θ_L and b are constants, $u_x(L, t)$ is constant with respect to time; thus, $u_{xt}(L, t) = 0$. From the Hopf–Cole transformation, we get

$$w_x = -\frac{\alpha}{D} u_x w,$$

resulting in

$$w_x(L, t) + \frac{\alpha}{D}(\theta_L + b)w = 0. \quad (\text{A2})$$

Equation (A2) is (11d) for $C = \frac{\alpha}{D}(\theta_L + b)$.

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