

Article

Exponential Stability of Traveling Waves with Application to Structured Population Dynamics

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Abstract

We investigate the dynamical stability of traveling-wave solutions in a scalar, age-structured model that incorporates nonlocal dispersal. By combining a comparison argument with a weighted-energy framework tailored to the convolution-type operator, we prove that traveling waves of non-critical speed are exponentially stable under appropriately weighted perturbations after integration over age. Numerical experiments illustrate the qualitative content of the decay result and the structure of the resulting rate bound.

Keywords: stability; traveling wave; weighted-energy method; age-structured model; nonlocal dispersal

MSC: 37M05; 76D33; 34K20

1. Introduction

Traveling waves are a fundamental solution concept in reaction–diffusion theory, and they have long been used to describe coherent invasion fronts and propagation phenomena in population genetics, ecological spread, and epidemic dynamics [1,2]. In many of these settings, the assumption of a homogeneous population is not realistic: demographic or infection processes depend strongly on individual age. Age structure therefore enters naturally, because survival, fertility, and transmission (or susceptibility) typically vary across the life cycle [3]. General age-structured epidemic models and their qualitative dynamics have been studied extensively [4], and traveling waves for age-structured models with local diffusion have been developed in several works [5,6].

A representative example is the simplified scalar age-structured epidemic framework considered in [5], where the existence of traveling waves connecting disease-free and endemic equilibria was established under local diffusion. Beyond existence, stability is essential if traveling waves are to be interpreted as observable long-time patterns. For the same local-diffusion setting, Zhang et al. [7] recently proved exponential stability of non-critical (non-minimal-speed) waves by combining a comparison argument with a weighted-energy method. While nonlinear stability of traveling fronts is well developed for several reaction–diffusion equations with local diffusion [8,9], corresponding results for age-structured models are comparatively scarce, with [7] being a notable step.

In parallel, nonlocal dispersal has become a standard modeling choice when movement cannot be captured by short-range diffusion. A common formulation replaces the



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Laplacian by a convolution-type operator of the form $(J * u - u)$, which represents long-range interactions such as seed dispersal over distance or human mobility via travel [10,11]. Compared with local diffusion, nonlocal dispersal can modify both the admissible wave speeds and the qualitative spreading behavior, including acceleration effects [12]. For age-structured equations with nonlocal dispersal, the existence of traveling waves and spreading speeds has been addressed in several directions: for instance, the authors in [13] treated an age-structured Fisher–KPP-type model, ref. [14] studied traveling waves and spreading speeds for a closely related scalar equation, and the study in Ref. [15] analyzed traveling waves in an age-structured nonlocal SIRS model.

Despite this progress, the stability theory for traveling waves in scalar age-structured models with nonlocal dispersal remains largely undeveloped, especially when one seeks exponential convergence without introducing extra delay structure. Existing stability results for nonlocal dispersal fronts are predominantly for models without age structure [16,17] or for delayed systems. Moreover, although weighted-energy techniques have been successfully adapted to certain nonlocal equations (in non-age-structured contexts) [18], their systematic use in age-structured nonlocal models has not been established.

Very recently, the authors in [19] proved exponential stability of traveling-wave solutions for a scalar age-structured model with nonlocal dispersal, building on the existence and spreading-speed framework in [14]. The present work is the SI epidemic formulation of [5], in which the logistic nonlinearity $p(\tau)u(1 - u)$ is directly coupled to the renewal boundary condition through $g(\tau) = b(\tau)p(\tau)$, and our analysis yields an explicit, directly computable bound on the decay rate in terms of the kernel exponential moment $\hat{J}(\eta)$ and the model parameters.

Motivated by the strategy in [7,9], the goal of this paper is to prove exponential stability of non-critical traveling waves for a scalar age-structured equation with nonlocal dispersal. The nonlocal operator introduces technical difficulties absent in the local-diffusion case, including:

- The lack of a standard maximum principle in the convolution setting, which complicates comparison-based arguments;
- Spectral and coercivity features of the linearized nonlocal operator that differ from those of the Laplacian, affecting decay estimates in weighted spaces;
- The need to choose a spatial weight $\omega(\cdot)$ that is compatible with the kernel J , balancing spatial decay against the interaction range encoded by J .

By combining a comparison principle with a weighted-energy framework tailored to the nonlocal term, we show that perturbations (after integration over age) decay exponentially to the traveling-wave profile for speeds above the critical threshold.

The remainder of the paper is organized as follows. Section 2 introduces the model, recalls existence results for traveling waves, and states the main exponential-stability theorem. Section 3 develops the comparison and weighted-energy estimates that yield the decay result. Section 4 presents numerical simulations illustrating the theoretical conclusions.

2. Preliminaries and Main Result

2.1. Definitions

Throughout, $C > 0$ denotes a generic constant whose value may change from line to line. For an interval $I \subset \mathbb{R}$, we write $L^2(I)$ for the usual square-integrable space and $H^k(I)$ ($k \in \mathbb{N}_0$) for the Sobolev space of functions whose weak derivatives up to order k belong to $L^2(I)$.

Fix a weight $\omega : \mathbb{R} \rightarrow (0, \infty)$. Define

$$L^2_\omega(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{R} \mid \|f\|_{L^2_\omega}^2 := \int_{\mathbb{R}} \omega(z) |f(z)|^2 dz < \infty \right\},$$

and the weighted Sobolev space

$$H^1_\omega(\mathbb{R}) := \left\{ f \in L^2_\omega(\mathbb{R}) \mid f_z \in L^2_\omega(\mathbb{R}) \right\}, \quad \|f\|_{H^1_\omega}^2 := \|f\|_{L^2_\omega}^2 + \|f_z\|_{L^2_\omega}^2.$$

For a kernel $J : \mathbb{R} \rightarrow \mathbb{R}$ and a function $v = v(z)$, convolution in z is written

$$(J * v)(z) := \int_{\mathbb{R}} J(y) v(z - y) dy, \quad \mathcal{L}v := J * v - v.$$

2.2. Model and Assumptions

Let $A \in (0, \infty)$ be the maximal age and let $u = u(t, \tau, x)$ denote the age density at time $t > 0$, age $\tau \in [0, A)$ and position $x \in \mathbb{R}$. We consider

$$\begin{cases} \partial_t u + \partial_\tau u = \mathcal{L}u(t, \tau, \cdot)(x) + p(\tau) u(1 - u), & t > 0, \tau \in [0, A), x \in \mathbb{R}, \\ u(t, 0, x) = \int_0^A g(\tau) u(t, \tau, x) d\tau, & t > 0, x \in \mathbb{R}, \\ u(0, \tau, x) = u_0(\tau, x) \geq 0, & \tau \in [0, A), x \in \mathbb{R}. \end{cases} \quad (1)$$

Here $p(\tau) > 0$ is continuous on $[0, A)$, $b(\tau) \geq 0$ is bounded on $[0, A)$, and we set $g(\tau) := b(\tau) p(\tau)$. We assume the normalization $\int_0^A g(\tau) d\tau = 1$, so that $u \equiv 0$ and $u \equiv 1$ are equilibria of (1); these are the states connected by the wave. We work with bounded mild solutions (cf. [3]) and take $0 \leq u_0 \leq 1$; the order interval $[0, 1]$ is then invariant by Lemma 2, so $0 \leq u \leq 1$ for all $t \geq 0$.

The dispersal kernel J satisfies:

- $J \geq 0$ and J is even;
- $J \in C^1(\mathbb{R})$ and $\int_{\mathbb{R}} J(y) dy = 1$;
- There exists $\eta_J > 0$ such that $\int_{\mathbb{R}} J(y) e^{\eta_J |y|} dy < \infty$.

For $0 < \eta < \eta_J$, the exponential moment of J is

$$\hat{J}(\eta) := \int_{\mathbb{R}} J(y) e^{\eta y} dy \in (0, \infty). \quad (2)$$

We use the bounds

$$\bar{b} := \sup_{\tau \in [0, A)} b(\tau), \quad \bar{p} := \sup_{\tau \in [0, A)} p(\tau), \quad \bar{g} := \sup_{\tau \in [0, A)} g(\tau), \quad \Lambda := \bar{p} + \bar{g}.$$

We seek traveling waves moving with speed $c > 0$ in the form $u(t, \tau, x) = \Phi(\tau, z)$, $z := x + ct$. The profile Φ satisfies

$$\begin{cases} \partial_\tau \Phi + c \partial_z \Phi = \mathcal{L}\Phi(\tau, \cdot)(z) + p(\tau) \Phi(1 - \Phi), \\ \Phi(0, z) = \int_0^A g(\tau) \Phi(\tau, z) d\tau, \\ \Phi(\tau, -\infty) = 0, \quad \Phi(\tau, +\infty) = 1. \end{cases} \quad (3)$$

(convolution in (3) is with respect to z .) For closely related scalar age-structured equations with nonlocal dispersal, the existence of a critical speed $c^* > 0$ and of a monotone traveling-wave profile Φ with $0 < \Phi < 1$ for every $c > c^*$ has been established in [13,14]. Since the

nonlinearity $p(\tau)\Phi(1 - \Phi)$ and the renewal condition in (3) differ from the precise forms treated there, we take the existence of such a monotone profile Φ , with the limits attained uniformly in τ , as a standing hypothesis of Theorem 1.

For a function $v = v(t, \tau, x)$, define the *age integral operator*

$$\mathcal{I}[v](t, x) := \int_0^A v(t, \tau, x) d\tau. \quad (4)$$

In the co-moving frame $z = x + ct$ we equip the spatial variable with the exponential weight

$$\omega(z) := e^{-\eta z}, \quad \eta > 0 \text{ (chosen sufficiently small, see (6))}. \quad (5)$$

Here ω decays as $z \rightarrow +\infty$ and grows as $z \rightarrow -\infty$, the natural choice for a front joining the unstable state 0 at $z = -\infty$ to the endemic state 1 at $z = +\infty$.

Theorem 1 (Exponential stability). *Let $c > c^*$ and assume that there exists $0 < \eta < \eta_f$ such that the speed condition*

$$\frac{c\eta}{2} > \frac{\hat{J}(\eta) - 1}{2} + \Lambda \quad (6)$$

holds, where $\hat{J}$ is the exponential moment defined in (2) and $\Lambda = \bar{p} + \bar{g}$. If the initial perturbation obeys

$$u_0(\tau, \cdot) - \Phi(\tau, \cdot) \in L^1((0, A); H_\omega^1(\mathbb{R})),$$

and $u_0(\tau, x) \rightarrow 1$ as $x \rightarrow +\infty$, uniformly in τ , then the Cauchy problem (1) admits a unique global solution $u(t, \tau, x)$ and

$$\mathcal{I}[u - \Phi](t, \cdot) \in C([0, \infty); H_\omega^1(\mathbb{R})) \cap L^2([0, \infty); H_\omega^1(\mathbb{R})).$$

Moreover, there exist constants $C > 0$ and $\sigma > 0$ such that

$$\sup_{x \in \mathbb{R}} \int_0^A |u(t, \tau, x) - \Phi(\tau, x + ct)| d\tau \leq Ce^{-\sigma t}, \quad t > 0. \quad (7)$$

One may take any decay rate σ satisfying

$$0 < \sigma < \min\{\sigma_1, \sigma_2\}, \quad \sigma_1 := \frac{c\eta}{2} - \frac{\hat{J}(\eta) - 1}{2} - \Lambda, \quad \sigma_2 := p_* - \bar{g}, \quad (8)$$

where $p_ := \inf_{\tau \in [0, A)} p(\tau)$, provided $p_* > \bar{g}$.*

Remark 1. *The two rates have distinct interpretations. The rate σ_1 is produced by the weighted-energy estimate and combines (i) the transport contribution $c\eta/2$ from the wave speed and the weight, (ii) the dispersive contribution $(\hat{J}(\eta) - 1)/2$ from the kernel's exponential moment, and (iii) the reaction–renewal contribution $\Lambda = \bar{p} + \bar{g}$ from the logistic term and the renewal boundary condition. The rate $\sigma_2 = p_* - \bar{g}$ is produced by the age-asymptotic argument at $z = +\infty$ and measures the net contraction at the endemic state: the maximum renewal flux $\bar{g}$ is offset by the minimum loss rate p_* acting through the logistic damping. The positivity assumption $p_* > \bar{g}$ thus expresses that the loss mechanism dominates the renewal mechanism at the endemic equilibrium. For age-independent $p(\tau) \equiv p$ one has $\bar{g} = p\bar{b}$ with $\bar{b} = \sup_\tau b(\tau)$ (Section 2.2), so the formula reads $\sigma_2 = p(1 - \bar{b})$; this reduction fails for variable $p(\tau)$, where $\sigma_2 = p_* - \bar{g}$ is the correct form. The condition $p_* > \bar{g}$ is restrictive but holds when loss dominates renewal at the endemic state; the example of Section 4 has $p_* = 0.2 > \bar{g} \approx 0.067$.*

3. Proof of Exponential Stability

Lemma 1. If $u_0(\tau, x) \geq 0$, then any bounded solution $u(t, \tau, x)$ of (1) remains non-negative for all $t \geq 0, \tau \in [0, A), x \in \mathbb{R}$.

Proof. Apply Lemma 2 below with $\bar{u} \equiv u$ and the trivial subsolution $\underline{u} \equiv 0$. At $t = 0$ and $\tau = 0$ both inequalities hold by assumption on u_0 and by the positivity of the renewal kernel g , and the right-hand side of (1) vanishes at $u \equiv 0$. Hence $u(t, \tau, x) \geq 0$. $\square$

Lemma 2 (Comparison principle). Assume that $\bar{u}(t, \tau, x)$ and $u(t, \tau, x)$ are bounded functions satisfying

$$\begin{cases} \partial_t \bar{u} + \partial_\tau \bar{u} - (J * \bar{u} - \bar{u}) - p(\tau) \bar{u}(1 - \bar{u}) \\ \geq \partial_t u + \partial_\tau u - (J * u - u) - p(\tau) u(1 - u), & t > 0, \tau \in [0, A), x \in \mathbb{R}, \\ \bar{u}(t, 0, x) \geq u(t, 0, x), & t > 0, x \in \mathbb{R}, \\ \bar{u}(0, \tau, x) \geq u(0, \tau, x), & \tau \in [0, A), x \in \mathbb{R}. \end{cases} \quad (9)$$

Then $\bar{u}(t, \tau, x) \geq u(t, \tau, x)$ for all $t \geq 0, \tau \in [0, A), x \in \mathbb{R}$.

Proof of Lemma 2. Because the operator $\mathcal{L}v = J * v - v$ is nonlocal, the classical pointwise maximum principle for the Laplacian is not directly applicable; we therefore provide a self-contained argument.

Set $w := \bar{u} - u$. The hypotheses give $w(0, \tau, x) \geq 0$ and $w(t, 0, x) \geq 0$. Using the factorisation $\bar{u}(1 - \bar{u}) - u(1 - u) = (1 - \bar{u} - u)w$, subtracting the two inequalities in (9) yields

$$\partial_t w + \partial_\tau w \geq (J * w - w) + B(t, \tau, x)w, \quad (10)$$

where $B := p(\tau)(1 - \bar{u} - u)$. As $\bar{u}$ and u are bounded by hypothesis and p is continuous, B is bounded; set $B_0 := \|B\|_{L^\infty} < \infty$. (We do not use $\bar{u}, u \in [0, 1]$ here to avoid circularity with the bound the comparison principle establishes.)

Fix $\lambda > B_0$ and set $\varphi(t, \tau, x) := e^{-\lambda t} w(t, \tau, x)$. Then $\varphi(0, \tau, x) \geq 0, \varphi(t, 0, x) \geq 0$, and

$$\partial_t \varphi + \partial_\tau \varphi \geq (J * \varphi - \varphi) + B_\lambda(t, \tau, x) \varphi, \quad (11)$$

where $B_\lambda := B - \lambda \leq B_0 - \lambda =: -\delta < 0$.

Define

$$m(t) := \operatorname{ess\,inf}_{\tau \in [0, A), x \in \mathbb{R}} \varphi(t, \tau, x).$$

We claim $m(t) \geq 0$ for all $t > 0$. Since $\varphi(0, \cdot, \cdot) \geq 0$, we have $m(0) \geq 0$. Suppose for contradiction that $m(t_1) < 0$ for some $t_1 > 0$, and let

$$t^* := \inf\{t > 0 : m(t) < 0\}.$$

By continuity, $m(t^*) = 0$. For $t > t^*$, there exists a sequence (τ_n, x_n) with $\varphi_n := \varphi(t, \tau_n, x_n) \rightarrow m(t) < 0$.

Nonlocal estimate. Since $J \geq 0$ and $\int_{\mathbb{R}} J dy = 1$, and since $\varphi(t, \tau_n, \cdot) \geq m(t)$ pointwise,

$$(J * \varphi)(t, \tau_n, x_n) = \int_{\mathbb{R}} J(y) \varphi(t, \tau_n, x_n - y) dy \geq m(t). \quad (12)$$

This is the analog of the minimum-point property of the Laplacian: the convolution operator automatically “sees” the global minimum because J is a positive probability kernel. Hence

$$(J * \varphi - \varphi)(t, \tau_n, x_n) \geq m(t) - \varphi_n \longrightarrow 0. \quad (13)$$

Reaction term. Since $B_\lambda \leq -\delta < 0$ and $\varphi_n \rightarrow m(t) < 0$:

$$B_\lambda(t, \tau_n, x_n) \varphi_n \geq (-\delta) \varphi_n \longrightarrow \delta |m(t)| > 0. \quad (14)$$

Characteristic argument. Combining (11)–(14) gives, for large n ,

$$(\partial_t \varphi + \partial_\tau \varphi)(t, \tau_n, x_n) \geq \frac{1}{2} \delta |m(t)| > 0. \quad (15)$$

Integrating backward along the characteristic through (t, τ_n, x_n) over $[t - \varepsilon, t]$, for $\varepsilon > 0$ small with $\tau_n - \varepsilon > 0$:

$$\varphi(t - \varepsilon, \tau_n - \varepsilon, x_n) \leq \varphi_n - \frac{\varepsilon}{2} \delta |m(t)| < 0. \quad (16)$$

Choosing $\varepsilon < t - t^*$, so that $t - \varepsilon > t^*$, (16) shows $m(t - \varepsilon) < 0$, contradicting the minimality of t^* . This contradiction establishes $m(t) \geq 0$, hence $w = e^{\lambda t} \varphi \geq 0$. $\square$

Remark 2. When the infimum in $x \in \mathbb{R}$ is not attained, the argument is made rigorous by first working on a truncated domain $[-R, R]$ with Neumann boundary conditions (where compactness ensures attainment) and then passing to the limit $R \rightarrow \infty$ using the spatial decay of φ inherited from the weighted initial data.

Let $\Phi(\tau, x + ct)$ be a traveling wave with speed $c > c^*$. Define the upper and lower envelopes of the initial datum:

$$\bar{v}_0(\tau, x) := \max\{u_0(\tau, x), \Phi(\tau, x)\}, \quad \underline{v}_0(\tau, x) := \min\{u_0(\tau, x), \Phi(\tau, x)\}. \quad (17)$$

Let $\bar{v}(t, \tau, x)$ and $\underline{v}(t, \tau, x)$ be the solutions of (1) with initial data $\bar{v}_0$ and $\underline{v}_0$, respectively. By Lemma 2,

$$\underline{v}(t, \tau, x) \leq u(t, \tau, x) \leq \bar{v}(t, \tau, x), \quad \underline{v}(t, \tau, x) \leq \Phi(\tau, x + ct) \leq \bar{v}(t, \tau, x). \quad (18)$$

Shift to the co-moving frame $z = x + ct$ and introduce

$$\begin{aligned} \Psi^+(t, \tau, z) &:= \bar{v}(t, \tau, x) - \Phi(\tau, z) \geq 0, \\ \Gamma^+(t, z) &:= \int_0^A \Psi^+(t, \tau, z) d\tau \geq 0. \end{aligned} \quad (19)$$

Lemma 3. Under the assumptions of Theorem 1,

$$\sup_{z \in \mathbb{R}} |\Gamma^+(t, z)| \leq C e^{-\sigma t}, \quad t > 0, \quad (20)$$

for any $0 < \sigma < \min\{\sigma_1, \sigma_2\}$ as in (8).

Proof. From the equations for $\bar{v}$ and Φ ,

$$\partial_t \Psi^+ + c \partial_z \Psi^+ + \partial_\tau \Psi^+ \leq (J * \Psi^+ - \Psi^+) + p(\tau) \Psi^+, \quad (21)$$

where we used the factorisation $\bar{v}(1 - \bar{v}) - \Phi(1 - \Phi) = (1 - \bar{v} - \Phi) \Psi^+$ together with the bound $(1 - \bar{v} - \Phi) \Psi^+ \leq \Psi^+$. The solution $\bar{v}$ evolves from the non-negative datum $\bar{v}_0 \geq 0$ in (17), so $\bar{v} \geq 0$ by Lemma 1, while the traveling-wave profile satisfies $0 < \Phi < 1$; hence

$\bar{v} + \Phi \geq 0$ and therefore $1 - \bar{v} - \Phi \leq 1$. Since $\Psi^+ = \bar{v} - \Phi \geq 0$ by (19), multiplying the inequality $1 - \bar{v} - \Phi \leq 1$ by the non-negative factor Ψ^+ preserves its direction and yields $(1 - \bar{v} - \Phi) \Psi^+ \leq \Psi^+$.

Integrate (21) over $\tau \in [0, A)$. Since $\Psi^+(t, A, z) = 0$ and

$$\Psi^+(t, 0, z) = \int_0^A g(\tau) \Psi^+(t, \tau, z) d\tau \leq \bar{g} \Gamma^+(t, z),$$

we obtain

$$\partial_t \Gamma^+ + c \partial_z \Gamma^+ \leq (J * \Gamma^+ - \Gamma^+) + \Lambda \Gamma^+. \quad (22)$$

Multiply (22) by $e^{2\sigma t} \omega(z) \Gamma^+(t, z)$ and integrate over $z \in \mathbb{R}$. For the transport term, integration by parts together with $\omega'(z) = -\eta \omega(z)$ gives

$$c \int_{\mathbb{R}} \omega(z) \Gamma^+(z) \partial_z \Gamma^+(z) dz = \frac{c\eta}{2} \|\Gamma^+\|_{L_{\omega}^2}^2,$$

so that

$$\frac{d}{dt} \left(\frac{1}{2} e^{2\sigma t} \|\Gamma^+(t)\|_{L_{\omega}^2}^2 \right) + e^{2\sigma t} \left(\frac{c\eta}{2} - \sigma \right) \|\Gamma^+\|_{L_{\omega}^2}^2 \leq e^{2\sigma t} \int_{\mathbb{R}} \omega \Gamma^+ (J * \Gamma^+ - \Gamma^+ + \Lambda \Gamma^+) dz. \quad (23)$$

For the nonlocal term, Young's inequality $2ab \leq a^2 + b^2$ applied to $a = \omega(z)^{1/2} \Gamma^+(z)$, $b = \omega(z)^{1/2} \Gamma^+(z - y)$, together with the change in variables $z \mapsto z - y$ and the symmetry $J(-y) = J(y)$, gives

$$\int_{\mathbb{R}} \omega(z) \Gamma^+(z) (J * \Gamma^+)(z) dz \leq \frac{1}{2} (1 + \hat{J}(\eta)) \|\Gamma^+\|_{L_{\omega}^2}^2, \quad (24)$$

where we used the translation identity $\int_{\mathbb{R}} \omega(z) \Gamma^+(z - y)^2 dz = e^{-\eta y} \|\Gamma^+\|_{L_{\omega}^2}^2$ (by the substitution $z' = z - y$ and $\omega(z) = e^{-\eta z}$), the symmetry of J to write $\int J(y) e^{-\eta y} dy = \int J(y) e^{\eta y} dy = \hat{J}(\eta)$, and the definition $\hat{J}(\eta) = \int J(y) e^{\eta y} dy$. In effect the weight turns the convolution into multiplication by the moment: the shift $z \mapsto z - y$ rescales the weighted norm by $e^{-\eta y}$, and averaging against J yields $\hat{J}(\eta)$. Inserting (24) into (23):

$$\frac{d}{dt} \left(\frac{1}{2} e^{2\sigma t} \|\Gamma^+\|_{L_{\omega}^2}^2 \right) + e^{2\sigma t} \underbrace{\left[\frac{c\eta}{2} - \sigma - \frac{\hat{J}(\eta) - 1}{2} - \Lambda \right]}_{>0 \text{ when } \sigma < \sigma_1} \|\Gamma^+\|_{L_{\omega}^2}^2 \leq 0. \quad (25)$$

Hence

$$e^{2\sigma t} \|\Gamma^+(t)\|_{L_{\omega}^2}^2 \leq \|\Gamma^+(0)\|_{L_{\omega}^2}^2. \quad (26)$$

Differentiating (21) in z and repeating the weighted-energy argument yields

$$e^{2\sigma t} \|\partial_z \Gamma^+(t)\|_{L_{\omega}^2}^2 \leq C \|\Gamma^+(0)\|_{H_{\omega}^1}^2. \quad (27)$$

For any $z_0 \gg 1$, since $\omega(z) \geq e^{-\eta z_0}$ on $(-\infty, z_0]$, the Sobolev embedding $H^1 \hookrightarrow C^0$ gives

$$\sup_{z \leq z_0} |\Gamma^+(t, z)| \leq C \|\Gamma^+(t)\|_{H_{\omega}^1((-\infty, z_0])} \leq C e^{-\sigma t}. \quad (28)$$

We first record a two-sided bound. Since $u \equiv 0$ and $u \equiv 1$ are equilibria of (1) and $0 \leq \bar{v}_0 \leq 1$, the comparison principle (Lemma 2) gives $0 \leq \bar{v} \leq 1$; as $0 < \Phi < 1$, this yields

$$0 \leq \Psi^+ = \bar{v} - \Phi \leq 1 - \Phi.$$

Because $\Phi(\tau, +\infty) = 1$ uniformly in τ , the squeeze shows that $\Psi^+(t, \tau, z) \rightarrow 0$ as $z \rightarrow +\infty$, uniformly in $t \geq 0$ and $\tau \in [0, A)$; in particular $\Gamma^+(t, z) \rightarrow 0$ as $z \rightarrow +\infty$ for each t .

To obtain the decay rate on the tail we use the sharper form of the reaction near the endemic state. Fix $\beta \in (0, \frac{1}{2})$ with $(1 - 2\beta)p_* > \bar{g}$, which is possible because $p_* > \bar{g}$. Since $\Phi(\tau, +\infty) = 1$ uniformly, choose z_0 so large that $\Phi(\tau, z) \geq 1 - \beta$ for all $z \geq z_0$ and $\tau \in [0, A)$. Then $\bar{v} \geq \Phi \geq 1 - \beta$ on $\{z \geq z_0\}$, so $1 - \bar{v} - \Phi \leq -(1 - 2\beta)$ there; using this in the exact identity for Ψ^+ (whose reaction is $p(\tau)(1 - \bar{v} - \Phi)\Psi^+$, before the bound leading to (21)), we obtain on $\{z \geq z_0\}$

$$\partial_t \Psi^+ + c \partial_z \Psi^+ + \partial_\tau \Psi^+ \leq (J * \Psi^+ - \Psi^+) - (1 - 2\beta)p(\tau)\Psi^+.$$

Integrating over $\tau \in [0, A)$, using $\Psi^+(t, A, z) = 0$ and the renewal bound $\Psi^+(t, 0, z) = \int_0^A g(\tau)\Psi^+(t, \tau, z) d\tau \leq \bar{g}\Gamma^+(t, z)$, yields

$$\partial_t \Gamma^+ + c \partial_z \Gamma^+ \leq (J * \Gamma^+ - \Gamma^+) - \kappa \Gamma^+ \quad \text{on } \{z \geq z_0\}, \quad \kappa := (1 - 2\beta)p_* - \bar{g} > 0. \quad (29)$$

Let $m(t) := \sup_{z \geq z_0} \Gamma^+(t, z) \geq 0$. For $z \geq z_0$, since $J \geq 0$ and $\int_{\mathbb{R}} J = 1$,

$$(J * \Gamma^+)(t, z) = \int_{\mathbb{R}} J(y) \Gamma^+(t, z - y) dy \leq m(t) + \sup_{z \leq z_0} \Gamma^+(t, z) \leq m(t) + Ce^{-\sigma t},$$

the last inequality by (28). Inserting this into (29) and arguing as in the maximum-principle step of Lemma 2, now on the half-line $\{z \geq z_0\}$ whose boundary value at $z = z_0$ is controlled by (28), we obtain

$$\frac{d}{dt} m(t) \leq Ce^{-\sigma t} - \kappa m(t), \quad \text{whence} \quad m(t) \leq Ce^{-\min\{\sigma, \kappa\}t}.$$

As $\beta \downarrow 0$ we have $\kappa \uparrow p_* - \bar{g} = \sigma_2$, so the tail decays at any rate below $\min\{\sigma_1, \sigma_2\}$. Combining $\sup_{z \geq z_0} \Gamma^+(t, z) = m(t) \leq Ce^{-\sigma t}$ with the bound (28) on $\{z \leq z_0\}$ gives (20). $\square$

Lemma 4. Under the assumptions of Theorem 1,

$$\sup_{z \in \mathbb{R}} \int_0^A |\Phi(\tau, z) - \underline{v}(t, \tau, x)| d\tau \leq Ce^{-\sigma t}, \quad t > 0, \quad (30)$$

for any $0 < \sigma < \min\{\sigma_1, \sigma_2\}$.

Proof. Define the lower perturbation and its age integral:

$$\Psi^-(t, \tau, z) := \Phi(\tau, z) - \underline{v}(t, \tau, x) \geq 0, \quad \Gamma^-(t, z) := \int_0^A \Psi^-(t, \tau, z) d\tau \geq 0. \quad (31)$$

A direct computation using the factorisation $\Phi(1 - \Phi) - \underline{v}(1 - \underline{v}) = (1 - \Phi - \underline{v})\Psi^-$ together with $\Phi, \underline{v} \in [0, 1]$, which gives $|1 - \Phi - \underline{v}| \leq 1$ and hence $p(\tau)(1 - \Phi - \underline{v})\Psi^- \leq p(\tau)\Psi^-$, shows that Ψ^- satisfies the same inequality (21) as Ψ^+ . Consequently Γ^- satisfies (22), and Steps 1–4 of the proof of Lemma 3 apply verbatim to Γ^- . Unlike $\bar{v} \geq \Phi$, the lower envelope is not automatically near 1 at $z = +\infty$; here the hypothesis $u_0 \rightarrow 1$ as $x \rightarrow +\infty$ gives $\underline{v}_0 \rightarrow 1$ there, and comparison with the spatially homogeneous endemic dynamics (whose equilibrium 1 is attracting when $p_* > \bar{g}$) yields $\underline{v} \rightarrow 1$ as $z \rightarrow +\infty$, uniformly in t . Thus $\underline{v} \geq 1 - \beta$ on $\{z \geq z_0\}$, so $1 - \Phi - \underline{v} \leq -(1 - 2\beta)$ and the sharp-

reaction estimate of Step 5 applies to Γ^- as well. This gives $\sup_z |\Gamma^-(t, z)| \leq Ce^{-\sigma t}$, which is exactly (30). $\square$

Lemma 5 (Convergence to the traveling wave). *Under the assumptions of Theorem 1,*

$$\sup_{x \in \mathbb{R}} \int_0^A |u(t, \tau, x) - \Phi(\tau, x + ct)| d\tau \leq Ce^{-\sigma t}, \quad t > 0, \quad (32)$$

for any $0 < \sigma < \min\{\sigma_1, \sigma_2\}$.

Proof. From the two-sided ordering (18) and the triangle inequality,

$$\int_0^A |u(t, \tau, x) - \Phi(\tau, x + ct)| d\tau \leq \Gamma^+(t, z) + \Gamma^-(t, z).$$

Applying Lemmas 3 and 4 to each term gives (32). $\square$

Lemma 5 completes the proof of Theorem 1.

4. Numerical Simulations

In this section, we utilize numerical simulations to illustrate the stability of traveling waves of the nonlocal age-structured model (1). The parameter values are selected as follows: $A = 25$, $p(\tau) \equiv 0.2$, $b(\tau) = \frac{1}{3}$ on $[5, 20]$ and zero elsewhere, so that $g(\tau) = b(\tau)p(\tau)$ satisfies $\int_0^A g(\tau) d\tau = 1$ with $\bar{b} = \frac{1}{3} < 1$. The dispersal kernel is $J(y) = \frac{1}{2}e^{-|y|}$, for which $\hat{J}(\eta) = 1/(1 - \eta^2)$ for $|\eta| < 1$, and we fix $\eta = 0.15$ and wave speed $c = 8.0$. With these choices, the speed condition (6) is satisfied since $c\eta/2 = 0.600 > (\hat{J}(0.15) - 1)/2 + \Lambda \approx 0.0115 + 0.267 \approx 0.279$, where $\Lambda = \bar{p} + \bar{g} \approx 0.267$. The decay-rate bounds in (8) evaluate to

$$\sigma_1 = \frac{c\eta}{2} - \frac{\hat{J}(\eta) - 1}{2} - \Lambda \approx 0.321, \quad \sigma_2 = p_* - \bar{g} \approx 0.133, \quad (33)$$

so that the conclusion of Theorem 1 holds with any $\sigma < \min\{\sigma_1, \sigma_2\} = 0.133$. To better perform numerical simulations, we set $N(t, x) = \mathcal{I}[u](t, x)$ and use a fourth-order Runge–Kutta scheme with $\Delta x = 0.5$, $\Delta \tau = 0.1$, $\Delta t = 0.05$ on $x \in [-200, 200]$; the convolution $J * u$ is evaluated by FFT on each age row.

We construct solutions $u(t, \tau, x)$ of (1) with $u(t, \tau, -\infty) = 0$ and $u(t, \tau, +\infty) = 1$, corresponding to $N(t, -\infty) = 0$ and $N(t, +\infty) = A$, and set the non-monotone initial data

$$N(0, x) = A \cdot \begin{cases} 0.8(1 - \tanh^2(0.15(x + 50))), & x < -30, \\ 0.4(1 + \tanh^2(0.10(x - 40))), & x > 30, \\ 0, & \text{otherwise,} \end{cases} \quad (34)$$

lifted uniformly in age as $u_0(\tau, x) = A^{-1}N(0, x)$, so that $\int_0^A u_0(\tau, x) d\tau \equiv N(0, x)$ and $u_0(\tau, \cdot) \in [0, 1]$.

By Theorem 1, since the speed condition (6) is satisfied for our parameter choice, any initial perturbation of the traveling wave Φ in the weighted space $L^1((0, A); H_w^1(\mathbb{R}))$ decays at a rate $\sigma < \min\{\sigma_1, \sigma_2\}$. The simulations below illustrate the qualitative content of this result. Figure 1 shows that the two-bump initial datum (34) evolves toward the endemic state $N \equiv A$ as t increases; by $t = 10$ each bump has grown to $N \approx A$ and the profile is monotonised in x , in qualitative agreement with the decay estimate (7). Figure 2 displays the full spacetime evolution of $N(t, x)$ in Figure 2a and of the age slices $u(t, \tau=5, x)$ and $u(t, \tau=20, x)$ in Figure 2b,c. The spatial shape of the solution is essentially the same

across ages, consistent with the τ -uniform structure of the profile $\Phi(\tau, z)$ used in the proof. Figure 2d contrasts the fat-tailed kernel $J(y)$ with a Dirac-like local diffusion: the unbounded support of J is the structural feature behind the loss of a pointwise maximum principle handled by Lemma 2. Figure 3 illustrates how the decay bound $\min\{\sigma_1, \sigma_2\}$ depends on the model parameters: Figure 3a plots the contours of $\sigma_1(\eta, c)$ on the (η, c) -plane together with the boundary $\sigma_1 = 0$ of the admissible region defined by (6); Figure 3b compares $\sigma_1(\eta)$, $\sigma_2 = p_* - \bar{g}$ and their minimum at fixed $c = 8.0$. The bound is controlled by σ_1 for small η and by σ_2 otherwise, separating the contribution of nonlocal dispersal from that of the age-renewal mechanism.

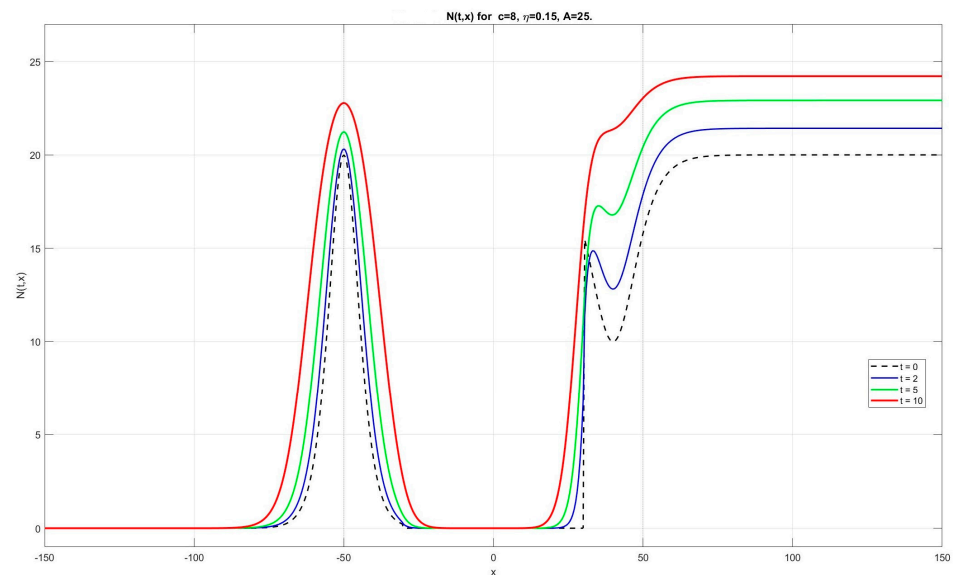


Figure 1. Snapshots of $N(t, x) = \mathcal{I}[u](t, x)$ subject to the initial data (34) at $t = 0, 2, 5, 10$. The two-bump profile grows toward the endemic state $N \equiv A$ and becomes monotonised in x , illustrating the qualitative content of Theorem 1.

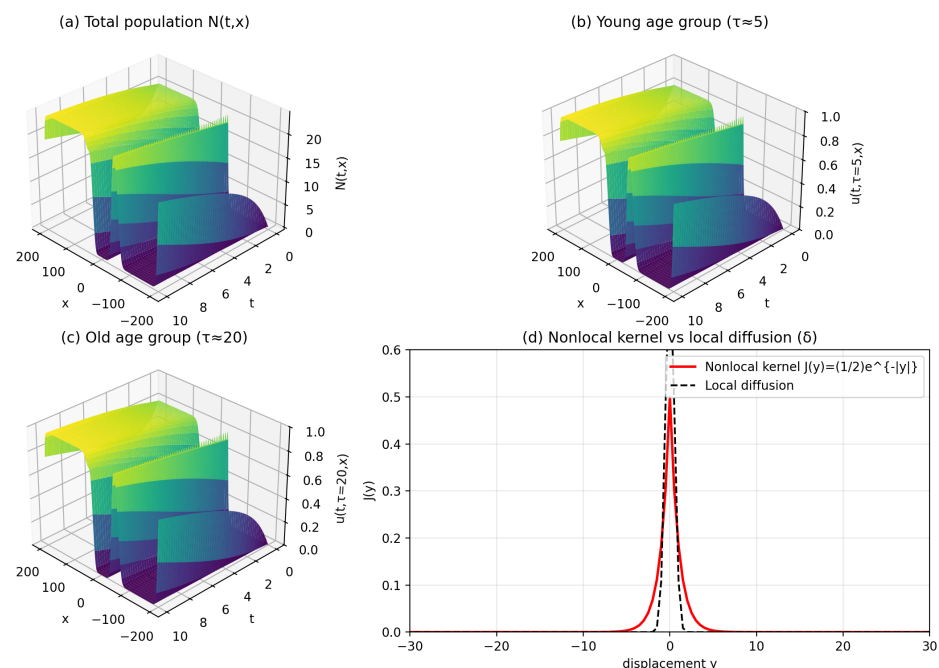


Figure 2. Spacetime structure of the solution. (a) Age-integrated population $N(t, x) = \mathcal{I}[u](t, x)$. (b) Young age-slice $u(t, \tau=5, x)$ (interior of $\text{supp } g$). (c) Old age-slice $u(t, \tau=20, x)$ (endpoint of $\text{supp } g$). The two age slices share the same spatial shape, illustrating the τ -uniformity of $\Phi(\tau, z)$. (d) Kernel $J(y) = \frac{1}{2}e^{-|y|}$ compared with a Dirac-like local diffusion.

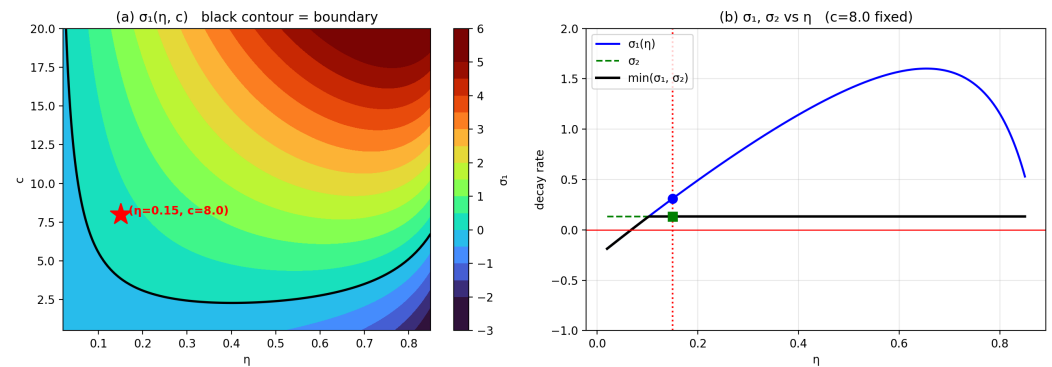


Figure 3. Parameter-space structure of the decay bound (8). (a) Contours of $\sigma_1(\eta, c)$; (b) the black curve marks $\sigma_1 = 0$, the boundary of the admissible region given by (6).

5. Conclusions

In this paper, we have established the exponential stability of non-critical traveling waves for a scalar age-structured equation with nonlocal dispersal. The main result, Theorem 1, asserts that for every admissible weight exponent η and wave speed c satisfying the condition $c\eta/2 > (\hat{J}(\eta) - 1)/2 + \Lambda$, every initial perturbation $u_0 - \Phi \in L^1((0, A); H_\omega^1(\mathbb{R}))$ generates a unique global solution whose age integral $\mathcal{I}[u - \Phi]$ decays in time at any rate $\sigma < \min\{\sigma_1, \sigma_2\}$, with σ_1 and σ_2 given explicitly by (8).

The proof combines a comparison argument with a weighted-energy framework tailored to the nonlocal operator $\mathcal{L} = J * -\text{id}$. The standard pointwise maximum principle is unavailable in this setting, and a self-contained comparison principle is therefore established from first principles in Lemma 2 through an exponential rescaling combined with a characteristic argument. The two-sided envelope ordering provided by this comparison principle is then propagated by a weighted-energy estimate, which yields the rate σ_1 from the spatial-decay structure, and by an age-asymptotic argument at $z = +\infty$, which yields the complementary rate σ_2 from the renewal mechanism. The effective bound $\min\{\sigma_1, \sigma_2\}$ thus reflects the joint contribution of nonlocal dispersal and age structure to the stability rate.

Numerical experiments illustrate these conclusions: the two-bump configuration (34) evolves toward the endemic state with monotonising profiles, the age slices share the same spatial shape (in agreement with the τ -uniformity of $\Phi(\tau, z)$), and the parameter-space analysis identifies the regimes in which the decay bound is governed by σ_1 or σ_2 . Possible extensions of the framework developed here include multi-group epidemic models, kernels with heterogeneous coefficients, and stability of critical waves at the threshold $c = c^*$, which we leave to future work.

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