

Article

Coefficient Estimates for Bi-Univalent Functions Associated with a Third-Order Logarithmic-Type Operator

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Abstract

In this paper, we introduce a new class of bi-univalent functions defined by a third-order logarithmic-type differential operator. By using the subordination principle and Carathéodory functions, we investigate the coefficient estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Furthermore, we derive the Fekete–Szegő inequality and obtain bounds for the second Hankel determinant $H_2(2)$ associated with this class. Several consequences of the main results are also discussed.

Keywords: bi-univalent functions; subordination; third-order logarithmic operator; coefficient estimates; Fekete–Szegő problem

MSC: 30C45

1. Introduction

Let $\mathcal{A}$ denote the family of analytic functions defined in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

normalized by the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

A function belonging to $\mathcal{A}$ is called univalent if it is injective in $\mathbb{D}$. The geometric theory of such functions plays a fundamental role in complex analysis, particularly in connection with coefficient problems and geometric properties of analytic mappings.

The concept of bi-univalent functions, in which both a function and its inverse are univalent in $\mathbb{D}$, was first systematically investigated by Lewin [1], who initiated the study of coefficient bounds for this class. His work opened a new direction in geometric function theory concerned with inverse coefficients and their estimates.

Building on this foundation, Brannan and Taha [2] introduced structured subclasses of bi-univalent functions and developed early results on coefficient estimates for both functions and their inverses. Their approach established a framework that has since been widely adopted in subsequent research on the class Σ of bi-univalent functions.



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Further developments were presented by Srivastava et al. [3], who employed convolution techniques and operator-based methods to construct and analyze new subclasses of analytic and bi-univalent functions. Their results demonstrated the effectiveness of unified operator approaches in deriving coefficient bounds under various geometric constraints.

In a more recent contribution, Ma [4] applied subordination principles to obtain improved coefficient estimates for initial Taylor coefficients of bi-univalent functions. This approach highlighted the usefulness of differential subordination techniques in refining earlier results.

During the last decade, significant attention was given to the construction of new subclasses of bi-univalent functions using linear and nonlinear operators. These operators have proven to be effective tools for generating generalized function classes and deriving associated coefficient inequalities.

In this direction, some authors [5–8] introduced several operator-based subclasses of bi-univalent functions associated with special polynomials and convolution structures. Their investigations provided a variety of coefficient estimates and structural properties that extended classical results in a unified manner. In 2024, Krushkal [9] discussed a variational technique for biunivalent functions, which provides a powerful tool for solving the general extremal problems on the classes of biunivalent holomorphic functions, while in 2025 Krushkal [10] considered the connection between biunivalence and the geometry of Teichmüller balls and provided some sufficient conditions for biunivalence of holomorphic functions on the disk. Murugusundaramoorthy et al. [11] presented and examined a family of bi-starlike functions with respect to symmetric conjugate points associated with a Lucas balancing polynomial. More recently, higher-order differential operators have been used to further generalize existing frameworks. For example, Dash et al. [12] studied a new class of bi-univalent functions defined via generalized Sălăgean-type operators and obtained coefficient bounds using subordination techniques together with functions that have a positive real part.

Similarly, Wanas et al. [13] investigated coefficient estimates and symmetry properties for families of bi-univalent functions associated with polynomial structures, illustrating the continued relevance of operator-driven approaches in modern geometric function theory.

Motivated by these developments, we introduce in this paper a new third-order logarithmic-type differential operator acting on analytic functions in the unit disk. This operator extends earlier logarithmic and Sălăgean-type constructions and provides a flexible framework for studying coefficient problems in the class Σ .

The main purpose of this work is to investigate coefficient bounds for the initial coefficients $|a_2|$ and $|a_3|$, establish a Fekete–Szegő type inequality, and derive estimates for the second Hankel determinant associated with the newly defined subclass.

The proposed operator $\mathcal{L}_{\alpha,\beta,\gamma}$ allows finer control over the geometric behavior of analytic mappings through additional higher-order terms, which enhances its ability to capture structural properties of bi-univalent functions in a more general setting.

Relation with Classical Operators

The third-order logarithmic-type operator introduced in this paper serves as a unifying framework that encompasses several well-known operators studied in geometric function theory.

Specifically, when $\gamma = 0$, the operator reduces to

$$\mathcal{L}_{\alpha,\beta}(f)(z) = \frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha} \frac{zf''(z)}{f'(z)}, \quad \alpha \neq 0.$$

which coincides with previously investigated logarithmic-type operators associated with starlike and convex functions.

Moreover, if $\beta = \gamma = 0$, then

$$\mathcal{L}_{\alpha,0,0}(f)(z) = \frac{zf'(z)}{f(z)},$$

one recovers the classical starlike function criterion

$$\frac{zf'(z)}{f(z)}.$$

The increasing use of generalized differential operators and subordination methods has led to the introduction of many new subclasses of bi-univalent functions and has provided a unified framework for obtaining coefficient estimates and related geometric properties [12]. These developments further motivate the introduction of higher-order operators capable of extending and unifying existing results.

2. Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

defined in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

A function $f \in \mathcal{A}$ is said to be univalent in $\mathbb{D}$ if it is one-to-one. The class of all univalent functions is denoted by $\mathcal{S}$.

If $f \in \mathcal{S}$, then f admits an inverse function $g = f^{-1}$ defined in a subdisk of $\mathbb{D}$ with the expansion

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \quad (2)$$

A function $f \in \mathcal{A}$ is called bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. The class of all such functions is denoted by Σ .

We recall that a function p belongs to the Carathéodory class $\mathcal{P}$ if

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \quad \Re(p(z)) > 0 \quad (z \in \mathbb{D}),$$

and it satisfies

$$|p_n| \leq 2, \quad n \geq 1.$$

We assume that the parameters $\alpha, \beta, \gamma \in \mathbb{R}$ are chosen such that $\alpha \neq 0$, ensuring that the coefficients $\frac{\beta}{\alpha}$ and $\frac{\gamma}{\alpha}$ are well-defined.

Moreover, since the operator involves logarithmic derivatives, we assume that f is locally univalent in $\mathbb{D}$, which guarantees that $f'(z) \neq 0$ in the domain of definition.

This restriction avoids singular behavior of the operator, particularly in limiting cases such as $\alpha \rightarrow 0$, where the operator becomes ill-defined.

Remark 1. The series expansions of the components of the operator $\mathcal{L}_{\alpha,\beta,\gamma}(f)$ are obtained by direct differentiation of the normalized analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and standard series division formulas. These expansions are valid in $\mathbb{D}$ due to local univalence of f .

3. Definition of the Operator and the Function Class

Definition 1. A function $f \in \Sigma$ belongs to the class $\mathcal{B}_\Sigma(\alpha, \beta, \gamma)$ if it satisfies the subordination conditions:

$$\frac{1}{\alpha} \mathcal{L}_{\alpha, \beta, \gamma}(f)(z) \prec p(z) \quad \text{and} \quad \frac{1}{\alpha} \mathcal{L}_{\alpha, \beta, \gamma}(g)(w) \prec q(w),$$

where $p, q \in \mathcal{P}$ (the Carathéodory class), $g = f^{-1}$, and the operator is defined by:

$$\mathcal{L}_{\alpha, \beta, \gamma}(f)(z) = \frac{zf'(z)}{f(z)} + \frac{\beta}{\alpha} \frac{zf''(z)}{f'(z)} + \frac{\gamma}{\alpha} z^2 \left(\frac{f''(z)}{f'(z)} \right)'.$$

Remark 2. The operator $\mathcal{L}_{\alpha, \beta, \gamma}$ generalizes several well-known logarithmic-type operators. In particular:

- If $\gamma = 0$, then it reduces to a second-order logarithmic operator.
- If $\beta = \gamma = 0$, then it reduces to the classical starlike function operator

$$\frac{zf'(z)}{f(z)}.$$

4. Series Expansion of the Operator

Let

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots$$

Then we have:

1. Expansion of $\frac{zf'(z)}{f(z)}$

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2 a_3 + a_2^3)z^3 + \dots$$

2. Expansion of $\frac{zf''(z)}{f'(z)}$

$$\frac{zf''(z)}{f'(z)} = 2a_2 z + (6a_3 - 4a_2^2)z^2 + (12a_4 - 18a_2 a_3 + 8a_2^3)z^3 + \dots$$

3. Expansion of $z^2 \left(\frac{f''}{f'} \right)'$

$$z^2 \left(\frac{f''(z)}{f'(z)} \right)' = (6a_3 - 4a_2^2)z^2 + 2(12a_4 - 18a_2 a_3 + 8a_2^3)z^3 + \dots$$

4. Combined Operator
Substituting into

$$\mathcal{L}_{\alpha, \beta, \gamma}(f)(z) = \frac{zf'}{f} + \frac{\beta}{\alpha} \frac{zf''}{f'} + \frac{\gamma}{\alpha} z^2 \left(\frac{f''}{f'} \right)', \quad (3)$$

we obtain:

$$\mathcal{L}_{\alpha, \beta, \gamma}(f)(z) = 1 + A_1 a_2 z + (A_2 a_3 + A_3 a_2^2)z^2 + (A_4 a_4 + A_5 a_2 a_3 + A_6 a_2^3)z^3 + \dots$$

where

$$\begin{aligned}
A_1 &= 1 + \frac{2\beta}{\alpha}, \\
A_2 &= 2 + \frac{6\beta}{\alpha} + \frac{6\gamma}{\alpha}, \\
A_3 &= -\left(1 + \frac{4\beta}{\alpha} + \frac{4\gamma}{\alpha}\right), \\
A_4 &= 3 + \frac{12\beta}{\alpha} + \frac{24\gamma}{\alpha}, \\
A_5 &= -3 - \frac{18\beta}{\alpha} - \frac{36\gamma}{\alpha}, \\
A_6 &= 1 + \frac{8\beta}{\alpha} + \frac{16\gamma}{\alpha}.
\end{aligned}$$

Correct Expansion of the Inverse Operator

Let

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots$$

Then we obtain the following expansions:

$$\frac{wg'(w)}{g(w)} = 1 - a_2 w + (3a_2^2 - 2a_3)w^2 + (-10a_2^3 + 12a_2 a_3 - 3a_4)w^3 + \dots, \quad (4)$$

$$\frac{wg''(w)}{g'(w)} = -2a_2 w + (8a_2^2 - 6a_3)w^2 + (-32a_2^3 + 42a_2 a_3 - 12a_4)w^3 + \dots, \quad (5)$$

$$w^2 \left(\frac{g''(w)}{g'(w)} \right)' = (8a_2^2 - 6a_3)w^2 + (-64a_2^3 + 84a_2 a_3 - 24a_4)w^3 + \dots \quad (6)$$

These expansions follow from the standard inverse function theory (see, for example, Srivastava et al. [3]).

5. Coefficient System Derived from the Operator Expansion

Since

$$\Re(\mathcal{L}_{\alpha, \beta, \gamma}(f)(z)) > 0$$

and

$$\Re(\mathcal{L}_{\alpha, \beta, \gamma}(g)(w)) > 0,$$

there exist functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \dots$$

which belong to the Carathéodory class $\mathcal{P}$ such that

$$\mathcal{L}_{\alpha, \beta, \gamma}(f)(z) = p(z)$$

and

$$\mathcal{L}_{\alpha,\beta,\gamma}(g)(w) = q(w).$$

Lemma 1. The analytic expansion of the operator $\mathcal{L}_{\alpha,\beta,\gamma}(f)(z)$ can be written in the form

$$\mathcal{L}_{\alpha,\beta,\gamma}(f)(z) = 1 + A_1 a_2 z + (A_2 a_3 + A_3 a_2^2) z^2 + (A_4 a_4 + A_5 a_2 a_3 + A_6 a_2^3) z^3 + \dots,$$

where the coefficients A_i are defined by

$$\begin{aligned} A_1 &= 1 + \frac{2\beta}{\alpha}, \\ A_2 &= 2 + \frac{6\beta}{\alpha} + \frac{6\gamma}{\alpha}, \\ A_3 &= -\left(1 + \frac{4\beta}{\alpha} + \frac{4\gamma}{\alpha}\right), \\ A_4 &= 3 + \frac{12\beta}{\alpha} + \frac{24\gamma}{\alpha}, \\ A_5 &= -3 - \frac{18\beta}{\alpha} - \frac{36\gamma}{\alpha}, \\ A_6 &= 1 + \frac{8\beta}{\alpha} + \frac{16\gamma}{\alpha}. \end{aligned}$$

$$A_1 a_2 = p_1 \quad (7)$$

$$A_2 a_3 + A_3 a_2^2 = p_2 \quad (8)$$

$$-A_1 a_2 = q_1 \quad (9)$$

$$2A_2 a_2^2 - A_2 a_3 + A_3 a_2^2 = q_2 \quad (10)$$

$$A_4 a_4 + A_5 a_2 a_3 + A_6 a_2^3 = p_3. \quad (11)$$

From (7) and (9):

$$p_1 = -q_1 \Rightarrow p_1^2 = q_1^2.$$

This shows consistency between direct and inverse subordination.

6. Main Results and Proofs

Theorem 1. Let the function $f(z)$ defined by (1) belong to the class $\mathcal{B}_\Sigma(\alpha, \beta, \gamma)$. Let the parameters α, β, γ satisfy the conditions $|A_1| > 0$ and $|A_2 + A_3| \neq 0$. Then:

$$|a_2| \leq \min \left\{ \frac{2}{|A_1|}, \sqrt{\frac{2}{|A_2 + A_3|}} \right\}.$$

Proof. By the definition of the class $\mathcal{B}_\Sigma(\alpha, \beta, \gamma)$, we have:

$$\mathcal{L}_{\alpha,\beta,\gamma}(f)(z) = p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

$$\mathcal{L}_{\alpha,\beta,\gamma}(g)(w) = q(w) = 1 + q_1 w + q_2 w^2 + \dots$$

where $p, q \in \mathcal{P}$ (the Carathéodory class), which implies $|p_n| \leq 2$ and $|q_n| \leq 2$ for all $n \geq 1$.

Step 1: From (7)

$$a_2 = \frac{p_1}{A_1}.$$

Hence:

$$|a_2| \leq \frac{|p_1|}{|A_1|} \leq \frac{2}{|A_1|}.$$

Step 2: Add Equations (8) and (10):

$$(A_2a_3 + A_3a_2^2) + (2A_2a_2^2 - A_2a_3 + A_3a_2^2) = p_2 + q_2.$$

Simplify:

$$2(A_2 + A_3)a_2^2 = p_2 + q_2.$$

Take modulus:

$$2|A_2 + A_3||a_2|^2 \leq |p_2| + |q_2| \leq 4.$$

Thus:

$$|a_2|^2 \leq \frac{2}{|A_2 + A_3|}.$$

Combine both estimates. $\square$

7. Coefficient a_3

Theorem 2. Let the function $f(z)$ belong to the class $\mathcal{B}_\Sigma(\alpha, \beta, \gamma)$. Suppose the parameters $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy $|A_1| > 0$ and $|A_2 + A_3| \neq 0$. Then:

$$|a_3| \leq \frac{2}{|A_2|} + \frac{4|A_3|}{|A_2||A_1|^2}.$$

Proof. From (8):

$$a_3 = \frac{p_2 - A_3a_2^2}{A_2}.$$

Substitute $a_2 = p_1/A_1$ from (7):

$$a_3 = \frac{p_2}{A_2} - \frac{A_3}{A_2} \left(\frac{p_1}{A_1} \right)^2.$$

Taking the modulus and using the triangle inequality:

$$|a_3| \leq \frac{|p_2|}{|A_2|} + \frac{|A_3|}{|A_2||A_1|^2} |p_1|^2.$$

Applying the bounds $|p_1| \leq 2$ and $|p_2| \leq 2$:

$$|a_3| \leq \frac{2}{|A_2|} + \frac{4|A_3|}{|A_2||A_1|^2}.$$

This establishes the result. $\square$

8. Fekete–Szegő Inequality

Theorem 3. Let $f \in \mathcal{B}_\Sigma(\alpha, \beta, \gamma)$ be of the form (1). If $\mu \in \mathbb{R}$, then the Fekete–Szegő functional satisfies the following inequality:

$$|a_3 - \mu a_2^2| \leq \frac{2}{|A_2|} + \frac{4|A_3 + \mu A_2|}{|A_2||A_1|^2}.$$

Proof. From (8), we have

$$a_3 = \frac{p_2 - A_3a_2^2}{A_2}.$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{p_2}{A_2} - \frac{A_3 + \mu A_2}{A_2} a_2^2.$$

Using (7),

$$a_2 = \frac{p_1}{A_1},$$

and hence

$$a_2^2 = \frac{p_1^2}{A_1^2}.$$

Substituting this into the above expression gives

$$a_3 - \mu a_2^2 = \frac{p_2}{A_2} - \frac{A_3 + \mu A_2}{A_2 A_1^2} p_1^2.$$

Applying the triangle inequality, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{|p_2|}{|A_2|} + \frac{|A_3 + \mu A_2|}{|A_2||A_1|^2} |p_1|^2.$$

Since $p \in \mathcal{P}$, the Carathéodory lemma yields

$$|p_1| \leq 2, \quad |p_2| \leq 2.$$

Therefore,

$$|a_3 - \mu a_2^2| \leq \frac{2}{|A_2|} + \frac{4|A_3 + \mu A_2|}{|A_2||A_1|^2}.$$

This completes the proof.

□

9. Second Hankel Determinant

Theorem 4. Let the function $f(z)$ belong to the class $\mathcal{B}_\Sigma(\alpha, \beta, \gamma)$. Assume that

$$A_1 A_2 A_4 \neq 0.$$

Then the second Hankel determinant

$$H_2(2) = a_2 a_4 - a_3^2$$

satisfies

$$|H_2(2)| \leq \frac{2}{|A_1 A_4|} \left(2 + |A_5||a_2||a_3| + |A_6||a_2|^3 \right) + \left(\frac{2 + |A_3||a_2|^2}{|A_2|} \right)^2.$$

Proof. The second Hankel determinant is defined by

$$H_2(2) = a_2 a_4 - a_3^2.$$

From (7), (8), and (11), we have

$$a_2 = \frac{p_1}{A_1},$$

$$a_3 = \frac{p_2 - A_3 a_2^2}{A_2},$$

and

$$a_4 = \frac{p_3 - A_5 a_2 a_3 - A_6 a_2^3}{A_4}.$$

Substituting these expressions into the definition of $H_2(2)$, we obtain

$$H_2(2) = a_2 \left(\frac{p_3 - A_5 a_2 a_3 - A_6 a_2^3}{A_4} \right) - \left(\frac{p_2 - A_3 a_2^2}{A_2} \right)^2.$$

Taking absolute values and applying the triangle inequality,

$$|H_2(2)| \leq |a_2| \left| \frac{p_3 - A_5 a_2 a_3 - A_6 a_2^3}{A_4} \right| + \left| \frac{p_2 - A_3 a_2^2}{A_2} \right|^2.$$

Therefore,

$$|H_2(2)| \leq \frac{|a_2|}{|A_4|} (|p_3| + |A_5||a_2||a_3| + |A_6||a_2|^3) + \frac{(|p_2| + |A_3||a_2|^2)^2}{|A_2|^2}.$$

Since $p \in \mathcal{P}$, the Carathéodory lemma implies

$$|p_n| \leq 2, \quad n \geq 1.$$

Hence,

$$|H_2(2)| \leq \frac{|a_2|}{|A_4|} (2 + |A_5||a_2||a_3| + |A_6||a_2|^3) + \left(\frac{2 + |A_3||a_2|^2}{|A_2|} \right)^2.$$

Using

$$|a_2| = \left| \frac{p_1}{A_1} \right| \leq \frac{2}{|A_1|},$$

we finally obtain

$$|H_2(2)| \leq \frac{2}{|A_1 A_4|} (2 + |A_5||a_2||a_3| + |A_6||a_2|^3) + \left(\frac{2 + |A_3||a_2|^2}{|A_2|} \right)^2.$$

This completes the proof. $\square$

10. Examples and Verification

Example 1. Let us consider the function

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \dots,$$

which is a well-known analytic and starlike function in $\mathbb{D}$.

For the special choice of parameters

$$\alpha = 1, \quad \beta = 0, \quad \gamma = 0,$$

our operator reduces to the classical logarithmic starlike operator

$$\mathcal{L}_{1,0,0}(f)(z) = \frac{zf'(z)}{f(z)}.$$

In this case, we obtain

$$A_1 = 1, \quad A_2 = 2, \quad A_3 = -1.$$

Hence, Theorem 1 yields

$$|a_2| \leq \min\{2, \sqrt{2}\} = \sqrt{2}.$$

Since for this function $a_2 = 1$, the estimate is satisfied.

This confirms that the obtained bounds are consistent with classical starlike-type functions and demonstrate the validity of the proposed operator framework in the limiting case.

11. Discussion and Future Work

11.1. Discussion

The present work introduces a third-order logarithmic-type differential operator $\mathcal{L}_{\alpha,\beta,\gamma}(f)(z)$ and applies it to construct a new subclass of bi-univalent functions. The main idea is to extend classical first and second-order logarithmic operators by incorporating a higher-order term

$$z^2 \left(\frac{f''(z)}{f'(z)} \right)',$$

which allows for finer control over the geometric behavior of analytic functions in the unit disk.

The obtained coefficient estimates for a_2 and a_3 , as well as the Fekete–Szegő inequality and Hankel determinant bounds, show that the proposed framework generalizes several known results in the literature. In particular, by suitable choices of parameters α , β , and γ , the operator reduces to previously studied differential operators associated with starlike and convex functions.

The use of Carathéodory functions and subordination principles provides a systematic and unified approach for deriving coefficient bounds for both the function f and its inverse g . This ensures consistency between direct and inverse analytic structures, which is essential in the theory of bi-univalent functions.

However, it is worth noting that the sharpness of the obtained bounds is not addressed in this work, and the results are mainly of a non-sharp nature, as is common in the study of bi-univalent function classes.

11.2. Future Work

The present study opens several directions for further research:

- **Sharp bounds:** Determining sharp coefficient estimates for a_2 , a_3 , and higher-order coefficients within the proposed class.
- **Higher-order coefficients:** Extending the analysis to coefficients a_n for $n \geq 4$, and studying their dependence on the operator parameters.
- **Other geometric subclasses:** Investigating starlike, convex, close-to-convex, and strongly bi-univalent subclasses associated with the proposed operator.
- **Alternative subordination frameworks:** Replacing Carathéodory functions with other dominant functions such as Janowski or Sakaguchi-type functions.
- **Hankel determinants:** Generalizing the study to higher-order Hankel determinants $H_q(n)$ for $q \geq 2$.
- **Special functions and polynomials:** Exploring connections of the operator with orthogonal polynomials such as Chebyshev, Gegenbauer, or Horadam polynomials.
- **q-analogue and fractional extensions:** Developing q-calculus or fractional-order versions of the proposed operator.

These directions may lead to deeper insights into the structure of bi-univalent functions and further generalizations of the current theory.

12. Conclusions

In this paper, we introduced a new third-order logarithmic-type differential operator, which provides a unified framework for studying various subclasses of bi-univalent functions. By using the Carathéodory lemma and coefficient comparison techniques, we obtained explicit coefficient estimates for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Furthermore, we established a generalized Fekete–Szegő inequality and obtained an explicit bound for the second Hankel determinant $H_2(2)$. These results successfully generalize several existing classes in the literature, recovering classical results as special cases by choosing specific parameter values for α , β , and γ . Future research will focus on investigating higher-order Hankel determinants and applying these operators within the context of q -calculus and fractional-order function theory.

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