

Article

On Orbit Tangent Graphs for Lie Group Actions Through Hypergraph Incidence Structures and Separating Tangent Frameworks

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Abstract

This paper introduces a graphical framework for smooth Lie group actions based on tangent orbit interactions. In contrast with classical intersection graphs, where vertices usually represent algebraic subobjects and edges record set-theoretic intersections, the present construction uses non-trivial orbits as vertices and creates edges from common nonzero tangent directions inside the fixed ambient embedding. Starting from infinitesimal tangent spaces generated by the action, we construct Lie orbit tangent graphs and analyze their adjacency structure, connectedness, completeness, degrees and diameter estimates. To describe local and global interactions, tangent fibers, local tangent orbit cliques, tangent orbit hypergraphs and incidence structures are introduced. We further develop separating tangent paths and use them to construct neighborhood systems and tangent-separating topologies. The framework gives a unified way to encode orbit-level tangent interactions and may be useful in geometric analysis, symmetry-based dynamical systems, differential topology and mathematical physics, where orbits and infinitesimal directions describe invariant motions, constraints or symmetry-reduced configurations. Several examples are included to illustrate how Lie group actions, graph structures, hypergraphs and tangent geometry interact within the proposed scheme.

Keywords: Lie group actions; orbit tangent graphs; tangent fibers; hypergraph structures; incidence structures; separating tangent paths

MSC: 22E46; 05C25; 05C65; 57S25; 05C38

1. Introduction

Graph theory provides a mathematical framework for representing interactions among objects through vertices and edges. Owing to its flexibility, graph-theoretical methods have become fundamental tools for modeling complex relationships arising in various disciplines, including computer science, biology, engineering, social sciences, algebra and topology. In recent years, combining graph structures with algebraic and topological frameworks has become an effective approach for investigating structural, combinatorial and interaction-based properties.

Graph theory and algebra are closely related, since many graphs arise naturally from algebraic structures such as groups, rings and semigroups. One important direction in this interaction is the study of intersection graphs, where adjacency is determined by non-trivial intersections between algebraic objects. Chakrabarty et al. [1] extended this idea to ideals of



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rings. Zelinka [2] introduced intersection graphs for finite Abelian groups, where vertices correspond to subgroups and edges represent non-trivial intersections. Shen [3] studied intersection graphs of subgroups of finite groups. Akbari et al. [4] further investigated structural properties of group intersection graphs. Several studies have focused on graph-theoretical properties of intersection graphs. Zhao et al. [5] investigated connectivity and robustness in random intersection graphs. Kayacan [6] studied connectedness properties of intersection graphs of finite groups. Aproz and Fathima [7] examined intersection power graphs of finite groups. Beheshtipour and Jafarian Amiri [8,9] investigated clique structures and clique numbers associated with intersection graphs.

The concept of intersection graphs has also been extended beyond groups. Ramanathan [10] introduced projective intersection graphs associated with ideals of commutative rings. Moh'd and Ahmed [11] studied simple-intersection graphs of rings, while Ahmed and Moh'd [12] proposed new class of intersection graphs associated with modules.

More recently, intersection graph techniques have been extended to actions and semigroup acts. Rasouli and Tehranian [13] introduced intersection graphs of $\mathcal{S}$ -acts and investigated their basic properties. Delfan et al. [14] further developed that framework and obtained additional structural results. Khosravi and Roueentan [15] studied algebraic properties of $\mathcal{S}$ -acts through congruence structures. More recently, Alshammari et al. [16] introduced graphical homotopy constructions for monoid intersection graphs and demonstrated applications involving graphical monoids.

On other hand, graph theory has strong interactions with topology. Several studies have explored how graphs generate topological structures and how topological methods can be interpreted graphically. Tucker [17] investigated graph-theoretic methods in topology through covering-space constructions. Sari and Kopuzlu [18] studied topological spaces generated by simple graphs. Noll [19] investigated graph-theoretic viewpoints related to closed graph structures, while Branman et al. [20] developed graphical models associated with topological groups.

Recent developments have focused on graphical topological structures and their applications. Damag et al. [21] introduced out monodirected topologies and applied them to human nervous systems. Alzubaidi et al. [22] studied topologies generated from graphs and their applications. Damag et al. [23] proposed upper α -graphical topological spaces together with applications. More recently, Damag et al. [24] introduced monophonic sets and rough directed topological spaces and demonstrated their applicability to directed networks. Some contributions also show the usefulness of hypergraph and path-based graph methods in modern mathematical modelling. For example, Zhao et al. [25] investigated vulnerability measures for uniform hypergraph networks through scattering numbers, while Kalampakas [26] developed path-based hyperoperations on fuzzy graphs. These studies support the broader view that graph and hypergraph constructions can encode higher-order relations that are not visible in ordinary pairwise models.

The graph introduced in this work (Figure 1) differs from the above intersection-graph constructions in two essential ways. First, its vertices are not subgroups, ideals, modules or semigroup acts but non-trivial orbits of a smooth Lie group action. Second, adjacency is not determined by a direct set-theoretic intersection of the vertices themselves; it is determined by the existence of a common nonzero tangent direction between tangent spaces of two orbits after these tangent spaces are realized in the ambient Euclidean space. Thus, the proposed graph is an extrinsic tangent-interaction graph associated with infinitesimal orbit geometry, rather than a classical intersection graph of algebraic objects.

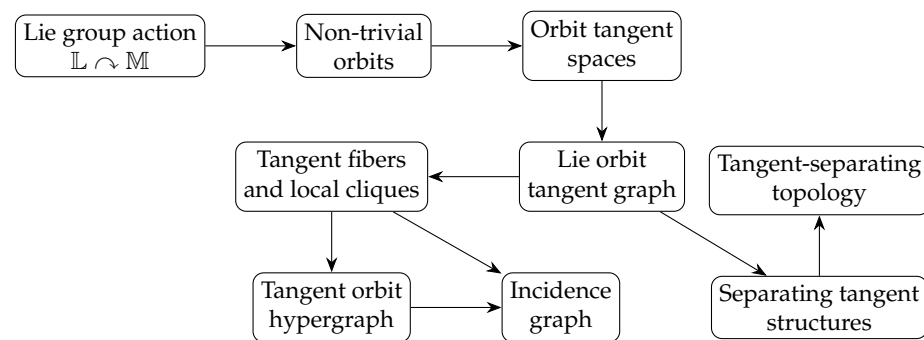


Figure 1. Schematic relationship among the main objects used in the paper. The Lie group action produces orbit tangent spaces; common tangent directions generate the Lie orbit tangent graph, tangent fibers and local cliques. These fibers produce hypergraph and incidence structures, while separating tangent directions generate tangent neighborhoods and the associated tangent separating topology.

This distinction also provides practical motivation. In geometry and differential topology, the construction can be used as a combinatorial summary of how orbit tangent spaces overlap across different orbit families. In dynamical systems and symmetry analysis, it identifies pairs or clusters of symmetry-generated motions that share infinitesimal directions. In mathematical physics, such tangent interactions may describe common infinitesimal modes in configuration spaces with symmetries, constrained systems or symmetry-reduced phase spaces. The associated hypergraph and incidence structures then record higher-order common tangent directions among several orbits, while separating tangent directions detect edges generated by directions unique to a given pair of orbit vertices.

The interaction between graph theory, Lie group actions and geometric structures has attracted considerable attention due to its ability to reveal hidden relations between algebraic, topological and dynamical systems. Although intersection graphs and graphical topologies associated with algebraic structures have been extensively studied, little attention has been devoted to constructing graph structures directly from tangent orbit interactions generated by Lie group actions. Moreover, relationships between tangent fibers, hypergraph structures, incidence methods and orbit connectivity remain largely unexplored. Motivated by these observations, this work develops new framework that combines graph-theoretical, geometric and hypergraph techniques for studying orbit interactions through tangent structures.

The main contributions of this work can be summarized as follows. We introduce Lie orbit tangent graphs generated from infinitesimal tangent interactions associated with smooth Lie group actions and investigate their fundamental structural properties. We define tangent fibers and local tangent orbit cliques, and we establish how they decompose the edge structure and control connectivity and domination. We construct tangent orbit hypergraphs and incidence graphs to record higher-order common tangent directions and to characterize global orbit interactions. We introduce separating tangent paths and tangent-separating neighborhood systems, leading to a topology generated by tangent interaction data. Finally, we prove connectedness results linking tangent-separating paths, hypergraph fiber chains, incidence relations and connectedness properties of the induced topological space.

Section 2 introduces Lie orbit tangent graphs together with their basic constructions, examples, and structural properties. Section 3 studies tangent orbit fibers, local tangent orbit cliques, and their effects on connectivity and domination. Section 4 develops hypergraph and incidence structures associated with $Lie_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ and investigates their graph-theoretical properties. Section 5 introduces tangent-separating structures and develops the tangent-separating topology together with its connectedness properties.

Throughout this work, a graph is written in the form $(\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$, where $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ represent the collections of vertices and edges, respectively. The cardinality of $\mathcal{V}(\mathcal{G})$ is called the order of $\mathcal{G}$ and is denoted by $\mathcal{O}(\mathcal{G})$. For a vertex $q \in \mathcal{V}(\mathcal{G})$, the quantity ∂_q denotes the number of neighboring vertices of q . A path in $\mathcal{G}$ refers to an ordered sequence of distinct vertices such that every two successive vertices are adjacent. The graph $\mathcal{G}$ is said to be connected whenever arbitrary vertices $p, q \in \mathcal{V}(\mathcal{G})$ can be linked through a path. For vertices p and q , the distance $\partial(p, q)$ is defined as the smallest length among all possible paths connecting them; if no connecting path exists, we set $\partial(p, q) = \infty$. The diameter of $\mathcal{G}$ is given by $\text{diam}(\mathcal{G}) = \sup\{\partial(p, q) : p, q \in \mathcal{V}(\mathcal{G})\}$.

A clique in $\mathcal{G}$ is a subgraph in which every two distinct vertices are adjacent. A path component of $\mathcal{G}$ refers to a connected subgraph that is maximal under set inclusion. Let $A \subseteq \mathcal{V}(\mathcal{G})$ be nonempty. The set A is called dominating whenever each vertex belonging to $\mathcal{V}(\mathcal{G}) \setminus A$ is adjacent to at least one element of A .

A hypergraph $\mathcal{H}$ is represented by the pair $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, where $\mathcal{V}(\mathcal{H})$ denotes the vertex collection, and $\mathcal{E}(\mathcal{H}) \subseteq \mathcal{P}(\mathcal{V}(\mathcal{H}))$ is a family of subsets called hyperedges. The associated two-section graph of $\mathcal{H}$ is constructed by connecting distinct vertices whenever they appear together in at least one hyperedge.

The fundamental concepts and results from topology that are required throughout this work can be found in standard references; see [27]. Recall [28] that a chart on a topological space $\mathbb{M}$ consists of a pair (U, φ) , where $U \subseteq \mathbb{M}$ is an open subset, and φ maps U homeomorphically onto a subset of $\mathbb{R}^n$. A smooth manifold of dimension n is a Hausdorff and second-countable topological space $\mathbb{M}$ equipped with an atlas $\{(U_i, \varphi_i)\}_{i \in I}$ satisfying the compatibility condition that whenever $U_i \cap U_j \neq \emptyset$, the coordinate change maps

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$$

are smooth mappings. Recall [28] that a Lie group is a smooth manifold $\mathbb{L}$ equipped with a group operation such that both multiplication and inversion are smooth mappings. More precisely, the multiplication map $\mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L}$ given by $(l_1, l_2) \mapsto l_1 l_2$ and the inversion map $\mathbb{L} \rightarrow \mathbb{L}$ defined by $l \mapsto l^{-1}$ are smooth.

2. Lie Orbit Tangent Graphs

This section introduces the construction of Lie orbit tangent graphs associated with smooth Lie group actions. The necessary geometric preliminaries are first presented, followed by proposed graph construction and illustrative examples. Several structural properties including connectedness, completeness, degree characterization and diameter estimates are then established.

A smooth action of a Lie group $\mathbb{L}$ on a smooth manifold $\mathbb{M}$ is a smooth map $\mathbb{L} \times \mathbb{M} \rightarrow \mathbb{M}$ defined by $(l, m) \mapsto l \cdot m$ such that $e \cdot m = m$ and $(l_1 l_2) \cdot m = l_1 \cdot (l_2 \cdot m)$ for all $l_1, l_2 \in \mathbb{L}$ and $m \in \mathbb{M}$.

For $m \in \mathbb{M}$, the orbit generated by m is given by $\mathbb{L} \cdot m = \{l \cdot m : l \in \mathbb{L}\}$. Let $\mathbb{M}$ be a smooth manifold and fix $m \in \mathbb{M}$. The tangent space associated with m , denoted by $T_m \mathbb{M}$, is defined by

$$T_m \mathbb{M} = \{\gamma'(0) : \gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{M}, \gamma(0) = m\}.$$

Let $e \in \mathbb{L}$ denote the identity element. The Lie algebra corresponding to $\mathbb{L}$, denoted by $\mathfrak{g}$, is defined by $\mathfrak{g} = T_e \mathbb{L}$. Assume that $\mathbb{L}$ is a Lie group with Lie algebra $\mathfrak{g}$. The exponential mapping is written as $\exp : \mathfrak{g} \rightarrow \mathbb{L}$. For each $X \in \mathfrak{g}$, the curve $t \mapsto \exp(tX)$ generates a one-parameter subgroup. For each $X \in \mathfrak{g}$, define the associated fundamental vector field by

$$X_{\mathbb{M}}(m) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX) \cdot m).$$

Given $m \in \mathbb{M}$, denote by $T_m(\mathbb{L} \cdot m)$ the tangent space of the orbit $\mathbb{L} \cdot m$ at the point m . The orbit tangent space admits the following representation:

$$T_m(\mathbb{L} \cdot m) = \{X_{\mathbb{M}}(m) : X \in \mathfrak{g}\}.$$

An orbit $\mathbb{L} \cdot m$ is called non-trivial if $T_x(\mathbb{L} \cdot m) \neq \{0\}$ for every $x \in \mathbb{L} \cdot m$. Since $\mathbb{M} \subseteq \mathbb{R}^N$ is an embedded manifold, every tangent space $T_x(\mathbb{M})$ is canonically identified with a linear subspace of $\mathbb{R}^N$. Therefore, tangent spaces associated with different points are compared through their ambient realization inside $\mathbb{R}^N$.

Definition 1. Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$. We define the Lie orbit tangent graph $Lie_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ as the graph whose vertex set consists of all non-trivial orbits arising from the action. Two distinct orbit vertices $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ are connected by an edge whenever there are points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ satisfying

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\},$$

where the corresponding tangent spaces are viewed as linear subspaces of $\mathbb{R}^N$.

The construction is also meaningful when the action has infinitely many non-trivial orbits. In that case, $Lie_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ is understood as a possibly infinite simple graph whose vertices are all non-trivial orbits and whose edges are determined by the same tangent-intersection condition. Results based only on finite paths, such as the existence of connecting paths or fiber chains, continue to have the same interpretation. Results involving degrees, domination numbers, diameters or incidence counts require the corresponding local finiteness or finite-incidence assumptions, or else they must be read in terms of cardinal invariants. Throughout the remainder of this paper, unless otherwise stated, we work under the standing assumption that $Lie_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ has finitely many vertices and that the associated tangent orbit hypergraphs and incidence graphs have finitely many incidences.

It is important to emphasize that the present definition is extrinsic: the embedding $\mathbb{M} \subseteq \mathbb{R}^N$ is part of the data. Tangent spaces at different points are compared after being realized as linear subspaces of the same ambient Euclidean space. Therefore, two different embeddings may in principle produce different adjacency relations unless they preserve the relevant tangent intersections, for example under an ambient linear isometry or under an embedding equivalence that carries common tangent directions to common tangent directions. An intrinsic variant would require an additional comparison device, such as a connection, parallel transport or a chosen trivialization; this lies beyond the present paper.

Example 1. Consider the Lie group $\mathbb{L} = (\mathbb{R}, +)$ acting smoothly on the embedded manifold $\mathbb{M} = \mathbb{R}^2$ by horizontal translations, that is, $l \cdot (x, y) = (x + l, y)$ for all $l \in \mathbb{L}$ and $(x, y) \in \mathbb{M}$. For each $m = (x, y) \in \mathbb{M}$, the orbit of m is $\mathbb{L} \cdot m = \{(x + l, y) : l \in \mathbb{R}\} = \mathbb{R} \times \{y\}$. Thus, the non-trivial orbits are the horizontal lines $\mathbb{L} \cdot m_y = \mathbb{R} \times \{y\}$, where $m_y = (0, y)$ and $y \in \mathbb{R}$. The Lie algebra of $\mathbb{L}$ is $\mathfrak{g} = \mathbb{R}$. For $X \in \mathfrak{g}$, the fundamental vector field is given by

$$X_{\mathbb{M}}(x, y) = \left. \frac{d}{dt} \right|_{t=0} ((tX) \cdot (x, y)) = \left. \frac{d}{dt} \right|_{t=0} (x + tX, y) = (X, 0).$$

Hence, for every $m = (x, y) \in \mathbb{M}$, we have $T_m(\mathbb{L} \cdot m) = \text{span}\{(1, 0)\}$. Therefore, for any two distinct orbits $\mathbb{L} \cdot m_y$ and $\mathbb{L} \cdot m_z$ with $y \neq z$, choose points $x \in \mathbb{L} \cdot m_y$ and $w \in \mathbb{L} \cdot m_z$. Since

$$T_x(\mathbb{L} \cdot m_y) = \text{span}\{(1, 0)\} = T_w(\mathbb{L} \cdot m_z),$$

we obtain

$$T_x(\mathbb{L} \cdot m_y) \cap T_w(\mathbb{L} \cdot m_z) = \text{span}\{(1, 0)\} \neq \{0\}.$$

Consequently, every two distinct vertices are adjacent and hence the Lie orbit tangent graph $\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ is a complete graph.

Example 2. Let $\mathbb{L} = (\mathbb{R}^3, +)$ be a Lie group and let $\{e_1, e_2, \dots, e_{12}\}$ be the standard basis of $\mathbb{R}^{12}$. Define the following three-dimensional subspaces:

$$\begin{aligned} V_1 &= \text{span}\{e_1, e_2, e_3\}, & V_2 &= \text{span}\{e_1, e_4, e_5\}, & V_3 &= \text{span}\{e_2, e_4, e_6\}, \\ V_4 &= \text{span}\{e_3, e_6, e_7\}, & V_5 &= \text{span}\{e_5, e_8, e_9\}, & V_6 &= \text{span}\{e_7, e_8, e_{10}\}, \\ V_7 &= \text{span}\{e_9, e_{10}, e_{11}\}. \end{aligned}$$

For each $i = 1, 2, \dots, 7$, define $c_i = ie_{12}$ and $\mathbb{M}_i = c_i + V_i$. Let

$$\mathbb{M} = \bigcup_{i=1}^7 \mathbb{M}_i.$$

Define a smooth action of $\mathbb{L}$ on $\mathbb{M}$ as follows. If $l = (s_1, s_2, s_3) \in \mathbb{L}$ and $m = c_i + v \in \mathbb{M}_i$, define $l \cdot m = c_i + v + A_i(l)$, where $A_i: \mathbb{R}^3 \rightarrow V_i$ is the linear isomorphism mapping the standard basis vectors of $\mathbb{R}^3$ onto the generators of V_i . Hence, each $\mathbb{M}_i$ is one orbit, that is, $\mathbb{L} \cdot m_i = \mathbb{M}_i$ for every $m_i \in \mathbb{M}_i$. Since $\mathfrak{g} = \mathbb{R}^3$, for every $X \in \mathfrak{g}$, we obtain

$$X_{\mathbb{M}}(m) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX) \cdot m) = \left. \frac{d}{dt} \right|_{t=0} (c_i + v + A_i(tX)) = A_i(X).$$

Therefore, for every point $x \in \mathbb{M}_i$, we have $T_x(\mathbb{L} \cdot x) = V_i$. In particular, $T_x(\mathbb{L} \cdot x) = V_i$ for every $x \in \mathbb{L} \cdot m_i$ and $i = 1, 2, \dots, 7$. Hence,

$$\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})] = \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_2, \mathbb{L} \cdot m_3, \mathbb{L} \cdot m_4, \mathbb{L} \cdot m_5, \mathbb{L} \cdot m_6, \mathbb{L} \cdot m_7\}.$$

Moreover, let $x_i \in \mathbb{L} \cdot m_i$ and $x_j \in \mathbb{L} \cdot m_j$. Since $T_{x_i}(\mathbb{L} \cdot m_i) = V_i$ and $T_{x_j}(\mathbb{L} \cdot m_j) = V_j$, we obtain

$$\begin{aligned} V_1 \cap V_2 &= \text{span}\{e_1\}, & V_1 \cap V_3 &= \text{span}\{e_2\}, & V_1 \cap V_4 &= \text{span}\{e_3\}, & V_2 \cap V_3 &= \text{span}\{e_4\}, \\ V_2 \cap V_5 &= \text{span}\{e_5\}, & V_3 \cap V_4 &= \text{span}\{e_6\}, & V_4 \cap V_6 &= \text{span}\{e_7\}, & V_5 \cap V_6 &= \text{span}\{e_8\}, \\ V_5 \cap V_7 &= \text{span}\{e_9\}, & V_6 \cap V_7 &= \text{span}\{e_{10}\}. \end{aligned}$$

All remaining pairwise intersections are equal to $\{0\}$, and hence adjacency follows; see the Lie orbit tangent graph $\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ in Figure 2.

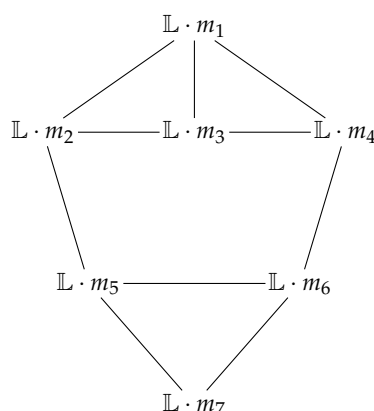


Figure 2. The Lie orbit tangent graph $\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ in Example 2.

Theorem 1. Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$. For each $x \in \mathbb{M}$, define the infinitesimal action map $\rho_x : \mathfrak{g} \rightarrow \mathbb{R}^N$ by $\rho_x(X) = X_{\mathbb{M}}(x)$. If, for any two distinct non-trivial orbits $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$, and for all points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$, we have $\rho_x(\mathfrak{g}) \cap \rho_y(\mathfrak{g}) = \{0\}$, then $\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ is edgeless.

Proof. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be two distinct non-trivial orbits. Let $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$. For each $X \in \mathfrak{g}$, the smooth curve $\gamma_x^X(t) = \exp(tX) \cdot x$ lies in the orbit $\mathbb{L} \cdot m$. Hence, its velocity vector at $t = 0$ is

$$(\gamma_x^X)'(0) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX) \cdot x) = X_{\mathbb{M}}(x) = \rho_x(X).$$

Thus, $\rho_x(\mathfrak{g}) \subseteq T_x(\mathbb{L} \cdot m)$. Conversely, every tangent vector to the orbit $\mathbb{L} \cdot m$ at x is generated by some one-parameter subgroup of $\mathbb{L}$ and therefore, $T_x(\mathbb{L} \cdot m) = \rho_x(\mathfrak{g})$. Similarly, $T_y(\mathbb{L} \cdot n) = \rho_y(\mathfrak{g})$. By the hypothesis, $\rho_x(\mathfrak{g}) \cap \rho_y(\mathfrak{g}) = \{0\}$. Hence,

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) = \{0\}.$$

Since this holds for all $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$, there are no points in the two orbits whose tangent spaces have a nonzero intersection. Therefore, $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ are not adjacent. Since this holds for every two distinct non-trivial orbits, $\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ is edgeless. $\square$

Theorem 2. Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$ and let

$$\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})] = \{\mathbb{L} \cdot m_i : i = 1, 2, \dots, n\}.$$

For each $x \in \mathbb{M}$, define the infinitesimal action map $\rho_x : \mathfrak{g} \rightarrow \mathbb{R}^N$ by $\rho_x(X) = X_{\mathbb{M}}(x)$. Then,

$$\deg(\mathbb{L} \cdot m_i) = |\{j \in \{1, 2, \dots, n\} \setminus \{i\} : \exists x \in \mathbb{L} \cdot m_i, \exists y \in \mathbb{L} \cdot m_j \text{ such that } \rho_x(\mathfrak{g}) \cap \rho_y(\mathfrak{g}) \neq \{0\}\}|.$$

In particular, if for every $i \neq j$, there exist $x \in \mathbb{L} \cdot m_i$ and $y \in \mathbb{L} \cdot m_j$ such that $\rho_x(\mathfrak{g}) \cap \rho_y(\mathfrak{g}) \neq \{0\}$, then $\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ is complete, and $\deg(\mathbb{L} \cdot m_i) = n - 1$ for every $i = 1, 2, \dots, n$.

Proof. Fix $i \in \{1, 2, \dots, n\}$. Let $j \neq i$. For each $X \in \mathfrak{g}$, the smooth curve $\gamma_x^X(t) = \exp(tX) \cdot x$ lies in the orbit $\mathbb{L} \cdot m_i$. Hence,

$$(\gamma_x^X)'(0) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX) \cdot x) = X_{\mathbb{M}}(x) = \rho_x(X).$$

Thus, $T_x(\mathbb{L} \cdot m_i) = \rho_x(\mathfrak{g})$. Similarly, $T_y(\mathbb{L} \cdot m_j) = \rho_y(\mathfrak{g})$. Therefore, the adjacency condition is equivalent to

$$\rho_x(\mathfrak{g}) \cap \rho_y(\mathfrak{g}) \neq \{0\}$$

for some $x \in \mathbb{L} \cdot m_i$ and $y \in \mathbb{L} \cdot m_j$. Hence,

$$\deg(\mathbb{L} \cdot m_i) = |\{j \in \{1, 2, \dots, n\} \setminus \{i\} : \exists x \in \mathbb{L} \cdot m_i, \exists y \in \mathbb{L} \cdot m_j \text{ such that } \rho_x(\mathfrak{g}) \cap \rho_y(\mathfrak{g}) \neq \{0\}\}|.$$

If for every $i \neq j$ there exist such points x and y , then every two distinct vertices are adjacent. Hence, $\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ is complete, and each vertex is adjacent to the remaining $n - 1$ vertices. Thus, $\deg(\mathbb{L} \cdot m_i) = n - 1$ for every $i = 1, 2, \dots, n$. $\square$

Theorem 3. Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$ and let

$$\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})] = \{\mathbb{L} \cdot m_i : i = 1, 2, \dots, n\}.$$

Suppose that there is a non-trivial orbit $\mathbb{L} \cdot m_0$ such that, for every $i = 1, 2, \dots, n$, there exist points $x_0 \in \mathbb{L} \cdot m_0$ and $x_i \in \mathbb{L} \cdot m_i$, together with $X_i, Y_i \in \mathfrak{g}$ satisfying $(X_i)_{\mathbb{M}}(x_0) = (Y_i)_{\mathbb{M}}(x_i) \neq 0$. Then, $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is connected, and $\text{diam}(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) \leq 2$.

Proof. For each $i = 1, 2, \dots, n$, choose points $x_0 \in \mathbb{L} \cdot m_0$ and $x_i \in \mathbb{L} \cdot m_i$ together with $X_i, Y_i \in \mathfrak{g}$ such that $(X_i)_{\mathbb{M}}(x_0) = (Y_i)_{\mathbb{M}}(x_i) \neq 0$. Define the two smooth curves $\gamma_0^{X_i}(t) = \exp(tX_i) \cdot x_0$ and $\gamma_i^{Y_i}(t) = \exp(tY_i) \cdot x_i$. Since $\gamma_0^{X_i}(t)$ lies in the orbit $\mathbb{L} \cdot m_0$, its velocity vector at $t = 0$ belongs to $T_{x_0}(\mathbb{L} \cdot m_0)$. Hence,

$$(\gamma_0^{X_i})'(0) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX_i) \cdot x_0) = (X_i)_{\mathbb{M}}(x_0) \in T_{x_0}(\mathbb{L} \cdot m_0).$$

Similarly, since $\gamma_i^{Y_i}(t)$ lies in the orbit $\mathbb{L} \cdot m_i$, we have

$$(\gamma_i^{Y_i})'(0) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tY_i) \cdot x_i) = (Y_i)_{\mathbb{M}}(x_i) \in T_{x_i}(\mathbb{L} \cdot m_i).$$

By hypothesis, $(X_i)_{\mathbb{M}}(x_0) = (Y_i)_{\mathbb{M}}(x_i) \neq 0$. Therefore,

$$T_{x_0}(\mathbb{L} \cdot m_0) \cap T_{x_i}(\mathbb{L} \cdot m_i) \neq \{0\}.$$

Thus, every vertex $\mathbb{L} \cdot m_i$ is adjacent to the vertex $\mathbb{L} \cdot m_0$. Now, let $\mathbb{L} \cdot m_r$ and $\mathbb{L} \cdot m_s$ be any two distinct vertices. If they are adjacent, then their distance is one. If they are not adjacent, then both are adjacent to $\mathbb{L} \cdot m_0$ and hence, $\mathbb{L} \cdot m_r - \mathbb{L} \cdot m_0 - \mathbb{L} \cdot m_s$ is a path of length two. Hence, every two vertices are connected by a path of length at most two. Consequently, $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is connected, and $\text{diam}(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) \leq 2$. $\square$

Theorem 4. Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$ and let

$$\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] = \{\mathbb{L} \cdot m_i : i = 1, 2, \dots, n\}.$$

Suppose that there exist nonzero vectors $v_1, v_2, \dots, v_r \in \mathbb{R}^N$ such that

$$\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] = \bigcup_{q=1}^r \mathcal{C}_q, \quad \mathcal{C}_q = \{\mathbb{L} \cdot m_i : \exists x_i \in \mathbb{L} \cdot m_i \text{ such that } v_q \in T_{x_i}(\mathbb{L} \cdot m_i)\},$$

and for every $q = 1, 2, \dots, r-1$, we have $\mathcal{C}_q \cap \mathcal{C}_{q+1} \neq \emptyset$. Then, $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is connected.

Proof. For each $q = 1, 2, \dots, r$, we first show that $\mathcal{C}_q$ induces a complete subgraph. Let $\mathbb{L} \cdot m_i$ and $\mathbb{L} \cdot m_j$ be two distinct vertices in $\mathcal{C}_q$. Then, there exist points $x_i \in \mathbb{L} \cdot m_i$ and $x_j \in \mathbb{L} \cdot m_j$ such that $v_q \in T_{x_i}(\mathbb{L} \cdot m_i)$ and $v_q \in T_{x_j}(\mathbb{L} \cdot m_j)$. Since $v_q \neq 0$, we have

$$T_{x_i}(\mathbb{L} \cdot m_i) \cap T_{x_j}(\mathbb{L} \cdot m_j) \neq \{0\}.$$

Hence, the two vertices are adjacent. Thus, $\mathcal{C}_q$ is a clique. Now, let $\mathbb{L} \cdot m_a$ and $\mathbb{L} \cdot m_b$ be any two vertices. By the covering assumption, there exist $p, s \in \{1, 2, \dots, r\}$ such that $\mathbb{L} \cdot m_a \in \mathcal{C}_p$ and $\mathbb{L} \cdot m_b \in \mathcal{C}_s$. If $p = s$, then both vertices lie in the same clique $\mathcal{C}_p$, so they are adjacent or equal. Assume that $p < s$. Since $\mathcal{C}_q \cap \mathcal{C}_{q+1} \neq \emptyset$ for every $q = p, p+1, \dots, s-1$, choose a vertex $O_q \in \mathcal{C}_q \cap \mathcal{C}_{q+1}$. Since $\mathcal{C}_p$ is a clique, $\mathbb{L} \cdot m_a$ is connected to O_p . Since $O_q \in \mathcal{C}_{q+1}$ and $O_{q+1} \in \mathcal{C}_{q+1}$, the vertices O_q and O_{q+1} are adjacent or equal for each $q = p, p+1, \dots, s-2$. Finally, since $O_{s-1} \in \mathcal{C}_s$ and $\mathbb{L} \cdot m_b \in \mathcal{C}_s$, they are adjacent or equal. Thus, there is a path from $\mathbb{L} \cdot m_a$ to $\mathbb{L} \cdot m_b$. If $p > s$, the same argument applies

after interchanging p and s . Therefore, every two vertices are connected by a path. Hence, $Lie_{\mathfrak{g}}(\mathbb{LM})$ is connected. $\square$

Theorem 5. Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$. Suppose that there is a nonzero vector $v \in \mathbb{R}^N$ such that for every non-trivial orbit $\mathbb{L} \cdot m$, there exist a point $x \in \mathbb{L} \cdot m$ and an element $X_x \in \mathfrak{g}$ satisfying $(X_x)_{\mathbb{M}}(x) = v$. Then, $Lie_{\mathfrak{g}}(\mathbb{LM})$ is complete.

Proof. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be two distinct non-trivial orbits. By hypothesis, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ together with $X_x, X_y \in \mathfrak{g}$ such that $(X_x)_{\mathbb{M}}(x) = v$ and $(X_y)_{\mathbb{M}}(y) = v$. Define the smooth curves $\gamma_x(t) = \exp(tX_x) \cdot x$ and $\gamma_y(t) = \exp(tX_y) \cdot y$. Since $\gamma_x(t)$ lies in the orbit $\mathbb{L} \cdot m$, its velocity vector at $t = 0$ belongs to $T_x(\mathbb{L} \cdot m)$. Hence,

$$\gamma'_x(0) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX_x) \cdot x) = (X_x)_{\mathbb{M}}(x) = v \in T_x(\mathbb{L} \cdot m).$$

Similarly,

$$\gamma'_y(0) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX_y) \cdot y) = (X_y)_{\mathbb{M}}(y) = v \in T_y(\mathbb{L} \cdot n).$$

Thus, $v \in T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n)$. Since $v \neq 0$, we obtain

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Therefore, $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ are adjacent. Since this holds for every two distinct non-trivial orbits, $Lie_{\mathfrak{g}}(\mathbb{LM})$ is complete. $\square$

3. Global Properties of Lie Orbit Tangent Graphs

This section investigates the local tangent structure of $Lie_{\mathfrak{g}}(\mathbb{LM})$ through tangent orbit fibers and local tangent orbit cliques. We study how common tangent directions generate adjacency relations and determine global graph properties. In particular, we establish decomposition results, derive connectivity and domination properties and present applications illustrating how local tangent interactions control the overall graph structure. For each nonzero vector $v \in \mathbb{R}^N$, define

$$\mathcal{L}_T(v) = \{\mathbb{L} \cdot m \in \mathcal{V}[Lie_{\mathfrak{g}}(\mathbb{LM})] : \exists x \in \mathbb{L} \cdot m \text{ such that } v \in T_x(\mathbb{L} \cdot m)\}.$$

The set $\mathcal{L}_T(v)$ is called the tangent orbit fiber determined by v . The number $\mu_T(v) = |\mathcal{L}_T(v)|$ is called the tangent orbit multiplicity of v . If $\mu_T(v) \geq 2$, then v is called a common tangent direction. For $v \neq 0$, the subgraph induced by $\mathcal{L}_T(v)$ is called the local tangent orbit clique at v and is denoted by $K_v^T = Lie_{\mathfrak{g}}(\mathbb{LM})[\mathcal{L}_T(v)]$.

For example, in Example 2, the common direction e_1 gives the tangent fiber $\mathcal{L}_T(e_1) = \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_2\}$; hence, the local tangent orbit clique $K_{e_1}^T$ consists of the edge joining these two vertices. Similarly, e_4 gives $\mathcal{L}_T(e_4) = \{\mathbb{L} \cdot m_2, \mathbb{L} \cdot m_3\}$. Thus, a tangent fiber records all orbit vertices that share a specified infinitesimal direction, while the associated local clique records the pairwise adjacencies forced by that direction.

Theorem 6. The Lie orbit tangent graph $Lie_{\mathfrak{g}}(\mathbb{LM})$ is the edge union of its local tangent orbit cliques K_v^T , where v runs over all nonzero vectors of $\mathbb{R}^N$.

Proof. We prove that every edge of $Lie_{\mathfrak{g}}(\mathbb{LM})$ appears in some local tangent orbit clique and conversely every edge appearing in a local tangent orbit clique is an edge of $Lie_{\mathfrak{g}}(\mathbb{LM})$.

Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be two distinct vertices of $Lie_{\mathfrak{g}}(\mathbb{LM})$. Suppose first that they are adjacent. Then, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ such that

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Hence, there is a nonzero vector $v \in \mathbb{R}^N$ such that $v \in T_x(\mathbb{L} \cdot m)$ and $v \in T_y(\mathbb{L} \cdot n)$. Therefore, $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in \mathcal{L}_T(v)$ and so the edge $\mathbb{L} \cdot m - \mathbb{L} \cdot n$ belongs to the local tangent orbit clique K_v^T . To see the infinitesimal origin of this common direction, use the characterization $T_x(\mathbb{L} \cdot m) = \{X_{\mathbb{M}}(x) : X \in \mathfrak{g}\}$ and $T_y(\mathbb{L} \cdot n) = \{Y_{\mathbb{M}}(y) : Y \in \mathfrak{g}\}$. Thus, there exist $X, Y \in \mathfrak{g}$ such that $v = X_{\mathbb{M}}(x) = Y_{\mathbb{M}}(y)$. Equivalently, for the curves $\gamma_x^X(t) = \exp(tX) \cdot x$ and $\gamma_y^Y(t) = \exp(tY) \cdot y$, we have

$$(\gamma_x^X)'(0) = \frac{d}{dt} \Big|_{t=0} (\exp(tX) \cdot x) = X_{\mathbb{M}}(x) = v = Y_{\mathbb{M}}(y) = \frac{d}{dt} \Big|_{t=0} (\exp(tY) \cdot y) = (\gamma_y^Y)'(0).$$

Hence, the edge is generated by a common nonzero infinitesimal direction. Conversely, suppose that $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ are adjacent inside some local tangent orbit clique K_v^T . Then, $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in \mathcal{L}_T(v)$. By the definition of $\mathcal{L}_T(v)$, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ such that $v \in T_x(\mathbb{L} \cdot m)$ and $v \in T_y(\mathbb{L} \cdot n)$. Since $v \neq 0$, it follows that

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Therefore, $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ are adjacent in $Lie_{\mathfrak{g}}(\mathbb{LM})$. Hence, the edge set of $Lie_{\mathfrak{g}}(\mathbb{LM})$ is exactly the union of the edge sets of all local tangent orbit cliques K_v^T . That is,

$$Lie_{\mathfrak{g}}(\mathbb{LM}) = \bigcup_{v \in \mathbb{R}^N \setminus \{0\}} K_v^T$$

is an edge union. $\square$

Theorem 7. Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$. Let Θ be a clique in $Lie_{\mathfrak{g}}(\mathbb{LM})$. If there exist points $x_j \in \mathbb{L} \cdot m_j$ for each vertex $\mathbb{L} \cdot m_j \in \Theta$ such that

$$\bigcap_{\mathbb{L} \cdot m_j \in \Theta} T_{x_j}(\mathbb{L} \cdot m_j) \neq \{0\},$$

then there is a nonzero vector $v \in \mathbb{R}^N$ such that $\Theta \subseteq \mathcal{L}_T(v)$. Consequently, Θ is contained in the local tangent orbit clique K_v^T . In particular, if every clique of $Lie_{\mathfrak{g}}(\mathbb{LM})$ has a nonzero common tangent direction, then $\omega(Lie_{\mathfrak{g}}(\mathbb{LM})) = \max_{v \in \mathbb{R}^N \setminus \{0\}} \mu_T(v)$.

Proof. Let $\Theta = \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_2, \dots, \mathbb{L} \cdot m_r\}$ be a clique and assume that there exist points $x_j \in \mathbb{L} \cdot m_j$ such that

$$\bigcap_{j=1}^r T_{x_j}(\mathbb{L} \cdot m_j) \neq \{0\}.$$

Then, there is a nonzero vector $v \in \mathbb{R}^N$ satisfying $v \in T_{x_j}(\mathbb{L} \cdot m_j)$ for every $j = 1, 2, \dots, r$. By definition of $\mathcal{L}_T(v)$, this implies $\mathbb{L} \cdot m_j \in \mathcal{L}_T(v)$ for all j . Hence, $\Theta \subseteq \mathcal{L}_T(v)$. Therefore, Θ is contained in $K_v^T = Lie_{\mathfrak{g}}(\mathbb{LM})[\mathcal{L}_T(v)]$. Now, we show compatibility with Theorem 6. By Theorem 6, every edge of $Lie_{\mathfrak{g}}(\mathbb{LM})$ is generated by at least one nonzero common tangent direction. In the present case, the same vector v belongs to all tangent spaces associated with the vertices of Θ . Hence, every edge inside Θ is generated by the same local tangent fiber $\mathcal{L}_T(v)$. Therefore, Θ is contained in one local tangent orbit clique. It remains to prove the equality for the clique number. For every nonzero vector $v \in \mathbb{R}^N$, the subgraph K_v^T

is complete. Indeed, if $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in \mathcal{L}_T(v)$ with distinct vertices, then there exist points $x \in \mathbb{L} \cdot m, y \in \mathbb{L} \cdot n$ such that $v \in T_x(\mathbb{L} \cdot m)$ and $v \in T_y(\mathbb{L} \cdot n)$. Hence,

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Therefore, the vertices are adjacent, and K_v^T is a clique of order $\mu_T(v)$. Consequently,

$$\omega(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) \geq \max_{v \in \mathbb{R}^N \setminus \{0\}} \mu_T(v).$$

Conversely, let Θ be a clique of maximum cardinality. By assumption, Θ possesses a nonzero common tangent direction. Hence, there is $v \neq 0$ such that $\Theta \subseteq \mathcal{L}_T(v)$. Therefore,

$$|\Theta| \leq |\mathcal{L}_T(v)| = \mu_T(v) \leq \max_{u \in \mathbb{R}^N \setminus \{0\}} \mu_T(u).$$

Since $|\Theta| = \omega(\text{Lie}_{\mathfrak{g}}(\mathbb{LM}))$, we obtain

$$\omega(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) \leq \max_{u \in \mathbb{R}^N \setminus \{0\}} \mu_T(u).$$

Combining both inequalities gives $\omega(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) = \max_{v \in \mathbb{R}^N \setminus \{0\}} \mu_T(v)$. $\square$

Theorem 8. Let $\mathcal{C} = \{K_v^T : v \in \mathbb{R}^N \setminus \{0\}\}$ be the family of all local tangent orbit cliques. Suppose that for every two cliques $K_u^T, K_v^T \in \mathcal{C}$, there is a finite sequence $K_u^T = K_{v_0}^T, K_{v_1}^T, \dots, K_{v_r}^T = K_v^T$ such that $K_{v_i}^T \cap K_{v_{i+1}}^T \neq \emptyset$ for all $i = 0, 1, \dots, r-1$. Then, $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is connected.

Proof. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be arbitrary vertices of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Choose local tangent orbit cliques K_u^T and K_v^T such that $\mathbb{L} \cdot m \in K_u^T$ and $\mathbb{L} \cdot n \in K_v^T$. If $K_u^T = K_v^T$, then K_u^T is complete. Hence, $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ are connected. Assume now that $K_u^T \neq K_v^T$. By hypothesis, there is a sequence

$$K_u^T = K_{v_0}^T, K_{v_1}^T, \dots, K_{v_r}^T = K_v^T.$$

Since $K_{v_i}^T \cap K_{v_{i+1}}^T \neq \emptyset$, choose a vertex $O_i \in K_{v_i}^T \cap K_{v_{i+1}}^T$ for every $i = 0, 1, \dots, r-1$. Since $\mathbb{L} \cdot m, O_0 \in K_{v_0}^T$, and $K_{v_0}^T$ is complete, $\mathbb{L} \cdot m$ is adjacent to O_0 . Similarly, since $O_i, O_{i+1} \in K_{v_{i+1}}^T$, and $K_{v_{i+1}}^T$ is complete, O_i is adjacent to O_{i+1} for every $i = 0, 1, \dots, r-2$. Finally, since $O_{r-1}, \mathbb{L} \cdot n \in K_{v_r}^T$, and $K_{v_r}^T$ is complete, O_{r-1} is adjacent to $\mathbb{L} \cdot n$. Therefore,

$$\mathbb{L} \cdot m - O_0 - O_1 - \dots - O_{r-1} - \mathbb{L} \cdot n$$

is a path joining $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$. Hence, $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is connected. $\square$

Theorem 9. Let $\mathcal{S} = \{K_{v_1}^T, K_{v_2}^T, \dots, K_{v_r}^T\}$ be a family of local tangent orbit cliques such that $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] \subseteq \bigcup_{i=1}^r V(K_{v_i}^T)$. Choose one vertex $O_i \in V(K_{v_i}^T)$ for each $i = 1, 2, \dots, r$. Then, $D = \{O_1, O_2, \dots, O_r\}$ is a dominating set of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Consequently, $\gamma(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) \leq r$.

Proof. Let $\mathbb{L} \cdot m$ be an arbitrary vertex of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. If $\mathbb{L} \cdot m \in D$, there is nothing to prove. Assume that $\mathbb{L} \cdot m \notin D$. Since

$$\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] \subseteq \bigcup_{i=1}^r V(K_{v_i}^T),$$

there is $j \in \{1, 2, \dots, r\}$ such that $\mathbb{L} \cdot m \in K_{v_j}^T$. Since $O_j \in K_{v_j}^T$, both vertices $\mathbb{L} \cdot m$ and O_j belong to the same local tangent orbit clique. Since $K_{v_j}^T$ is complete, $\mathbb{L} \cdot m$ is adjacent to O_j .

Hence, every vertex outside D is adjacent to some vertex of D . Therefore, D is a dominating set of $Lie_g(\mathbb{LM})$. Consequently, $\gamma(Lie_g(\mathbb{LM})) \leq |D| = r$. $\square$

Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$. Define the tangent fiber incidence graph $\mathcal{B}_T(\mathbb{LM})$ as the bipartite graph whose first part is $\mathcal{V}[Lie_g(\mathbb{LM})]$ and whose second part is $\{v \in \mathbb{R}^N \setminus \{0\} : \mu_T(v) \geq 2\}$. A vertex $\mathbb{L} \cdot m \in \mathcal{V}[Lie_g(\mathbb{LM})]$ is adjacent to a vector v if and only if there is a point $x \in \mathbb{L} \cdot m$ such that $v \in T_x(\mathbb{L} \cdot m)$. In this bipartite graph, one side consists of orbit vertices, and the other side consists of common tangent directions. For instance, in Example 2, the direction e_1 is joined to the orbit vertices $\mathbb{L} \cdot m_1$ and $\mathbb{L} \cdot m_2$, while the direction e_{10} is joined to $\mathbb{L} \cdot m_6$ and $\mathbb{L} \cdot m_7$. Thus, incidence edges explicitly record which orbit vertices participate in each common tangent direction.

Theorem 10. *Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$. A vertex $\mathbb{L} \cdot m$ is adjacent to a vector v in $\mathcal{B}_T(\mathbb{LM})$ whenever there is $x \in \mathbb{L} \cdot m$ such that $v \in T_x(\mathbb{L} \cdot m)$. Then, two non-trivial orbits $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ belong to the same connected component of $Lie_g(\mathbb{LM})$ if and only if they belong to the same connected component of $\mathcal{B}_T(\mathbb{LM})$. Moreover, whenever the distance in $\mathcal{B}_T(\mathbb{LM})$ is finite,*

$$d_{Lie}(\mathbb{L} \cdot m, \mathbb{L} \cdot n) \leq \frac{1}{2} d_{\mathcal{B}_T}(\mathbb{L} \cdot m, \mathbb{L} \cdot n).$$

Proof. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be two non-trivial orbits. Assume first that they belong to the same connected component of $Lie_g(\mathbb{LM})$. Then, there is a path

$$\mathbb{L} \cdot m = \mathbb{L} \cdot m_0 - \mathbb{L} \cdot m_1 - \cdots - \mathbb{L} \cdot m_r = \mathbb{L} \cdot n.$$

For each $i = 1, 2, \dots, r$, the vertices $\mathbb{L} \cdot m_{i-1}$ and $\mathbb{L} \cdot m_i$ are adjacent. Hence, there exist points $x_{i-1} \in \mathbb{L} \cdot m_{i-1}$ and $x_i \in \mathbb{L} \cdot m_i$ such that

$$T_{x_{i-1}}(\mathbb{L} \cdot m_{i-1}) \cap T_{x_i}(\mathbb{L} \cdot m_i) \neq \{0\}.$$

Choose a nonzero vector $v_i \in T_{x_{i-1}}(\mathbb{L} \cdot m_{i-1}) \cap T_{x_i}(\mathbb{L} \cdot m_i)$. Then, $\mu_T(v_i) \geq 2$, so v_i is a vertex of $\mathcal{B}_T(\mathbb{LM})$. Since $v_i \in T_{x_{i-1}}(\mathbb{L} \cdot m_{i-1})$ and $v_i \in T_{x_i}(\mathbb{L} \cdot m_i)$, the incidence path

$$\mathbb{L} \cdot m = \mathbb{L} \cdot m_0 - v_1 - \mathbb{L} \cdot m_1 - v_2 - \cdots - v_r - \mathbb{L} \cdot m_r = \mathbb{L} \cdot n$$

is in $\mathcal{B}_T(\mathbb{LM})$. Hence, the two orbit vertices belong to the same connected component. Conversely, assume that $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ belong to the same connected component of $\mathcal{B}_T(\mathbb{LM})$. Since $\mathcal{B}_T(\mathbb{LM})$ is bipartite, there is a path

$$\mathbb{L} \cdot m = \mathbb{L} \cdot m_0 - v_1 - \mathbb{L} \cdot m_1 - v_2 - \cdots - v_r - \mathbb{L} \cdot m_r = \mathbb{L} \cdot n.$$

For every i , there exist points $x_{i-1} \in \mathbb{L} \cdot m_{i-1}$ and $x_i \in \mathbb{L} \cdot m_i$ such that $v_i \in T_{x_{i-1}}(\mathbb{L} \cdot m_{i-1})$ and $v_i \in T_{x_i}(\mathbb{L} \cdot m_i)$. Therefore,

$$T_{x_{i-1}}(\mathbb{L} \cdot m_{i-1}) \cap T_{x_i}(\mathbb{L} \cdot m_i) \neq \{0\}.$$

Then, the vertices $\mathbb{L} \cdot m_{i-1}$ and $\mathbb{L} \cdot m_i$ are adjacent. Hence,

$$\mathbb{L} \cdot m = \mathbb{L} \cdot m_0 - \mathbb{L} \cdot m_1 - \cdots - \mathbb{L} \cdot m_r = \mathbb{L} \cdot n$$

is a path in $Lie_g(\mathbb{LM})$. Finally, suppose $d_{\mathcal{B}_T}(\mathbb{L} \cdot m, \mathbb{L} \cdot n) < \infty$. Since $\mathcal{B}_T(\mathbb{LM})$ is bipartite, every shortest path has even length, say, $d_{\mathcal{B}_T}(\mathbb{L} \cdot m, \mathbb{L} \cdot n) = 2r$. Then, the previous argument yields a path in $Lie_g(\mathbb{LM})$ of length r . Hence,

$$d_{Lie}(\mathbb{L} \cdot m, \mathbb{L} \cdot n) \leq r = \frac{1}{2} d_{\mathcal{B}_T}(\mathbb{L} \cdot m, \mathbb{L} \cdot n).$$

The proof is complete. $\square$

Proposition 1. Let $\mathbb{L}$ be the three-dimensional Abelian Lie group

$$\mathbb{L} = \left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : s_1, s_2, s_3 \in \mathbb{R} \right\}.$$

Let $\{e_1, e_2, \dots, e_{13}\}$ be the standard basis of $\mathbb{R}^{13}$ and define

$$\begin{aligned} V_1 &= \text{span}\{e_1, e_2, e_3\}, & V_2 &= \text{span}\{e_1, e_4, e_5\}, & V_3 &= \text{span}\{e_1, e_6, e_7\}, \\ V_4 &= \text{span}\{e_2, e_4, e_8\}, & V_5 &= \text{span}\{e_3, e_9, e_{10}\}, & V_6 &= \text{span}\{e_5, e_8, e_{11}\}, \\ V_7 &= \text{span}\{e_7, e_{10}, e_{11}\}. \end{aligned}$$

For $i = 1, 2, \dots, 7$, let $c_i = ie_{13}$, define $\mathbb{M}_i = c_i + V_i$, put $\mathbb{M} = \bigcup_{i=1}^7 \mathbb{M}_i$ and let $A_i : \mathbb{R}^3 \rightarrow V_i$ be the linear isomorphism sending the standard basis of $\mathbb{R}^3$ onto the generators of V_i . Define the action by

$$l \cdot (c_i + v) = c_i + v + A_i(s_1, s_2, s_3),$$

where

$$l = \begin{pmatrix} 1 & s_1 & s_2 & s_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, $\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ has seven vertices and ten edges, contains a triangle, is connected and is neither complete nor cyclic.

Proof. The Lie algebra of $\mathbb{L}$ is

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Let

$$X = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}.$$

Since $X^2 = 0$, we have $\exp(tX) = I_4 + tX$. Let $m = c_i + v \in \mathbb{M}_i$ and define $\gamma_i^X(t) = \exp(tX) \cdot m$. Then, $\gamma_i^X(t) = c_i + v + A_i(tx_1, tx_2, tx_3)$. Therefore,

$$(\gamma_i^X)'(0) = \frac{d}{dt} \Big|_{t=0} (c_i + v + A_i(tx_1, tx_2, tx_3)) = A_i(x_1, x_2, x_3).$$

Hence, $X_{\mathbb{M}}(m) = A_i(x_1, x_2, x_3)$. Since A_i is onto V_i , we obtain

$$T_x(\mathbb{L} \cdot m_i) = \{X_{\mathbb{M}}(x) : X \in \mathfrak{g}\} = V_i \quad \text{for every } x \in \mathbb{L} \cdot m_i.$$

Because each A_i is surjective, the action is transitive on every component $\mathbb{M}_i$. Therefore, $\mathbb{L} \cdot m_i = \mathbb{M}_i$ and

$$\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] = \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_2, \dots, \mathbb{L} \cdot m_7\}.$$

Now,

$$\begin{aligned} V_1 \cap V_2 = V_1 \cap V_3 = V_2 \cap V_3 = \text{span}\{e_1\}, & \quad V_1 \cap V_4 = \text{span}\{e_2\}, & \quad V_1 \cap V_5 = \text{span}\{e_3\}, \\ V_2 \cap V_4 = \text{span}\{e_4\}, & \quad V_2 \cap V_6 = \text{span}\{e_5\}, & \quad V_3 \cap V_7 = \text{span}\{e_7\}, & \quad V_4 \cap V_6 = \text{span}\{e_8\}, \end{aligned}$$

and $V_5 \cap V_7 = \text{span}\{e_{10}\}$. All remaining pairwise intersections equal $\{0\}$. Since the tangent space along each orbit is constant and equal to V_i , Definition 1 implies that

$$\mathcal{E}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] = \{\mathbb{L} \cdot m_1 \mathbb{L} \cdot m_2, \mathbb{L} \cdot m_1 \mathbb{L} \cdot m_3, \mathbb{L} \cdot m_2 \mathbb{L} \cdot m_3, \mathbb{L} \cdot m_1 \mathbb{L} \cdot m_4,$$

$$\mathbb{L} \cdot m_1 \mathbb{L} \cdot m_5, \mathbb{L} \cdot m_2 \mathbb{L} \cdot m_4, \mathbb{L} \cdot m_2 \mathbb{L} \cdot m_6, \mathbb{L} \cdot m_3 \mathbb{L} \cdot m_7, \mathbb{L} \cdot m_4 \mathbb{L} \cdot m_6, \mathbb{L} \cdot m_5 \mathbb{L} \cdot m_7\}.$$

Hence, the graph has seven vertices and ten edges. Moreover, $\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_2, \mathbb{L} \cdot m_3$ form a triangle since every pair shares the direction e_1 . The graph is connected because every vertex outside this triangle is adjacent to one of its vertices. The graph is not complete since $V_4 \cap V_5 = \{0\}$. Hence, for every $x \in \mathbb{L} \cdot m_4$ and $y \in \mathbb{L} \cdot m_5$, we have $T_x(\mathbb{L} \cdot m_4) \cap T_y(\mathbb{L} \cdot m_5) = \{0\}$ and therefore, $\mathbb{L} \cdot m_4$ and $\mathbb{L} \cdot m_5$ are not adjacent. Also, the graph is not cyclic because it contains a triangle and vertices of degree greater than two. Therefore, the proposition follows. $\square$

4. Hypergraph and Incidence Structures of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$

This section investigates the global structure of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ through tangent fiber incidence methods and hypergraph techniques.

Definition 2. Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$. The tangent orbit hypergraph associated with $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is denoted by $\mathcal{H}_T(\mathbb{LM})$ and defined by $\mathcal{H}_T(\mathbb{LM}) = (\mathcal{V}_T, \mathcal{E}_T)$, where $\mathcal{V}_T = \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$ and $\mathcal{E}_T = \{\mathcal{L}_T(v) : v \in \mathbb{R}^N \setminus \{0\}, \mu_T(v) \geq 2\}$. Each hyperedge $\mathcal{L}_T(v)$ consists of all orbit vertices for which there is a point on the orbit whose tangent space contains the common nonzero direction v .

To illustrate this definition, Example 2 gives hyperedges such as $\mathcal{L}_T(e_1) = \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_2\}$, $\mathcal{L}_T(e_6) = \{\mathbb{L} \cdot m_3, \mathbb{L} \cdot m_4\}$ and $\mathcal{L}_T(e_8) = \{\mathbb{L} \cdot m_5, \mathbb{L} \cdot m_6\}$. Each hyperedge groups all orbits sharing one common tangent direction. In examples where a direction is shared by more than two orbits, the corresponding hyperedge has cardinality greater than two and records a genuinely higher-order tangent interaction rather than only a pairwise edge.

Theorem 11. The graph $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is exactly the 2-section graph of the tangent orbit hypergraph $\mathcal{H}_T(\mathbb{LM})$.

Proof. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be two distinct vertices of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Suppose first that they are adjacent in $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Then, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ such that

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Hence, there is a nonzero vector $v \in \mathbb{R}^N$ such that $v \in T_x(\mathbb{L} \cdot m)$ and $v \in T_y(\mathbb{L} \cdot n)$. Therefore, $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in \mathcal{L}_T(v)$ and since $\mu_T(v) \geq 2$, the set $\mathcal{L}_T(v)$ is a hyperedge of $\mathcal{H}_T(\mathbb{LM})$. Thus, the two vertices are contained in a common hyperedge, so they are adjacent in the 2-section graph of $\mathcal{H}_T(\mathbb{LM})$. Conversely, suppose that $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ are adjacent in the 2-section graph of $\mathcal{H}_T(\mathbb{LM})$. Then, there is a hyperedge $\mathcal{L}_T(v) \in \mathcal{E}_T$ such

that $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in \mathcal{L}_T(v)$. By the definition of $\mathcal{L}_T(v)$, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ such that $v \in T_x(\mathbb{L} \cdot m)$ and $v \in T_y(\mathbb{L} \cdot n)$. Since $v \neq 0$, we obtain

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Hence, $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ are adjacent in $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Therefore, both graphs have the same vertex set and the same edge set, so $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is exactly the 2-section graph of $\mathcal{H}_T(\mathbb{LM})$. $\square$

Theorem 12. Let $S \subseteq \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$. Then, S is an independent set of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ if and only if for every two distinct vertices $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in S$, every points $x \in \mathbb{L} \cdot m, y \in \mathbb{L} \cdot n$, and every $X, Y \in \mathfrak{g}$, we have $X_{\mathbb{M}}(x) \neq Y_{\mathbb{M}}(y)$ whenever $X_{\mathbb{M}}(x) \neq 0$ and $Y_{\mathbb{M}}(y) \neq 0$. Equivalently, S is independent if and only if $|S \cap \mathcal{L}_T(v)| \leq 1$ for every hyperedge $\mathcal{L}_T(v) \in \mathcal{E}_T$.

Proof. Assume first that S is an independent set of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be two distinct vertices in S . Suppose, to the contrary, that there exist points $x \in \mathbb{L} \cdot m, y \in \mathbb{L} \cdot n$, and elements $X, Y \in \mathfrak{g}$ such that $X_{\mathbb{M}}(x) = Y_{\mathbb{M}}(y) \neq 0$. Define the smooth curves $\gamma_x^X(t) = \exp(tX) \cdot x$ and $\gamma_y^Y(t) = \exp(tY) \cdot y$. Since γ_x^X lies in $\mathbb{L} \cdot m$ and γ_y^Y lies in $\mathbb{L} \cdot n$, we have

$$(\gamma_x^X)'(0) = \frac{d}{dt} \Big|_{t=0} (\exp(tX) \cdot x) = X_{\mathbb{M}}(x) \in T_x(\mathbb{L} \cdot m),$$

and

$$(\gamma_y^Y)'(0) = \frac{d}{dt} \Big|_{t=0} (\exp(tY) \cdot y) = Y_{\mathbb{M}}(y) \in T_y(\mathbb{L} \cdot n).$$

Put $v = X_{\mathbb{M}}(x) = Y_{\mathbb{M}}(y)$. Since $v \neq 0$, we obtain $v \in T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n)$. Therefore,

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Hence, the two vertices are adjacent in $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$, contradicting independence. Thus, no two distinct vertices of S can share the same nonzero fundamental tangent direction. Conversely, assume that for every two distinct vertices $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in S$, every points $x \in \mathbb{L} \cdot m, y \in \mathbb{L} \cdot n$, and every $X, Y \in \mathfrak{g}$, the equality $X_{\mathbb{M}}(x) = Y_{\mathbb{M}}(y) \neq 0$ never occurs. Suppose that S is not independent. Then, there exist two adjacent vertices $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in S$. Then, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ such that

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Hence, there is $0 \neq v \in T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n)$. By the tangent characterization of orbits, there exist $X, Y \in \mathfrak{g}$ satisfying $v = X_{\mathbb{M}}(x)$ and $v = Y_{\mathbb{M}}(y)$. Therefore, $X_{\mathbb{M}}(x) = Y_{\mathbb{M}}(y) \neq 0$, contradicting the hypothesis. Hence, S is independent. It remains to prove the hypergraph formulation. By Theorem 11, $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is the 2-section graph of $\mathcal{H}_T(\mathbb{LM})$. Hence, two vertices are adjacent if and only if they belong to a common hyperedge $\mathcal{L}_T(v)$. If S is independent and $|S \cap \mathcal{L}_T(v)| \geq 2$ for some hyperedge, then two vertices of S belong to the same hyperedge and are therefore adjacent, a contradiction. Thus, $|S \cap \mathcal{L}_T(v)| \leq 1$ for every $\mathcal{L}_T(v) \in \mathcal{E}_T$. Conversely, assume $|S \cap \mathcal{L}_T(v)| \leq 1$ for every hyperedge. If two vertices of S were adjacent, then Theorem 6 would provide a nonzero tangent direction v generating this edge, so both vertices would belong to $\mathcal{L}_T(v)$, contradicting $|S \cap \mathcal{L}_T(v)| \leq 1$. Therefore, S is independent. $\square$

Theorem 13. Let $\mathcal{D} \subseteq \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$. Then, $\mathcal{D}$ is a dominating set of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ if and only if for every vertex $\mathbb{L} \cdot m \notin \mathcal{D}$, there is a nonzero vector $v \in \mathbb{R}^N$ such that $\mathbb{L} \cdot m \in \mathcal{L}_T(v)$ and $\mathcal{D} \cap \mathcal{L}_T(v) \neq \emptyset$. Equivalently, every vertex outside $\mathcal{D}$ shares a nonzero infinitesimal tangent direction with at least one vertex of $\mathcal{D}$.

Proof. Assume first that $\mathcal{D}$ is a dominating set of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Let $\mathbb{L} \cdot m \notin \mathcal{D}$. Since $\mathcal{D}$ is dominating, there is a vertex $\mathbb{L} \cdot n \in \mathcal{D}$ such that $\mathbb{L} \cdot m$ is adjacent to $\mathbb{L} \cdot n$. Then, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ satisfying

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Hence, there is a nonzero vector $v \in T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n)$. Therefore, $\mathbb{L} \cdot m \in \mathcal{L}_T(v)$ and $\mathbb{L} \cdot n \in \mathcal{D} \cap \mathcal{L}_T(v)$. Thus, $\mathcal{D} \cap \mathcal{L}_T(v) \neq \emptyset$. Conversely, suppose that for every vertex $\mathbb{L} \cdot m \notin \mathcal{D}$, there is $v \in \mathbb{R}^N \setminus \{0\}$ such that $\mathbb{L} \cdot m \in \mathcal{L}_T(v)$ and $\mathcal{D} \cap \mathcal{L}_T(v) = \emptyset$. Choose $\mathbb{L} \cdot n \in \mathcal{D} \cap \mathcal{L}_T(v)$. Since both vertices belong to $\mathcal{L}_T(v)$, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ such that $v \in T_x(\mathbb{L} \cdot m)$ and $v \in T_y(\mathbb{L} \cdot n)$. Since $v \neq 0$, we obtain

$$T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n) \neq \{0\}.$$

Hence, $\mathbb{L} \cdot m$ is adjacent to $\mathbb{L} \cdot n$. Therefore, every vertex outside $\mathcal{D}$ is adjacent to a vertex of $\mathcal{D}$, so $\mathcal{D}$ is dominating. To express this infinitesimally, let $\mathbb{L} \cdot m \notin \mathcal{D}$ and choose $\mathbb{L} \cdot n \in \mathcal{D}$ as above. Since $v \in T_x(\mathbb{L} \cdot m)$ and $v \in T_y(\mathbb{L} \cdot n)$, the tangent characterization gives elements $X, Y \in \mathfrak{g}$ satisfying $v = X_{\mathbb{M}}(x) = Y_{\mathbb{M}}(y)$. Equivalently, for the curves $\gamma_x^X(t) = \exp(tX) \cdot x$ and $\gamma_y^Y(t) = \exp(tY) \cdot y$, we have

$$(\gamma_x^X)'(0) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX) \cdot x) = X_{\mathbb{M}}(x) = v = Y_{\mathbb{M}}(y) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tY) \cdot y) = (\gamma_y^Y)'(0).$$

Therefore, every vertex outside $\mathcal{D}$ is controlled by a vertex of $\mathcal{D}$ through a common nonzero infinitesimal tangent direction. $\square$

Theorem 14. Let $\mathcal{Q} \subseteq \mathbb{R}^N \setminus \{0\}$ be a family of tangent directions such that $\{\mathcal{L}_T(v) : v \in \mathcal{Q}\}$ covers $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$. If for each $v \in \mathcal{Q}$, there is a vertex $O_v \in \mathcal{L}_T(v)$, then the set $D_{\mathcal{Q}} = \{O_v : v \in \mathcal{Q}\}$ is a dominating set of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. In particular, if τ_T denotes the minimum number of tangent fibers needed to cover $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$, then $\gamma(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) \leq \tau_T$.

Proof. Let $\mathbb{L} \cdot m$ be any vertex of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Since $\{\mathcal{L}_T(v) : v \in \mathcal{Q}\}$ covers $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$, there is $v \in \mathcal{Q}$ such that $\mathbb{L} \cdot m \in \mathcal{L}_T(v)$. By construction, $O_v \in \mathcal{L}_T(v)$. If $\mathbb{L} \cdot m = O_v$, then $\mathbb{L} \cdot m \in D_{\mathcal{Q}}$. Assume now that $\mathbb{L} \cdot m \neq O_v$. Since both vertices belong to $\mathcal{L}_T(v)$, there exist points $x \in \mathbb{L} \cdot m$ and $y \in O_v$ such that $v \in T_x(\mathbb{L} \cdot m)$ and $v \in T_y(O_v)$. Since $v \neq 0$, we obtain $T_x(\mathbb{L} \cdot m) \cap T_y(O_v) \neq \{0\}$. Hence, $\mathbb{L} \cdot m$ is adjacent to O_v . Therefore, every vertex is either contained in $D_{\mathcal{Q}}$ or adjacent to a vertex of $D_{\mathcal{Q}}$. Thus, $D_{\mathcal{Q}}$ is a dominating set. Now, let τ_T be the minimum cardinality of a family of tangent fibers covering $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$. Choose a covering family $\{\mathcal{L}_T(v) : v \in \mathcal{Q}_0\}$ with $|\mathcal{Q}_0| = \tau_T$. By the first part, choosing one vertex from each fiber gives a dominating set $D_{\mathcal{Q}_0}$ satisfying $|D_{\mathcal{Q}_0}| \leq |\mathcal{Q}_0| = \tau_T$. Since $\gamma(\text{Lie}_{\mathfrak{g}}(\mathbb{LM}))$ is the minimum cardinality of a dominating set, we obtain $\gamma(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) \leq \tau_T$. $\square$

Theorem 15. Let $\mathbb{L}$ act smoothly on an embedded manifold $\mathbb{M} \subseteq \mathbb{R}^N$, and let $\mathcal{H}_T(\mathbb{LM})$ be the tangent orbit hypergraph. Suppose that there exists an integer $R \in \mathbb{N}$ such that for every two vertices $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$, there is a sequence of hyperedges $\mathcal{L}_T(v_1), \mathcal{L}_T(v_2), \dots, \mathcal{L}_T(v_r)$, where $r \leq R$, satisfying $\mathbb{L} \cdot m \in \mathcal{L}_T(v_1)$, $\mathbb{L} \cdot n \in \mathcal{L}_T(v_r)$, and $\mathcal{L}_T(v_i) \cap \mathcal{L}_T(v_{i+1}) \neq \emptyset$ for every $i = 1, 2, \dots, r-1$. Then, $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is connected and $\text{diam}(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) \leq R$.

Proof. Let $\mathbb{L} \cdot m, \mathbb{L} \cdot n \in \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$. By hypothesis, there is a sequence of hyperedges $\mathcal{L}_T(v_1), \mathcal{L}_T(v_2), \dots, \mathcal{L}_T(v_r)$, where $r \leq R$, such that $\mathbb{L} \cdot m \in \mathcal{L}_T(v_1)$, $\mathbb{L} \cdot n \in \mathcal{L}_T(v_r)$, and $\mathcal{L}_T(v_i) \cap \mathcal{L}_T(v_{i+1}) \neq \emptyset$ for every $i = 1, 2, \dots, r-1$. Choose vertices $O_i \in \mathcal{L}_T(v_i) \cap \mathcal{L}_T(v_{i+1})$

for $i = 1, 2, \dots, r-1$. Since $\mathbb{L} \cdot m, O_1 \in \mathcal{L}_T(v_1)$, Theorem 11 implies that either $\mathbb{L} \cdot m = O_1$, or they are adjacent in $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Similarly, for every $i = 1, 2, \dots, r-2$, the vertices O_i and O_{i+1} belong to the same hyperedge $\mathcal{L}_T(v_{i+1})$, so they are equal or adjacent by Theorem 11. Finally, since $O_{r-1}, \mathbb{L} \cdot n \in \mathcal{L}_T(v_r)$, they are equal or adjacent. Consequently,

$$\mathbb{L} \cdot m - O_1 - O_2 - \dots - O_{r-1} - \mathbb{L} \cdot n$$

gives a path after removing repeated consecutive vertices. Hence, $d(\mathbb{L} \cdot m, \mathbb{L} \cdot n) \leq r \leq R$. Since the choice of vertices was arbitrary, every two vertices are connected. Therefore, $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is connected, and $\text{diam}(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) \leq R$. $\square$

Theorem 16. Under the standing finiteness assumptions, let $I(\mathbb{LM})$ denote the number of incidences in the tangent fiber incidence graph $\mathcal{B}_T(\mathbb{LM})$. Then,

$$I(\mathbb{LM}) = \sum_{v \in \mathcal{V}_T^*} \mu_T(v) = \sum_{\mathbb{L} \cdot m \in \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]} d_{\mathcal{B}_T}(\mathbb{L} \cdot m),$$

where $\mathcal{V}_T^* = \{v \in \mathbb{R}^N \setminus \{0\} : \mu_T(v) \geq 2\}$ denotes the tangent-direction part of $\mathcal{B}_T(\mathbb{LM})$.

Proof. The tangent fiber incidence graph $\mathcal{B}_T(\mathbb{LM})$ is bipartite with parts $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$ and $\mathcal{V}_T^* = \{v \in \mathbb{R}^N \setminus \{0\} : \mu_T(v) \geq 2\}$. An incidence in $\mathcal{B}_T(\mathbb{LM})$ is an edge joining an orbit vertex $\mathbb{L} \cdot m$ to a tangent direction v whenever there is $x \in \mathbb{L} \cdot m$ such that $v \in T_x(\mathbb{L} \cdot m)$. For a fixed tangent direction $v \in \mathcal{V}_T^*$, the number of orbit vertices incident with v is exactly $|\mathcal{L}_T(v)| = \mu_T(v)$. Summing over all tangent directions in $\mathcal{V}_T^*$ gives

$$I(\mathbb{LM}) = \sum_{v \in \mathcal{V}_T^*} \mu_T(v).$$

On the other hand, counting the same incidences from the orbit-vertex side, the number of tangent directions incident with a fixed vertex $\mathbb{L} \cdot m$ is precisely its degree $d_{\mathcal{B}_T}(\mathbb{L} \cdot m)$ in the incidence graph. Hence,

$$I(\mathbb{LM}) = \sum_{\mathbb{L} \cdot m \in \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]} d_{\mathcal{B}_T}(\mathbb{L} \cdot m).$$

Since both sums count the same incidence edges of $\mathcal{B}_T(\mathbb{LM})$, once by tangent directions and once by orbit vertices, we obtain the desired result. $\square$

Proposition 2. Let $\mathbb{L}$ be the Lie group

$$\mathbb{L} = \left\{ \begin{pmatrix} 1 & s_1 & s_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : s_1, s_2 \in \mathbb{R} \right\}.$$

Let $\{e_1, e_2, \dots, e_{11}\}$ be the standard basis of $\mathbb{R}^{11}$ and define

$$\begin{aligned} V_1 &= \text{span}\{e_1, e_2\}, & V_2 &= \text{span}\{e_1, e_3\}, & V_3 &= \text{span}\{e_2, e_4\}, \\ V_4 &= \text{span}\{e_3, e_4\}, & V_5 &= \text{span}\{e_4, e_5\}, & V_6 &= \text{span}\{e_5, e_6\}. \end{aligned}$$

For $i = 1, \dots, 6$, let $c_i = ie_{11}$, define $\mathbb{M}_i = c_i + V_i$, put $\mathbb{M} = \bigcup_{i=1}^6 \mathbb{M}_i$ and define the action by

$$l \cdot (c_i + v) = c_i + v + A_i(s_1, s_2),$$

where $A_i : \mathbb{R}^2 \rightarrow V_i$ is the natural linear isomorphism. Then, $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ has six vertices and seven edges, $\mathcal{H}_T(\mathbb{LM})$ contains five hyperedges, $\text{diam}(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) = 3$ and $\gamma(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) = 2$.

Proof. The Lie algebra is

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$

Let

$$X = \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $X^2 = 0$, we have $\exp(tX) = I + tX$. Let $m = c_i + v \in \mathbb{M}_i$. Define $\gamma_i^X(t) = \exp(tX) \cdot m$. Then, $\gamma_i^X(t) = c_i + v + A_i(tx_1, tx_2)$. Therefore,

$$(\gamma_i^X)'(0) = \frac{d}{dt} \Big|_{t=0} (c_i + v + A_i(tx_1, tx_2)) = A_i(x_1, x_2).$$

Hence, $X_{\mathbb{M}}(m) = A_i(x_1, x_2)$ and therefore,

$$T_x(\mathbb{L} \cdot m_i) = \{X_{\mathbb{M}}(x) : X \in \mathfrak{g}\} = V_i \quad \text{for every } x \in \mathbb{L} \cdot m_i.$$

Since every A_i is surjective, each component $\mathbb{M}_i$ is one orbit. Therefore,

$$\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] = \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_2, \mathbb{L} \cdot m_3, \mathbb{L} \cdot m_4, \mathbb{L} \cdot m_5, \mathbb{L} \cdot m_6\}.$$

Now

$$V_1 \cap V_2 = \text{span}\{e_1\}, \quad V_1 \cap V_3 = \text{span}\{e_2\}, \quad V_2 \cap V_4 = \text{span}\{e_3\}, \quad V_3 \cap V_4 = \text{span}\{e_4\},$$

$$V_3 \cap V_5 = \text{span}\{e_4\}, \quad V_4 \cap V_5 = \text{span}\{e_4\},$$

and $V_5 \cap V_6 = \text{span}\{e_5\}$. All remaining intersections are equal to $\{0\}$. Since the tangent spaces are constant along every orbit and equal to V_i , Definition 1 implies that

$$\begin{aligned} \mathcal{E}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] = & \{\mathbb{L} \cdot m_1 \mathbb{L} \cdot m_2, \mathbb{L} \cdot m_1 \mathbb{L} \cdot m_3, \mathbb{L} \cdot m_2 \mathbb{L} \cdot m_4, \mathbb{L} \cdot m_3 \mathbb{L} \cdot m_4, \\ & \mathbb{L} \cdot m_3 \mathbb{L} \cdot m_5, \mathbb{L} \cdot m_4 \mathbb{L} \cdot m_5, \mathbb{L} \cdot m_5 \mathbb{L} \cdot m_6\}. \end{aligned}$$

Therefore, the graph has seven edges. Moreover,

$$\mathcal{L}_T(e_1) = \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_2\}, \quad \mathcal{L}_T(e_2) = \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_3\}, \quad \mathcal{L}_T(e_3) = \{\mathbb{L} \cdot m_2, \mathbb{L} \cdot m_4\},$$

$$\mathcal{L}_T(e_4) = \{\mathbb{L} \cdot m_3, \mathbb{L} \cdot m_4, \mathbb{L} \cdot m_5\}.$$

and $\mathcal{L}_T(e_5) = \{\mathbb{L} \cdot m_5, \mathbb{L} \cdot m_6\}$. Thus, $\mathcal{H}_T(\mathbb{LM})$ has five hyperedges. Observe that

$$\mathbb{L} \cdot m_2 - \mathbb{L} \cdot m_4 - \mathbb{L} \cdot m_5 - \mathbb{L} \cdot m_6$$

is a path joining $\mathbb{L} \cdot m_2$ and $\mathbb{L} \cdot m_6$, which has length three. Also,

$$\mathbb{L} \cdot m_1 - \mathbb{L} \cdot m_3 - \mathbb{L} \cdot m_5 - \mathbb{L} \cdot m_6$$

shows that some vertex pairs have distance at least three. Since every pair of vertices can be connected by a path containing at most three edges, we obtain $\text{diam}(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) = 3$. Finally, $D = \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_5\}$ dominates every vertex. Since no single vertex dominates $\mathbb{L} \cdot m_6$ together with $\{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_2\}$, we obtain $\gamma(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) = 2$. $\square$

5. Tangent-Separating Structures and Tangent Topologies

This section introduces tangent-separating structures in $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ and investigates how separating tangent directions determine local and global properties of the graph. We define tangent-separating paths and construct tangent topologies generated by their neighborhood systems. These structures provide a refined description of connectivity and tangent interactions between orbit vertices.

Definition 3. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be adjacent vertices of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. A nonzero vector v is called a separating tangent direction if there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ such that $v \in T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n)$, and for every vertex $\mathbb{L} \cdot p \notin \{\mathbb{L} \cdot m, \mathbb{L} \cdot n\}$ and every point $z \in \mathbb{L} \cdot p$, we have $v \notin T_z(\mathbb{L} \cdot p)$.

Definition 4. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be two vertices of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. A path

$$\mathbb{L} \cdot m = \mathbb{L} \cdot m_0 - \mathbb{L} \cdot m_1 - \cdots - \mathbb{L} \cdot m_r = \mathbb{L} \cdot n$$

is called a tangent-separating path (short, ts-path) if for every $i = 1, 2, \dots, r$, there is a separating tangent direction v_i together with points $x_{i-1} \in \mathbb{L} \cdot m_{i-1}$ and $x_i \in \mathbb{L} \cdot m_i$ such that $v_i \in T_{x_{i-1}}(\mathbb{L} \cdot m_{i-1}) \cap T_{x_i}(\mathbb{L} \cdot m_i)$.

Definition 5. The tangent-separating neighborhood of a vertex $\mathbb{L} \cdot m$, denoted by $\Omega_T[\mathbb{L} \cdot m]$, is defined as the set consisting of $\mathbb{L} \cdot m$ together with all vertices $\mathbb{L} \cdot n$ such that there is a ts-path of length at most two connecting $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$.

For example, in Example 3, the vector u_{12} is a separating tangent direction for the edge joining $\mathbb{L} \cdot m_1$ and $\mathbb{L} \cdot m_2$, because it belongs to $V_1 \cap V_2$ and does not belong to any other orbit tangent space in that construction. Hence, this edge itself is a ts-path of length one. The neighborhood $\Omega_T[\mathbb{L} \cdot m]$ then collects vertices that can be reached from $\mathbb{L} \cdot m$ by at most two such separating steps.

Define

$$\mathcal{N}_T(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) = \{\Omega_T[O] : O \in \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]\}.$$

A subset $U \subseteq \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$ is called $\tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM}))$ -open whenever for every $O \in U$ we have $\Omega_T[O] \subseteq U$. The resulting topology on $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$ is called the tangent-separating topology and is denoted by $\tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM}))$.

Example 3. Let $\mathbb{L}$ be the five-dimensional Abelian Lie group

$$\mathbb{L} = \left\{ \begin{pmatrix} 1 & s_1 & s_2 & s_3 & s_4 & s_5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : s_1, s_2, s_3, s_4, s_5 \in \mathbb{R} \right\}.$$

Let

$$\{u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{25}, u_{26}, u_{34}, u_{35}, u_{36}, u_{45}, u_{46}, u_{56}, u_{57}, u_{67}, p_1, p_2, q_1, q_2, q_3, w\}$$

be a basis of $\mathbb{R}^{21}$, where each symbol u_{ij} denotes the basis vector indexed by the pair (i, j) . Define

$$\begin{aligned} V_1 &= \text{span}\{u_{12}, u_{13}, u_{14}, p_1, p_2\}, & V_2 &= \text{span}\{u_{12}, u_{23}, u_{24}, u_{25}, u_{26}\}, \\ V_3 &= \text{span}\{u_{13}, u_{23}, u_{34}, u_{35}, u_{36}\}, & V_4 &= \text{span}\{u_{14}, u_{24}, u_{34}, u_{45}, u_{46}\}, \\ V_5 &= \text{span}\{u_{25}, u_{35}, u_{45}, u_{56}, u_{57}\}, & V_6 &= \text{span}\{u_{26}, u_{36}, u_{46}, u_{56}, u_{67}\}, \\ V_7 &= \text{span}\{u_{57}, u_{67}, q_1, q_2, q_3\}. \end{aligned}$$

For $i = 1, 2, \dots, 7$, let $c_i = iw$, define $\mathbb{M}_i = c_i + V_i$ and put $\mathbb{M} = \bigcup_{i=1}^7 \mathbb{M}_i \subseteq \mathbb{R}^{21}$. Let $A_i : \mathbb{R}^5 \rightarrow V_i$ be the linear isomorphism sending the standard basis of $\mathbb{R}^5$ onto the five displayed generators of V_i . For $l \in \mathbb{L}$ written as

$$l = \begin{pmatrix} 1 & s_1 & s_2 & s_3 & s_4 & s_5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and for $m = c_i + v \in \mathbb{M}_i$, define

$$l \cdot m = c_i + v + A_i(s_1, s_2, s_3, s_4, s_5).$$

Since $w \notin V_i$ for all $i = 1, \dots, 7$, the affine sets $\mathbb{M}_i = c_i + V_i$ are pairwise disjoint. Also, since each A_i is onto V_i , each $\mathbb{M}_i$ is a non-trivial orbit, say, $\mathbb{L} \cdot m_i = \mathbb{M}_i$. The Lie algebra $\mathfrak{g}$ consists of matrices

$$X = \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $X^2 = 0$, we have $\exp(tX) = I_6 + tX$. Hence, for $m = c_i + v \in \mathbb{M}_i$,

$$X_{\mathbb{M}}(m) = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX) \cdot m) = A_i(x_1, x_2, x_3, x_4, x_5).$$

Thus, $T_x(\mathbb{L} \cdot m_i) = V_i$ for every $x \in \mathbb{L} \cdot m_i$ and every $i = 1, 2, \dots, 7$. The nonzero intersections are exactly

$$\begin{aligned} V_1 \cap V_2 &= \text{span}\{u_{12}\}, & V_1 \cap V_3 &= \text{span}\{u_{13}\}, & V_1 \cap V_4 &= \text{span}\{u_{14}\}, & V_2 \cap V_3 &= \text{span}\{u_{23}\}, \\ V_2 \cap V_4 &= \text{span}\{u_{24}\}, & V_2 \cap V_5 &= \text{span}\{u_{25}\}, & V_2 \cap V_6 &= \text{span}\{u_{26}\}, & V_3 \cap V_4 &= \text{span}\{u_{34}\}, \\ V_3 \cap V_5 &= \text{span}\{u_{35}\}, & V_3 \cap V_6 &= \text{span}\{u_{36}\}, & V_4 \cap V_5 &= \text{span}\{u_{45}\}, & V_4 \cap V_6 &= \text{span}\{u_{46}\}, \\ V_5 \cap V_6 &= \text{span}\{u_{56}\}, & V_5 \cap V_7 &= \text{span}\{u_{57}\}, & V_6 \cap V_7 &= \text{span}\{u_{67}\}. \end{aligned}$$

All remaining pairwise intersections are equal to $\{0\}$. Since tangent spaces are constant along each orbit and equal to the corresponding subspaces V_i , Definition 1 implies that $\text{Lie}_{\mathfrak{g}}(\mathbb{L}\mathbb{M})$ has seven vertices and exactly fifteen edges. Moreover, every displayed edge has a separating tangent direction, because each vector u_{ij} belongs only to the two corresponding orbit tangent spaces associated with V_i and V_j ; see Figure 3. Since there is no ts-path of length at most two joining $\mathbb{L} \cdot m_1$ and

$\mathbb{L} \cdot m_7$, while every other vertex is joined to $\mathbb{L} \cdot m_1$ by a ts -path of length at most two, we get $\Omega_T[\mathbb{L} \cdot m_1] = \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] \setminus \{\mathbb{L} \cdot m_7\}$. Similarly,

$$\Omega_T[\mathbb{L} \cdot m_7] = \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] \setminus \{\mathbb{L} \cdot m_1\}.$$

For $i = 2, 3, 4, 5, 6$, every vertex is connected to $\mathbb{L} \cdot m_i$ by a ts -path of length at most two. Hence,

$$\Omega_T[\mathbb{L} \cdot m_i] = \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})].$$

Therefore,

$$\tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})) = \{\emptyset, \mathcal{V} \setminus \{\mathbb{L} \cdot m_1, \mathbb{L} \cdot m_7\}, \mathcal{V} \setminus \{\mathbb{L} \cdot m_1\}, \mathcal{V} \setminus \{\mathbb{L} \cdot m_7\}, \mathcal{V}\},$$

where $\mathcal{V} := \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$.

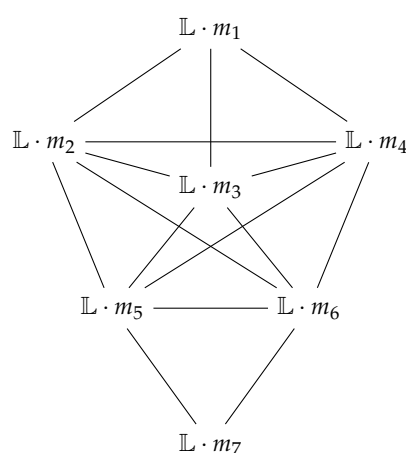


Figure 3. $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ corresponding to Example 3.

Theorem 17. Suppose that $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is connected and every edge of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ admits a separating tangent direction. Then, every two vertices of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ are connected by a ts -path.

Proof. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be arbitrary vertices of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. Since $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ is connected, there is a path

$$\mathbb{L} \cdot m = \mathbb{L} \cdot m_0 - \mathbb{L} \cdot m_1 - \dots - \mathbb{L} \cdot m_r = \mathbb{L} \cdot n.$$

For each $i = 1, 2, \dots, r$, the pair $\mathbb{L} \cdot m_{i-1}$ and $\mathbb{L} \cdot m_i$ is an edge of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$. By hypothesis, this edge admits a separating tangent direction. Hence, there is a nonzero vector v_i together with points $x_{i-1} \in \mathbb{L} \cdot m_{i-1}$ and $x_i \in \mathbb{L} \cdot m_i$ such that $v_i \in T_{x_{i-1}}(\mathbb{L} \cdot m_{i-1}) \cap T_{x_i}(\mathbb{L} \cdot m_i)$, and $v_i \notin T_z(\mathbb{L} \cdot p)$ for every vertex $\mathbb{L} \cdot p \notin \{\mathbb{L} \cdot m_{i-1}, \mathbb{L} \cdot m_i\}$ and every point $z \in \mathbb{L} \cdot p$. Therefore, every edge of the path admits a separating tangent direction. Hence,

$$\mathbb{L} \cdot m_0 - \mathbb{L} \cdot m_1 - \dots - \mathbb{L} \cdot m_r$$

is a ts -path. Since $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ were arbitrary, every two vertices are connected by a ts -path. $\square$

Theorem 18. If every two vertices of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ are connected by a ts -path, then the topological space $(\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})], \tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})))$ is connected.

Proof. Suppose that $(\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})], \tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})))$ is disconnected. Then, there exist nonempty disjoint open sets U and V such that $U \cup V = \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$. Choose $\mathbb{L} \cdot m \in U$ and $\mathbb{L} \cdot n \in V$. Since every two vertices are connected by a ts-path, there is a ts-path

$$\mathbb{L} \cdot m = \mathbb{L} \cdot m_0 - \mathbb{L} \cdot m_1 - \cdots - \mathbb{L} \cdot m_r = \mathbb{L} \cdot n.$$

Since $\mathbb{L} \cdot m_0 \in U$ and $\mathbb{L} \cdot m_r \in V$, there exists an index k such that $\mathbb{L} \cdot m_k \in U$ and $\mathbb{L} \cdot m_{k+1} \notin U$. Since U is open and $\mathbb{L} \cdot m_k \in U$, the definition of τ_t implies that $\Omega_T[\mathbb{L} \cdot m_k] \subseteq U$. Because $\mathbb{L} \cdot m_k - \mathbb{L} \cdot m_{k+1}$ is part of a ts-path, we have $\mathbb{L} \cdot m_{k+1} \in \Omega_T[\mathbb{L} \cdot m_k]$. Hence, $\mathbb{L} \cdot m_{k+1} \in U$, which contradicts the choice of $\mathbb{L} \cdot m_{k+1}$. Therefore,

$$(\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})], \tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})))$$

is connected. $\square$

Theorem 19. Let $\mathcal{G}_t$ be the graph whose vertex set is $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$ and whose edges are exactly the edges of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ admitting separating tangent directions. Then, the connected components of $(\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})], \tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})))$ coincide with the ts-path components of $\mathcal{G}_t$. In particular, if $\mathcal{G}_t$ is connected, then $(\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})], \tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})))$ is connected.

Proof. For a vertex $O \in \mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$, let $C_t(O)$ denote the set of all vertices connected to O by a ts-path in $\mathcal{G}_t$. We first prove that $C_t(O)$ is open. Let $P \in C_t(O)$. If $Q \in \Omega_T[P]$, then P and Q are connected by a ts-path of length at most two. Since O is connected to P by a ts-path, concatenating the two paths gives a ts-path from O to Q . Hence, $Q \in C_t(O)$ and therefore, $\Omega_T[P] \subseteq C_t(O)$. By the definition of τ_t , this implies that $C_t(O)$ is open. Now, we show that $C_t(O)$ is closed. The complement $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})] \setminus C_t(O)$ is a union of ts-path components different from $C_t(O)$. By the first part, each such component is open. Hence, the complement of $C_t(O)$ is open and therefore $C_t(O)$ is closed. Assume that $C_t(O) = U \cup V$, where U and V are nonempty disjoint subsets open in the subspace topology on $C_t(O)$. Choose $P \in U$ and $Q \in V$. Since $P, Q \in C_t(O)$, there exists a ts-path

$$P = P_0 - P_1 - \cdots - P_r = Q.$$

Since $P_0 \in U$ and $P_r \in V$, there exists an index k such that $P_k \in U$ and $P_{k+1} \notin U$. Since U is open in the subspace topology and $P_k \in U$, we obtain $\Omega_T[P_k] \cap C_t(O) \subseteq U$. Because $P_k - P_{k+1}$ is part of a ts-path, $P_{k+1} \in \Omega_T[P_k]$. Since also $P_{k+1} \in C_t(O)$, we obtain $P_{k+1} \in \Omega_T[P_k] \cap C_t(O) \subseteq U$, contradicting the choice of P_{k+1} . Therefore, $C_t(O)$ is connected. Thus, every ts-path component is connected, open, and closed. Since different ts-path components are disjoint and cover $\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})]$, they are exactly the connected components of $(\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})], \tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})))$. In particular, if $\mathcal{G}_t$ is connected, then there is only one ts-path component. Hence, the whole space is connected. $\square$

Theorem 20. Let $\mathcal{H}_T(\mathbb{LM})$ be the tangent orbit hypergraph and let $\mathcal{G}_t$ be the graph whose edges are exactly the edges of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$ admitting separating tangent directions. Suppose that for every two vertices of $\text{Lie}_{\mathfrak{g}}(\mathbb{LM})$, there is a finite sequence of tangent fibers $\mathcal{L}_T(v_1), \mathcal{L}_T(v_2), \dots, \mathcal{L}_T(v_r)$ such that the first vertex belongs to $\mathcal{L}_T(v_1)$, the second vertex belongs to $\mathcal{L}_T(v_r)$, $\mathcal{L}_T(v_i) \cap \mathcal{L}_T(v_{i+1}) \neq \emptyset$ for every $i = 1, 2, \dots, r-1$ and every edge in the 2-section graph induced by these fibers admits a separating tangent direction. Then, the topological space $(\mathcal{V}[\text{Lie}_{\mathfrak{g}}(\mathbb{LM})], \tau_t(\text{Lie}_{\mathfrak{g}}(\mathbb{LM})))$ is connected.

Proof. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be arbitrary vertices of $Lie_{\mathfrak{g}}(\mathbb{LM})$. By hypothesis, there is a finite sequence of tangent fibers

$$\mathcal{L}_T(v_1), \mathcal{L}_T(v_2), \dots, \mathcal{L}_T(v_r)$$

such that $\mathbb{L} \cdot m \in \mathcal{L}_T(v_1)$, $\mathbb{L} \cdot n \in \mathcal{L}_T(v_r)$, and $\mathcal{L}_T(v_i) \cap \mathcal{L}_T(v_{i+1}) \neq \emptyset$ for every $i = 1, 2, \dots, r-1$. Choose $O_i \in \mathcal{L}_T(v_i) \cap \mathcal{L}_T(v_{i+1})$ for $i = 1, 2, \dots, r-1$. By Theorem 11, $Lie_{\mathfrak{g}}(\mathbb{LM})$ is the 2-section graph of $\mathcal{H}_T(\mathbb{LM})$. Hence, any two distinct vertices contained in the same tangent fiber are adjacent. Therefore, after removing repeated consecutive vertices if necessary, we obtain

$$\mathbb{L} \cdot m - O_1 - O_2 - \dots - O_{r-1} - \mathbb{L} \cdot n.$$

By hypothesis, every edge in this path admits a separating tangent direction. Therefore, the obtained path is a ts-path. Since the vertices were arbitrary, every two vertices are connected by a ts-path. Hence, by Theorem 18, $(\mathcal{V}[Lie_{\mathfrak{g}}(\mathbb{LM})], \tau_t(Lie_{\mathfrak{g}}(\mathbb{LM})))$ is connected. $\square$

Theorem 21. Let $\mathbb{L} \cdot m - \mathbb{L} \cdot n$ be an edge of $Lie_{\mathfrak{g}}(\mathbb{LM})$ admitting a separating tangent direction v . Then, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ and an element $X \in \mathfrak{g}$ such that $\left. \frac{d}{dt} \right|_{t=0} (\exp(tX) \cdot x) = v$. Moreover, $\mathbb{L} \cdot n \in \Omega_T[\mathbb{L} \cdot m]$.

Proof. Since $\mathbb{L} \cdot m - \mathbb{L} \cdot n$ is an edge admitting a separating tangent direction, there exist points $x \in \mathbb{L} \cdot m$ and $y \in \mathbb{L} \cdot n$ such that

$$v \in T_x(\mathbb{L} \cdot m) \cap T_y(\mathbb{L} \cdot n).$$

Since $v \in T_x(\mathbb{L} \cdot m)$, by the characterization of orbit tangent spaces, there is $X \in \mathfrak{g}$ such that

$$v = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX) \cdot x).$$

Because v is separating, the edge $\mathbb{L} \cdot m - \mathbb{L} \cdot n$ itself forms a ts-path of length one. Therefore, $[\mathbb{L} \cdot n \in \Omega_T[\mathbb{L} \cdot m]]$. $\square$

Corollary 1. Suppose that every edge of $Lie_{\mathfrak{g}}(\mathbb{LM})$ admits a separating tangent direction. Then, the following are equivalent:

1. $Lie_{\mathfrak{g}}(\mathbb{LM})$ is connected.
2. Every two vertices are connected by a ts-path.
3. The tangent orbit hypergraph $\mathcal{H}_T(\mathbb{LM})$ is fiber-connected.
4. $(\mathcal{V}[Lie_{\mathfrak{g}}(\mathbb{LM})], \tau_t(Lie_{\mathfrak{g}}(\mathbb{LM})))$ is connected.

Proof. (1) $\Rightarrow$ (2) Assume that $Lie_{\mathfrak{g}}(\mathbb{LM})$ is connected. Since every edge admits a separating tangent direction, Theorem 17 implies that every two vertices are connected by a ts-path. Hence, (2) holds. (2) $\Rightarrow$ (3) Assume that every two vertices are connected by a ts-path. Let $\mathbb{L} \cdot m$ and $\mathbb{L} \cdot n$ be arbitrary vertices. Then, there is a ts-path

$$\mathbb{L} \cdot m = \mathbb{L} \cdot m_0 - \mathbb{L} \cdot m_1 - \dots - \mathbb{L} \cdot m_r = \mathbb{L} \cdot n.$$

For each edge $\mathbb{L} \cdot m_{i-1} - \mathbb{L} \cdot m_i$, there is a separating tangent direction v_i and points $x_{i-1} \in \mathbb{L} \cdot m_{i-1}$ and $x_i \in \mathbb{L} \cdot m_i$ such that $v_i \in T_{x_{i-1}}(\mathbb{L} \cdot m_{i-1}) \cap T_{x_i}(\mathbb{L} \cdot m_i)$. Therefore, $\mathbb{L} \cdot m_{i-1}, \mathbb{L} \cdot m_i \in \mathcal{L}_T(v_i)$. Hence, the sequence

$$\mathcal{L}_T(v_1), \mathcal{L}_T(v_2), \dots, \mathcal{L}_T(v_r)$$

forms a chain of tangent fibers joining the two vertices. Thus, the tangent orbit hypergraph is fiber-connected. (3) $\Rightarrow$ (4) Suppose that the tangent orbit hypergraph is fiber-connected. Then, Theorem 20 implies that every two vertices are connected by a ts-path. Therefore, by Theorem 18, $(\mathcal{V}[Lie_g(\mathbb{LM})], \tau_t(Lie_g(\mathbb{LM})))$ is connected. (4) $\Rightarrow$ (1) Suppose that $(\mathcal{V}[Lie_g(\mathbb{LM})], \tau_t(Lie_g(\mathbb{LM})))$ is connected. Assume that $Lie_g(\mathbb{LM})$ is disconnected. Then, the vertex set can be written as a disjoint union $\mathcal{V} = A \cup B$ where there are no graph edges joining a vertex of A to a vertex of B . Consequently, no ts-path joins a vertex of A to a vertex of B . By Theorem 19, ts-path components coincide with connected components of the topological space. Hence, A and B determine two disjoint nonempty clopen subsets of $(\mathcal{V}[Lie_g(\mathbb{LM})], \tau_t(Lie_g(\mathbb{LM})))$ contradicting connectedness. Therefore, $Lie_g(\mathbb{LM})$ is connected. $\square$

6. Conclusions

In this work, we introduced Lie orbit tangent graphs as an extrinsic graph-theoretical framework associated with smooth Lie group actions. The main novelty is that orbit vertices are connected through common nonzero ambient tangent directions, rather than through ordinary set-theoretic intersections of algebraic subobjects. This makes the construction sensitive to infinitesimal orbit geometry and separates it from classical intersection graphs. We developed tangent fibers, local tangent orbit cliques, tangent orbit hypergraphs and incidence graphs in order to describe pairwise and higher-order tangent interactions among orbit vertices. We also introduced separating tangent directions, tangent-separating paths and tangent-separating neighborhood systems and used them to construct a topology generated by tangent-interaction data. The obtained results establish connectedness criteria, clique and domination descriptions, incidence-count identities, diameter estimates and relationships between fiber-connected hypergraphs and connected tangent separating topologies.

The framework may be useful in geometric analysis, differential topology, dynamical systems, symmetry analysis and mathematical physics. In these settings, Lie group orbits often represent symmetry classes, invariant motions or reduced configurations, while tangent directions represent infinitesimal transformations. The proposed graph and hypergraph structures provide a combinatorial language for detecting shared infinitesimal modes and higher-order orbit interactions. The present study is intentionally foundational. The construction depends on the chosen ambient embedding used to compare tangent spaces, and most global counting statements were formulated under finite-vertex or finite-incidence assumptions. Future work may develop intrinsic versions using connections or parallel transport, study locally finite infinite orbit graphs, investigate weighted or directed versions based on the dimension or orientation of tangent intersections, and apply the framework to concrete symmetry-reduced models in geometry and mathematical physics.

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