

Article

Computational Lower Bounds for Wheel Ramsey Numbers

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Abstract

We study small off-diagonal Ramsey numbers involving wheel graphs. Following Radziszowski's convention, we write $W_n = K_1 + C_{n-1}$ for the wheel on n vertices. A lower bound for $R(W_m, W_n)$ is certified by a graph G such that G is W_m -free and $\bar{G}$ is W_n -free. We conduct computational experiments using several complementary search techniques, including partial Ramsey colorings and local search procedures. The computations exploit the structure of wheels: G contains W_n if and only if some vertex neighborhood contains a cycle C_{n-1} . We report new lower-bound constructions for three off-diagonal wheel Ramsey numbers and provide the corresponding Ramsey graphs as explicit certificates.

Keywords: wheel Ramsey numbers; Ramsey graphs; computational lower bounds

MSC: 05C55; 05D10

1. Introduction

Ramsey theory is concerned with the appearance of unavoidable structures in sufficiently large discrete systems. It originated with Ramsey's remarkable theorem [1].

Theorem 1. *For every pair of positive integers s, t , there exists an integer N such that every red–blue coloring of the edges of K_N contains either a red copy of K_s or a blue copy of K_t .*

The smallest such integer N is called the *Ramsey number* of the pair (K_s, K_t) and is denoted by $R(K_s, K_t)$, or simply $R(s, t)$. In general, for two graphs G and H , the Ramsey number $R(G, H)$ is the smallest positive integer N such that every red–blue coloring of the edges of the complete graph K_N contains either a red copy of G or a blue copy of H . Equivalently, $R(G, H)$ is the least integer N such that every graph on N vertices contains G as a subgraph or its complement contains H as a subgraph.

The existence of $R(G, H)$ follows from Theorem 1. Indeed, let g and h be the number of vertices in graphs G and H , respectively. By Theorem 1, $R(K_g, K_h)$ exists. Let $N = R(K_g, K_h)$. Then, every red–blue coloring of the edges of K_N contains either a red K_g or a blue K_h . Since G is a subgraph of K_g and H is a subgraph of K_h , such a coloring contains either a red copy of G or a blue copy of H . Therefore, $R(G, H)$ exists for every pair of finite graphs G, H and $R(G, H) \leq R(K_g, K_h)$.

By the definition of $R(G, H)$, for any $N < R(G, H)$, there exists a graph F on N vertices such that G is not a subgraph of F and H is not a subgraph of its complement $\bar{F}$. We call such a graph a $(G, H; N)$ -Ramsey graph. It is a *witness* for the lower bound $R(G, H) \geq N + 1$.

In this paper, we study Ramsey numbers involving wheel graphs. Let C_k denote the cycle on k vertices. Following Radziszowski [2], we write $W_n = K_1 + C_{n-1}$ for the wheel on n vertices. Some authors instead use $W_n = K_1 + C_n$, indexing wheels by the length of the



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cycle. The added vertex is called the *hub*, while the cycle C_{n-1} is called the *rim* of the wheel. Ramsey problems for wheels are closely related to Ramsey problems for cycles, but they are more restrictive because the existence of a cycle alone is not enough; one must also force a vertex adjacent to all vertices of that cycle. Thus,

$$R(W_m, W_n) \geq R(C_{m-1}, W_n) \geq R(C_{m-1}, C_{n-1}).$$

1.1. Diagonal Wheel Ramsey Numbers

The diagonal case, where both forbidden monochromatic graphs are the same wheel, is especially interesting. We write $R(W_n) := R(W_n, W_n)$. Thus, $R(W_n)$ is the smallest integer N such that every red–blue coloring of the edges of K_N contains a monochromatic copy of W_n . Exact values are known for some small wheels, while for larger wheels the problem is often treated through upper and lower bounds. In particular, the case of even wheels and odd wheels may behave differently, since the parity of the rim cycle affects the structure of extremal colorings. It has shown that the problem remains challenging even for wheels of moderate order [3–5].

Previous work has established several exact values and bounds for diagonal wheel Ramsey numbers. For example, Faudree and McKay [3] proved that $R(W_6) = 17$, thereby disproving a conjecture of Erdős. Later computational work gave additional small diagonal values and bounds; for instance, Lidický and Pfender [6] obtained $R(W_7) = 19$ and the upper bound $R(W_8) \leq 26$. Recently, DeBiasio and Wimbish [5] and Zhang and Chen [7] improved several general bounds for diagonal wheel Ramsey numbers.

DeBiasio and Wimbish [5] use the convention $W_k = K_1 + C_k$. Translating their bounds into our notation, we obtain

$$\begin{cases} 3n - 2 \leq R(W_n) \leq 6n - 10, & \text{if } n \text{ is even,} \\ 5k - \frac{1 + (-1)^k}{2} \leq R(W_n) \leq 8k + 664, & \text{if } n = 2k + 1 \text{ is odd.} \end{cases}$$

Zhang and Chen [7] proved that, for $n \geq 8$,

$$\begin{cases} 3n - 2 \leq R(W_n) \leq \min\left\{6n - 10, \frac{16n + 1978}{3}\right\}, & \text{if } n \text{ is even,} \\ \frac{5n - 6 + \operatorname{sgn}(n)}{2} \leq R(W_n) \leq 4n + \min\left\{\frac{n - 9}{2}, 660\right\}, & \text{if } n \text{ is odd,} \end{cases}$$

where $\operatorname{sgn}(4k + 1) = -1$ and $\operatorname{sgn}(4k + 3) = 1$. Then, $25 \leq R(W_{11}) \leq 45$.

1.2. Off-Diagonal Wheel Ramsey Numbers

Burr and Erdős [8] determined infinitely many exact values $R(W_m, W_n)$ in the case where one of the two parameters is 3, where $W_3 = C_3 = K_3$. In particular, $R(W_3, W_n) = 2n - 1$ for all $n \geq 6$. Several Ramsey numbers $R(W_m, W_n)$ are known exactly for $4 \leq m < n$. For other pairs (m, n) , only lower and upper bounds are known. Lidický and Pfender [6] applied flag algebras and semidefinite programming to finite Ramsey problems. Their computations yielded several wheel-related upper bounds, including

$$\begin{aligned} R(W_4, W_7) &\leq 21, & R(W_4, W_8) &\leq 26, & R(W_5, W_7) &\leq 16, & R(W_5, W_8) &\leq 17, \\ R(W_6, W_7) &\leq 19, & R(W_6, W_8) &\leq 26, & R(W_7, W_7) &\leq 19, & R(W_7, W_8) &\leq 21, \\ R(W_8, W_8) &\leq 26. \end{aligned}$$

Lidický et al. [9] described a range of different computational methods applied for bounds on Ramsey numbers including flag algebras, local search, bottom-up generation,

and enumeration of polycirculant graphs. Several bounds were obtained for Ramsey numbers involving wheel graphs, both in the diagonal and off-diagonal cases. More generally, Radziszowski's dynamic survey [2] tabulates many small off-diagonal wheel-wheel values and bounds.

1.3. Computational Methods

Computational methods can be applied to problems on Ramsey numbers to obtain both lower and upper bounds. Lower bounds are usually obtained by constructing explicit Ramsey graphs, while upper bounds require proving that no such graph exists.

One common lower-bound method is heuristic local search. Starting from a random coloring of the edges of K_N , one repeatedly changes edge colors in order to reduce the number of forbidden monochromatic subgraphs. Variants include simulated annealing, tabu search, genetic algorithms, and more recent reinforcement-learning approaches. These methods are effective at finding Ramsey graphs with little apparent structure.

Another approach is to search within structured graph families. Circulant, block-circulant, and more generally polycirculant graphs have been used for many Ramsey lower-bound constructions. In a circulant graph, adjacency is determined by cyclic distance; in a block-circulant graph, the adjacency matrix is partitioned into circulant blocks. Such restrictions greatly reduce the search space while still producing strong constructions.

Ramsey problems can also be encoded as satisfiability problems. In SAT encoding, each edge of K_N is assigned a Boolean variable representing its color, and clauses are added to forbid monochromatic copies of the target graphs. A satisfying assignment gives a lower-bound construction, while an unsatisfiable instance proves an upper bound. SAT solvers are often combined with symmetry breaking and cube-and-conquer techniques.

For upper bounds, exhaustive generation and isomorph rejection are classical tools. One generates all graphs avoiding one forbidden graph and tests whether their complements avoid the other. Canonical-labeling tools such as nauty are commonly used to avoid generating isomorphic copies multiple times. This approach is powerful for small and medium-sized cases but becomes expensive as the number of vertices grows.

Another upper-bound method is the use of semidefinite programming and flag algebras. In this approach one derives inequalities between densities of small colored graphs. A semidefinite-programming certificate can then show that a hypothetical Ramsey graph on N vertices cannot exist. Such methods have been applied successfully to several finite Ramsey problems.

Finally, some computations use exact integer-programming or constraint programming formulations. These are similar in spirit to SAT encodings but allow additional linear constraints, objective functions, or specialized branch-and-bound strategies.

1.4. Results

In this paper we study small off-diagonal Ramsey numbers involving wheel graphs. Known values and bounds for Ramsey numbers involving wheel graphs are summarized in Radziszowski's dynamic survey [2]. In particular, Table VIII in that survey summarizes the currently known bounds on $R(W_m, W_n)$ for $m \leq n \leq 10$. We implemented a heuristic search for edge colorings of K_N which avoid a red W_m and a blue W_n . Any such coloring is a certificate for $R(W_m, W_n) > N$, equivalently $R(W_m, W_n) \geq N + 1$. We report new lower bounds for $m = 5, 7, 9$ and $n = 11$ (Theorem 2). The new lower bounds and the currently known bounds on $R(W_m, W_n)$ for $m \leq n \leq 11$ are shown in Table 1.

Table 1. The bounds on Ramsey numbers $R(W_m, W_n)$ for $m \leq n \leq 11$. A single integer denotes an exact value. An interval denotes the best lower and upper bounds listed in the table. An entry of the form $\geq L$ denotes a known lower bound, with no corresponding special upper bound recorded in the table beyond general Ramsey-type bounds. The new bounds are marked in bold.

m	n	3	4	5	6	7	8	9	10	11
3		6	9	11	11	13	15	17	19	21
4			18	17	19	21	22–26	≥ 25	≥ 28	≥ 31
5				15	17	15	17	18	≥ 21	≥ 22
6					17	19	22–26	≥ 25	≥ 28	≥ 31
7						19	19–21	≥ 19	≥ 21	≥ 22
8							22–25	≥ 25	≥ 28	≥ 31
9								≥ 21	≥ 25	≥ 23
10									≥ 28	≥ 31
11										≥ 25

2. Heuristic Search

We use the following notation throughout this section. Let $G = (V, E)$ be a graph. For each vertex $v \in V$, let $N_G(v)$ denote its neighborhood in a graph G . For a set $V' \subseteq V$, the subgraph induced by V' is denoted by $G[V']$.

We use a local search heuristic to construct candidate $(W_m, W_n; N)$ -Ramsey graphs. The goal is to find a graph G on N vertices such that W_m is not a subgraph of G and W_n is not a subgraph of $\overline{G}$. Recall that $W_m = K_1 + C_{m-1}$. Hence, a copy of W_m in G is detected by a vertex v such that the induced graph on its neighborhood $G[N_G(v)]$ contains a cycle of length $m - 1$. We call such a vertex a *bad hub*. Similarly, a bad hub in $\overline{G}$ is a vertex v such that $\overline{G}[N_{\overline{G}}(v)]$ contains a cycle of length $n - 1$.

We now describe an algorithm for testing whether a given graph G is a $(W_m, W_n; N)$ -Ramsey graph (Algorithm 1). The algorithm tests separately whether G contains a copy of W_m and whether $\overline{G}$ contains a copy of W_n .

Algorithm 1: Testing whether G is a $(W_m, W_n; N)$ -Ramsey graph

Input: A graph $G = (V, E)$ on N vertices

Output: TRUE if G is a $(W_m, W_n; N)$ -Ramsey graph, and FALSE otherwise

foreach $v \in V$ **do**

if $G[N_G(v)]$ contains a cycle C_{m-1} **then**
 return FALSE;

Let $\overline{G}$ be the complement of G ;

foreach $v \in V$ **do**

if $\overline{G}[N_{\overline{G}}(v)]$ contains a cycle C_{n-1} **then**
 return FALSE;

return TRUE;

Since $W_m = K_1 + C_{m-1}$, the vertex v can serve as the hub of a copy of W_m if and only if the induced subgraph $G[N_G(v)]$ contains a cycle of length $m - 1$. Therefore, to test whether W_m is a subgraph of G , we inspect all vertices v and search for a C_{m-1} in $G[N_G(v)]$. If such a cycle is found for some v , then graph G contains W_m .

The same test is applied to the complement graph. Namely, for each vertex v , we consider its neighborhood $N_{\bar{G}}(v)$ in $\bar{G}$ and test whether $\bar{G}[N_{\bar{G}}(v)]$ contains a cycle of length $n - 1$. If such a cycle is found, then graph $\bar{G}$ contains W_n .

Thus, G is a $(W_m, W_n; N)$ -Ramsey graph if and only if both tests fail: no vertex of G is a bad hub for W_m , and no vertex of $\bar{G}$ is a bad hub for W_n .

This reduces testing a Ramsey graph to the problem of cycle detection.

Cycle Detection Problem. Given a graph G and an integer k , decide whether G contains a cycle of length exactly k .

This problem is NP-complete if k is part of the input. However, if k is a fixed constant, then the problem is polynomial-time solvable by brute force in $O(n^k)$ time (by checking ordered k -tuples of distinct vertices).

Setting	Complexity
k fixed	polynomial time
k part of input	NP-complete
k as parameter	FPT

Several algorithms have been developed for the cycle detection problem; see the summary in Table 2.

Table 2. Some algorithms for detecting a cycle of prescribed length. Here, $n = |V|$, and ω denotes the matrix-multiplication exponent.

Algorithm	Setting	Running Time
Naive enumeration	fixed k	$O(n^k)$
Monien [10]	fixed k	$O(k! V E)$
Alon–Yuster–Zwick [11]	fixed or parameterized k	$2^{O(k)} n^{O(1)}$
Alon–Yuster–Zwick [12]	fixed k	$O(V ^\omega \log V)$
Yuster–Zwick [13]	fixed even cycle C_{2k}	$O(V ^2)$

The testing algorithm above only needs to detect whether a forbidden cycle exists. In the local search heuristic, however, we use a score function based on cycle counts rather than mere cycle detection.

Setting	Complexity of Counting C_k
k fixed	polynomial time
k part of input	#P-complete
$k = V(G) $	counting Hamiltonian cycles, #P-complete

We use two related score functions. The first score counts bad hubs:

$$s_{\text{hub}}(G) = \#\{v : G[N_G(v)] \text{ contains a } C_{m-1}\} + \#\{v : \bar{G}[N_{\bar{G}}(v)] \text{ contains a } C_{n-1}\}. \quad (1)$$

Thus, $s_{\text{hub}}(G) = 0$ if and only if G is a $(W_m, W_n; N)$ -Ramsey graph.

Let $n_{\text{sub}}(G, H)$ be the number of subgraphs of a graph G isomorphic to H . Let $\text{Inj}(G, H)$ denote the number of injective homomorphisms from H to G and let $\text{Aut}(H)$ denote the number of automorphisms of H . Then, $n_{\text{sub}}(G, H) = \text{Inj}(G, H) / \text{Aut}(H)$.

For $n \geq 5$, the hub of $W_n = K_1 + C_{n-1}$ is the unique vertex of degree $n - 1$, and hence it is fixed by every automorphism. Therefore, the automorphisms of W_n are precisely the automorphisms of the rim cycle C_{n-1} . Hence, $\text{Aut}(W_n) \cong \text{Aut}(C_{n-1}) \cong D_{n-1}$, and $|\text{Aut}(W_n)| = 2(n - 1)$, $n \geq 5$. If $n = 4$, then $W_4 = K_4$, so $|\text{Aut}(W_4)| = |\text{Aut}(K_4)| = 4! = 24$. Thus,

$$|\text{Aut}(W_n)| = \begin{cases} 24, & n = 4, \\ 2(n - 1), & n \geq 5. \end{cases}$$

The second score counts the total number of forbidden wheel copies:

$$s_{\text{wh}}(G) = n_{\text{sub}}(G, W_m) + n_{\text{sub}}(\overline{G}, W_n). \quad (2)$$

The search starts from a random graph G , or optionally from a structured graph found by a preliminary search. At each iteration, we identify bad hubs in G and in $\overline{G}$. For each bad hub, we find one or more corresponding rim cycles. If v is the hub and $x_1 x_2 \cdots x_{m-1} x_1$ is a rim cycle in $G[N_G(v)]$, then the associated wheel uses the spoke edges $vx_1, vx_2, \dots, vx_{m-1}$ and the rim edges $x_1 x_2, x_2 x_3, \dots, x_{m-1} x_1$.

All these edges are natural candidates for flipping, since changing any one of them destroys this particular wheel. We form a candidate list from edges appearing in currently detected forbidden wheels, both in G and in $\overline{G}$. To avoid making the search too greedy, we also add a small number of random edges to the candidate list.

Each candidate move consists of flipping one edge of G . The move selection combines simulated annealing and tabu search. Rather than always choosing the best move, the algorithm randomly chooses one of the best T admissible moves, where T is a temperature parameter. The temperature T is gradually decreased during the search, so that early iterations are more exploratory and later iterations are more greedy. To prevent short cycles in the search trajectory, recently flipped edges are declared tabu for τ iterations and are not allowed to be flipped again during that period.

Whenever a new best score is found, the corresponding graph is saved. If the search stagnates, we reset the temperature T , or restart from the best graph found so far after applying a small random perturbation. The overall heuristic is summarized as follows.

1. Start from a random graph G , or from a good graph found by a preliminary search.
2. Find all bad hubs in G and in $\overline{G}$, or sample bad hubs if there are too many.
3. For each bad hub, find one or several bad rim cycles.
4. Add the spoke and rim edges of these wheels to the candidate flip list.
5. Add several random edges to the candidate list.
6. Evaluate all admissible candidate flips using s_{hub} and s_{wh} .
7. Choose randomly among the best T non-tabu flips.
8. Flip the chosen edge and mark it tabu for τ iterations.
9. Save the graph whenever a new best score is obtained.
10. If the search stagnates, reset T or restart from a perturbed copy of the best graph.

Degree Pruning Before Cycle Detection

A useful preliminary pruning step is to reduce the search graph to its 2-core. Suppose that we want to test whether a graph H contains a cycle C_k . Since every vertex of a cycle has degree 2 within the cycle, no vertex of degree 0 or 1 in H can be part of any cycle. Moreover, after deleting such vertices, new vertices of degree 0 or 1 may appear, and these

also cannot be part of a cycle. We repeatedly delete all vertices of at most degree 1 until the 2-core is found.

In the wheel-search setting, to test whether a vertex v can be the hub of a copy of $W_m = K_1 + C_{m-1}$, we search for C_{m-1} in the induced subgraph $G[N(v)]$. Before running the cycle detection, we compute the 2-core of $G[N(v)]$. If this core has fewer than $m - 1$ vertices, then v cannot be the hub of a W_m . Otherwise, the cycle detection is run only on the remaining vertices of the core. This pruning is often very effective in our experiments.

3. New Lower Bounds for Ramsey Numbers $R(W_m, W_n)$

We ran experiments using the heuristic described in the previous section. Our computations produced three Ramsey graphs $R(W_m, W_n)$ for $n = 11$ providing new lower bounds.

Theorem 2. *The following lower bounds hold: $R(W_5, W_{11}) \geq 22$, $R(W_7, W_{11}) \geq 22$, and $R(W_9, W_{11}) \geq 23$.*

Proof. For each listed pair $(m_i, n_i), i = 1, 2, 3$, we give an explicit graph G_i on N_i vertices. Their edge sets are listed in Tables 3–5. The corresponding strings in the graph6 format are as follows.

- The graph G_1 from Table 3 with 106 edges:

T]q{‘fDxUYDVENpXdRHfoImZDEyCLuILxwxF

- The graph G_2 from Table 4 with 104 edges:

Tr‘kkljIrUSvvMdIHWz[MkMHJDJaMS_rkKwu

- The graph G_3 from Table 5 with 120 edges:

UYKE[z]BDn‘tQvAZd@Fmg|ofv‘NfcwQ~MG]GthW

Each graph G_i was checked using a verification program. The program tests all possible hubs v by performing the following steps:

1. Checking whether $G_i[N_{G_i}(v)]$ contains a cycle of length $m_i - 1$.
2. Checking whether $\overline{G_i}[N_{\overline{G_i}}(v)]$ contains a cycle of length $n_i - 1$.

In every case, no such cycle was found. Hence, $R(W_{m_i}, W_{n_i}) \geq N_i + 1$.

The graphs witnessing the lower bounds were all independently verified using the Graph-package in SageMath [14]. $\square$

Table 3. A $(W_5, W_{11}; 21)$ -Ramsey graph with 106 edges.

(0,2), (0,3), (0,4), (0,5), (0,6), (0,8), (0,9), (0,10), (0,13), (0,17), (0,19), (1,2), (1,3), (1,4), (1,8), (1,9), (1,10), (1,13), (1,17), (1,18), (1,19), (1,20), (2,5), (2,7), (2,9), (2,11), (2,14), (2,15), (2,16), (2,20), (3,5), (3,6), (3,7), (3,12), (3,15), (3,20), (4,5), (4,10), (4,11), (4,12), (4,14), (4,16), (4,20), (5,8), (5,9), (5,10), (5,13), (5,17), (5,19), (6,7), (6,11), (6,14), (6,15), (6,16), (6,18), (7,8), (7,9), (7,10), (7,13), (7,15), (7,17), (7,19), (8,11), (8,12), (8,13), (8,15), (8,16), (8,20), (9,11), (9,12), (9,14), (9,15), (9,16), (9,20), (10,11), (10,12), (10,14), (10,15), (10,16), (10,20), (11,12), (11,13), (11,17), (11,18), (11,19), (12,13), (12,17), (12,18), (12,19), (13,14), (13,16), (13,20), (14,16), (14,17), (14,18), (14,19), (15,17), (15,18), (15,19), (16,17), (16,18), (16,19), (17,19), (17,20), (18,20), (19,20).
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Table 4. A $(W_7, W_{11}; 21)$ -Ramsey graph with 104 edges.

(0,1), (0,2), (0,4), (0,6), (0,7), (0,11), (0,12), (0,13), (0,16), (0,19), (1,3), (1,5), (1,8), (1,10), (1,12), (1,14), (1,15), (1,17), (1,18), (2,3), (2,5), (2,8), (2,9), (2,10), (2,11), (2,15), (2,16), (2,18), (3,6), (3,7), (3,12), (3,13), (3,16), (3,18), (3,19), (4,5), (4,8), (4,9), (4,10), (4,12), (4,14), (4,15), (4,17), (4,20), (5,6), (5,7), (5,11), (5,12), (5,13), (5,15), (5,20), (6,7), (6,8), (6,9), (6,10), (6,11), (6,14), (6,15), (6,17), (7,8), (7,9), (7,10), (7,14), (7,17), (7,18), (8,11), (8,12), (8,13), (8,16), (8,20), (9,11), (9,12), (9,16), (9,19), (9,20), (10,11), (10,12), (10,13), (10,16), (10,19), (10,20), (11,14), (11,15), (11,17), (11,18), (12,14), (12,15), (12,18), (13,14), (13,15), (13,17), (13,18), (13,19), (14,16), (14,19), (14,20), (15,19), (15,20), (16,17), (16,18), (17,19), (17,20), (18,19), (18,20).

Table 5. A $(W_9, W_{11}; 22)$ -Ramsey graph with 120 edges.

(0,2), (0,6), (0,7), (0,8), (0,9), (0,12), (0,13), (0,14), (0,16), (0,19), (0,20), (1,2), (1,3), (1,6), (1,8), (1,9), (1,10), (1,11), (1,12), (1,13), (1,14), (1,20), (2,4), (2,9), (2,10), (2,11), (2,15), (2,16), (2,17), (2,18), (2,21), (3,4), (3,7), (3,8), (3,9), (3,11), (3,12), (3,13), (3,14), (3,19), (4,6), (4,7), (4,8), (4,9), (4,11), (4,13), (4,14), (4,19), (4,20), (5,6), (5,7), (5,8), (5,11), (5,12), (5,13), (5,14), (5,17), (5,18), (5,19), (5,20), (6,10), (6,15), (6,16), (6,17), (6,18), (6,20), (6,21), (7,12), (7,15), (7,16), (7,17), (7,18), (7,21), (8,10), (8,14), (8,15), (8,16), (8,17), (8,18), (9,10), (9,15), (9,16), (9,17), (9,18), (9,21), (10,11), (10,12), (10,13), (10,14), (10,19), (10,20), (11,15), (11,16), (11,17), (11,21), (12,15), (12,16), (12,17), (12,18), (12,21), (13,15), (13,16), (13,17), (13,18), (13,19), (14,17), (14,18), (14,21), (15,18), (15,19), (15,20), (16,19), (16,20), (17,19), (17,20), (17,21), (18,19), (18,20), (19,21), (20,21).

4. Discussion

This paper continues the computational effort to improve lower bounds for wheel–wheel Ramsey numbers. Three new lower bounds were computed (Theorem 2). The search was distributed across several machines and run in parallel for several days until $(W_m, W_n; N)$ -Ramsey graphs were found. The computations were performed on Windows 11 PCs with an Intel(R) Core(TM) i7-12700 CPU, 64 GB RAM. The programs were compiled with gcc. The computational bottleneck is the detection and counting cycles C_k in the relevant neighborhood subgraphs. It would be interesting to design faster algorithms for this problem.

Several questions remain unanswered. First, it is natural to ask whether the Ramsey graphs reported here, as well as earlier constructions, can be extended to larger values of N , or whether they are close to optimal. Table 1 summarizes Ramsey numbers $R(W_m, W_n)$ for all $m \leq n \leq 11$. Second, the role of symmetry in such constructions is not yet fully understood. Many successful searches use circulant, block-circulant, or otherwise structured graphs, but it remains unclear when such restrictions preserve the possibility of finding extremal examples. Finally, it would be interesting to investigate whether the graphs obtained by computation can be used to prove lower bounds, or to formulate conjectures for infinite families of wheel–wheel Ramsey numbers.

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Abbreviations

The following abbreviations are used in this manuscript:

K_n	the complete graph on n vertices
C_n	the cycle on n vertices
W_n	the wheel graph $K_1 + C_{n-1}$, obtained by joining one hub vertex to all vertices of C_{n-1}
$\overline{G}$	the complement of a graph G
$R(G, H)$	the Ramsey number for G versus H , i.e., the least integer N such that every graph on N vertices contains G or its complement contains H
$R(G)$	the diagonal Ramsey number $R(G, G)$
$(G, H; N)$ -Ramsey graph	a graph F on N vertices such that F contains no copy of G and $\overline{F}$ contains no copy of H
$N_G(v)$	the neighborhood of a vertex v in G
$s_{\text{hub}}(G)$	the number of bad hubs; see Equation (1)
$n_{\text{sub}}(G, H)$	the number of induced subgraphs of a graph G isomorphic to a graph H
$\text{Inj}(G, H)$	the number of injective homomorphisms from H to G
$\text{Aut}(H)$	the number of automorphisms of H
$s_{\text{wh}}(G)$	the second score; see Equation (2)

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