

Article

Topological Structures of Fuzzy Modal Logics Based on Residuated Lattices

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Abstract

The purpose of this paper is to interpret fuzzy Kripke models as fuzzy information systems with objects and attributes. We introduce topological structures (interior, closure operators, Alexandrov pretopologies, Alexandrov precotopologies, fuzzy rough set) to the formulas of fuzzy Kripke models based on complete residuated lattices. We study the relations of possible worlds as objects and formulas as attributes in a fuzzy information system. Using the properties of residuated and Galois connections, we can obtain fuzzy concept lattices and formal fuzzy concept lattices. We give their examples.

Keywords: complete residuated lattices; fuzzy Kripke model; fuzzy modal logics; residuated (Galois) connections; interior (closure) operators; (formal) fuzzy concept lattices

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1. Introduction

Kripke [1] introduced Kripke semantics for modal logics with necessarily true and possibly true as non-classical logic systems. Modal logic is one of the most developed and studied non-classical logics [2]. The two operators of necessity and possibility from a partially ordered set to a partially ordered set form a residuated connection [3]. The two operators of sufficiency form a Galois connection. Two connections play an important role in solving the structures, which are topologies, rough sets, concept lattices and modal operators [4–8]. Generalizations of the concepts of necessity and possibility offer a rich setting to model and study notions from many different areas and software applications [9–11].

Fuzzy structures in the real world provide much more adequate information than classical structures [12,13]. Using fuzzy Galois connections, Bělohlávek [12] introduced the notion of formal fuzzy concepts on a complete residuated lattice [14]. Using residuated connections, we can find other fuzzy concepts. Using these connections, we can solve fuzzy relation equations and the theory of interior and closure operations, fuzzy rough sets and fuzzy concept lattices in complete residuated lattices [5–19]. It is used in signal and image processing, data analysis and neural network approaches [12,15,16].

A fuzzy Kripke model [20] is a generalization of a classical Kripke model used in modal logic. Fitting [21] introduced modal many-valued logics based on finite Heyting algebras. Many researchers [22–29] developed fuzzy modal logics based on various algebraic structures (MTL algebras [20,23], Gödel algebra [24,26]).



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In recent times, there has been significant development in the field of bisimulations for fuzzy automata and fuzzy modal logic based on various algebraic structures (Heyting algebras [28,29], residuated lattice [27]).

The purpose of this paper is to interpret fuzzy Kripke models as fuzzy information systems with objects and attributes.

In Section 2, we introduce complete residuated lattices as extensions of left continuous t-norms, Heyting algebras, Gödel t-norms and product t-norms. We define the fuzzy (necessity, possibility and sufficiency) modal operators for the formulas of fuzzy Kripke models based on complete residuated lattices.

In Section 3, two formulas are defined in Theorem 1: the degree of semantical equivalence and fuzzy partial orders. In Theorems 3 and 4, we investigate the properties of the fuzzy (necessity, possibility and sufficiency) modal operators for the formulas of fuzzy Kripke models based on the residuated lattice. We show that the modal order-preserving operators of necessity and possibility form residuated connections. The modal order-reversing operators of sufficiency form Galois connections. Moreover, we can solve the solutions for fuzzy relation equations with fuzzy modal operations of formulas.

In Theorems 5, 7 and 8, as topological structures, we suggest interior (closure) operators, Alexandrov pretopologies, Alexandrov precotopologies and fuzzy rough sets for the formulas of fuzzy Kripke models. Moreover, we investigate their properties. From these results, big data with a fuzzy Kripke model can also be variously classified into small groups with viewpoints of interior (closure) operators, fuzzy rough sets and Alexandrov pre-(co)topologies.

In Example 2, we give various interior (closure) operators, Alexandrov (pre-)topologies, Alexandrov precotopologies and fuzzy rough sets for a fuzzy Kripke model.

In Section 4, we investigate the relations between possible worlds as objects and formulas as attributes in two fuzzy Kripke models. Using the properties of a residuated connection, we can obtain a fuzzy concept of possible worlds and formulas for two fuzzy Kripke models based on a residuated lattice. We show that it is a complete lattice. Using the properties of a Galois connection, we can obtain a formal fuzzy concept of possible worlds and formulas. Also, it is a complete lattice. We give their examples. Big data with two fuzzy Kripke models can also be variously classified into small groups with a (formal) fuzzy concept of possible worlds and formulas.

2. Preliminaries

Definition 1 ([12,14,17]). An algebra $(L, \wedge, \vee, \odot, \Rightarrow, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

- (L1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;
 - (L2) $(L, \odot, \top)$ is a commutative monoid;
 - (L3) $x \odot y \leq z$ iff $x \leq y \Rightarrow z$ for $x, y, z \in L$.
- If $\odot = \wedge$, then it is called a Heyting algebra.

In this paper, we always assume that $(L, \wedge, \vee, \odot, \Rightarrow, \perp, \top)$ is a complete residuated lattice. For each $x \in L$, $x^* = x \Rightarrow \perp$ and the condition $x^{**} = x$ is called a double negative law.

Lemma 1 ([17]). For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

- (1) $\top \Rightarrow x = x$, $\perp \odot x = \perp$.
- (2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \Rightarrow y \leq x \Rightarrow z$ and $z \Rightarrow x \leq y \Rightarrow x$.
- (3) $x \leq y$ iff $x \Rightarrow y = \top$.
- (4) $x \Rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \Rightarrow y_i)$.
- (5) $(\bigvee_i x_i) \Rightarrow y = \bigwedge_i (x_i \Rightarrow y)$.
- (6) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$.

- (7) $(x \odot y) \Rightarrow z = x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z)$.
 (8) $(x \Rightarrow y) \odot (z \Rightarrow w) \leq (x \odot z) \Rightarrow (y \odot w)$ and $x \Rightarrow y \leq (x \odot z) \Rightarrow (y \odot z)$.
 (9) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z$.
 (10) $\bigvee_{i \in \Gamma} x_i \Rightarrow \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \Rightarrow y_i)$ and $\bigwedge_{i \in \Gamma} x_i \Rightarrow \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \Rightarrow y_i)$.
 (11) $x \Rightarrow y \leq (y \Rightarrow z) \Rightarrow (x \Rightarrow z)$ and $x \Rightarrow y \leq (z \Rightarrow x) \Rightarrow (z \Rightarrow y)$.

Definition 2 ([12]). Let X be a set. A function $e_X : X \times X \rightarrow L$ is called:

- (E1) Reflexive if $e_X(x, x) = \top$ for all $x \in X$;
 (E2) Transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$.

If $e_X(x, y) = e_X(y, x) = \top$, then $x = y$. If e_X satisfies (E1) and (E2), e_X is a fuzzy preorder. If e_X satisfies (E1), (E2) and (E3), e_X is a fuzzy partial order.

The set of formulas Φ_L of a many-valued modal logic is the smallest set containing propositional symbols and truth constants, and is closed under logical connectives and modal operators as follows:

- (1) An enumerable set of propositional symbols $P = \{p_1, p_2, \dots, p_n \mid n \in \mathbb{N}\}$.
 (2) A set of truth constants $\bar{L} = \{\bar{t} \mid t \in L\}$.
 (3) If $\phi_i, \phi, \psi \in \Phi_L$ for $i \in I$, then $\bigwedge_{i \in I} \phi_i, \bigvee_{i \in I} \phi_i \in \Phi_L, \phi \& \psi, \phi \rightarrow \psi, \neg \phi = \phi \rightarrow \bar{\perp} \in \Phi_L$.
 (4) For $\phi \in \Phi_L$, we have six modal operators such that $[\phi], [\phi]^{-1} \in \Phi_L$ (possibility operators); $\langle \phi \rangle, \langle \phi \rangle^{-1} \in \Phi_L$ (necessity operators); and $\{\phi\}, \{\phi\}^{-1} \in \Phi_L$ (sufficiency operators).

Definition 3 ([25,29]). A fuzzy Kripke frame is a pair $F = (W, R)$ where W is a non-empty set of possible worlds and $R : W \times W \rightarrow L$ is the fuzzy accessibility function.

A fuzzy Kripke model is a tuple $M = (W, R, V)$ where (W, R) is a fuzzy Kripke frame and $V : W \times P \rightarrow L$ is a fuzzy valuation function defined by $V(w, p) = p^V(w)$ for each $p \in P$ and $w \in W$.

The function V can be inductively extended to a function $V : W \times \Phi_L \rightarrow L$ defined by $V(w, \phi) = \phi^V(w)$ for each $\phi \in \Phi_L$ and $w \in W$, as follows (V1)–(V11): for each $\phi_i, \phi, \psi \in \Phi_L$ for $i \in I$ and $t \in L$,

- (V1) $\bar{t}^V(w) = t$,
 (V2) $(\bigwedge_{i \in I} \phi_i)^V(w) = \bigwedge_{i \in I} (\phi_i)^V(w)$,
 (V3) $(\bigvee_{i \in I} \phi_i)^V(w) = \bigvee_{i \in I} (\phi_i)^V(w)$,
 (V4) $(\phi \& \psi)^V(w) = \phi^V(w) \odot \psi^V(w)$,
 (V5) $(\phi \rightarrow \psi)^V(w) = \phi^V(w) \Rightarrow \psi^V(w)$,
 (V6) $[\phi]^V(w) = \bigwedge_{v \in W} (R(w, v) \Rightarrow \phi^V(v))$,
 (V7) $([\phi]^{-1})^V(w) = \bigwedge_{v \in W} (R(v, w) \Rightarrow \phi^V(v))$,
 (V8) $\langle \phi \rangle^V(w) = \bigvee_{v \in W} (R(w, v) \odot \phi^V(v))$,
 (V9) $(\langle \phi \rangle^{-1})^V(w) = \bigvee_{v \in W} (R(v, w) \odot \phi^V(v))$,
 (V10) $\{\phi\}^V(w) = \bigwedge_{v \in W} (\phi^V(v) \Rightarrow R(w, v))$,
 (V11) $(\{\phi\}^{-1})^V(w) = \bigwedge_{v \in W} (\phi^V(v) \Rightarrow R(v, w))$.

We denote $(\neg \phi)^V(w) = \phi^V(w) \Rightarrow \perp$.

Notation 1. Let $M = (W, R, V)$ be a fuzzy Kripke model.

- (1) We denote $\phi_i, \phi, \psi, \psi_i, \bigwedge_{i \in I} \phi_i, \bigvee_{i \in I} \phi_i, \phi \& \psi, \phi \rightarrow \psi, \neg \phi = \phi \rightarrow \bar{\perp}, [\phi], [\phi]^{-1}, \langle \phi \rangle, \langle \phi \rangle^{-1}, \{\phi\}, \{\phi\}^{-1}, p_i \in \Phi_L$. Moreover, $\bar{t} \in \Phi_L$ for each $t \in L$.

(2) The maps $[-], [-]^{-1}, \langle - \rangle, \langle - \rangle^{-1}, \{- \}, \{- \}^{-1} : \Phi_L \rightarrow \Phi_L$ are defined by $[-](\phi) = [\phi], [-]^{-1}(\phi) = [\phi]^{-1}, \langle - \rangle(\phi) = \langle \phi \rangle, \langle - \rangle^{-1}(\phi) = \langle \phi \rangle^{-1}, \{- \}(\phi) = \{\phi\}, \{- \}^{-1}(\phi) = \{\phi\}^{-1}$, for each $\phi \in \Phi_L$.

(3) $V(w, \phi) = \phi^V(w)$ for each $\phi \in \Phi_L$ and $w \in W$.

3. Fuzzy Modal Operators, Alexandrov L -Topologies and Fuzzy Rough Sets

In this section, we study fuzzy partial orders and investigate the properties of the fuzzy (necessity, possibility and sufficiency) modal operators for the formulas of Kripke models based on a residuated lattice.

Theorem 1. Let $M = (W, R, V)$ be a fuzzy Kripke model. For $\phi, \psi \in \Phi_L$, we define

$$\begin{aligned} e_M(\phi^V, \psi^V) &= \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \psi^V(w)), \\ e_M(\phi^V \equiv \psi^V) &= e_M(\phi^V, \psi^V) \wedge e_M(\psi^V, \phi^V). \end{aligned}$$

Then, for each $\phi, \phi_i, \psi, \psi_i, \phi_1, \psi_1 \in \Phi_L$, the following properties hold.

- (1) A map $e_M : \Phi_L \times \Phi_L \rightarrow L$ is a fuzzy partial order on Φ_L .
- (2) $e_M(\phi^V \equiv \psi^V) \odot e_M(\phi_1^V \equiv \psi_1^V) \leq e_M((\phi \& \phi_1)^V \equiv (\psi \& \psi_1)^V)$.
- (3) $e_M(\phi^V \equiv \psi^V) \leq e_M((\bar{t} \& \phi)^V \equiv (\bar{t} \& \psi)^V)$.
- (4) $e_M(\phi^V \equiv \psi^V) \leq e_M((\bar{t} \rightarrow \phi)^V \equiv (\bar{t} \rightarrow \psi)^V)$.
- (5) $e_M(\phi^V \equiv \psi^V) \leq e_M((\psi \rightarrow \bar{t})^V \equiv (\phi \rightarrow \bar{t})^V)$.
- (6) $\bigwedge_{i \in \Gamma} e_M(\phi_i^V \equiv \psi_i^V) \leq e_M(\bigvee_{i \in \Gamma} \phi_i^V \equiv \bigvee_{i \in \Gamma} \psi_i^V)$.
- (7) $\bigwedge_{i \in \Gamma} e_M(\phi_i^V \equiv \psi_i^V) \leq e_M(\bigwedge_{i \in \Gamma} \phi_i^V \equiv \bigwedge_{i \in \Gamma} \psi_i^V)$.

Proof. (1) For $\phi, \psi, \varphi \in \Phi_L$,

$$\begin{aligned} &e_M(\phi^V, \psi^V) \odot e_M(\psi^V, \varphi^V) \\ &= \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \psi^V(w)) \odot \bigwedge_{w \in W} (\psi^V(w) \Rightarrow \varphi^V(w)) \\ &\leq \bigwedge_{w \in W} ((\phi^V(w) \Rightarrow \psi^V(w)) \odot (\psi^V(w) \Rightarrow \varphi^V(w))) \\ &\leq \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \varphi^V(w)) \text{ (by Lemma 1 (9))} \\ &= e_M(\phi^V, \varphi^V). \end{aligned}$$

Other cases are easily proved.

(2) For each $\phi, \phi_1, \psi, \psi_1 \in \Phi_L$,

$$\begin{aligned} &e_M((\phi \& \phi_1)^V, (\psi \& \psi_1)^V) \\ &= \bigwedge_{u \in W} ((\phi^V(u) \odot \phi_1^V(u)) \Rightarrow (\psi^V(u) \odot \psi_1^V(u))) \\ &\text{(by Lemma 1 (8))} \\ &\geq \bigwedge_{u \in W} ((\phi^V(u) \Rightarrow \psi^V(u)) \odot (\phi_1^V(u) \Rightarrow \psi_1^V(u))) \\ &\geq \bigwedge_{u \in W} ((\phi^V(u) \Rightarrow \psi^V(u)) \odot \bigwedge_{u \in W} (\phi_1^V(u) \Rightarrow \psi_1^V(u))) \\ &= e_M(\phi^V, \psi^V) \odot e_M(\phi_1^V, \psi_1^V). \end{aligned}$$

Similarly, $e_M((\psi \& \psi_1)^V, (\phi \& \phi_1)^V) \geq e_M(\psi^V, \phi^V) \odot e_M(\psi_1^V, \phi_1^V)$. Thus, by Lemma 1 (2),

$$\begin{aligned} &e_M(\phi^V \equiv \psi^V) \odot e_M(\phi_1^V \equiv \psi_1^V) \\ &= (e_M(\phi^V, \psi^V) \wedge e_M(\psi^V, \phi^V)) \\ &\odot (e_M(\phi_1^V, \psi_1^V) \wedge e_M(\psi_1^V, \phi_1^V)) \\ &\leq (e_M(\phi^V, \psi^V) \odot e_M(\phi_1^V, \psi_1^V)) \\ &\wedge (e_M(\psi^V, \phi^V) \odot e_M(\psi_1^V, \phi_1^V)) \\ &\leq e_M(\phi^V \& \phi_1^V, \psi^V \& \psi_1^V) \wedge e_M(\psi^V \& \psi_1^V, \phi^V \& \phi_1^V) \\ &= e_M(\phi^V \& \phi_1^V \equiv \psi^V \& \psi_1^V). \end{aligned}$$

(3) For each $\phi, \psi, \bar{t} \in \Phi_L$ and $t \in L$, by Lemma 1 (8),

$$\begin{aligned}
& e_M((\bar{t} \& \phi)^V, (\bar{t} \& \psi)^V) \\
&= \bigwedge_{w \in W} ((\bar{t}^V(w) \odot \phi^V(w)) \Rightarrow (\bar{t}^V(w) \odot \psi^V(w))) \\
&\geq \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \psi^V(w)) = e_M(\phi^V, \psi^V).
\end{aligned}$$

Similarly, $e_M((\bar{t} \& \psi)^V, (\bar{t} \& \phi)^V) \geq e_M(\psi^V, \phi^V)$.

(4) For each $\phi, \psi, \bar{t} \in \Phi_L$ and $t \in L$, by Lemma 1 (11),

$$\begin{aligned}
& e_M((\bar{t} \rightarrow A)^V, (\bar{t} \rightarrow \psi)^V) \\
&= \bigwedge_{u \in W} ((\bar{t}^V(u) \Rightarrow \phi^V(u)) \Rightarrow (\bar{t}^V(u) \Rightarrow \psi^V(u))) \\
&\geq \bigwedge_{u \in W} (\phi^V(u) \Rightarrow \psi^V(u)) = e_M(\phi^V, \psi^V).
\end{aligned}$$

Similarly, $e_M((\bar{t} \rightarrow \psi)^V, (\bar{t} \rightarrow \phi)^V) \geq e_M(\psi^V, \phi^V)$.

(5) For each $\phi, \psi, \bar{t} \in \Phi_L$ and $t \in L$, by Lemma 1 (11),

$$\begin{aligned}
& e_M((\phi \rightarrow \bar{t})^V, (\psi \rightarrow \bar{t})^V) \\
&= \bigwedge_{u \in W} ((\phi^V(u) \Rightarrow \bar{t}^V(u)) \Rightarrow (\psi^V(u) \Rightarrow \bar{t}^V(u))) \\
&\geq \bigwedge_{u \in W} (\psi^V(u) \Rightarrow \phi^V(u)) = e_M(\psi^V, \phi^V).
\end{aligned}$$

Similarly, $e_M((\psi \rightarrow \bar{t})^V, (\phi \rightarrow \bar{t})^V) \geq e_M(\phi^V, \psi^V)$.

(6) For each $\phi_i, \psi_i \in \Phi_L$ and $i \in \Gamma$, by Lemma 1 (10),

$$\begin{aligned}
& e_M(\bigvee_{i \in \Gamma} \phi_i^V, \bigvee_{i \in \Gamma} \psi_i^V) \\
&= \bigwedge_{u \in W} (\bigvee_{i \in \Gamma} \phi_i^V(u) \Rightarrow \bigvee_{i \in \Gamma} \psi_i^V(u)) \\
&\geq \bigwedge_{u \in W} \bigwedge_{i \in \Gamma} (\phi_i^V(u) \Rightarrow \psi_i^V(u)) \\
&= \bigwedge_{i \in \Gamma} \bigwedge_{u \in W} (\phi_i^V(u) \Rightarrow \psi_i^V(u)) \\
&= \bigwedge_{i \in \Gamma} e_M(\phi_i^V, \psi_i^V).
\end{aligned}$$

Hence, $\bigwedge_{i \in \Gamma} e_M(\phi_i^V, \psi_i^V) \leq e_M(\bigvee_{i \in \Gamma} \phi_i^V, \bigvee_{i \in \Gamma} \psi_i^V)$. Similarly, $\bigwedge_{i \in \Gamma} e_M(\psi_i^V, \phi_i^V) \leq e_M(\bigvee_{i \in \Gamma} \psi_i^V, \bigvee_{i \in \Gamma} \phi_i^V)$.

(7) For each $\phi_i, \psi_i \in \Phi_L$ and $i \in \Gamma$, by Lemma 1 (10),

$$\begin{aligned}
& e_M(\bigwedge_{i \in \Gamma} \phi_i^V, \bigwedge_{i \in \Gamma} \psi_i^V) \\
&= \bigwedge_{u \in W} (\bigwedge_{i \in \Gamma} \phi_i^V(u) \Rightarrow \bigwedge_{i \in \Gamma} \psi_i^V(u)) \\
&\geq \bigwedge_{u \in W} \bigwedge_{i \in \Gamma} (\phi_i^V(u) \Rightarrow \psi_i^V(u)) \\
&= \bigwedge_{i \in \Gamma} \bigwedge_{u \in W} (\phi_i^V(u) \Rightarrow \psi_i^V(u)) \\
&= \bigwedge_{i \in \Gamma} e_M(\phi_i^V, \psi_i^V).
\end{aligned}$$

□

Lemma 2. Let L follow a double negative law and be defined by $x^* = x \Rightarrow \perp$, $x \oplus y = (x^* \odot y^*)^*$.

(1) For each $x_i \in L$ and $i \in \Gamma$, $\bigwedge_{i \in \Gamma} x_i^* = (\bigvee_{i \in \Gamma} x_i)^*$ and $\bigvee_{i \in \Gamma} x_i^* = (\bigwedge_{i \in \Gamma} x_i)^*$.

(2) For each $x, y \in L$, $(x \Rightarrow y)^* = x \odot y^*$ and $x^* \Rightarrow y^* = y \Rightarrow x$.

(3) For each $x, y, z, w \in L$, $(x \Rightarrow y) \odot (z \Rightarrow w) \leq (x \oplus z) \Rightarrow (y \oplus w)$.

Proof. (1) By Lemma 1 (5), $(\bigvee_{i \in \Gamma} x_i)^* = \bigvee_{i \in \Gamma} x_i \Rightarrow \perp = \bigwedge_{i \in \Gamma} (x_i \Rightarrow \perp) = \bigwedge_{i \in \Gamma} x_i^*$. Moreover, since $(\bigvee_{i \in \Gamma} x_i^*)^* = \bigwedge_{i \in \Gamma} x_i^{**}$, $\bigvee_{i \in \Gamma} x_i^* = (\bigwedge_{i \in \Gamma} x_i)^*$.

(2) Since $(x \odot y^*)^* = x \Rightarrow y$ from Lemma 1 (7), $(x \Rightarrow y)^* = x \odot y^*$.

Since $(y \Rightarrow x) \odot (x \Rightarrow \perp) \leq y \Rightarrow \perp$, $y \Rightarrow x \leq x^* \Rightarrow y^*$. Moreover, $x^* \Rightarrow y^* \leq y^{**} \Rightarrow x^{**} = y \Rightarrow x$.

(3) For each $x, y, z, w \in L$,

$$\begin{aligned}
& (x \oplus z) \Rightarrow (y \oplus w) \\
& = (x^* \odot z^*)^* \Rightarrow (y^* \odot w^*)^* \\
& = (y^* \odot w^*) \Rightarrow (x^* \odot z^*) \text{ (by (2))} \\
& \geq (y^* \Rightarrow x^*) \odot (w^* \Rightarrow z^*) \text{ (by Lemma 1 (8))} \\
& = (x \Rightarrow y) \odot (z \Rightarrow w).
\end{aligned}$$

□

Example 1. Let $L = [0, 1]$ be a unit interval.

(1) Define Gödel t -norm $\odot, \Rightarrow: [0, 1] \times [0, 1]$ as

$$x \odot y = x \wedge y, \quad x \Rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y. \end{cases}$$

Since $(x \Rightarrow 0) \Rightarrow 0 = 0 \Rightarrow 0 = 1 \neq x$ for $0 \leq x < 1$, it does not follow a double negation law.

(2) Define product t -norm $\odot, \Rightarrow: [0, 1] \times [0, 1]$ as

$$x \odot y = x \cdot y, \quad x \Rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y/x, & \text{if } x > y. \end{cases}$$

Since $(x \Rightarrow 0) \Rightarrow 0 = 0 \Rightarrow 0 = 1 \neq x$ for $0 \leq x < 1$, it does not follow a double negation law.

(3) Define t -norm $\odot, \Rightarrow: [0, 1] \times [0, 1]$ as

$$\begin{aligned}
x \odot y &= ((x^p + y^p - 1) \vee 0)^{1/p}, \quad p > 0, \\
x \Rightarrow y &= (1 - x^p + y^p)^{1/p} \wedge 1, \\
x^* &= x \Rightarrow 0 = (1 - x^p)^{1/p}.
\end{aligned}$$

It follows a double negation law. If $p = 1$, it is called a Łukasiewicz t -norm.

Theorem 2. Let $M = (W, R, V)$ be a fuzzy Kripke model. For $\phi, \psi \in \Phi_L$, let L follow a double negative law and be defined by $x \oplus y = (x^* \odot y^*)^*$.

- (1) For $\phi, \psi \in \Phi_L$, $e_M(\phi^V, \psi^V) = e_M((\neg\psi)^V, (\neg\phi)^V)$.
- (2) $e_M(\phi^V \equiv \psi^V) \odot e_M(\phi_1^V \equiv \psi_1^V) \leq e_M(\phi^V \oplus \phi_1^V \equiv \psi^V \oplus \psi_1^V)$.
- (3) $(\neg((\neg\phi)))^V = \langle \phi \rangle$.

Proof. (1) For $\phi, \psi \in \Phi_L$, by Lemma 2 (2),

$$\begin{aligned}
e_M(\phi^V, \psi^V) &= \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \psi^V(w)) \\
&= \bigwedge_{w \in W} (\psi^{V*}(w) \Rightarrow \phi^{V*}(w)) \\
&= e_M((\neg\psi)^V, (\neg\phi)^V)
\end{aligned}$$

(2) For $\phi, \psi, \phi_1, \psi_2 \in \Phi_L$,

$$\begin{aligned}
& e_M((\neg(\neg\phi \& \neg\phi_1))^V, (\neg(\neg\psi \& \neg\psi_1))^V) \\
&= \bigwedge_{w \in W} ((\phi^{V*}(w) \odot \phi_1^{V*}(w))^* \Rightarrow (\psi^{V*}(w) \odot \psi_1^{V*}(w))^*) \\
&= \bigwedge_{w \in W} ((\phi^V(w) \oplus \phi_1^V(w)) \Rightarrow (\psi^V(w) \oplus \psi_1^V(w))) \\
& \text{(by Lemma 2(3))} \\
&\geq \bigwedge_{w \in W} ((\phi^V(w) \Rightarrow \psi^V(w)) \odot (\phi_1^V(w) \Rightarrow \psi_1^V(w))) \\
&\geq \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \psi^V(w)) \odot \bigwedge_{w \in W} (\phi_1^V(w) \Rightarrow \psi_1^V(w)) \\
&= e_M(\phi^V, \psi^V) \odot e_M(\phi_1^V, \psi_1^V).
\end{aligned}$$

Similarly, $e_M((\neg(\neg\psi \& \neg\psi_1))^V, (\neg(\neg\phi \& \neg\phi_1))^V) \geq e_M(\psi^V, \phi^V) \odot e_M(\psi_1^V, \phi_1^V)$.

$$\begin{aligned}
& e_M(\phi^V \equiv \psi^V) \odot e_M(\phi_1^V \equiv \psi_1^V) \\
&= (e_M(\phi^V, \psi^V) \wedge e_M(\psi^V, \phi^V)) \odot (e_M(\phi_1^V, \psi_1^V) \wedge e_M(\psi_1^V, \phi_1^V)) \\
&\leq (e_M(\phi^V, \psi^V) \odot e_M(\phi_1^V, \psi_1^V)) \wedge (e_M(\psi^V, \phi^V) \odot e_M(\psi_1^V, \phi_1^V)) \\
&\leq e_M((\neg(\neg\phi \& \neg\phi_1))^V, (\neg(\neg\psi \& \neg\psi_1))^V) \\
&\quad \wedge e_M((\neg(\neg\psi \& \neg\psi_1))^V, (\neg(\neg\phi \& \neg\phi_1))^V) \\
&= e_M((\neg(\neg\phi \& \neg\phi_1))^V \equiv (\neg(\neg\psi \& \neg\psi_1))^V).
\end{aligned}$$

(3) For $\phi \in \Phi_L$, by Lemma 2 (1 and 2),

$$\begin{aligned}
& (\neg([\neg\phi]))^V(w) = (\bigwedge_{v \in W} (R(w, v) \Rightarrow (\neg\phi)^V))^* \\
&= \bigvee_{v \in W} (R(w, v) \Rightarrow (\phi^V(v) \Rightarrow \perp))^* \\
&= \bigvee_{v \in W} (R(w, v) \odot \phi^V(v)) = \langle \phi \rangle(w).
\end{aligned}$$

□

Definition 4. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be fuzzy Kripke models and $F : \Phi_L \rightarrow \Phi'_L$ and $G : \Phi'_L \rightarrow \Phi_L$ maps.

(1) $(e_M, F, G, e_{M'})$ is called a residuated connection if for all $\phi \in \Phi_L, \psi \in \Phi'_L$,

$$e_{M'}(F(\phi)^{V'}, \psi^{V'}) = e_M(\phi^V, G(\psi)^V).$$

(2) $(e_M, F, G, e_{M'})$ is called a Galois connection if for all $\phi \in \Phi_L, \psi \in \Phi'_L$,

$$e_{M'}(\psi^{V'}, F(\phi)^{V'}) = e_M(\phi^V, G(\psi)^V).$$

Theorem 3. Let $M = (W, R, V)$ be a fuzzy Kripke model. The maps $[-], [-]^{-1}, \langle - \rangle, \langle - \rangle^{-1} : \Phi_L \rightarrow \Phi_L$ are defined by $[-](\phi) = [\phi]$, for each $\phi \in \Phi_L$, similar to other cases. Then, the following statements hold:

(1) $(e_M, \langle - \rangle, [-]^{-1}, e_M)$ is a residuated connection.

(2) For each $\phi_i, \psi_i \in \Phi_L$,

$$\begin{aligned}
& e_M(\phi^V, ([\langle \phi \rangle]^{-1})^V) = \top, \\
& e_M(\langle [\psi]^{-1} \rangle^V, \psi^V) = \top, \\
& e_M(\phi^V, \psi^V) \leq e_M(\langle \phi \rangle^V, \langle \psi \rangle^V), \\
& e_M(\phi^V, \psi^V) \leq e_M(([\phi]^{-1})^V, ([\psi]^{-1})^V).
\end{aligned}$$

(3) $(e_M, \langle - \rangle^{-1}, [-], e_M)$ is a residuated connection.

(4) For each $\phi_i, \psi_i \in \Phi_L$,

$$\begin{aligned}
& e_M(\phi^V, [\langle \phi \rangle^{-1}]^V) = \top, \\
& e_M(\langle [\psi]^{-1} \rangle^V, \psi^V) = \top, \\
& e_M(\phi^V, \psi^V) \leq e_M((\langle \phi \rangle^{-1})^V, (\langle \psi \rangle^{-1})^V), \\
& e_M(\phi^V, \psi^V) \leq e_M([\phi]^V, [\psi]^V).
\end{aligned}$$

(5) R is reflexive and transitive, then

$$\begin{aligned}
& [\langle \phi \rangle^{-1}]^V = (\langle \phi \rangle^{-1})^V, \\
& ([\langle \phi \rangle]^{-1})^V = \langle \phi \rangle^V, \\
& \langle [\phi]^{-1} \rangle^V = ([\phi]^{-1})^V, \\
& (\langle [\phi] \rangle^{-1})^V = [\phi]^V.
\end{aligned}$$

(6) For each $\phi_i, \psi_i \in \Phi_L$,

$$\begin{aligned} [\langle \phi \rangle^{-1}]^V &= [\phi]^V, \\ (\langle [\psi]^{-1} \rangle^{-1})^V &= (\langle \psi \rangle^{-1})^V, \\ [\langle [\phi]^{-1} \rangle]^V &= [\phi]^{-1}, \\ \langle [\langle \psi \rangle^{-1}]^V &= \langle \psi \rangle^V. \end{aligned}$$

(7) If, for $\psi \in \Phi_L$, there exists a solution $\phi_0 \in \Phi_L$ such that $\langle \phi_0 \rangle^V = \psi^V$, then there exists a solution $[\langle \phi_0 \rangle]^{-1} \in \Phi_L$ such that $\langle [\langle \phi_0 \rangle]^{-1} \rangle^V = \psi^V$ and $([\langle \phi_0 \rangle]^{-1})^V \geq \phi_0^V$.

(8) If, for $\psi \in \Phi_L$, there exists a solution $\phi_0 \in \Phi_L$ such that $[\phi_0]^V = \psi^V$, then there exists a solution $\langle [\phi_0] \rangle^{-1} \in \Phi_L$ such that $[\langle [\phi_0] \rangle^{-1}]^V = \psi^V$ and $(\langle [\phi_0] \rangle^{-1})^V \leq \phi_0^V$.

Proof. (1) For $\phi, \psi \in \Phi_L$, by Lemma 1 (5 and 7),

$$\begin{aligned} e_M(\langle \phi \rangle^V, \psi^V) &= \bigwedge_{w \in W} (\langle \phi \rangle^V(w) \Rightarrow \psi^V(w)) \\ &= \bigwedge_{w \in W} (\bigvee_{u \in W} (R(w, u) \odot \phi^V(u)) \Rightarrow \psi^V(w)) \\ &= \bigwedge_{u \in W} (\phi^V(u) \Rightarrow \bigwedge_{w \in W} (R(w, u) \Rightarrow \psi^V(w))) \\ &= \bigwedge_{u \in W} (\phi^V(u) \Rightarrow ([\psi]^{-1})^V(u)) \\ &= e_M(\phi^V, ([\psi]^{-1})^V). \end{aligned}$$

(2) For each $\phi, \psi \in \Phi_L$, by (1),

$$\begin{aligned} \top &= e_M(\langle \phi \rangle^V, \langle \phi \rangle^V) = e_M(\phi^V, ([\langle \phi \rangle]^{-1})^V), \\ \top &= e_M(([\psi]^{-1})^V, ([\psi]^{-1})^V) = e_M(\langle [\psi]^{-1} \rangle^V, \psi^V). \end{aligned}$$

For each $\phi, \psi \in \Phi_L$,

$$\begin{aligned} e_M(\langle \phi \rangle^V, \langle \psi \rangle^V) &= e_M(\phi^V, ([\langle \psi \rangle]^{-1})^V) \\ &\geq e_M(\phi^V, \psi^V) \odot e_M(\psi^V, ([\langle \psi \rangle]^{-1})^V) \\ &= e_M(\phi^V, \psi^V). \end{aligned}$$

$$\begin{aligned} e_M([\phi]^{-1})^V, ([\psi]^{-1})^V &= e_M(\langle [\phi]^{-1} \rangle^V, \psi^V) \\ &\geq e_M(\langle [\phi]^{-1} \rangle^V, \phi^V) \odot e_M(\phi^V, \psi^V) \\ &= e_M(\phi^V, \psi^V). \end{aligned}$$

(4) R is reflexive and transitive, then

$$\begin{aligned} [\langle \phi \rangle^{-1}]^V(w) &= \bigwedge_{v \in W} (R(w, v) \Rightarrow (\langle \phi \rangle^{-1})^V(v)) \\ &= \bigwedge_{v \in W} (R(w, v) \Rightarrow \bigvee_{u \in W} (R(u, v) \odot \phi^V(u))) \\ &= \bigwedge_{v \in W} (R(w, v) \Rightarrow \bigvee_{u \in W} (\bigvee_{s \in W} (R(u, s) \odot R(s, v)) \odot \phi^V(u))) \\ &= \bigwedge_{v \in W} (R(w, v) \Rightarrow \bigvee_{u \in W} ((R(u, w) \odot R(w, v)) \odot \phi^V(u))) \\ &\geq \bigvee_{u \in W} (R(u, w) \odot \phi^V(u)) = (\langle \phi \rangle^{-1})^V(w) \end{aligned}$$

$$\begin{aligned} \langle [\phi] \rangle^{-V}(w) &= \bigvee_{v \in W} (R(v, w) \odot [\phi]^V(v)) \\ &= \bigvee_{v \in W} (R(v, w) \odot \bigwedge_{u \in W} (R(v, u) \Rightarrow \phi^V(u))) \\ &= \bigvee_{v \in W} (R(v, w) \odot \bigwedge_{u \in W} (\bigvee_{s \in W} (R(v, s) \odot R(s, u)) \Rightarrow \phi^V(u))) \\ &\leq \bigvee_{v \in W} (R(v, w) \odot \bigwedge_{u \in W} ((R(v, w) \odot R(w, u)) \Rightarrow \phi^V(u))) \\ &= \bigvee_{v \in W} (R(v, w) \odot (R(v, w) \Rightarrow \bigwedge_{u \in W} (R(w, u) \Rightarrow \phi^V(u)))) \\ &\leq \bigwedge_{u \in W} (R(w, u) \Rightarrow \phi^V(u)) = [\phi]^V(w). \end{aligned}$$

(6) By (2), $\phi^V \leq ([\langle \phi \rangle]^{-1})^V$ implies $\langle \phi \rangle^V \leq \langle [\langle \phi \rangle]^{-1} \rangle^V$. Moreover, $\langle [\langle \phi \rangle]^{-1} \rangle^V \leq \langle \phi \rangle^V$. Thus, $\langle [\langle \phi \rangle]^{-1} \rangle^V = \langle \phi \rangle^V$. Other cases are similarly proved.

(7) By (2), $([\langle \phi_0 \rangle]^{-1})^V = \langle \phi_0 \rangle^V = \psi^V$. Hence, $[\langle \phi_0 \rangle]^{-1}$ is a solution with $\phi_0^V \leq ([\langle \phi_0 \rangle]^{-1})^V$.

(8) By (2), $([[\phi_0]^{-1}]]^{-1})^V = ([\phi_0]^{-1})^V = \psi^V$. Hence, $([\phi_0]^{-1})^V$ is a solution with $([\phi_0]^{-1})^V \leq \phi_0^V$. $\square$

Theorem 4. Let $M = (W, R, V)$ be a fuzzy Kripke model. The maps $\{-\}, \{-\}^{-1} : \Phi_L \rightarrow \Phi_L$ are defined by $\{-\}(\phi) = \{\phi\}$ and $\{-\}^{-1}(\phi) = \{\phi\}^{-1}$, for each $\phi \in \Phi_L$. Then, the following statements hold:

- (1) $(e_M, \{-\}, \{-\}^{-1}, e_M)$ is a Galois connection.
- (2) For each $\phi, \psi \in \Phi_L$,

$$\begin{aligned} e_M(\phi^V, (\{\{\phi\}\}^{-1})^V) &= \top = e_M(\psi^V, \{\{\psi\}^{-1}\}^V), \\ e_M(\phi^V, \psi^V) &\leq e_M(\{\psi\}^V, \{\phi\}^V), \\ e_M(\phi^V, \psi^V) &\leq e_M((\{\psi\}^{-1})^V, (\{\phi\}^{-1})^V). \end{aligned}$$

(3) Let L follow a double negative law. If $R(w, w) = \perp$ and $R^*(w, u) \odot R^*(u, v) \leq R^*(w, v)$ for all $w, u, v \in W$, then $\phi \in \Phi_L$ and

$$\{\neg\{\phi\}\}^V = \{\phi\}^V, (\{\neg\{\phi\}^{-1}\}^{-1})^V = (\{\phi\}^{-1})^V.$$

- (4) For each $\phi_i, \psi_i \in \Phi_L$,

$$\begin{aligned} \{\{\{\phi\}\}^{-1}\}^V &= \{\phi\}^V, \\ (\{\{\{\phi\}^{-1}\}\}^{-1})^V &= (\{\phi\}^{-1})^V. \end{aligned}$$

(5) If, for $\psi \in \Phi_L$, there exists a solution $\phi_0 \in \Phi_L$ such that $\psi^V = \{\phi_0\}^V$, then there exists a solution $\{\{\phi_0\}\}^{-1} \in \Phi_L$ such that $\psi^V = \{\phi_0\}^V = \{\{\{\phi_0\}\}^{-1}\}^V$ and $(\{\{\phi_0\}\}^{-1})^V \geq \phi_0^V$.

(6) If, for $\psi \in \Phi_L$, there exists a solution $\phi_0 \in \Phi_L$ such that $\psi^V = (\{\phi_0\}^{-1})^V$, then there exists a solution $\{\{\phi_0\}\} \in \Phi_L$ such that $\psi^V = (\{\phi_0\}^{-1})^V = (\{\{\{\phi_0\}\}^{-1}\}^{-1})^V$ and $\{\{\phi_0\}^{-1}\}^V \geq \phi_0^V$.

Proof. (1) For each $\phi, \psi, \{\psi\}, \{\phi\}^{-1} \in \Phi_L$, by Lemma 1 (4 and 7),

$$\begin{aligned} e_M(\phi^V, \{\psi\}^V) &= \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \{\psi\}^V(w)) \\ &= \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \bigwedge_{u \in W} (\psi^V(u) \Rightarrow R(w, u))) \\ &= \bigwedge_{u \in W} (\psi^V(u) \Rightarrow \bigwedge_{w \in W} (\phi^V(w) \Rightarrow R(w, u))) \\ &= \bigwedge_{u \in W} (\psi^V(u) \Rightarrow (\{\phi\}^{-1})^V(u)) \\ &= e_M(\psi^V, \{\phi\}^{-V}). \end{aligned}$$

- (2) For each $\phi, \psi \in \Phi_L$,

$$\begin{aligned} \top &= e_M(\{\phi\}^V, \{\phi\}^V) = e_M(\phi^V, (\{\{\phi\}\}^{-1})^V), \\ \top &= e_M((\{\psi\}^{-1})^V, (\{\psi\}^{-1})^V) = e_M(\psi^V, \{\{\psi\}^{-1}\}^V). \end{aligned}$$

For each $\phi, \psi \in \Phi_L$,

$$\begin{aligned} e_M(\{\phi\}^V, \{\psi\}^V) &= e_M(\psi^V, (\{\{\phi\}\}^{-1})^V) \\ &\geq e_M(\psi^V, \phi^V) \odot e_M(\phi^V, (\{\{\phi\}\}^{-1})^V) \\ &= e_M(\psi^V, \phi^V). \end{aligned}$$

$$\begin{aligned} e_M((\{\phi\}^{-1})^V, (\{\psi\}^{-1})^V) &= e_M(\psi^V, \{\{\phi\}^{-1}\}^V) \\ &\geq e_M(\psi^V, \phi^V) \odot e_M(\phi^V, \{\{\phi\}^{-1}\}^V) \\ &= e_M(\psi^V, \phi^V). \end{aligned}$$

(3) Since $R(w, w) = \perp$ and $R^*(w, u) \odot R^*(u, v) \leq R^*(w, v)$ for all $w, u, v \in W$,

$$\begin{aligned} \top \odot R^*(w, v) &= R^*(w, w) \odot R^*(w, v) \\ &\leq \bigvee_{u \in W} (R^*(w, u) \odot R^*(u, v)) \leq R^*(w, v). \end{aligned}$$

Thus, $\bigvee_{u \in W} (R^*(w, u) \odot R^*(u, v)) = R^*(w, v)$.

$$\begin{aligned} \{\neg\{\phi\}\}^V(w) &= \bigwedge_{u \in W} (\neg\{\phi\}^V(u) \Rightarrow R(w, u)) \\ &= \bigwedge_{u \in W} ((\bigwedge_{v \in W} (\phi^V(v) \Rightarrow R(u, v)))^* \Rightarrow R(w, u)) \\ &= \bigwedge_{u \in W} (\bigvee_{v \in W} (\phi^V(v) \odot R^*(u, v)) \Rightarrow R(w, u)) \\ &= \bigwedge_{u \in W} \bigwedge_{v \in W} (\phi^V(v) \Rightarrow (R^*(u, v) \Rightarrow R(w, u))) \\ &= \bigwedge_{v \in W} (\phi^V(v) \Rightarrow \bigwedge_{u \in W} (R^*(u, v) \odot R^*(u, u))^*) \\ &= \bigwedge_{v \in W} (\phi^V(v) \Rightarrow R(w, v)) \\ &= \{\phi\}^V(w). \end{aligned}$$

(4) By (2), $\phi^V \leq (\{\{\phi\}\}^{-1})^V$ implies $\{\phi\}^V \geq \{\{\{\phi\}\}^{-1}\}^V$. Moreover, $\{\phi\}^V \leq \{\{\{\phi\}\}^{-1}\}^V$. Thus, $\{\phi\}^V = \{\{\{\phi\}\}^{-1}\}^V$. The other case is similarly proved.

(5) By (2), $\psi^V = \{\phi_0\}^V = \{\{\{\phi_0\}\}^{-1}\}^V$. Hence, $(\{\{\phi_0\}\}^{-1})^V$ is a solution with $\phi_0^V \leq (\{\{\phi_0\}\}^{-1})^V$.

(6) It is similarly proved as (5). $\square$

Definition 5. Let $M = (W, R, V)$ be a fuzzy Kripke model.

(1) A map $I : \Phi_L \rightarrow \Phi_L$ is called an interior operator iff it satisfies the following conditions; for each $\phi, \psi \in \Phi_L$,

- (I1) $I(\phi)^V \leq \phi^V$;
- (I2) $I(I(\phi))^V = I(\phi)^V$;
- (I3) $e_M(I(\phi)^V, I(\psi)^V) \geq e_M(\phi^V, \psi^V)$.

(2) A map $C : \Phi_L \rightarrow \Phi_L$ is called a closure operator iff it satisfies the following conditions; for each $\phi, \psi \in \Phi_L$,

- (C1) $C(\phi)^V \geq \phi^V$;
- (C2) $C(C(\phi))^V = C(\phi)^V$;
- (C3) $e_M(C(\phi)^V, C(\psi)^V) \geq e_M(\phi^V, \psi^V)$.

The pair $(I(\phi)^V, C(\phi)^V)$ is called a fuzzy rough set for $\phi \in \Phi_L$.

Theorem 5. Let $M = (W, R, V)$ be a fuzzy Kripke model.

- (1) The maps $[\langle - \rangle^{-1}]$, $[\langle - \rangle]^{-1}$, $\{\{\{-\}^{-1}\}\}$, $\{\{\{-\}\}^{-1}\} : \Phi_L \rightarrow \Phi_L$ are closure operators.
- (2) $\langle [-] \rangle^{-1}$, $\langle [-] \rangle^{-1} : \Phi_L \rightarrow \Phi_L$ are interior operators.

Proof. (1) (C1) Since $(e_M, \langle - \rangle^{-1}, [-], e_M)$ is a residuated connection from Theorem 3 (3), $\top = e_M((\langle \phi \rangle^{-1})^V, (\langle \phi \rangle^{-1})^V) = e_M(\phi^V, [\langle \phi \rangle^{-1}]^V)$. By Lemma 1 (3), $\phi^V \leq [\langle \phi \rangle^{-1}]^V$.

(C2) Since $(\langle \phi \rangle^{-1})^V = (\langle [\langle \phi \rangle^{-1}] \rangle^{-1})^V$ from Theorem 3 (6), $[\langle \phi \rangle^{-1}]^V = [\langle [\langle \phi \rangle^{-1}] \rangle^{-1}]^V$.

(C3) By Theorem 3 (4), we have

$$\begin{aligned} &e_M([\langle \phi \rangle^{-1}]^V, [\langle \psi \rangle^{-1}]^V) \\ &\geq e_M((\langle \phi \rangle^{-1})^V, (\langle \psi \rangle^{-1})^V) \\ &\geq e_M(\phi^V, \psi^V). \end{aligned}$$

Hence, $[\langle - \rangle^{-1}]$ is a closure operator. Other cases are similarly proved.

(2) (I1) Since $(e_M, \langle - \rangle^{-1}, [-], e_M)$ is a residuated connection from Theorem 3 (3), $\top = e_M([\phi]^V, [\phi]^V) = e_M((\langle [\phi] \rangle^{-1})^V, \phi^V)$. By Lemma 1(3), $(\langle [\phi] \rangle^{-1})^V \leq \phi^V$.

(I2) Since $[\langle [\phi] \rangle^{-1}]^V = [\phi]^V$ from Theorem 3(6), $(\langle [\langle [\phi] \rangle^{-1}] \rangle^{-1})^V = (\langle [\phi] \rangle^{-1})^V$.

(I3) By Theorem 3 (4), we have

$$\begin{aligned}
& e_M(\langle [\phi] \rangle^{-1})^V, (\langle [\psi] \rangle^{-1})^V) \\
& \geq e_M([\phi]^V, [\psi]^V) \\
& \geq e_M(\phi^V, \psi^V).
\end{aligned}$$

Hence, $\langle [-] \rangle^{-1}$ is an interior operator. Other cases are similarly proved.

□

Definition 6. Let $M = (W, R, V)$ be a fuzzy Kripke model.

(1) A subset $\tau \subset L^W$ is called an Alexandrov pretopology on W iff it satisfies the following conditions:

- (O1) $\perp^V \in \tau$.
- (O2) If $\phi_i^V \in \tau$ for all $i \in I$, then $(\bigvee_{i \in I} \phi_i)^V \in \tau$.
- (O3) If $\phi_i^V \in \tau$ and $t \in L$, then $(\bar{t} \& \phi)^V \in \tau$.

(2) A subset $\eta \subset L^W$ is called an Alexandrov precotopology on W iff it satisfies the following conditions:

- (F1) $\top^V \in \eta$.
- (F2) If $\phi_i^V \in \eta$ for all $i \in I$, then $(\bigwedge_{i \in I} \phi_i)^V \in \eta$.
- (F3) If $\phi^V \in \eta$ and $t \in L$, then $(\bar{t} \rightarrow \phi)^V \in \eta$.

A subset $\tau \subset L^X$ is called an Alexandrov topology on X if it is both an Alexandrov pretopology and Alexandrov precotopology on X .

Theorem 6. Let $M = (W, R, V)$ be a fuzzy Kripke model. Let $D : \Phi_L \rightarrow \Phi_L$ be a map. The following statements are equivalent:

- (1) $e_M(\phi^V, \psi^V) \leq e_M(D(\phi)^V, D(\psi)^V)$ for all $\phi, \psi \in \Phi_L$.
- (2) $\bar{t}^V \odot D(\phi)^V \leq D(\bar{t} \& \phi)^V$ for each $t \in L, \phi \in \Phi_L$ and $D(\phi)^V \leq D(\psi)^V$ for $\phi^V \leq \psi^V$.
- (3) $D(\bar{t} \rightarrow \phi)^V \leq \bar{t}^V \Rightarrow D(\phi)^V$ for each $t \in L, \phi \in \Phi_L$ and $D(\phi)^V \leq D(\psi)^V$ for $\phi^V \leq \psi^V$.

Proof. (1) $\Rightarrow$ (2). By Lemma 1 (3), if $\phi^V \leq \psi^V$, then $e_M(\phi^V, \psi^V) = \top$ and $e_M(D(\phi)^V, D(\psi)^V) = \top$. Thus, $D(\phi)^V \leq D(\psi)^V$. Since $t \leq \phi^V(w) \Rightarrow (\bar{t} \& \phi)^V(w)$,

$$t \leq e_M(\phi^V, (\bar{t} \& \phi)^V) \leq e_M(D(\phi)^V, D(\bar{t} \& \phi)^V).$$

Hence, $\bar{t}^V \odot D(\phi)^V \leq D(\bar{t} \& \phi)^V$.

(2) $\Rightarrow$ (1). Let $t = e_M(\phi^V, \psi^V)$. Then, $(\bar{t} \& \phi)^V(w) = e_M(\phi^V, \psi^V) \odot \phi^V(w) \leq \psi^V(w)$. Moreover, since $(\bar{t} \& \phi)^V \leq \psi^V$, by (2),

$$e_M(\phi^V, \psi^V) \odot D(\phi)^V \leq D(\bar{t} \& \phi)^V \leq D(\psi)^V.$$

Hence, $e_M(\phi^V, \psi^V) \leq e_M(D(\phi)^V, D(\psi)^V)$.

(1) $\Rightarrow$ (3). From (1), $D(\phi)^V \leq D(\psi)^V$ for $\phi^V \leq \psi^V$. Since $t \leq e_M((\bar{t} \rightarrow \phi)^V, \phi^V) \leq e_M(D(\bar{t} \rightarrow \phi)^V, D(\phi)^V)$, we have $D(\bar{t} \rightarrow \phi)^V \leq \bar{t}^V \Rightarrow D(\phi)^V$.

(3) $\Rightarrow$ (1). Let $t = e_M(\phi^V, \psi^V)$. Since $\phi^V(w) \odot e_M(\phi^V, \psi^V) \leq \psi^V(w)$, $\phi^V(w) \leq e_M(\phi^V, \psi^V) \Rightarrow \psi^V(w)$. Then,

$$D(\phi)^V \leq D(\bar{t} \rightarrow \psi^V)^V \leq e_M(\phi^V, \psi^V) \Rightarrow D(\psi)^V.$$

Thus, $e_M(\phi^V, \psi^V) \leq e_M(D(\phi)^V, D(\psi)^V)$

□

Theorem 7. Let $M = (W, R, V)$ be a fuzzy Kripke model.

(1) Let (W, I) be an interior space. Define $\tau_I = \{\phi^V \in L^W \mid \phi^V = I(\phi)^V\}$. Then, τ_I is an Alexandrov pretopology on W such that $\tau_I = \{I(\phi)^V \in L^W \mid \phi \in \Phi_L\}$.

(2) Let (W, C) be a closure space. Define $\eta_C \subset L^X$ as $\eta_C = \{\phi^V \in L^W \mid \phi^V = C(\phi)^V\}$. Then, η_C is an Alexandrov precotopology on W such that $\eta_C = \{C(\phi)^V \in L^W \mid \phi \in \Phi_L\}$.

Proof. (1) (O1) Since $\overline{\perp}^V \geq I(\overline{\perp})^V$, $I(\overline{\perp})^V = \overline{\perp}^V$. Hence, $\overline{\perp}^V \in \tau_I$.

(O2) For $\phi_i^V \in \tau$ for all $i \in I$, then $(\bigvee_{i \in I} \phi_i)^V \in \tau_I$ from

$$\begin{aligned} (\bigvee_i \phi_i)^V &\geq I(\bigvee_i \phi_i)^V \\ &\geq \bigvee_i I(\phi_i)^V = \bigvee_i \phi_i^V. \end{aligned}$$

(O3) For $\phi_i^V \in \tau_I$ and $t \in L$, then $(\bar{t} \& \phi)^V \in \tau_I$ from

$$\begin{aligned} (\bar{t} \& \phi)^V &\geq I(\bar{t} \& \phi)^V \\ &\geq \bar{t}^V \odot I(\phi)^V = (\bar{t} \& \phi)^V. \end{aligned}$$

Therefore, τ_I is an Alexandrov pretopology on W . Moreover, since $I(I(\phi))^V = I(\phi)^V$, $\{I(\phi)^V \in L^W \mid \phi \in \Phi_L\} \subset \tau_I$. Trivially, $\tau_I \subset \{I(\phi)^V \in L^W \mid \phi \in \Phi_L\}$.

(2) (F1) Since $\overline{\top}^V \geq C(\overline{\top})^V \geq \overline{\top}^V$, $C(\overline{\top})^V = \overline{\top}^V$. Hence, $\overline{\top}^V \in \eta_C$.

(F2) For $\phi_i^V \in \eta_C$ for all $i \in I$, then $(\bigwedge_{i \in I} \phi_i)^V \in \eta_C$ from

$$\begin{aligned} (\bigwedge_i \phi_i)^V &\leq C(\bigwedge_i \phi_i)^V \\ &\leq \bigwedge_i C(\phi_i)^V = \bigwedge_i \phi_i^V. \end{aligned}$$

(F3) For $\phi_i^V \in \eta_C$ and $t \in L$, then $(\bar{t} \rightarrow \phi)^V \in \eta_C$ from

$$\begin{aligned} (\bar{t} \rightarrow \phi)^V &\leq C(\bar{t} \rightarrow \phi)^V \\ &\leq \bar{t}^V \Rightarrow C(\phi)^V = (\bar{t} \rightarrow \phi)^V. \end{aligned}$$

Therefore, η_C is an Alexandrov precotopology on W .

□

Remark 1. Let $M = (W, R, V)$ be a fuzzy Kripke model.

(1) By (V1–V5) in Definition 3, $\tau_M = \{\phi^V \in L^W \mid \phi \in \Phi_L\}$ is an Alexandrov topology on W .

(2) If (W, I) is an interior space, $\tau_I = \{I(\phi)^V \in L^W \mid \phi \in \Phi_L\}$ is an Alexandrov pretopology on W for each $I \in \{[\langle - \rangle]^{-1}, \langle [-]^{-1} \rangle\}$ in Theorem 7.

(3) If (W, C) is a closure space, $\eta_C = \{C(\phi)^V \in L^W \mid \phi \in \Phi_L\}$ is an Alexandrov precotopology on W for each $C \in \{[\langle - \rangle]^{-1}, \langle [-]^{-1} \rangle, \{\{-\}^{-1}\}, \{\{-\}\}^{-1}\}$ in Theorem 7.

(4) By (2) and (3), the pair $(I(\phi)^V, C(\phi)^V)$ is a fuzzy rough set for $\phi \in \Phi_L$ where $I \in \{[\langle - \rangle]^{-1}, \langle [-]^{-1} \rangle\}$ and $C \in \{[\langle - \rangle]^{-1}, \langle [-]^{-1} \rangle, \{\{-\}^{-1}\}, \{\{-\}\}^{-1}\}$.

Theorem 8. Let $M = (W, R, V)$ be a fuzzy Kripke model.

(1) For $e_W \in L^{W \times W}$, we define $\tau_{e_W} = \{\phi^V \in L^W \mid \phi^V(x) \odot e_W(x, y) \leq \phi^V(y), \phi \in \Phi_L\}$. Then, τ_{e_W} is an Alexandrov topology on W .

If e_W is reflexive, $\tau_{e_W} = \{\phi^V \in L^W \mid \bigvee_{x \in X} (\phi^V(x) \odot e_W(x, y)) = \phi^V(y), \phi \in \Phi_L\}$.

If e_W is a preorder, $\tau_{e_W} = \{\bigvee_{x \in X} (\phi^V(x) \odot e_W(x, y)) \in L^W \mid \phi \in \Phi_L\}$.

(2) Let (W, τ_M) be an Alexandrov topological space. Define $e_{\tau_M}(x, y) = \bigwedge_{\phi^V \in \tau_M} (\phi^V(x) \Rightarrow \phi^V(y))$. Then, e_{τ_M} is a fuzzy preorder with $\tau_M = \tau_{e_{\tau_M}}$.

Proof. (1) (O1, F1) $\perp_W, \top_W \in \tau_{e_W}$.

(O2,F2) If $\phi_i^V \in \tau_{e_W}$ for all $i \in I$, then

$$\begin{aligned} (\bigvee_{i \in I} \phi_i^V(x)) \odot e_W(x, y) &= \bigvee_{i \in I} (\phi_i^V(x) \odot e_W(x, y)) \\ &\leq \bigvee_{i \in I} \phi_i^V(y), \\ (\bigwedge_{i \in I} \phi_i^V(x)) \odot e_W(x, y) &\leq \bigwedge_{i \in I} (\phi_i^V(x) \odot e_W(x, y)) \\ &\leq \bigwedge_{i \in I} \phi_i^V(y). \end{aligned}$$

(O3, F3) Let $\phi^V \in \tau_{e_W}$ and $t \in L$. Since $(t \odot \phi^V(x)) \odot e_W(x, y) = t \odot (\phi^V(x) \odot e_W(x, y)) \leq t \odot \phi^V(y)$, we have

$$\begin{aligned} t \odot (t \Rightarrow \phi^V)(x) \odot e_W(x, y) &\leq \phi^V(x) \odot e_W(x, y) \leq \phi^V(y) \\ \text{if and only if } (t \Rightarrow \phi^V)(x) \odot e_W(x, y) &\leq (t \Rightarrow \phi^V)(y). \end{aligned}$$

Thus, $(\bar{t} \& \phi)^V, (\bar{t} \rightarrow \phi)^V \in \tau_{e_W}$

Therefore, τ_{e_W} is an Alexandrov topology on W .

If e_W is reflexive, then $\bigvee_{x \in X} (\phi^V(x) \odot e_W(x, y)) \geq \phi^V(y) \odot e_W(y, y) = \phi^V(y)$. Hence, the result holds.

Let e_W be a fuzzy preorder and $\tau = \{\bigvee_{x \in X} (\phi^V(x) \odot e_W(x, -)) \in L^W \mid \phi \in \Phi_L\}$.

Let $\phi^V \in \tau_{e_W}$. Since e_W is reflexive, $\phi^V = \bigvee_{x \in X} (\phi^V(x) \odot e_W(x, -)) \in \tau$.

Let $\bigvee_{x \in W} (\phi^V(x) \odot e_W(x, -)) \in \tau$. Then,

$$\begin{aligned} \bigvee_{y \in W} (\bigvee_{x \in W} (\phi^V(x) \odot e_W(x, y)) \odot e_W(y, z)) \\ = \bigvee_{x \in W} (\phi^V(x) \odot \bigvee_{y \in W} (e_W(x, y) \odot e_W(y, z))) \\ = \bigvee_{x \in X} (\phi^V(x) \odot e_W(x, z)). \end{aligned}$$

Hence, $\bigvee_{x \in W} (\phi^V(x) \odot e_W(x, -)) \in \tau_{e_W}$.

(2) (E1) $e_{\tau_M}(x, x) = \bigwedge_{\phi^V \in \tau_M} (\phi^V(x) \Rightarrow \phi^V(x)) = \top$ for all $x \in X$;

(E2) $e_{\tau_M}(x, y) \odot e_{\tau_M}(y, z) = \bigwedge_{\phi^V \in \tau_M} (\phi^V(x) \Rightarrow \phi^V(y)) \odot \bigwedge_{\phi^V \in \tau_M} (\phi^V(y) \Rightarrow \phi^V(z)) \leq \bigwedge_{\phi^V \in \tau_M} (\phi^V(x) \Rightarrow \phi^V(z)) = e_{\tau_M}(x, z)$, for all $x, y, z \in W$. Hence, e_{τ_M} is a fuzzy preorder. Let $\psi^V \in \tau_M$. Then,

$$\begin{aligned} \bigvee_{x \in W} (\psi^V(x) \odot \bigwedge_{\phi^V \in \tau_M} (\phi^V(x) \Rightarrow \phi^V(y))) \\ \leq \bigvee_{x \in W} (\psi^V(x) \odot (\psi^V(x) \Rightarrow \psi^V(y))) \leq \psi^V(y). \end{aligned}$$

Hence, $\psi^V \in \tau_{e_{\tau_M}}$.

Since τ_M is an Alexandrov topology on X , $\psi^V(x) \odot \bigwedge_{\phi^V \in \tau_M} (\phi^V(x) \Rightarrow \phi^V(-)) \in \tau_M$. For all $\psi^V \in \tau_{e_{\tau_M}}$, we have $\psi^V = \bigvee_{x \in W} (\psi^V(x) \odot \bigwedge_{\phi^V \in \tau_M} (\phi^V(x) \Rightarrow \phi^V(-))) \in \tau_M$, and so $\psi^V \in \tau_M$. Hence, $\tau_M = \tau_{e_{\tau_M}}$. $\square$

Remark 2. Let $M = (W, R, V)$ be a fuzzy Kripke model. For $e_W = R \in L^{W \times W}$, by Theorem 8 (1), we define

$$\begin{aligned} \tau_R &= \{\phi^V \in L^W \mid \phi^V(x) \odot R(x, y) \leq \phi^V(y), \phi \in \Phi_L\} \\ &= \{\phi^V \in L^W \mid (\langle \phi \rangle^{-1})^V \leq \phi^V\} \\ &= \{\phi^V \in L^W \mid \phi^V \leq [\phi]^V\}, \end{aligned}$$

$$\begin{aligned} \tau_{R^{-1}} &= \{\phi^V \in L^W \mid \phi^V(x) \odot R(y, x) \leq \phi^V(y), \phi \in \Phi_L\} \\ &= \{\phi^V \in L^W \mid \langle \phi \rangle^V \leq \phi^V\} \\ &= \{\phi^V \in L^W \mid \phi^V \leq ([\phi]^{-1})^V\}. \end{aligned}$$

Then, τ_R and $\tau_{R^{-1}}$ are Alexandrov topologies on W .

Example 2. Let $(L = [0, 1], \odot, \Rightarrow, 0, 1)$ be a complete residuated lattice defined by

$$x \odot y = (x + y - 1) \vee 0, \quad x \Rightarrow y = (1 - x + y) \wedge 1.$$

Let $M = (W, R, V)$ be a fuzzy Kripke model with a set of propositional symbols $P = \{p_1, p_2, p_3\}$ and $W = \{w_1, w_2, w_3, w_4\}$, where $p_i^V(w_j) = V(w_j, p_i)$ and $R \in L^{W \times W}$ as follows:

	p_1^V	p_2^V	p_3^V
w_1	0.8	0.7	0.6
w_2	0.3	0.5	0.3
w_3	0.6	0.4	0.9
w_4	0.4	0.6	0.8

R	w_1	w_2	w_3	w_4
w_1	0.6	0.3	0.5	0.1
w_2	0.8	0.1	0.4	0.9
w_3	0.7	0.6	0.2	0.7
w_4	0.1	0.6	0.9	0.5

(1) We obtain a fuzzy partial order $e_W(w_i, w_j) = \bigwedge_{p \in P} (p^V(w_i) \Rightarrow p^V(w_j))$ as

e_W	w_1	w_2	w_3	w_4
w_1	1	0.5	0.7	0.6
w_2	1	1	0.9	1
w_3	0.7	0.4	1	0.8
w_4	0.8	0.5	0.8	1

By Theorem 8 (1), we can obtain an Alexandrov topology $\tau_{e_W} = \{\bigvee_{w_i \in W} (\phi^V(w_i) \odot e_W(w_i, w_j)) \in L^W \mid \phi \in \Phi_L\}$.

(2) Since $\langle \phi \rangle^V(w) = \bigvee_{v \in W} (R(w, v) \odot \phi^V(v))$, we have

	$\langle p_1 \rangle^V$	$\langle p_2 \rangle^V$	$\langle p_3 \rangle^V$
w_1	0.4	0.3	0.4
w_2	0.6	0.5	0.7
w_3	0.5	0.4	0.5
w_4	0.5	0.3	0.8

By Remark 2, $\tau_{R^{-1}} = \{\phi^V \in L^W \mid \langle \phi \rangle^V \leq \phi^V\}$ is an Alexandrov topology. Since $\langle p_2 \rangle^V \leq p_2^V$ and $\langle p_i \rangle^V \not\leq p_i^V$ for $i = 1, 3$, $p_2^V \in \tau_{R^{-1}}$ and $p_i^V \notin \tau_{R^{-1}}$ for $i = 1, 3$.

By Theorem 5 (1), $[\langle - \rangle]^{-1} : \Phi_L \rightarrow \Phi_L$ is a closure operator with

	$([\langle p_1 \rangle]^{-1})^V$	$([\langle p_2 \rangle]^{-1})^V$	$([\langle p_3 \rangle]^{-1})^V$
w_1	0.8	0.7	0.8
w_2	0.9	0.7	0.9
w_3	0.6	0.4	0.9
w_4	0.7	0.6	0.8

For $\langle p_1 \rangle^V = (0.4, 0.6, 0.5, 0.5) \in L^W$, $([\langle p_1 \rangle]^{-1})^V = \langle p_1 \rangle^V$ and $(0.8, 0.9, 0.6, 0.7) = ([\langle p_1 \rangle]^{-1})^V \geq p_1^V = (0.8, 0.3, 0.6, 0.4)$.

By Theorem 7 (2), $\eta_{[\langle - \rangle]^{-1}} = \{([\langle \phi \rangle]^{-1})^V \in L^W \mid \phi \in \Phi_L\}$ is an Alexandrov precotopology.

(3) Since $(\langle \phi \rangle^{-1})^V(w) = \bigvee_{v \in W} (R(v, w) \odot \phi^V(v))$, we have

	$(\langle p_1 \rangle^{-1})^V$	$(\langle p_2 \rangle^{-1})^V$	$(\langle p_3 \rangle^{-1})^V$
w_1	0.4	0.3	0.6
w_2	0.2	0.2	0.5
w_3	0.3	0.5	0.7
w_4	0.3	0.4	0.6

By Remark 2, $\tau_R = \{\phi^V \in L^W \mid (\langle \phi \rangle^{-1})^V \leq \phi^V\}$ is an Alexandrov topology. Since $(\langle p_1 \rangle^{-1})^V \leq p_1^V$ and $(\langle p_i \rangle^{-1})^V \not\leq p_i^V$ for $i = 2, 3$, $p_1^V \in \tau_R$ and $p_i^V \notin \tau_R$ for $i = 2, 3$. We obtain

	$[\langle p_1 \rangle^{-1}]^V$	$[\langle p_2 \rangle^{-1}]^V$	$[\langle p_3 \rangle^{-1}]^V$
w_1	0.8	0.7	1
w_2	0.4	0.5	0.7
w_3	0.6	0.6	0.9
w_4	0.4	0.6	0.8

Since $[\langle - \rangle^{-1}] : \Phi_L \rightarrow \Phi_L$ is a closure operator from Theorem 5 (1), by Theorem 7 (2), $\eta_{[\langle - \rangle^{-1}]} = \{[\langle \phi \rangle^{-1}]^V \in L^W \mid \phi \in \Phi_L\}$ is an Alexandrov precotopology.

(4) Since $[\phi]^V(w) = \bigvee_{v \in W} (R(w, v) \Rightarrow \phi^V(v))$, we have

	$[p_1]^V$	$[p_2]^V$	$[p_3]^V$
w_1	1	0.9	1
w_2	0.5	0.7	0.8
w_3	0.7	0.9	0.7
w_4	0.7	0.5	0.7

By Remark 2, $\tau_R = \{\phi^V \in L^W \mid \phi^V \leq [\phi]^V\}$ is an Alexandrov topology. Since $p_1^V \leq [p_1]^V$ and $p_i^V \not\leq [p_i]^V$ for $i = 2, 3$, $p_1^V \in \tau_R$ and $p_i^V \notin \tau_R$ for $i = 2, 3$.

By Theorem 5 (2), $\langle [-] \rangle^{-1} : \Phi_L \rightarrow \Phi_L$ is an interior operator with

	$(\langle [p_1] \rangle^{-1})^V$	$(\langle [p_2] \rangle^{-1})^V$	$(\langle [p_3] \rangle^{-1})^V$
w_1	0.6	0.6	0.6
w_2	0.3	0.5	0.3
w_3	0.6	0.4	0.6
w_4	0.4	0.6	0.7

By Theorem 7 (1), $\tau_{\langle [-] \rangle^{-1}} = \{(\langle [\phi] \rangle^{-1})^V \in L^W \mid \phi \in \Phi_L\}$ is an Alexandrov pretopology.

By Remark 1 (4), the pair $((\langle [p_1] \rangle^{-1})^V, [p_1]^V) = ((0.6, 0.3, 0.6, 0.4), (0.8, 0.4, 0.6, 0.4))$ is a rough set for p_1 . Moreover, the pair $((\langle [\phi] \rangle^{-1})^V, [\langle \phi \rangle^{-1}]^V)$ is a rough set for $\phi \in \Phi_L$.

In the other case, the pair $((\langle [p_1] \rangle^{-1})^V, ([p_1]^V)^V) = ((0.6, 0.3, 0.6, 0.4), (0.8, 0.9, 0.6, 0.7))$ is a rough set for p_1 . Moreover, the pair $((\langle [\phi] \rangle^{-1})^V, ([\langle \phi \rangle^{-1}]^V)^V)$ is a rough set for $\phi \in \Phi_L$.

(5) Since $([\phi]^{-1})^V(w) = \bigwedge_{v \in W} (R(v, w) \Rightarrow \phi^V(v))$, we have

	$([p_1]^{-1})^V$	$([p_2]^{-1})^V$	$([p_3]^{-1})^V$
w_1	0.5	0.7	0.5
w_2	0.8	0.8	1
w_3	0.5	0.7	0.9
w_4	0.4	0.6	0.4

By Remark 2, $\tau_{R^{-1}} = \{\phi^V \in L^W \mid \phi^V \leq ([\phi]^{-1})^V\}$ is an Alexandrov topology. Since $p_2^V \leq ([p_2]^{-1})^V$ and $p_i^V \not\leq ([p_i]^{-1})^V$ for $i = 1, 3$, $p_2^V \in \tau_{R^{-1}}$ and $p_i^V \notin \tau_{R^{-1}}$ for $i = 1, 3$.

By Theorem 5 (2), $\langle [-]^{-1} \rangle : \Phi_L \rightarrow \Phi_L$ is an interior operator with

	$\langle [p_1]^{-1} \rangle^V$	$\langle [p_2]^{-1} \rangle^V$	$\langle [p_3]^{-1} \rangle^V$
w_1	0.1	0.3	0.4
w_2	0.3	0.5	0.3
w_3	0.4	0.4	0.6
w_4	0.4	0.6	0.8

By Theorem 7 (1), $\tau_{\langle [-]^{-1} \rangle} = \{ \langle [\phi]^{-1} \rangle^V \in L^W \mid \phi \in \Phi_L \}$ is an Alexandrov pretopology.

By Remark 1 (4), the pair $(\langle [p_1]^{-1} \rangle^V, \langle [p_1]^{-1} \rangle^V) = ((0.1, 0.3, 0.4, 0.4), (0.8, 0.4, 0.6, 0.4))$ is a fuzzy rough set for p_1 . Moreover, the pair $(\langle [\phi]^{-1} \rangle^V, \langle [\phi]^{-1} \rangle^V)$ is a fuzzy rough set for $\phi \in \Phi_L$.

In the other case, the pair $(\langle [p_1]^{-1} \rangle^V, ([p_1]^{-1})^V) = ((0.1, 0.3, 0.4, 0.4), (0.8, 0.9, 0.6, 0.7))$ is a fuzzy rough set for p_1 . Moreover, the pair $(\langle [\phi]^{-1} \rangle^V, ([\phi]^{-1})^V)$ is a fuzzy rough set for $\phi \in \Phi_L$.

(6) Since $\{\phi\}^V(w) = \bigwedge_{v \in W} (\phi^V(v) \Rightarrow R(w, v))$ and $(\{\phi\}^{-1})^V(w) = \bigwedge_{v \in W} (\phi^V(v) \Rightarrow R(v, w))$, we have

	$\{p_1\}^V$	$\{p_2\}^V$	$\{p_3\}^V$
w_1	0.7	0.5	0.3
w_2	0.8	0.6	0.5
w_3	0.6	0.8	0.3
w_4	0.3	0.4	0.5

By Theorem 5 (1), $\{\{-\}\}^{-1} : \Phi_L \rightarrow \Phi_L$ is a closure operator with

	$(\{\{p_1\}\}^{-1})^V$	$(\{\{p_2\}\}^{-1})^V$	$(\{\{p_3\}\}^{-1})^V$
w_1	0.8	0.7	0.6
w_2	0.3	0.5	0.6
w_3	0.6	0.4	0.9
w_4	0.4	0.6	0.8

By Theorem 7 (2), $\eta_{\{\{-\}\}^{-1}} = \{ (\{\{\phi\}\}^{-1})^V \in L^W \mid \phi \in \Phi_L \}$ is an Alexandrov pretopology.

By Remark 1 (4), the pair $((\langle [p_1]^{-1} \rangle^V, (\{\{p_1\}\}^{-1})^V) = ((0.6, 0.3, 0.6, 0.4), (0.8, 0.3, 0.6, 0.4))$ is a fuzzy rough set for p_1 . Moreover, the pair $((\langle [\phi]^{-1} \rangle^V, (\{\{\phi\}\}^{-1})^V)$ is a fuzzy rough set for $\phi \in \Phi_L$.

In the other case, the pair $(\langle [p_1]^{-1} \rangle^V, (\{\{p_1\}\}^{-1})^V) = ((0.1, 0.3, 0.4, 0.4), (0.8, 0.3, 0.6, 0.4))$ is a fuzzy rough set for p_1 . Moreover, the pair $(\langle [\phi]^{-1} \rangle^V, (\{\{\phi\}\}^{-1})^V)$ is a fuzzy rough set for $\phi \in \Phi_L$.

4. Fuzzy Concept Lattice Based on Modal Logics

In this section, we investigate the relations between possible worlds as objects and formulas as attributes in a fuzzy information system.

Theorem 9. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be fuzzy Kripke models. Suppose that there exists $S \in L^{W \times W'}$ with $S(w, -) = \psi_w^{V'}$, $S(-, w') = \phi_{w'}^V$ where $\phi_{w'} \in \Phi_L$, $\psi_w \in \Phi'_L$.

(1) Define a map $F : \Phi_L \rightarrow \Phi'_L$ as

$$\begin{aligned} F(\phi)^{V'}(w') &= \bigvee_{w \in W} (\psi_w^V(w') \odot \phi^V(w)) \\ &= \bigvee_{w \in W} (S(w, w') \odot \phi^V(w)). \end{aligned}$$

Moreover, for $\phi, \phi_i \in \Phi_L, \psi, \psi_i \in \Phi'_L, t \in L$,

$$\begin{aligned} F(\bigvee_{i \in \Gamma} \phi_i)^{V'} &= \bigvee_{i \in \Gamma} F(\phi_i)^{V'}, \\ F(\bar{t} \& \phi)^{V'} &= (\bar{t} \& F(\phi))^{V'}. \end{aligned}$$

(2) Define $G(\psi) = \bigvee \{\phi \in \Phi_L \mid F(\phi)^{V'} \leq \psi^{V'}\}$. Then, there exists $G : \Phi'_L \rightarrow \Phi_L$ such that

$$\begin{aligned} G(\bigwedge_{i \in \Gamma} \psi_i)^V &= \bigwedge_{i \in \Gamma} G(\psi_i)^V, \\ G(\bar{t} \rightarrow \psi)^V &= (\bar{t} \rightarrow G(\psi))^V. \end{aligned}$$

Moreover, $(e_M, F, G, e_{M'})$ is a residuated connection, that is, $e_{M'}(F(\phi)^{V'}, \psi^{V'}) = e_M(\phi^V, G(\psi)^V)$ for each $\phi \in \Phi_L, \psi \in \Phi'_L$ and

$$G(\psi)^V(w) = \bigwedge_{w' \in W'} (S(w, w') \Rightarrow \psi^{V'}(w')).$$

Proof. (1) Since there exists $S \in L^{W \times W'}$ with $S(w, -) = \psi_w^{V'}$ and $\psi_w \in \Phi'_L$ by Definition 3 (V3) and (V4), for each $w \in W$,

$$\begin{aligned} (\psi_w \& \overline{\phi^V(w)}) &\in \Phi'_L, \\ \bigvee_{w \in W} (\psi_w \& \overline{\phi^V(w)}) &\in \Phi'_L. \end{aligned}$$

The map F is well-defined and $S(w, w') = \psi_w^{V'}(w')$ and

$$F(\phi)^{V'}(w') = \bigvee_{w \in W} (S(w, w') \odot \phi^V(w)).$$

For $\phi, \phi_i \in \Phi_L, t \in L$, by Lemma 1 (6),

$$\begin{aligned} F(\bigvee_{i \in \Gamma} \phi_i)^{V'}(w') &= \bigvee_{w \in W} (S(w, w') \odot \bigvee_{i \in \Gamma} \phi_i^V(w)) \\ &= \bigvee_{i \in \Gamma} \bigvee_{w \in W} (S(w, w') \odot \phi_i^V(w)) \\ &= \bigvee_{i \in \Gamma} F(\phi_i)^{V'}(w'), \end{aligned}$$

$$\begin{aligned} F(\bar{t} \& \phi)^{V'}(w') &= \bigvee_{w \in W} (S(w, w') \odot (\bar{t} \& \phi)^V(w)) \\ &= \bigvee_{w \in W} (S(w, w') \odot (t \odot \phi^V(w))) \\ &= t \odot \bigvee_{w \in W} (S(w, w') \odot \phi^V(w)) \\ &= (\bar{t} \& F(\phi))^{V'}(w'). \end{aligned}$$

Hence, $F(\bigvee_{i \in \Gamma} \phi_i)^{V'} = \bigvee_{i \in \Gamma} F(\phi_i)^{V'}$ and $F(\bar{t} \& \phi)^{V'} = (\bar{t} \& F(\phi))^{V'}$ for all $t \in L$.

(2) Since $G(\psi_1)^V \leq G(\psi_2)^V$ for $\psi_1^{V'} \leq \psi_2^{V'}$, $G(\bigwedge_{i \in \Gamma} \psi_i)^V \leq \bigwedge_{i \in \Gamma} G(\psi_i)^V$.

Since $F(\phi_1)^{V'} \leq F(\phi_2)^{V'}$ for $\phi_1^V \leq \phi_2^V$,

$$F(\bigwedge_{i \in \Gamma} G(\psi_i))^{V'} \leq F(G(\psi_i))^{V'} \leq \psi_i^{V'}.$$

Thus, $F(\bigwedge_{i \in \Gamma} G(\psi_i))^{V'} \leq \bigwedge_{i \in \Gamma} \psi_i^{V'}$. By the definition of G , $\bigwedge_{i \in \Gamma} G(\psi_i)^V \leq G(\bigwedge_{i \in \Gamma} \psi_i)^V$. Hence, $\bigwedge_{i \in \Gamma} G(\psi_i)^V = G(\bigwedge_{i \in \Gamma} \psi_i)^V$.

For $\psi \in \Phi'_L, t \in L$,

$$\begin{aligned} F(G(\bar{t} \rightarrow \psi))^{V'}(w') &= F(\bigvee \{\phi \in \Phi_L \mid F(\phi)^{V'} \leq (\bar{t} \rightarrow \psi)^{V'}\}) \\ &= \bigvee \{F(\phi)^{V'} \in \Phi'_L \mid F(\phi)^{V'} \leq (\bar{t} \rightarrow \psi)^{V'}\}(w') \\ &\leq t \Rightarrow \psi^{V'}(w'), \end{aligned}$$

$$\begin{aligned} F(\bar{t} \& G(\bar{t} \rightarrow \psi))^{V'}(w') &= t \odot F(G(\bar{t} \rightarrow \psi))^{V'}(w') \\ &\leq t \odot (t \Rightarrow \psi^{V'}(w')) \leq \psi^{V'}(w'). \end{aligned}$$

By the definition of G , $t \odot G(\bar{t} \rightarrow \psi)^V(w) \leq G(\psi)^V(w)$. Thus, $G(\bar{t} \rightarrow \psi)^V \leq (\bar{t} \rightarrow G(\psi))^V$.

On the other hand, since $(\bar{t} \& (\bar{t} \rightarrow G(\psi)))^V(w) = t \odot (t \Rightarrow G(\psi)^V(w)) \leq G(\psi)^V(w)$,

$$\begin{aligned} t \odot F(\bar{t} \rightarrow G(\psi))^{V'}(w') &= F(\bar{t} \& (\bar{t} \rightarrow G(\psi)))^V(w') \\ &\leq F(G(\psi))^{V'}(w') \leq \psi^{V'}(w'). \end{aligned}$$

Thus, $F(\bar{t} \rightarrow G(\psi))^{V'}(w') \leq (\bar{t} \rightarrow \psi)^{V'}(w')$.

By the definition of G , $(\bar{t} \rightarrow G(\psi))^V(w) \leq G(\bar{t} \rightarrow \psi)^V(w)$. Hence, $(\bar{t} \rightarrow G(\psi))^V(w) = G(\bar{t} \rightarrow \psi)^V(w)$. Thus, $G(\bar{t} \rightarrow \psi)^V = (\bar{t} \rightarrow G(\psi))^V$.

Put $t = e_{M'}(F(\phi)^{V'}, \psi^{V'})$. For $\phi \in \Phi_L$,

$$\begin{aligned} F(\bar{t} \& \phi)^{V'}(w') &= t \odot F(\phi)^{V'}(w') \\ &= e_{M'}(F(\phi)^{V'}, \psi^{V'}) \odot F(\phi)^{V'}(w') \\ &\leq \psi^{V'}(w'). \end{aligned}$$

Since $(\bar{t} \& \phi) \in \Phi_L$, $(\bar{t} \& \phi)^V \leq G(\psi)^V$. Then,

$$\begin{aligned} t \odot \phi^V(w) &\leq G(\psi)^V(w), \\ t &= e_{M'}(F(\phi)^{V'}, \psi^{V'}) \leq e_M(\phi^V, G(\psi)^V). \end{aligned}$$

Put $s = e_M(\phi^V, G(\psi)^V)$. For $\psi \in \Phi'_L$,

$$\begin{aligned} G(\bar{s} \rightarrow \psi)^V(w) &= s \Rightarrow G(\psi)^V(w) \\ &= e_M(\phi^V, G(\psi)^V) \Rightarrow G(\psi)^V(w) \\ &\geq (\phi^V(w) \Rightarrow G(\psi)^V(w)) \Rightarrow G(\psi)^V(w) \\ &\geq \phi^V(w). \end{aligned}$$

Then, $G(\bar{s} \rightarrow \psi)^V \geq \phi^V$ iff $F(\phi)^V \leq (\bar{s} \rightarrow \psi)^V$ iff $s \leq e_{M'}(F(\phi)^{V'}, \psi^{V'})$. Thus, $e_{M'}(F(\phi)^{V'}, \psi^{V'}) = e_M(\phi^V, G(\psi)^V)$.

For $\phi \in \Phi_L, \psi \in \Phi'_L$,

$$\begin{aligned} e_M(\phi^V, G(\psi)^V) &= \bigwedge_{w \in W} (\phi^V(w) \Rightarrow G(\psi)^V(w)) \\ &= e_{M'}(F(\phi)^{V'}, \psi^{V'}) \\ &= \bigwedge_{w' \in W'} (F(\phi)^{V'}(w') \Rightarrow \psi^{V'}(w')) \\ &= \bigwedge_{w' \in W'} (\bigvee_{w \in W} (S(w, w') \odot \phi^V(w)) \Rightarrow \psi^{V'}(w')) \\ &= \bigwedge_{w \in W} \bigwedge_{w' \in W'} (\phi^V(w) \Rightarrow (S(w, w') \Rightarrow \psi^{V'}(w'))) \\ &= \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \bigwedge_{w' \in W'} (S(w, w') \Rightarrow \psi^{V'}(w'))) \\ &= e_M(\phi^V, \bigwedge_{w' \in W'} (S(-, w') \Rightarrow \psi^{V'}(w'))). \end{aligned}$$

Since $S(-, w') \Rightarrow \psi^{V'}(w') = (\phi_{w'} \rightarrow \overline{\psi^{V'}(w')})^V$, by Definition 3 (V2) and (V5), for each $w' \in W'$,

$$\begin{aligned} (\phi_{w'} \rightarrow \overline{\psi^{V'}(w')}) &\in \Phi_L, \\ \bigwedge_{w' \in W'} (\phi_{w'} \rightarrow \overline{\psi^{V'}(w')}) &\in \Phi_L. \end{aligned}$$

Thus, $e_M(\phi^V, G(\psi)^V) = e_M(\phi^V, \bigwedge_{w' \in W'} (S(-, w') \Rightarrow \psi^{V'}(w')))$.

Put $\phi = \bigwedge_{w' \in W'} (\phi_{w'} \rightarrow \overline{\psi^{V'}(w')})$. Then, $e_M(\phi^V, G(\psi)^V) = \top$. So, $\phi^V = \bigwedge_{w' \in W'} (S(-, w') \Rightarrow \psi^{V'}(w')) \leq G(\psi)^V$.

Put $\phi^V = G(\psi)^V$. Then, $\top = e_M(\phi^V, G(\psi)^V) = \bigwedge_{w \in W} (G(\psi)^V(w) \Rightarrow \bigwedge_{w' \in W'} (S(w, w') \Rightarrow \psi^{V'}(w')))$. So, $G(\psi)^V(w) \leq \bigwedge_{w' \in W'} (S(w, w') \Rightarrow \psi^{V'}(w'))$. Hence, $G(\psi)^V(w) = \bigwedge_{w' \in W'} (S(w, w') \Rightarrow \psi^{V'}(w'))$. $\square$

Theorem 10. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be fuzzy Kripke models. Suppose that there exists $S \in L^{W \times W'}$ with $S(w, -) = \psi_w^{V'}$, $S(-, w') = \phi_{w'}^V$, where $\phi_{w'} \in \Phi_L$, $\psi_w \in \Phi'_L$.

(1) Define a map $P : \Phi_L \rightarrow \Phi'_L$ as

$$P(\phi)^{V'}(w') = \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \psi_w^{V'}(w')).$$

Then, there exists $S \in L^{W \times W'}$ with $S(w, w') = \psi_w^{V'}(w')$ and

$$P(\phi)^{V'}(w') = \bigwedge_{w \in W} (\phi^V(w) \Rightarrow S(w, w')).$$

Moreover, for $\phi, \phi_i \in \Phi_L$, $\psi, \psi_i \in \Phi'_L$, $t \in L$,

$$\begin{aligned} P(\bigvee_{i \in \Gamma} \phi_i)^{V'} &= \bigwedge_{i \in \Gamma} P(\phi_i)^{V'}, \\ P(\bar{t} \& \phi)^{V'} &= (\bar{t} \rightarrow P(\phi))^{V'}. \end{aligned}$$

(2) Define $Q(\psi) = \bigvee \{\phi \in \Phi_L \mid P(\phi)^{V'} \geq \psi^V\}$. Then, there exists $Q : \Phi'_L \rightarrow \Phi_L$ such that

$$\begin{aligned} Q(\psi)^V(w) &= \bigvee_{w' \in W'} (\psi^{V'}(w') \Rightarrow S(w, w')), \\ Q(\bigvee_{i \in \Gamma} \psi_i)^V &= \bigwedge_{i \in \Gamma} Q(\psi_i)^V, \\ Q(\bar{t} \& \psi)^V &= (\bar{t} \rightarrow Q(\psi))^V. \end{aligned}$$

Moreover, $(e_M, P, Q, e_{M'})$ is a Galois connection, that is, $e_{M'}(\psi^{V'}, P(\phi)^{V'}) = e_M(\phi^V, Q(\psi)^V)$ for each $\phi \in \Phi_L$, $\psi \in \Phi'_L$.

Proof. (1) By Definition 3 (V2) and (V5), $(\overline{\phi^V(w)} \rightarrow \psi_w) \in \Phi'_L$ for each $w \in W$, $\bigvee_{w \in W} (\overline{\phi^V(w)} \rightarrow \psi_w) \in \Phi'_L$. So, P is well-defined. Then, there exists $S \in L^{W \times W'}$ with $S(w, w') = \psi_w^{V'}(w')$ and

$$P(\phi)^{V'}(w') = \bigwedge_{w \in W} (\phi^V(w) \Rightarrow S(w, w')).$$

For $\phi, \phi_i \in \Phi_L$, $t \in L$,

$$\begin{aligned} P(\bigvee_{i \in \Gamma} \phi_i)^{V'}(w') &= \bigwedge_{w \in W} (\bigvee_{i \in \Gamma} \phi_i^V(w) \Rightarrow S(w, w')) \\ &= \bigwedge_{i \in \Gamma} \bigwedge_{w \in W} (\phi_i^V(w) \Rightarrow S(w, w')) \\ &= \bigwedge_{i \in \Gamma} P(\phi_i)^{V'}(w'), \\ P(\bar{t} \& \phi)^{V'}(w') &= \bigwedge_{w \in W} ((\bar{t} \& \phi)^V(w) \Rightarrow S(w, w')) \\ &= \bigwedge_{w \in W} ((t \odot \phi^V(w)) \Rightarrow S(w, w')) \\ &= \bigwedge_{w \in W} (t \Rightarrow (\phi^V(w) \Rightarrow S(w, w'))) \\ &= t \Rightarrow \bigwedge_{w \in W} (\phi^V(w) \Rightarrow S(w, w')) \\ &= (\bar{t} \rightarrow P(\phi))^{V'}. \end{aligned}$$

Hence, $P(\bigvee_{i \in \Gamma} \phi_i)^{V'} = \bigwedge_{i \in \Gamma} P(\phi_i)^{V'}$ and $P(\bar{t} \& \phi)^{V'} = (\bar{t} \rightarrow P(\phi))^{V'}$ for all $t \in L$.

(2) Since $Q(\psi_1)^V \leq Q(\psi_2)^V$ for $\psi_2^{V'} \leq \psi_1^{V'}$, $Q(\bigvee_{i \in \Gamma} \psi_i)^V \leq \bigwedge_{i \in \Gamma} Q(\psi_i)^V$.

Since $P(\phi_1)^{V'} \geq P(\phi_2)^{V'}$ for $\phi_1^V \leq \phi_2^V$,

$$\begin{aligned} & P(Q(\psi_i))^{V'}(w') \\ &= P(\bigvee \{\phi \in \Phi_L \mid P(\phi)^{V'} \geq \psi_i^V\})(w') \\ &= \bigwedge \{P(\phi)^{V'} \in \Phi'_L \mid P(\phi)^{V'} \geq \psi_i^V\}(w') \\ &\geq \psi_i^V(w'), \end{aligned}$$

$$P(\bigwedge_{i \in \Gamma} Q(\psi_i))^{V'} \geq P(Q(\psi_i))^{V'} \geq \psi_i^V.$$

Thus, $P(\bigwedge_{i \in \Gamma} Q(\psi_i))^{V'} \geq \bigvee_{i \in \Gamma} \psi_i^V$. By the definition of Q , $Q(\bigvee_{i \in \Gamma} \psi_i)^V \geq \bigwedge_{i \in \Gamma} Q(\psi_i)^V$. Hence, $Q(\bigvee_{i \in \Gamma} \psi_i)^V = \bigwedge_{i \in \Gamma} Q(\psi_i)^V$.

For $\psi \in \Phi'_L, t \in L$,

$$\begin{aligned} & P(\bar{t} \& Q(\bar{t} \& \psi))^{V'}(w') = t \Rightarrow P(Q(\bar{t} \& \psi))^{V'}(w') \\ & \geq t \Rightarrow (t \odot \psi^{V'}(w')) \geq \psi^{V'}(w'). \end{aligned}$$

By the definition of Q , $Q(\psi)^V(w) \geq (\bar{t} \& Q(\bar{t} \& \psi))(w) = t \odot Q(\bar{t} \& \psi)(w)$. Thus, $Q(\bar{t} \& \psi)^V(w) \leq (\bar{t} \rightarrow Q(\psi))^V(w)$.

On the other hand, since $(\bar{t} \& (\bar{t} \rightarrow Q(\psi)))^V(w) = t \odot (t \Rightarrow Q(\psi)^V(w)) \leq Q(\psi)^V(w)$,

$$\begin{aligned} & t \Rightarrow P(\bar{t} \rightarrow Q(\psi))^{V'}(w') \\ &= P(\bar{t} \& (\bar{t} \rightarrow Q(\psi)))^{V'}(w') \\ &\geq P(Q(\psi))^{V'}(w') \geq \psi^{V'}(w') \end{aligned}$$

Thus, $P(\bar{t} \rightarrow Q(\psi))^{V'}(w') \geq (\bar{t} \& \psi)^{V'}(w')$.

By the definition of Q , $Q(\bar{t} \& \psi)^V(w) \geq (\bar{t} \rightarrow Q(\psi))^V(w)$. Hence, $Q(\bar{t} \& \psi)^V(w) \geq (\bar{t} \rightarrow Q(\psi))^V(w)$.

Put $t = e_{M'}(\psi^{V'}, P(\phi)^{V'})$.

$$\begin{aligned} & P(\bar{t} \& \phi)^{V'}(w') = t \Rightarrow P(\phi)^{V'}(w') \\ &= e_{M'}(\psi^{V'}, P(\phi)^{V'}) \Rightarrow P(\phi)^{V'}(w') \\ &\geq (\psi^{V'}(w') \Rightarrow P(\phi)^{V'}(w')) \Rightarrow P(\phi)^{V'}(w') \\ &\geq \psi^{V'}(w'). \end{aligned}$$

Since $(\bar{t} \& \phi) \in \Phi_L$, $(\bar{t} \& \phi)^V \leq Q(\psi)^V$,

$$\begin{aligned} & t \odot \phi^V(w) \leq Q(\psi)^V(w), \\ & t = e_{M'}(\psi^{V'}, P(\phi)^{V'}) \leq e_M(\phi^V, Q(\psi)^V). \end{aligned}$$

Similarly, $e_M(\phi^V, Q(\psi)^V) \leq e_{M'}(\psi^{V'}, P(\phi)^{V'})$. Hence, $e_M(\phi^V, Q(\psi)^V) = e_{M'}(\psi^{V'}, P(\phi)^{V'})$.

For $\phi \in \Phi_L, \psi \in \Phi'_L$,

$$\begin{aligned} & e_M(\phi^V, Q(\psi)^V) = e_{M'}(\psi^{V'}, P(\phi)^{V'}) \\ &= \bigwedge_{w' \in W'} (\psi^{V'}(w') \Rightarrow \bigwedge_{w \in W} (\phi^V(w) \Rightarrow S(w, w'))) \\ &= \bigwedge_{w \in W} \bigwedge_{w' \in W'} (\psi^{V'}(w') \Rightarrow (\phi^V(w) \Rightarrow S(w, w'))) \\ & \text{(by Lemma 1(7))} \\ &= \bigwedge_{w \in W} \bigwedge_{w' \in W'} (\phi^V(w) \Rightarrow (\psi^{V'}(w') \Rightarrow S(w, w'))) \\ &= \bigwedge_{w \in W} (\phi^V(w) \Rightarrow \bigwedge_{w' \in W'} (\psi^{V'}(w') \Rightarrow S(w, w'))) \\ &= e_M(\phi^V, \bigwedge_{w' \in W'} (\psi^{V'}(w') \Rightarrow S(-, w'))). \end{aligned}$$

Since $\overline{\psi^V(w')} \Rightarrow \phi_{w'} \in \Phi_L$ and $\bigwedge_{w' \in W'} (\overline{\psi^V(w')} \Rightarrow \phi_{w'})^V \in \Phi_L$, by a similar proof of Theorem 9 (2),

$$Q(\psi)^V(w) = \bigwedge_{w' \in W'} (\psi^{V'}(w') \Rightarrow S(w, w')).$$

□

Definition 7. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be fuzzy Kripke models. Let $F : \Phi_L \rightarrow \Phi'_L$ and $G : \Phi'_L \rightarrow \Phi_L$ be two maps.

(1) Let $(e_M, F, G, e_{M'})$ be a residuated connection. A family $\mathcal{A}(F, G)$ is called a fuzzy concept lattice where

$$\begin{aligned} \mathcal{A}(F, G) = \{(\phi^V, \psi^{V'}) \in L^W \times L^{W'} \mid F(\phi)^{V'} = \psi^{V'}, G(\psi)^V = \phi^V\}, \\ (\phi_1^V, \psi_1^{V'}) \leq (\phi_2^V, \psi_2^{V'}) \text{ iff } \phi_1^V \leq \phi_2^V \text{ (or } \psi_1^{V'} \leq \psi_2^{V'} \text{)}. \end{aligned}$$

Moreover, $(\phi^V, \psi^{V'})$ is called a fuzzy concept.

(2) Let $(e_M, F, G, e_{M'})$ be a Galois connection. A family $\mathcal{F}(F, G)$ is called a formal fuzzy concept lattice where

$$\begin{aligned} \mathcal{F}(F, G) = \{(\phi^V, \psi^{V'}) \in L^W \times L^{W'} \mid F(\phi)^{V'} = \psi^{V'}, G(\psi)^V = \phi^V\}, \\ (\phi_1^V, \psi_1^{V'}) \leq (\phi_2^V, \psi_2^{V'}) \text{ iff } \phi_1^V \leq \phi_2^V \text{ (or } \psi_2^{V'} \leq \psi_1^{V'} \text{)}. \end{aligned}$$

Moreover, $(\phi^V, \psi^{V'})$ is called a formal fuzzy concept.

Theorem 11. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be fuzzy Kripke models. Let $F : \Phi_L \rightarrow \Phi'_L$ and $G : \Phi'_L \rightarrow \Phi_L$ be two maps and $(e_M, F, G, e_{M'})$ be a residuated connection. Then, the following properties hold.

(1) For $\phi, \phi_i \in \Phi_L, \psi, \psi_i \in \Phi'_L, t \in L$,

$$\begin{aligned} F(\bigvee_{i \in \Gamma} \phi_i)^{V'} &= \bigvee_{i \in \Gamma} F(\phi_i)^{V'}, \\ F(\bar{t} \& \phi)^{V'} &= (\bar{t} \& F(\phi))^{V'}, \\ G(\bigwedge_{i \in \Gamma} \psi_i)^V &= \bigwedge_{i \in \Gamma} G(\psi_i)^V, \\ G(\bar{t} \rightarrow \psi)^V &= (\bar{t} \rightarrow G(\psi))^V. \end{aligned}$$

(2) For each $\phi, \phi_1, \phi_2 \in \Phi_L, \psi, \psi_1, \psi_2 \in \Phi'_L$,

$$\begin{aligned} e_M(\phi_1^V, \phi_2^V) &\leq e_{M'}(F(\phi_1)^{V'}, F(\phi_2)^{V'}), \\ e_{M'}(\psi_1^{V'}, \psi_2^{V'}) &\leq e_M(G(\psi_1)^V, G(\psi_2)^V), \\ e_M(\phi_1^V, G(F(\phi_1))^{V'}) &= \top = e_{M'}(F(G(\psi_1))^{V'}, \psi_1^{V'}), \\ F(G(F(\phi)))^{V'} &= F(\phi)^{V'}, G(F(G(\psi)))^V = G(\psi)^V. \end{aligned}$$

(3) For each $\phi \in \Phi_L, \psi \in \Phi'_L$,

$$\begin{aligned} \mathcal{A}(F, G) &= \{(\phi^V, F(\phi)^{V'}) \in L^W \times L^{W'} \mid G(F(\phi))^{V'} = \phi^V\} \\ &= \{(G(\psi)^V, \psi^{V'}) \in L^W \times L^{W'} \mid F(G(\psi))^{V'} = \psi^{V'}\} \end{aligned}$$

(4) For each $(\phi_1^V, \psi_1^{V'}), (\phi_2^V, \psi_2^{V'}) \in \mathcal{A}(F, G)$,

$$\phi_1^V \leq \phi_2^V \text{ iff } \psi_1^{V'} \leq \psi_2^{V'}.$$

(5) For each $(\phi_1^V, \psi_1^{V'}), (\phi_2^V, \psi_2^{V'}), (\phi_i^V, \psi_i^{V'})_{i \in I} \in \mathcal{A}(F, G)$, we define

$$(\phi_1^V, \psi_1^{V'}) \leq (\phi_2^V, \psi_2^{V'}) \text{ iff } \phi_1^V \leq \phi_2^V \text{ (or } \psi_1^{V'} \leq \psi_2^{V'} \text{)},$$

$$\begin{aligned}\bigvee_{i \in I} (\phi_i^V, \psi_i^{V'}) &= (G(F(\bigvee_{i \in I} \phi_i))^V, \bigvee_{i \in I} \psi_i^{V'}) \\ &= (G(\bigvee_{i \in I} \psi_i)^V, \bigvee_{i \in I} \psi_i^{V'}).\end{aligned}$$

$$\begin{aligned}\bigwedge_{i \in I} (\phi_i^V, \psi_i^{V'}) &= (\bigwedge_{i \in I} \phi_i^V, F(G(\bigwedge_{i \in I} \psi_i))^V) \\ &= (\bigwedge_{i \in I} \phi_i^V, F(\bigwedge_{i \in I} \psi_i)^{V'}).\end{aligned}$$

Then, $\mathcal{A}(F, G)$ forms a complete lattice.

Proof. (1) For $\phi_i \in \Phi_L, \psi \in \Phi'_L, i \in I$, by Lemma 1 (5) and (V3),

$$\begin{aligned}e_{M'}(F(\bigvee_{i \in \Gamma} \phi_i)^{V'}, \psi^{V'}) &= e_M((\bigvee_{i \in \Gamma} \phi_i)^V, G(\psi)^V) \\ &= \bigwedge_{i \in \Gamma} e_M(\phi_i^V, G(\psi)^V) \\ &= \bigwedge_{i \in \Gamma} e_{M'}(F(\phi_i)^{V'}, \psi^{V'}) \\ &= e_{M'}(\bigvee_{i \in \Gamma} F(\phi_i)^{V'}, \psi^{V'}).\end{aligned}$$

Hence, $F(\bigvee_{i \in \Gamma} \phi_i)^{V'} = \bigvee_{i \in \Gamma} F(\phi_i)^{V'}$.

For $\phi \in \Phi_L, \psi \in \Phi'_L, t \in L$,

$$\begin{aligned}e_{M'}(F(\bar{t} \& \phi)^{V'}, \psi^{V'}) &= e_M((\bar{t} \& \phi)^V, G(\psi)^V) \\ &= \bigwedge_{w \in W} ((t \odot \phi^V(w)) \Rightarrow G(\psi)^V(w)) \\ &= \bigwedge_{w \in X} (t \Rightarrow (\phi^W(w) \Rightarrow G(\psi)^V(w))) \quad (\text{by Lemma 1 (7)}) \\ &= t \Rightarrow \bigwedge_{w \in X} (\phi^W(w) \Rightarrow G(\psi)^W) \\ &= t \Rightarrow e_{M'}(F(\phi)^{V'}, \psi^{V'}) \\ &= e_{M'}((t \odot F(\phi))^{V'}, \psi^{V'}).\end{aligned}$$

Hence, $F(\bar{t} \& \phi)^{V'} = (\bar{t} \& F(\phi))^{V'}$ for all $t \in L$.

For $\phi \in \Phi_L, \psi_i \in \Phi'_L, i \in I$, by Lemma 1 (4) and (V2),

$$\begin{aligned}e_M(\phi^V, G(\bigwedge_{i \in \Gamma} \psi_i)^V) &= e_{M'}(F(\phi)^{V'}, (\bigwedge_{i \in \Gamma} \psi_i)^{V'}) \\ &= \bigwedge_{i \in \Gamma} e_{M'}(F(\phi)^{V'}, \psi_i^{V'}) \\ &= \bigwedge_{i \in \Gamma} e_M(\phi^V, G(\psi_i)^V) = e_M(\phi^V, \bigwedge_{i \in \Gamma} G(\psi_i)^V),\end{aligned}$$

Hence, $G(\bigwedge_{i \in \Gamma} \psi_i)^V = \bigwedge_{i \in \Gamma} G(\psi_i)^V$.

For $\phi \in \Phi_L, \psi \in \Phi'_L, t \in L$, by Lemma 1 (7),

$$\begin{aligned}e_M(\phi^V, G(\bar{t} \rightarrow \psi)^V) &= e_{M'}(F(\phi)^{V'}, (\bar{t} \rightarrow \psi)^{V'}) \\ &= \bigwedge_{w' \in W'} (F(\phi)^{V'}(w') \Rightarrow (\bar{t} \Rightarrow \psi^{V'}(w'))) \\ &= t \Rightarrow \bigwedge_{w' \in W'} (F(\phi)^{V'}(w') \Rightarrow \psi^{V'}(w')) \\ &= t \Rightarrow e_M(\phi^V, G(\psi)^V) \\ &= e_M(\phi^V, (\bar{t} \rightarrow G(\psi))^V).\end{aligned}$$

Hence, $G(\bar{t} \rightarrow \psi)^V = (\bar{t} \rightarrow G(\psi))^V$ for all $t \in L$.

(2) For each $\phi_1 \in \Phi_L, \psi_1 \in \Phi'_L$,

$$\begin{aligned}\top &= e_{M'}(F(\phi_1)^{V'}, F(\phi_1)^{V'}) = e_M(\phi_1^V, G(F(\phi_1))^V), \\ \top &= e_M(G(\psi_1)^V, G(\psi_1)^V) = e_{M'}(F(G(\psi_1))^{V'}, \psi_1^{V'}).\end{aligned}$$

For each $\phi_1, \phi_2 \in \Phi_L$,

$$\begin{aligned}e_{M'}(F(\phi_1)^{V'}, F(\phi_2)^{V'}) &= e_M(\phi_1^V, G(F(\phi_2))^V) \\ &\geq e_M(\phi_1^V, \phi_2^V) \odot e_M(\phi_2^V, G(F(\phi_2))^V) \\ &= e_M(\phi_1^V, \phi_2^V).\end{aligned}$$

For each $\psi_1, \psi_2 \in \Phi'_L$,

$$\begin{aligned} e_M(G(\psi_1)^V, G(\psi_2)^V) &= e_{M'}(F(G(\psi_1))^{V'}, \psi_2^{V'}) \\ &\geq e_{M'}(F(G(\psi_1))^{V'}, \psi_1^{V'}) \odot e_{M'}(\psi_1^{V'}, \psi_2^{V'}) \\ &= e_{M'}(\psi_1^{V'}, \psi_2^{V'}). \end{aligned}$$

Since $G(F(\phi))^V \geq \phi^V$, $F(G(F(\phi)))^{V'} \geq F(\phi)^{V'}$. Moreover, $F(G(F(\phi)))^{V'} \leq F(\phi)^{V'}$. Hence, $F(G(F(\phi)))^{V'} = F(\phi)^{V'}$. Similarly, $G(F(G(\psi)))^V = G(\psi)^V$.

(3) Put $\mathcal{B} = \{(\phi^V, F(\phi)^{V'}) \in L^W \times L^{W'} \mid G(F(\phi))^V = \phi^V\}$.

Let $(\phi^V, \psi^{V'}) \in \mathcal{A}(F, G)$. Then, $\psi^{V'} = F(\phi)^{V'}$, $G(F(\phi))^V = G(\psi)^V = \phi^V$. Thus, $(\phi^V, F(\phi)^{V'}) \in \mathcal{B}$.

Let $(\phi^V, F(\phi)^{V'}) \in \mathcal{B}$. Since $G(F(\phi))^V = \phi^V$, $F(\phi)^{V'} = F(\phi)^{V'}$ and $(\phi^V, F(\phi)^{V'}) \in \mathcal{A}(F, G)$.

Put $\mathcal{A} = \{(G(\psi))^V, \psi^{V'} \in L^W \times L^{W'} \mid F(G(\psi))^{V'} = \psi^{V'}\}$.

Let $(\phi^V, \psi^{V'}) \in \mathcal{A}(F, G)$. Then, $\phi = G(\psi)^V$, $F(G(\psi))^{V'} = F(\phi)^{V'} = \psi^{V'}$. Thus, $(G(\psi))^V, \psi^{V'} \in \mathcal{A}$.

Let $(G(\psi))^V, \psi^{V'} \in \mathcal{A}$. Since $F(G(\psi))^{V'} = \psi^{V'}$, $G(\psi)^V = G(\psi)^V$ and $(G(\psi))^V, \psi^{V'} \in \mathcal{A}(F, G)$.

(4) Let $\phi_1^V \leq \phi_2^V$. By (2), $\top = e_M(\phi_1^V, \phi_2^V) \leq e_M(F(\phi_1)^{V'}, F(\phi_2)^{V'})$. Then, $\psi_1^{V'} = F(\phi_1)^{V'} \leq F(\phi_2)^{V'} = \psi_2^{V'}$. Let $\psi_1^{V'} \leq \psi_2^{V'}$. Then, $\top = e_{M'}(\psi_1^{V'}, \psi_2^{V'}) \leq e_M(G(\psi_1)^V, G(\psi_2)^V)$. Thus, $\phi_1^V = G(\psi_1)^V \leq G(\psi_2)^V = \phi_2^V$.

(5) For each $i \in I$, $(\phi_i^V, \psi_i^{V'}) \in \mathcal{A}(F, G)$,

$$\begin{aligned} F(G(\bigvee_{i \in I} \phi_i))^{V'} &= F(\bigvee_{i \in I} \phi_i)^{V'} \\ &= \bigvee_{i \in I} F(\phi_i)^{V'} = \bigvee_{i \in I} \psi_i^{V'}, \\ G(F(\bigvee_{i \in I} \phi_i))^V &= G(\bigvee_{i \in I} F(\phi_i))^V \\ &= G(\bigvee_{i \in I} \psi_i)^V. \end{aligned}$$

Hence, $\bigvee_{i \in I} (\phi_i^V, \psi_i^{V'}) \in \mathcal{A}(F, G)$.

Since $\psi_i^{V'} \leq \bigvee_{i \in I} \psi_i^{V'}$, $\phi_i^V = G(\psi_i)^V \leq G(\bigvee_{i \in I} \psi_i)^V = G(F(\bigvee_{i \in I} \phi_i))^V$. Thus, $(\phi_i^V, \psi_i^{V'}) \leq \bigvee_{i \in I} (\phi_i^V, \psi_i^{V'})$.

If $(\phi_i^V, \psi_i^{V'}) \leq (\phi^V, \psi^{V'})$ for each $i \in I$, then $\bigvee_{i \in I} \psi_i^{V'} \leq \psi^{V'}$ and $G(\bigvee_{i \in I} \psi_i)^V \leq G(\psi)^V = \phi^V$. Hence, $\bigvee_{i \in I} (\phi_i^V, \psi_i^{V'}) = (G(\bigvee_{i \in I} \psi_i)^V, \bigvee_{i \in I} \psi_i^{V'}) \leq (\phi^V, \psi^{V'})$. Thus, $\bigvee_{i \in I} (\phi_i^V, \psi_i^{V'})$ is a supremum of $(\phi_i^V, \psi_i^{V'})$ for each $i \in I$.

For each $i \in I$, $(\phi_i^V, \psi_i^{V'}) \in \mathcal{A}(F, G)$,

$$\begin{aligned} G(F(G(\bigwedge_{i \in I} \psi_i)))^V &= G(\bigwedge_{i \in I} \psi_i)^V \\ &= \bigwedge_{i \in I} G(\psi_i)^V = \bigwedge_{i \in I} \phi_i^V, \\ F(G(\bigwedge_{i \in I} \phi_i))^{V'} &= F(\bigwedge_{i \in I} G(\psi_i))^{V'} \\ &= F(\bigwedge_{i \in I} \phi_i)^{V'}. \end{aligned}$$

Hence, $\bigwedge_{i \in I} (\phi_i^V, \psi_i^{V'}) \in \mathcal{A}(F, G)$. Moreover, since $\bigwedge_{i \in I} \phi_i^V \leq \phi_i^V$, $F(G(\bigwedge_{i \in I} \psi_i))^{V'} \leq \bigwedge_{i \in I} F(G(\psi_i))^{V'} = \bigwedge_{i \in I} \psi_i^{V'} \leq \psi_i^{V'}$. Thus, $\bigwedge_{i \in I} (\phi_i^V, \psi_i^{V'}) \leq (\phi_i^V, \psi_i^{V'})$ for each $i \in I$.

If $(\phi^V, \psi^{V'}) \leq (\phi_i^V, \psi_i^{V'})$ for each $i \in I$, then $\phi^V \leq \bigwedge_{i \in I} \phi_i^V$ and $\psi^{V'} = F(\phi)^{V'} \leq F(\bigwedge_{i \in I} \phi_i)^{V'}$. Hence, $(\phi^V, \psi^{V'}) \leq \bigwedge_{i \in I} (\phi_i^V, \psi_i^{V'}) = (\bigwedge_{i \in I} \phi_i^V, F(\bigwedge_{i \in I} \phi_i)^{V'})$. Thus, $\bigwedge_{i \in I} (\phi_i^V, \psi_i^{V'})$ is an infimum of $(\phi_i^V, \psi_i^{V'})$ for each $i \in I$.

Theorem 12. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be fuzzy Kripke models. Let $P : \Phi_L \rightarrow \Phi'_L$ and $Q : \Phi'_L \rightarrow \Phi_L$ be two maps and $(e_M, P, Q, e_{M'})$ be a Galois connection. Then, the following properties hold:

(1) For $\phi, \phi_i \in \Phi_L, \psi, \psi_i \in \Phi'_L, t \in L$,

$$\begin{aligned} P(\bigvee_{i \in \Gamma} \phi_i)^{V'} &= \bigwedge_{i \in \Gamma} P(\phi_i)^{V'}, \\ P(\bar{t} \& \phi)^{V'} &= (\bar{t} \rightarrow P(\phi))^{V'}, \\ Q(\bigvee_{i \in \Gamma} \psi_i)^V &= \bigwedge_{i \in \Gamma} Q(\psi_i)^V, \\ Q(\bar{t} \& \psi)^V &= (\bar{t} \rightarrow Q(\psi))^V. \end{aligned}$$

(2) For each $\phi_1, \phi_2 \in \Phi_L, \psi_1, \psi_2 \in \Phi'_L$,

$$\begin{aligned} e_M(\phi_1^V, Q(P(\phi_1))^V) &= \top, \\ e_M(\psi_1^{V'}, P(Q(\psi_1))^{V'}) &= \top, \\ e_M(\phi_1^V, \phi_2^V) &\leq e_M(P(\phi_2)^{V'}, P(\phi_1)^{V'}), \\ e_M(\psi_1^{V'}, \psi_2^{V'}) &\leq e_M(Q(\psi_2)^V, Q(\psi_1)^V), \\ P(Q(P(\phi_1)))^{V'} &= P(\phi_1)^{V'}, \\ Q(P(Q(\psi_1)))^V &= Q(\psi_1)^V. \end{aligned}$$

(3) For each $\phi \in \Phi_L, \psi \in \Phi'_L$,

$$\begin{aligned} \mathcal{F}(P, Q) &= \{(\phi^V, P(\phi)^{V'}) \in L^W \times L^W \mid Q(P(\phi))^V = \phi^V\} \\ &= \{(Q(\psi)^V, \psi^{V'}) \in L^W \times L^W \mid P(Q(\psi))^{V'} = \psi^{V'}\} \end{aligned}$$

(4) For each $(\phi_1^V, \psi_1^{V'}), (\phi_2^V, \psi_2^{V'}) \in \mathcal{F}(P, Q)$,

$$\phi_1^V \leq \phi_2^V \text{ iff } \psi_2^{V'} \leq \psi_1^{V'}.$$

(5) For each $(\phi_1^V, \psi_1^{V'}), (\phi_2^V, \psi_2^{V'}), (\phi_i^V, \psi_i^{V'})_{i \in I} \in \mathcal{F}(P, Q)$, we define

$$(\phi_1^V, \psi_1^{V'}) \leq (\phi_2^V, \psi_2^{V'}) \text{ iff } \phi_1^V \leq \phi_2^V \text{ (or } \psi_2^{V'} \leq \psi_1^{V'}),$$

$$\begin{aligned} \bigvee_{i \in I} (\phi_i^V, \psi_i^{V'}) &= (Q(P(\bigvee_{i \in I} \phi_i))^{V'}, (\bigwedge_{i \in I} \psi_i)^{V'}) \\ &= (Q(\bigwedge_{i \in I} \psi_i)^V, (\bigwedge_{i \in I} \psi_i)^{V'}). \end{aligned}$$

$$\begin{aligned} \bigwedge_{i \in I} (\phi_i^V, \psi_i^{V'}) &= ((\bigwedge_{i \in I} \phi_i)^V, P(Q(\bigvee_{i \in I} \psi_i))^{V'}) \\ &= ((\bigwedge_{i \in I} \phi_i)^V, P(\bigwedge_{i \in I} \psi_i)^{V'}). \end{aligned}$$

Then, $\mathcal{F}(P, Q)$ forms a complete lattice.

Proof. (1) For $\phi_i \in \Phi_L, \psi \in \Phi'_L, i \in I$, by Lemma 1 (4) and (V2),

$$\begin{aligned} e_{M'}(\psi^{V'}, P(\bigvee_{i \in \Gamma} \phi_i)^{V'}) &= e_M((\bigvee_{i \in \Gamma} \phi_i)^V, Q(\psi)^V) \\ &= \bigwedge_{i \in \Gamma} e_M(\phi_i^V, Q(\psi)^V) = \bigwedge_{i \in \Gamma} (e_{M'}(\psi^{V'}, P(\phi_i)^{V'})) \\ &= e_M(\psi^{V'}, \bigwedge_{i \in \Gamma} P(\phi_i)^{V'}). \end{aligned}$$

Hence, $P(\bigvee_{i \in \Gamma} \phi_i)^{V'} = \bigwedge_{i \in \Gamma} P(\phi_i)^{V'}$.

For $\phi \in \Phi_L, \psi \in \Phi'_L, t \in L$, by Lemma 1 (7),

$$\begin{aligned} e_{M'}(\psi^{V'}, P(\bar{t} \& \phi)^{V'}) &= e_M((\bar{t} \& \phi)^V, Q(\psi)^V) \\ &= \bigwedge_{w \in W} (t \Rightarrow (\phi^V(w) \Rightarrow Q(\psi)^V(w))) \\ &= t \Rightarrow \bigwedge_{w \in W} (\phi^V(w) \Rightarrow Q(\psi)^V(w)) \\ &= t \Rightarrow e_{M'}(\psi^{V'}, P(\phi)^{V'}) \\ &= e_{M'}(\psi^{V'}, (\bar{t} \rightarrow P(\phi))^{V'}). \end{aligned}$$

Hence, $P(\bar{t} \& \phi)^{V'} = (\bar{t} \rightarrow P(\phi))^{V'}$ for all $t \in L$.

Other cases are similarly proved.

(2) For each $\phi_1 \in \Phi_L, \psi_1 \in \Phi'_L$,

$$\begin{aligned}\top &= e_{M'}(P(\phi_1)^{V'}, P(\phi_1)^{V'}) = e_M(\phi_1^V, Q(P(\phi_1))^V), \\ \top &= e_M(Q(\psi_1)^V, Q(\psi_1)^V) = e_{M'}(\psi_1^{V'}, P(Q(\psi_1)^{V'})).\end{aligned}$$

For each $\phi_1, \phi_2 \in \Phi_L, \psi_1, \psi_2 \in \Phi'_L$,

$$\begin{aligned}e_{M'}(P(\phi_2)^{V'}, P(\phi_1)^{V'}) &= e_M(\phi_1^V, Q(P(\phi_2))^V) \\ &\geq e_M(\phi_1^V, \phi_2^V) \odot e_M(\phi_2^V, Q(P(\phi_2))^V) \\ &= e_M(\phi_1^V, \phi_2^V).\end{aligned}$$

Similarly, $e_{M'}(\psi_1^{V'}, \psi_2^{V'}) \leq e_M(Q(\psi_2)^V, Q(\psi_1)^V)$.

Since $Q(P(\phi))^{V'} \geq \phi^V, P(Q(P(\phi)))^{V'} \leq P(\phi)^{V'}$. Moreover, $P(Q(P(\phi)))^{V'} \geq P(\phi)^{V'}$. Hence, $P(Q(P(\phi)))^{V'} = P(\phi)^{V'}$. Similarly, $Q(P(Q(\psi)))^V = Q(\psi)^V$.

(3) Put $\mathcal{B} = \{(\phi^V, P(\phi)^{V'}) \in L^W \times L^{W'} \mid Q(P(\phi))^V = \phi^V\}$.

Let $(\phi^V, \psi^{V'}) \in \mathcal{F}(P, Q)$. Then, $Q(P(\phi))^V = Q(\psi)^V = \phi^V, \psi^{V'} = P(\phi)^{V'}$. Thus, $(\phi^V, P(\phi)^{V'}) \in \mathcal{B}$.

Let $(\phi^V, P(\phi)^{V'}) \in \mathcal{B}$. Since $Q(P(\phi))^V = \phi^V, P(\phi)^{V'} = P(\phi)^{V'}$ and $(\phi^V, P(\phi)^{V'}) \in \mathcal{F}(P, Q)$.

Put $\mathcal{C} = \{(Q(\psi)^V, \psi^{V'}) \in L^W \times L^{W'} \mid P(Q(\psi))^{V'} = \psi^{V'}\}$.

Let $(\phi^V, \psi^{V'}) \in \mathcal{F}(P, Q)$. Then, $P(Q(\psi))^{V'} = P(\phi)^{V'} = \psi^{V'}, \phi^V = Q(\psi)^V$. Thus, $(Q(\psi)^V, \psi^{V'}) \in \mathcal{C}$.

Let $(Q(\psi)^V, \psi^{V'}) \in \mathcal{C}$. Since $P(Q(\psi))^{V'} = \psi^{V'}, Q(\psi)^V = Q(\psi)^V$ and $(Q(\psi)^V, \psi^{V'}) \in \mathcal{F}(P, Q)$.

(4) Let $\phi_1^V \leq \phi_2^V$. By (2), $\top = e_M(\phi_1^V, \phi_2^V) \leq e_M(P(\phi_2)^{V'}, P(\phi_1)^{V'})$. Then, $\psi_2^{V'} = P(\phi_2)^{V'} \leq P(\phi_1)^{V'} = \psi_1^{V'}$. Let $\psi_2^{V'} \leq \psi_1^{V'}$. Similarly, $\phi_1^V = Q(\psi_1)^V \leq Q(\psi_2)^V = \phi_2^V$.

(5) For each $i \in I, (\phi_i^V, \psi_i^{V'}) \in \mathcal{F}(P, Q)$,

$$\begin{aligned}P(Q(P(\bigvee_{i \in I} \phi_i)))^{V'} &= P(\bigvee_{i \in I} \phi_i)^{V'} \\ &= (\bigwedge_{i \in I} P(\phi_i))^{V'} = (\bigwedge_{i \in I} \psi_i)^{V'}, \\ Q(\bigwedge_{i \in I} \psi_i)^V &= Q(P(\bigvee_{i \in I} \psi_i))^V.\end{aligned}$$

Hence, $\bigvee_{i \in I} (\phi_i^V, \psi_i^{V'}) \in \mathcal{F}(P, Q)$.

Moreover, $\psi_i^{V'} \geq \bigwedge_{i \in I} \psi_i^{V'}$ and $\phi_i^V = Q(\psi_i)^V \leq Q(\bigwedge_{i \in I} \psi_i)^V = Q(P(\bigvee_{i \in I} \phi_i))^V$. Thus, $(\phi_i^V, \psi_i^{V'}) \leq \bigvee_{i \in I} (\phi_i^V, \psi_i^{V'})$.

If $(\phi_i^V, \psi_i^{V'}) \leq (\phi^V, \psi^{V'})$ for each $i \in I$, then $\psi_i^{V'} \geq \psi^{V'}$ implies $(\bigwedge_{i \in I} \psi_i)^{V'} \geq \psi^{V'}$. Thus,

$$Q(\bigwedge_{i \in I} \psi_i)^V = Q(P(\bigvee_{i \in I} \phi_i))^V \leq Q(\psi)^V = \phi^V.$$

Hence, $\bigvee_{i \in I} (\phi_i^V, \psi_i^{V'}) = (Q(P(\bigvee_{i \in I} \phi_i))^V, (\bigwedge_{i \in I} \psi_i)^{V'}) \leq (\phi^V, \psi^{V'})$. Thus, $\bigvee_{i \in I} (\phi_i^V, \psi_i^{V'})$ is a supremum of $(\phi_i^V, \psi_i^{V'})$ for each $i \in I$.

For each $i \in I, (\phi_i^V, \psi_i^{V'}) \in \mathcal{F}(P, Q)$,

$$\begin{aligned}Q(P(Q(\bigvee_{i \in I} \psi_i)))^V &= Q(\bigvee_{i \in I} \psi_i)^V \\ &= \bigwedge_{i \in I} Q(\psi_i)^V = \bigwedge_{i \in I} \phi_i^V, \\ P(Q(\bigvee_{i \in I} \psi_i))^{V'} &= P(\bigwedge_{i \in I} Q(\psi_i))^{V'} \\ &= P(\bigwedge_{i \in I} \phi_i)^{V'}.\end{aligned}$$

Hence, $\bigwedge_{i \in I} (\phi_i^V, \psi_i^{V'}) \in \mathcal{F}(P, Q)$.

Since $\bigvee_{i \in I} \psi_i^{V'} \geq \psi_i^{V'}$, $Q(\bigvee_{i \in I} \psi_i)^V \leq Q(\psi_i)^V$ implies $P(Q(\bigvee_{i \in I} \psi_i))^{V'} \geq P(Q(\psi_i))^{V'} \geq \psi_i^{V'}$. Thus, $\bigwedge_{i \in I} (\phi_i^V, \psi_i^{V'}) \leq (\phi_i^V, \psi_i^{V'})$ for each $i \in I$.

If $(\phi^V, \psi^{V'}) \leq (\phi_i^V, \psi_i^{V'})$ for each $i \in I$, then $\psi^{V'} \geq \bigvee_{i \in I} \psi_i^{V'}$ and $\phi^V = Q(\psi)^V \leq Q(\bigvee_{i \in I} \psi_i)^V$ implies $\psi^{V'} = P(\phi)^{V'} = P(Q(\psi))^{V'} \geq P(Q(\bigvee_{i \in I} \psi_i))^{V'}$. Hence, $(\phi^V, \psi^{V'}) \leq ((\bigwedge_{i \in I} \phi_i)^V, P(Q(\bigvee_{i \in I} \psi_i))^{V'})$. Thus, $\bigwedge_{i \in I} (\phi_i^V, \psi_i^{V'})$ is an infimum of $(\phi_i^V, \psi_i^{V'})$ for each $i \in I$.

□

Example 3. Let $(L = [0, 1], \odot, \Rightarrow, 0, 1)$ be a complete residuated lattice defined by

$$x \odot y = (x + y - 1) \vee 0, \quad x \Rightarrow y = (1 - x + y) \wedge 1.$$

Let $M = (W, R, V)$ and $M' = (W', R', V')$ be fuzzy Kripke models where $P = \{p_1, p_2, p_3\}$, $W = \{w_1, w_2, w_3, w_4\}$, $P' = \{p'_1, p'_2, p'_3\}$, $W' = \{w'_1, w'_2, w'_3\}$ and

V	p_1	p_2	p_3
w_1	0.9	0.7	0.4
w_2	0.5	0.4	0.6
w_3	0.7	0.2	0.8

V'	p'_1	p'_2	p'_3
w'_1	0.7	0.2	0.4
w'_2	0.4	0.8	0.6
w'_3	0.9	0.7	0.5

R	w_1	w_2	w_3
w_1	0.6	0.8	0.3
w_2	0.9	0.4	0.7
w_3	0.6	0.8	0.5

R'	w'_1	w'_2	w'_3
w'_1	0.7	0.9	0.4
w'_2	0.5	0.6	0.8
w'_3	0.9	0.5	0.3

Put

$$\begin{aligned} S(w_1, -) &= (p'_1)^{V'} = (0.7, 0.4, 0.9), \\ S(w_2, -) &= (p'_3)^{V'} = (0.4, 0.6, 0.5), \\ S(w_3, -) &= (p'_2)^{V'} = (0.2, 0.8, 0.7), \\ S(-, w'_1) &= (p_2)^V = (0.7, 0.4, 0.2), \\ S(-, w'_2) &= (p_3)^V = (0.4, 0.6, 0.8), \\ S(-, w'_3) &= (p_1)^V = (0.9, 0.5, 0.7). \end{aligned}$$

Then, it satisfies the condition in Theorem 9 for $S \in L^{W \times W'}$ as

S	w'_1	w'_2	w'_3
w_1	0.7	0.4	0.9
w_2	0.4	0.6	0.5
w_3	0.2	0.8	0.7

(1) By Theorem 9, $F : \Phi_L \rightarrow \Phi'_L$ and $G : \Phi'_L \rightarrow \Phi_L$ are defined by

$$\begin{aligned} F(\phi)^{V'}(w') &= \bigvee_{w \in W} (S(w, w') \odot \phi^V(w)), \\ G(\psi)^V(w) &= \bigwedge_{w' \in W'} (S(w, w') \Rightarrow \psi^{V'}(w')). \end{aligned}$$

$$\begin{aligned}
F(p_1)^{V'} &= (0.6, 0.5, 0.8), (p'_1)^{V'} = (0.7, 0.4, 0.9), \\
p_1^V &= (0.9, 0.5, 0.7), G(p'_1)^V = (1, 0.8, 0.6), \\
e_{M'}(F(p_1)^{V'}, (p'_1)^{V'}) &= e_M(p_1^V, G(p'_1)^V) = 0.9, \\
G(F(p_1))^V &= (0.9, 0.9, 0.7), \\
(G(F(p_1))^V, F(p_1)^{V'}) &\in \mathcal{A}(F, G), \\
F(G(p'_1))^{V'} &= (0.7, 0.4, 0.9) = (p'_1)^{V'}, \\
(G(p'_1)^V, (p'_1)^{V'}) &\in \mathcal{A}(F, G), \\
\langle p_2 \rangle^V &= (0.3, 0.6, 0.3), F(\langle p_2 \rangle)^{V'} = (0, 0.2, 0.2), \\
G(F(\langle p_2 \rangle))^V &= (0.3, 0.6, 0.4), \\
(G(F(\langle p_2 \rangle))^V, F(\langle p_2 \rangle)^{V'}) &\in \mathcal{A}(F, G), \\
([p_3]')^{V'} &= (0.7, 0.7, 0.5), G([p_3]')^V = (0.6, 1, 0.8), \\
F(G([p_3]'))^{V'} &= (0.4, 0.6, 0.5) \neq ([p_3]')^{V'}, \\
(G([p_3]')^V, ([p_3]')^{V'}) &\notin \mathcal{A}(F, G), \\
(G([p_3]')^V, F(G([p_3]'))^{V'}) &\in \mathcal{A}(F, G).
\end{aligned}$$

(2) By Theorem 10, $P : \Phi_L \rightarrow \Phi'_L$ and $Q : \Phi'_L \rightarrow \Phi_L$ are defined by

$$\begin{aligned}
P(\phi)^{V'}(w') &= \bigvee_{w \in W} (\phi^V(w) \Rightarrow S(w, w')), \\
Q(\psi)^V(w) &= \bigwedge_{w' \in W'} (\psi^{V'}(w') \Rightarrow S(w, w')).
\end{aligned}$$

$$\begin{aligned}
P(p_1)^{V'} &= (0.5, 0.5, 1), (p'_1)^{V'} = (0.7, 0.4, 0.9), \\
p_1^V &= (0.9, 0.5, 0.7), Q(p'_1)^V = (1, 0.6, 0.5), \\
e_{M'}((p'_1)^{V'}, P(p_1)^{V'}) &= e_M(p_1^V, Q(p'_1)^V) = 0.8, \\
Q(P(p_1))^V &= (0.9, 0.5, 0.7) = p_1^V, \\
(p_1^V, F(p_1)^{V'}) &\in \mathcal{F}(P, Q), \\
P(Q(p'_1))^{V'} &= (0.7, 0.4, 0.9) = (p'_1)^{V'}, \\
(Q(p'_1)^V, (p'_1)^{V'}) &\in \mathcal{F}(P, Q), \\
\langle p_2 \rangle^V &= (0.3, 0.6, 0.3), P(\langle p_2 \rangle)^{V'} = (0.8, 1, 0.9), \\
Q(P(\langle p_2 \rangle))^V &= (0.4, 0.6, 0.4) \neq \langle p_2 \rangle^V, \\
(\langle p_2 \rangle^V, P(\langle p_2 \rangle)^{V'}) &\notin \mathcal{F}(P, Q), \\
(Q(P(\langle p_2 \rangle))^V, P(\langle p_2 \rangle)^{V'}) &\in \mathcal{F}(P, Q), \\
([p_3]')^{V'} &= (0.7, 0.7, 0.5), Q([p_3]')^V = (0.7, 0.7, 0.5), \\
P(Q([p_3]'))^{V'} &= (0.7, 0.7, 0.8) \neq ([p_3]')^{V'}, \\
(Q([p_3]')^V, ([p_3]')^{V'}) &\notin \mathcal{F}(P, Q), \\
(Q([p_3]')^V, P(Q([p_3]'))^{V'}) &\in \mathcal{F}(P, Q).
\end{aligned}$$

5. Conclusions

In this paper, we have studied fuzzy partial orders and investigated the properties of the fuzzy (necessity, possibility and sufficiency) modal operators for the formulas of Kripke models based on the residuated lattice. We have introduced topological structures (interior, closure operators, Alexandrov pretopologies, Alexandrov pre-cotopologies, fuzzy rough sets) for the formulas of Kripke models.

Using the properties of residuated and Galois connections, we can obtain fuzzy concepts and formal fuzzy concepts of possible worlds and formulas for fuzzy Kripke models based on the residuated lattice. It has been shown that possible worlds and formulas for fuzzy Kripke models can also be variously classified into small groups.

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