

Article

Stability Criteria for the Control of Multi-Fractional-Order Systems with Multiple State Delays

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Abstract

This paper aims to investigate a multiple fractional derivative order system control with multiple delays. The generalized Lyapunov function and YkP lemma are used to obtain algebraic and frequency criteria for asymptotic stability for these systems. Linear fractional systems and linear systems with nonlinear components are analyzed. Frequency criteria are represented graphically by polar plots. In contrast to classical Popov-type polar plots, where the critical point is determined by the system's nonlinearity, in fractional-order systems, the critical point of the polar plots is determined by the fractional order. Numerical applications highlight the investigation techniques and the particularities of the presented criteria.

Keywords: fractional-order system; control; stability; delay**MSC:** 93D05

1. Introduction

This paper investigates a class of multiple fractional-order systems, with fractional orders ranging between 0 and 1, which, due to their complexity and wide scope of application, have attracted significant attention from researchers in recent decades. Within this class of systems, a special category is represented by systems containing time-delay components, the presence of which adds corresponding layers of difficulty. Nevertheless, investigations in this field have been numerous, making a complete overview difficult to achieve. A summary of these research efforts can be found in [1]. The aspects of the qualitative theory for multi-order fractional differential equation systems are treated in [2,3]. New Lyapunov stability theorems for fractional order systems and new ideas of V-asymptotic stability and V-stability by Lyapunov stability theorems are proposed in [4,5]. Fractional comparison principle for multi-variable fractional order systems is developed in [6]. The theorems of existence, uniqueness, dependence upon initial conditions, and asymptotic stability theorems for fractional order systems are presented in [7]. Vector Lyapunov functions and asymptotic stability by matrix criteria are studied in [8,9]. The stability analysis by Lyapunov methods and Krasovski approaches for fractional order time-varying delay systems are discussed in [10,11]. By using the fractional-order Razumikhin theorem, stability conditions for the asymptotic stability of fractional-order linear time-delay systems are derived in [12,13]. The investigation into the Hyers–Ulam stability of the fractional systems is presented in [14,15]. The sufficient conditions for finite-time stability of fractional-order systems are discussed in [16–19]. The fractional-order system approximation by complex



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fractional calculus equations with the estimation of the time-delays is studied in [20,21]. New delay-dependent stability criteria for fractional-order systems in terms of matrix inequalities are developed in [22–24].

The present work attempts to introduce several stability evaluation criteria for multiple fractional-order systems with multiple time delays. Furthermore, the presented methods fall within the scope of techniques that serve as investigation and design tools by expressing these criteria in the form of polar plots, which are familiar in the design of automated control systems. The main contributions of this paper, compared with the existing results, are listed as follows:

- The generalized Lyapunov function is proposed to investigate the asymptotic stability of multi-order fractional systems with multiple time delays;
- The application of the Yakubovich–Kalman–Popov lemma, adapted for this class of systems, allows us to obtain new stability criteria presented as frequency criteria and represented graphically by polar plots;
- In contrast to classical Popov-type polar plots, where the critical point is determined by the system's nonlinearity, in fractional-order systems, the critical point of the polar plots is determined by the fractional order γ , where $0 < \gamma < 1$;
- By using the generalized Lyapunov function, the results are extended to linear and nonlinear multi-order fractional systems with multiple time delays;
- Numerical applications highlight the investigation techniques and the particularities of the presented criteria;
- An analysis of the advantages of the proposed methods compared to procedures and techniques used in other papers is presented.

The paper is structured as follows: Section 2 presents mathematical preliminaries, and stability criteria, Section 3 presents applications, and Sections 4 and 5 are dedicated to discussion and conclusions.

Notations: The bold letters denote the vector or matrix variables and non-bold letters denote scalar variables.

2. Materials and Methods

2.1. Preliminaries

Definition 1. *The fractional order system*

$$D^\gamma \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) \quad (1)$$

where $D^\gamma x$ denotes Caputo fractional order derivative and $0 < \gamma < 1$ is asymptotically stable if [8–10]

$$\text{Arg}(\lambda_i) < \gamma \frac{\pi}{2}, \quad i = 1, 2, \dots, n \quad (2)$$

and λ_i are the eigenvalues of $\mathbf{A}$. (Denote $\mathbf{A}^{(\gamma)}$ the γ -fractional order stable matrix and $\mathbf{A}^H$ the Hurwitz stable matrix).

Definition 2. *The system*

$$D^{\gamma_i} \mathbf{x}_i(t) = \mathbf{A}_i \mathbf{x}_i(t) + \mathbf{A}_{id} \mathbf{x}_i(t - \tau_i) + \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{A}_{ij} \mathbf{x}_j(t) - \mathbf{b}_i u_i(t) \quad (3)$$

$$\mathbf{x}_i(t) = \varphi_i(t), \quad t \in [-\tau_i, 0], \quad i = 1, 2, \dots, N \quad (4)$$

$$\mathbf{y}_i(t) = \mathbf{c}_i^T \mathbf{x}_i(t), \quad i = 1, 2, \dots, N \quad (5)$$

where $\mathbf{x} = [x_1 x_2 \dots x_i \dots x_N]^T$, $\mathbf{x}_i = [x_{i1} x_{i2} \dots x_{iN_i}]^T$, is the state vector, $\mathbf{x} \in R^{\sum N_i}$, $\mathbf{x}_i \in R^{N_i}$, $D^{\gamma_i} \mathbf{x}_i(t)$ is the Caputo derivative of $\mathbf{x}_i(t)$ for $\gamma_i \in (0, 1]$, $\gamma_i \in \Gamma$, $\tau_k, (k = 1, 2, \dots, N)$, are the delay times, $\tau_k \in \mathfrak{J}$, $\mathbf{u} = [u_1, \dots, u_N]^T$ is the control vector, $u \in R^N$, $\mathbf{y} = [y_1, \dots, y_N]^T$, $\mathbf{y} \in R^N$ is the output vector, $\mathbf{A}_i, \mathbf{A}_{id} \in R^{N_i \times N_i}$, $\mathbf{A}_{ij} \in R^{N_i \times N_j}$, $\mathbf{b}_i, \mathbf{c}_i \in R^{N_i}$ is a Multivariable Fractional Order System with Multiple Time-Delays (MDFOS).

We consider that

$$\tau_k \in \mathfrak{J} \quad (6)$$

and $\mathfrak{J}$ is the time-delay space for which the stability conditions are ensured for any $\gamma_i \in \Gamma$. Also, $\varphi_i(t)$ are continuous specified functions on $t \in [-\tau_k, 0]$.

Definition 3. The generalized Lyapunov function of MDFOS is

$$V(\mathbf{x}) = \sum_{i=1}^N V_i(\mathbf{x}_i) \quad (7)$$

where $V_i(\mathbf{x}_i) = \mathbf{x}_i^T \mathbf{P}_i \mathbf{x}_i$, $\mathbf{P}_i = \mathbf{P}_i^T > 0$. The Multivariable Fractional Order Derivative of $V(\mathbf{x})$ is ($D^\gamma V$ is the notation):

$$D^\gamma V(\mathbf{x}) = \sum_{i=1}^N D^{\gamma_i} \left(\mathbf{x}_i^T \mathbf{P}_i \mathbf{x}_i \right) \quad (8)$$

Theorem 1. MDFOS is asymptotically stable if the generalized Lyapunov function satisfies the following conditions:

$$\xi_1 \|\mathbf{x}\|^2 \leq V(t) \leq \xi_2 \|\mathbf{x}\|^2, \quad \xi_i > 0, \quad i = 1, 2 \quad (9)$$

$$D^\gamma V(t) \leq -\eta_1 V(t) + \eta_2 V(t - \tau) \quad (10)$$

$$\eta_1 > \eta_2 > 0 \quad (11)$$

where $V(\mathbf{x}(t)) = V(t)$ and $V(\mathbf{x}(t - \tau)) = V(t - \tau)$.

Proof. See Appendix A. $\square$

Corollary 1. If $\mathbf{A}_{id} = \mathbf{0}$, $i = 1, 2, \dots, N$, the system is asymptotically stable if $\eta_1 > 0$.

2.2. Main Result

2.2.1. Fractional Order Linear System

Consider the system

$$D^\gamma \mathbf{x}(t) = \mathbf{A} \mathbf{x}(t) - \mathbf{b} u(t) \quad (12)$$

$$\mathbf{y}(t) = \mathbf{c}^T \mathbf{x}(t) \quad (13)$$

where $\mathbf{x} = [x_1 x_2 \dots x_i \dots x_N]^T$, is the state vector, $\mathbf{x} \in R^N$, $D^\gamma \mathbf{x}(t)$ is the Caputo derivative of $\mathbf{x}(t)$ for $\gamma \in (0, 1]$, u is the input, $u \in R^1$, $\mathbf{y} \in R^1$ is the output, $\mathbf{A} \in R^{N \times N}$, $\mathbf{b}, \mathbf{c} \in R^N$.

Assume that the following control law

$$u(t) = k y(t) \quad (14)$$

ensures as the matrix $\mathbf{A}$ either γ -fractional order stable ($k = k_\gamma$)

$$\mathbf{A}^{(\gamma)} = \mathbf{A} - k_\gamma \mathbf{b} \mathbf{c}^T \quad (15)$$

or Hurwitz stable ($k = k_H$)

$$\mathbf{A}^H = \mathbf{A} - k_H \mathbf{b} \mathbf{c}^T \quad (16)$$

Define the coefficient

$$\mu = \frac{k_H}{k_\gamma} \quad (17)$$

where μ satisfies the condition

$$\mu\gamma\frac{\pi}{2} > \frac{\pi}{2}; \mu > \frac{1}{\gamma} > 1; \quad (18)$$

The following relations can be inferred:

$$k_\gamma = \frac{k_H}{\mu} = k_H - \mu^*; \quad (19)$$

$$\mu^* = k_H \frac{\mu - 1}{\mu} \quad (20)$$

If we substitute the control law (14) and (15) into the system (12) and (13), and take (19) and (20) into account, this yields

$$D^\gamma x(t) = (A - k_H bc^T)x(t) + \mu^* bc^T x(t) \quad (21)$$

Considering that k_H from (16) determined such as $A^H = A - k_H bc^T$ is Hurwitz, this inequality becomes

$$D^\gamma x(t) = A^H x(t) + \mu^* bc^T x(t) \quad (22)$$

Theorem 2. The system (12), (13) with the control law (15) is asymptotically stable if

$$\operatorname{Re}\left(c^T(j\omega I - A^H)^{-1}b\right) \leq \frac{\alpha^2}{1 - \gamma} \quad (23)$$

where α is provided in the proof below.

Proof. Let $V(x) = x^T Px$ be the Lyapunov function that satisfies the condition (9). The γ -fractional order derivative of $V(x)$ is (Appendix B):

$$D^\gamma V(t) = D^\gamma (x^T Px) \leq (D^\gamma x)^T Px + x^T P(D^\gamma x) \quad (24)$$

Substituting (22) into (24) yields (for the sake of simplicity, throughout the remainder of this section, $x(t)$ is denoted as x)

$$D^\gamma V(x) \leq x^T (A^{HT} P + P A^H) x + 2x^T P \mu^* bc^T x \quad (25)$$

Considering that A^H is Hurwitz, adding and subtracting the term $\frac{1}{\mu} y^2$, after simple additional manipulations, it follows that

$$D^\gamma V(x) \leq -x^T q q^T x + 2x^T P \mu^* bc^T x + \frac{1}{\mu} y^2 - \frac{1}{\mu^2} y^2 \quad (26)$$

where

$$A^{HT} P + P A^H = -q q^T \quad (27)$$

Considering that

$$-\frac{1}{\mu} y^2 < -\frac{1}{\mu^2} y^2 \quad (28)$$

the inequality (26) becomes

$$D^\gamma V(x) \leq -x^T q q^T x - 2x^T \left(-P\mu\mu^* b c^T x - \frac{c}{2} \right) \frac{y}{\mu} - \frac{1}{\mu^2} y^2 \quad (29)$$

According to Kalman–Yakubovich–Popov Lemma [25], if A^H is Hurwitz stable, the pairs (A^H, b) , (A^H, c) are completely controllable and observable, respectively, a symmetric matrix P and the matrix $q q^T$, and a parameter α , solution of (27) and

$$-P\mu\mu^* b c^T x - \frac{c}{2} = \alpha q \quad (30)$$

exist if and only if

$$\alpha^2 + 2\operatorname{Re}\left(\frac{c^T}{2}(j\omega I - A^H)^{-1}(-\mu\mu^* b)\right) \geq 0 \quad (31)$$

$$0 < \alpha \leq 1 \quad (32)$$

or

$$\operatorname{Re}\left(c^T(j\omega I - A^H)^{-1}b\right) \leq \frac{\alpha^2}{\mu\mu^*} \quad (33)$$

Taking (18) and (20) into account yields

$$\operatorname{Re}\left(c^T(j\omega I - A^H)^{-1}b\right) \leq \frac{\alpha^2}{\gamma(\mu - 1)} \leq \frac{\alpha^2}{1 - \gamma} \quad (34)$$

If we denote

$$G(j\omega) = c^T(j\omega I - A^H)^{-1}b \quad (35)$$

This inequality becomes

$$\operatorname{Re}(G(j\omega)) \leq \frac{\alpha^2}{1 - \gamma} \quad (36)$$

If (34) is satisfied, taking (30) and (32) into account, the inequality (29) becomes

$$D^\gamma V(x) \leq -\left(1 - \alpha^2\right)x^T q q^T x - x^T \left(\alpha q + \frac{c}{\mu}\right) \left(\alpha q + \frac{c}{\mu}\right)^T x \quad (37)$$

$$D^\gamma V(x) \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} x^T P x \quad (38)$$

where

$$Q = \left(\alpha q + \frac{c}{\mu}\right) \left(\alpha q + \frac{c}{\mu}\right)^T \quad (39)$$

$$D^\gamma V(t) \leq -\eta_1^* V(t), \quad \eta_1^* = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > 0 \quad (40)$$

and according to Theorem 1, Corollary 1, if the condition (23) is verified, the system is asymptotically stable. $\square$

Remark 1. If $\gamma = 1$ then $k_\gamma = k_H$, $\mu = 1$, and the frequency condition (36) is always satisfied.

2.2.2. Fractional Order Linear System with State Delay

Consider the system

$$D^\gamma x(t) = Ax(t) + A_d x(t - \tau) - b u(t)$$

$$\mathbf{x}(t) = \varphi(t), \quad t \in [-\tau, 0] \quad (41)$$

$$\mathbf{y}(t) = \mathbf{c}^T \mathbf{x}(t)$$

where $\mathbf{x}$, u , y are the state vector, the input and output, respectively, as in the previous section, $\mathbf{x} \in \mathbb{R}^N$, $D^\gamma \mathbf{x}(t)$ is the Caputo derivative of $\mathbf{x}(t)$ for $\gamma \in (0, 1]$, τ is the delay times, $\tau \in \mathcal{J}$, $u \in \mathbb{R}^1$, $\mathbf{y} \in \mathbb{R}^1$, $\mathbf{A}\mathbf{A}_d \in \mathbb{R}^{N \times N}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^N$. Also, $\varphi(t)$ is a continuous specified function on $t \in [-\tau, 0]$. By following the same procedure as described in the preceding section, we consider the control law

$$u(t) = k y(t) \quad (42)$$

where the parameter $k = k_\gamma$ ensures as the matrix $\mathbf{A}$ is γ -fractional order stable.

Define the coefficients, μ , μ^* :

$$\mu = \frac{k_H}{k_\gamma} \quad (43)$$

that satisfies the conditions

$$\mu\gamma \frac{\pi}{2} > \frac{\pi}{2}; \mu > \frac{1}{\gamma} > 1; \quad (44)$$

$$\mu^* = k_H \frac{\mu - 1}{\mu} \quad (45)$$

If we substitute the control law (42) for $k = k_\gamma$ into the system (41)–(43) and take $(\mathbf{A}^H = \mathbf{A} - k_H \mathbf{b} \mathbf{c}^T)$ into account, this yields

$$D^\gamma \mathbf{x}(t) = \mathbf{A}^H \mathbf{x}(t) + \mu^* \mathbf{b} \mathbf{c}^T \mathbf{x}(t) + \mathbf{A}_d \mathbf{x}(t - \tau) \quad (46)$$

Theorem 3. The system (41) with the control law (42) ($k = k_\gamma$) is asymptotically stable if

$$\eta_1^* = \frac{\lambda_{\min}(\mathbf{Q})}{\lambda_{\max}(\mathbf{P})} - \frac{\lambda_{\max}(\mathbf{P}^2)}{\lambda_{\min}(\mathbf{P})} > 0, \quad \eta_2^* = \frac{\rho \tau_{\max}^2}{\lambda_{\min}(\mathbf{P})} \quad (47)$$

$$\eta_1^* > \eta_2^*; \tau \in \mathcal{J} \quad (48)$$

$$\operatorname{Re} \left(\mathbf{c}^T \left(j\omega \mathbf{I} - \mathbf{A}^H \right)^{-1} \mathbf{b} \right) \leq \frac{\alpha^2}{\mu \mu^*} \leq \frac{\alpha^2}{1 - \gamma} \quad (49)$$

where $\lambda_{\min}(\mathbf{Q})$, $\lambda_{\max}(\mathbf{P}^2)$, α , ρ are provided in the proof below.

Proof. Let $V(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x}$ be the Lyapunov function that satisfies the condition (9). The γ -fractional order derivative of $V(\mathbf{x})$ is (Appendix B)

$$D^\gamma V(t) = D^\gamma (\mathbf{x}^T \mathbf{P} \mathbf{x}) \leq (D^\gamma \mathbf{x})^T \mathbf{P} \mathbf{x} + \mathbf{x}^T \mathbf{P} (D^\gamma \mathbf{x}) \quad (50)$$

Substituting (23) into (27) yields (for the sake of simplicity, throughout the remainder of this paper, $\mathbf{x}(t)$ is denoted as $\mathbf{x}$, the time variable t is explicitly included only for delay-specific components, i.e., $\mathbf{x}(t - \tau)$)

$$D^\gamma V(\mathbf{x}) \leq \mathbf{x}^T (\mathbf{A}^H \mathbf{P} + \mathbf{P} \mathbf{A}^H) \mathbf{x} + 2 \mathbf{x}^T \mathbf{P} \mu^* \mathbf{b} \mathbf{c}^T \mathbf{x} + 2 \mathbf{x}^T \mathbf{P} \mathbf{A}_d \mathbf{x}(t - \tau) \quad (51)$$

Using inequality

$$2 \mathbf{x}^T \mathbf{P} \mathbf{A}_d \mathbf{x}(t - \tau) \leq \mathbf{x}^T \mathbf{P} \mathbf{P} \mathbf{x}(t) + \mathbf{x}^T(t - \tau) \mathbf{A}_d^T \mathbf{A}_d \mathbf{x}(t - \tau) \quad (52)$$

By following the same procedure as described in the preceding section

$$D^\gamma V(x) \leq -x^T q q^T x - 2x^T (-P\mu\mu^* b c^T x - \frac{c}{2}) \frac{y}{\mu} - \frac{1}{\mu^2} y^2 + x^T P P x(t) + x^T(t-\tau) A_d^T A_d x(t-\tau) \quad (53)$$

where

$$A^{HT} P + P A^H = -q q^T \quad (54)$$

According to the Kalman–Yakubovich–Popov Lemma [25], if A^H is Hurwitz stable, the pairs (A^H, b) , (A^H, c) are completely controllable and observable, respectively, a symmetric matrix P and the matrix $q q^T$, solution of (54) and

$$-P\mu\mu^* b c^T x - \frac{c}{2} = \alpha q \quad (55)$$

exist if and only if

$$\alpha^2 + 2\operatorname{Re}\left(\frac{c^T}{2} (j\omega I - A^H)^{-1} (-\mu\mu^* b)\right) \geq 0 \quad (56)$$

$$0 < \alpha \leq 1 \quad (57)$$

$$\operatorname{Re}\left(c^T (j\omega I - A^H)^{-1} b\right) \leq \frac{\alpha^2}{\mu\mu^*} \leq \frac{\alpha^2}{1-\gamma} \quad (58)$$

$$\operatorname{Re}(G(j\omega)) \leq \frac{\alpha^2}{1-\gamma} \quad (59)$$

Assume that the delay component satisfies the condition:

$$\|A_d^T A_d\| \leq \rho \tau^2 \quad (60)$$

If (59) is satisfied, taking (55) and (57) into account, the inequality (53) becomes

$$D^\gamma V(x) \leq -\left(1 - \alpha^2\right) x^T q q^T x - x^T \left(\alpha q + \frac{c}{\mu}\right) \left(\alpha q + \frac{c}{\mu}\right)^T x + x^T P P x + \rho \tau^2 x^T(t-\tau) x(t-\tau) \quad (61)$$

The first term is negative and this inequality becomes

$$D^\gamma V(x) \leq -\left(\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} - \frac{\lambda_{\max}(P^2)}{\lambda_{\min}(P)}\right) x^T P x + \frac{\rho \tau_{\max}^2}{\lambda_{\min}(P)} x^T(t-\tau) P x(t-\tau) \quad (62)$$

where

$$Q = \left(\alpha q + \frac{c}{\mu}\right) \left(\alpha q + \frac{c}{\mu}\right)^T \quad (63)$$

$$D^\gamma V(t) \leq -\eta_1^* V(t) + \eta_2^* V(t-\tau); \tau \in \mathfrak{I} \quad (64)$$

and according to Theorem 1, if the conditions (47)–(49) are verified, the system is asymptotically stable. $\square$

2.2.3. Multi-Fractional Order Linear Systems with Multiple State Delays

Consider MDFOS (3)–(5) (Definition 2), where according to (6) the variable τ_k is substituted by τ . Assume that the matrices A_{ij}, A_{id} satisfy the conditions:

$$\mathbf{a}. A_{id} A_{id}^T \leq \rho_{id} \tau_i^2 \mathbf{I}_{N_i}, \rho_{id} > 0; \quad (65)$$

$$\mathbf{b}. A_{ij} A_{ij}^T \leq \rho_{ij} \mathbf{I}_{N_i}, \rho_{ij} > 0; \rho_{ij} = \rho_{ji}; i \neq j, \mathbf{I}_{N_i} \text{ is identity matrix, } \mathbf{I}_{N_i} \in \mathbf{R}^{N_i \times N_i} \quad (66)$$

The following control law is applied to ensure the γ -fractional order stability:

$$u_i(t) = k_{\gamma i} y_i(t) = k_{\gamma i} \mathbf{c}_i^T \mathbf{x}_i(t), \quad k_{\gamma i} > 0; \quad i = 1, 2, \dots, N \quad (67)$$

and $\mathbf{A}_i$ become the γ -fractional order stable matrices

$$\mathbf{A}_i^{(\gamma)} = \mathbf{A}_i - k_{\gamma i} \mathbf{b}_i \mathbf{c}_i^T, \quad i = 1, 2, \dots, N \quad (68)$$

Let μ_i be a positive constant that satisfies inequality

$$\mu_i \gamma_i \frac{\pi}{2} > \frac{\pi}{2}; \quad \mu_i > \frac{1}{\gamma_i} > 1; \quad (69)$$

We define the coefficients

$$k_{Hi} = \mu_i k_{\gamma i}; \quad k_{Hi} > 1; \quad i = 1, 2, \dots, N \quad (70)$$

The following relations can be inferred:

$$k_{\gamma i} = \frac{k_{Hi}}{\mu_i} = k_{Hi} - \mu_i^*; \quad (71)$$

$$\mu_i^* = k_{Hi} \frac{\mu_i - 1}{\mu_i} \quad (72)$$

If we substitute the control law (67) into the system (3)–(5) and take (69) and (72) into account, this yields

$$D^{\gamma_i} \mathbf{x}_i(t) = (\mathbf{A}_i - k_{Hi} \mathbf{b}_i \mathbf{c}_i^T) \mathbf{x}_i(t) + \mu_i^* \mathbf{b}_i \mathbf{c}_i^T \mathbf{x}_i(t) + \mathbf{A}_{id} \mathbf{x}_i(t - \tau_i) + \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{A}_{ij} \mathbf{x}_j(t) \quad (73)$$

Considering that k_{Hi} from (70) determines as $\mathbf{A}_i^H = \mathbf{A}_i - k_{Hi} \mathbf{b}_i \mathbf{c}_i^T$ is Hurwitz, this inequality becomes

$$D^{\gamma_i} \mathbf{x}_i(t) = \mathbf{A}_i^H \mathbf{x}_i(t) + \mu_i^* \mathbf{b}_i \mathbf{c}_i^T \mathbf{x}_i(t) + \mathbf{A}_{id} \mathbf{x}_i(t - \tau_i) + \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{A}_{ij} \mathbf{x}_j(t) \quad (74)$$

Theorem 4. The system (3)–(5) that satisfies the conditions (65) and (66), with the control law (67) is asymptotically stable if

$$\eta_i = \frac{\lambda_{\min}(\mathbf{Q}_i)}{\lambda_{\max}(\mathbf{P}_i)} - \frac{1}{\lambda_{\min}(\mathbf{P}_i)} \left(N \lambda_{\max}(\mathbf{P}_i^2) + \rho_i \right) > 0, \quad \frac{\rho_{id} \tau_i^2}{\lambda_{\min}(\mathbf{P}_i)} > 0, \quad i = 1, 2, \dots, N \quad (75)$$

$$\eta_1^* > \eta_2^*; \quad \tau_i \in \mathcal{I}, \quad i = 1, 2, \dots, N \quad (76)$$

$$\operatorname{Re}(G_i(j\omega)) \leq \frac{\alpha_{\min}^2}{1 - \gamma_{\min}}, \quad i = 1, 2, \dots, N, \quad (77)$$

where $\lambda_{\min}(\mathbf{Q}_i)$, $\lambda_{\max}(\mathbf{P}_i^2)$, $\gamma_{\max}$, ρ_i , η_1^* , η_2^* , $\alpha_{\min}^2$ are provided in the proof below.

Proof. Consider the generalized Lyapunov function (17), $V(t) = \sum_{i=1}^N V_i(\mathbf{x}_i(t))$, $V_i(\mathbf{x}_i) = \mathbf{x}_i^T \mathbf{P}_i \mathbf{x}_i$, $\mathbf{P}_i = \mathbf{P}_i^T > 0$, that satisfies the conditions (19). According to Definition 2, the multivariable fractional order derivative of $V(\mathbf{x})$ is (Appendix B)

$$D^{\gamma} V(t) = \sum_{i=1}^N D^{\gamma_i} \left(\mathbf{x}_i^T \mathbf{P}_i \mathbf{x}_i \right) \leq \sum_{i=1}^N \left(\left(D^{\gamma_i} \mathbf{x}_i^T \right) \mathbf{P}_i \mathbf{x}_i + \mathbf{x}_i^T \mathbf{P}_i \left(D^{\gamma_i} \mathbf{x}_i \right) \right) \quad (78)$$

By following the same procedure as described in the preceding section:

$$D^\gamma V(t) \leq \sum_{i=1}^N (x_i^T (A_i^{HT} P_i + P_i A_i^H) x_i + 2x_i^T P_i \mu_i^* b_i c_i^T x_i) + 2 \sum_{i=1}^N x_i^T (t - \tau_i) A_{id}^T P_i x_i(t) + 2 \sum_{i=1}^N \sum_{j \neq i}^N x_j^T A_{ij}^T P_i x_i \quad (79)$$

Using inequalities

$$2x_i^T A_{ij}^T P_i x_i \leq x_i^T P_i P_i x_i + x_j^T A_{ij}^T A_{ij} x_j \quad (80)$$

$$2x_i^T (t - \tau_i) A_{id}^T P_i x_i(t) \leq x_i^T(t) P_i P_i x_i(t) + x_i^T(t - \tau_i) A_{id}^T A_{id} x_i(t - \tau_i) \quad (81)$$

considering that A_i^H is Hurwitz

$$A_i^{HT} P_i + P_i A_i^H = -q_i q_i^T \quad (82)$$

and the inequalities

$$-\frac{1}{\mu_i} y_i^2 < -\frac{1}{\mu_i^2} y_i^2 \quad (83)$$

after simple additional manipulations, it follows that

$$D^\gamma V(t) \leq \sum_{i=1}^N \left((-x_i^T q_i q_i^T x_i - 2x_i^T (-P_i \mu_i^* \mu_i b_i - \frac{c_i}{2}) \frac{y_i}{\mu_i} - \frac{1}{\mu_i^2} y_i^2) + x_i^T P_i^2 x_i + x_i^T (t - \tau_i) A_{id}^T A_{id} x_i(t - \tau_i) \right) + 2 \sum_{i=1}^N \sum_{j \neq i}^N x_j^T A_{ij}^T P_i x_i \quad (84)$$

According to Kalman-Yakubovich-Popov Lemma [25], if A_i^H is Hurwitz stable, the pairs (A_i^H, b_i) , (A_i^H, c_i) are completely controllable and observable, respectively, a symmetric matrix P_i and the matrix $q_i q_i^T$, solution of (82) and

$$-P_i \mu_i^* \mu_i b_i - \frac{c_i}{2} = \alpha_i q_i, i = 1, 2, \dots, N \quad (85)$$

$$0 < \alpha_i \leq 1 \quad (86)$$

exist if and only if

$$\alpha_i^2 + 2 \operatorname{Re} \left(\frac{c_i^T}{2} (j\omega I - A_i^H)^{-1} (-\mu_i^* \mu_i b_i) \right) \geq 0 \quad (87)$$

$$\operatorname{Re} \left(c_i^T (j\omega I - A_i^H)^{-1} b_i \right) \leq \frac{\alpha_{\min}^2}{\mu_i^* \mu_i} = \frac{\alpha_i^2}{\gamma_i (\mu_i - 1)}, i = 1, 2, \dots, N \quad (88)$$

Provided that (70), (72), (86), yields

$$\operatorname{Re} \left(c_i^T (j\omega I - A_i^H)^{-1} b_i \right) \leq \frac{\alpha_i^2}{1 - \gamma_i}, i = 1, 2, \dots, N \quad (89)$$

If we denote by

$$\gamma_{\min} = \min_i \gamma_i; \alpha_{\min} = \min_i \alpha_i \quad (90)$$

$$G_i(j\omega) = c_i^T (j\omega I - A_i^H)^{-1} b_i \quad (91)$$

This inequality becomes

$$\operatorname{Re}(G_i(j\omega)) \leq \frac{\alpha_{\min}^2}{1 - \gamma_{\min}}, i = 1, 2, \dots, N, \quad (92)$$

If (92) is satisfied, taking (80) and (81) into account, the inequality (84) becomes

$$D^\gamma V(t) \leq \sum_{i=1}^N -\left(1 - \alpha_i^2\right) x_i^T q_i q_i^T x_i - \sum_{i=1}^N \left(x_i^T \left(q_i + \frac{c_i}{\mu_i}\right) \left(q_i + \frac{c_i}{\mu_i}\right)^T x_i + x_i^T P_i^2 x_i + \rho_{id} \tau_i^2 x_i^T (t - \tau_i) x_i(t - \tau_i)\right) + \sum_{i=1}^N \sum_{j \neq i}^N \rho_{ij} x_j^T x_i \quad (93)$$

Provided that the first term is negative, this results in

$$D^\gamma V(t) \leq \sum_{i=1}^N (-\lambda_{\min(Q_i)} x_i^T x_i + N\lambda_{\max(P_i^2)} x_i^T x_i + \rho_{id} \tau_i^2 x_i^T (t - \tau_i) x_i (t - \tau_i)) + \sum_{i=1}^N \sum_{j \neq i}^N \rho_{ij} x_j^T x_i \quad (94)$$

or,

$$D^\gamma V(t) \leq \sum_{i=1}^N \left(-\frac{\lambda_{\min(Q_i)}}{\lambda_{\max(P_i)}} x_i^T P_i x_i + N \frac{\lambda_{\max(P_i^2)}}{\lambda_{\min(P_i)}} x_i^T P_i x_i + \frac{\rho_{id} \tau_i^2}{\lambda_{\min(P_i)}} x_i^T (t - \tau_i) P_i x_i (t - \tau_i) + \frac{\rho_i}{\lambda_{\min(P_i)}} x_i^T P_i x_i \right) \quad (95)$$

where

$$Q_i = (q_i + \frac{c_i}{\mu_i})(q_i + \frac{c_i}{\mu_i})^T; \rho_i = \sum_{j \neq i}^N \rho_{ij} \quad (96)$$

and $\lambda_{\min(Q_i)}$, $\lambda_{\max(P_i^2)}$, $\lambda_{\max(P_i)}$ are the minimum and maximum eigenvalues of Q_i , P_i^2 , P_i , respectively. This inequality becomes,

$$D^\gamma V(t) \leq \sum_{i=1}^N \left(-\frac{\lambda_{\min(Q_i)}}{\lambda_{\max(P_i)}} + \frac{1}{\lambda_{\min(P_i)}} (N\lambda_{\max(P_i^2)} + \rho_i) \right) V_i(t) + \frac{\rho_{id} \tau_i^2}{\lambda_{\min(P_i)}} V_i(t - \tau_i) \quad (97)$$

If we denote by

$$\eta_1^* = \min_i \left(\frac{\lambda_{\min(Q_i)}}{\lambda_{\max(P_i)}} - \frac{1}{\lambda_{\min(P_i)}} (N\lambda_{\max(P_i^2)} + \rho_i) \right); \eta_2^* = \max_i \frac{\rho_{id} \tau_i^2}{\lambda_{\min(P_i)}}; \tau_i \in \mathfrak{J} \quad (98)$$

this inequality becomes

$$D^\gamma V(t) \leq -\eta_1^* V(t) + \eta_2^* V(t - \tau); \tau \in \mathfrak{J} \quad (99)$$

and according to Theorem 3, if the conditions (75)–(77) are verified, the system is asymptotically stable. $\square$

2.2.4. Multi-Fractional Order Systems with State Delays and Nonlinear Components

Consider MDFOS (3)–(5) with nonlinear components $f_i(x_i)$

$$D^{\gamma_i} x_i(t) = A_i x_i(t) + A_{id} x_i(t - \tau_i) + \sum_{j=1, j \neq i}^N A_{ij} x_j(t) + f_i(x_i), -b_i u_i(t) \quad (100)$$

where $A_i, x_i A_{id}, A_{ij}$, the initial conditions, the output and the inputs are defined as in the previous sections and the nonlinear components satisfy the conditions:

$$\|f_i(x_i)\|^2 \leq \sum_{j=1}^{N_i} l_{ij} \|x_{ij}\|^2, l_{ij} > 0 \quad (101)$$

Theorem 5. The system (100) that satisfies the conditions (65) and (66), with the control law (67) is asymptotically stable if

$$\eta_i = \frac{\lambda_{\min(Q_i)}}{\lambda_{\max(P_i)}} - \frac{1}{\lambda_{\min(P_i)}} (N\lambda_{\max(P_i^2)} + \rho_i + l_i^*) > 0, \eta_2^* = \max_i \frac{\rho_{id} \tau_i^2}{\lambda_{\min(P_i)}}; \tau_i \in \mathfrak{J}; i = 1, 2, \dots, N \quad (102)$$

$$\eta_1^* > \eta_2^*; \tau_i \in \mathfrak{J}, i = 1, 2, \dots, N \quad (103)$$

$$Re(G_i(j\omega)) \leq \frac{\alpha_{\min}^2}{1 - \gamma_{\min}}, i = 1, 2, \dots, N, \quad (104)$$

where $l_i^* = \max_j l_{ij}$.

Proof. The proof is similar to previous Section. $\square$

3. Results

3.1. Numerical Example 1

Consider the system

$$D^{0.8}x(t) = Ax(t) + A_d x(t - \tau) - bu(t) \quad (105)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -60 & 8 \end{bmatrix}; A_d = \tau \begin{bmatrix} 0.01 & 0 \\ 0 & 0.015 \end{bmatrix}; b = \begin{bmatrix} 0 \\ 3 \end{bmatrix}; c = \begin{bmatrix} 0.1 \\ 7.7 \end{bmatrix}; \quad (106)$$

$$\tau \in \mathcal{T} = [0.1 : 0.4]s; \quad (107)$$

The matrix A is 0.8-fractional order unstable with $\lambda_A = 4 \pm j 8$. The control law (42) ensures that matrix A is either 0.8-stable or Hurwitz stable. From Figure 1, it can be observed that for $k_{0.8} \in [3 : 4.5]$, A becomes 0.8-stable ($\lambda_{A^{0.8}} = 1.22 \pm j 7.65$), whereas for $k_H \in [5:7]$, it becomes Hurwitz stable ($\lambda_{A^H} = -1.42 \pm j 9.45$). According to (43) and (45), if we consider $k_{0.5} = 3.6$ and $k_H = 7$ yields $\mu = 1.94$, $\mu^* = 3.39$ and the control law (14), $A^{(\gamma)} = A - k_\gamma bc^T$. The pairs (A^H, b) , (A^H, c) are completely controllable and observable, respectively, ($A^H = A - k_H bc^T$) and solving the system in MATLAB-R2020a.

$$A^{HT}P + PA^H = -qq^T - \delta I \quad (108)$$

$$-P\mu^*\mu b - \frac{c}{2} = \alpha q \quad (109)$$

for $\delta = 57.7$, yields

$$P = \begin{bmatrix} 22.5 & 0.474 \\ 0.474 & 0.474 \end{bmatrix}; q = \begin{bmatrix} -4 \\ -14.8 \end{bmatrix}; \lambda_P = \begin{bmatrix} 22.5 \\ 0.47 \end{bmatrix}; \alpha = 1; \quad (110)$$

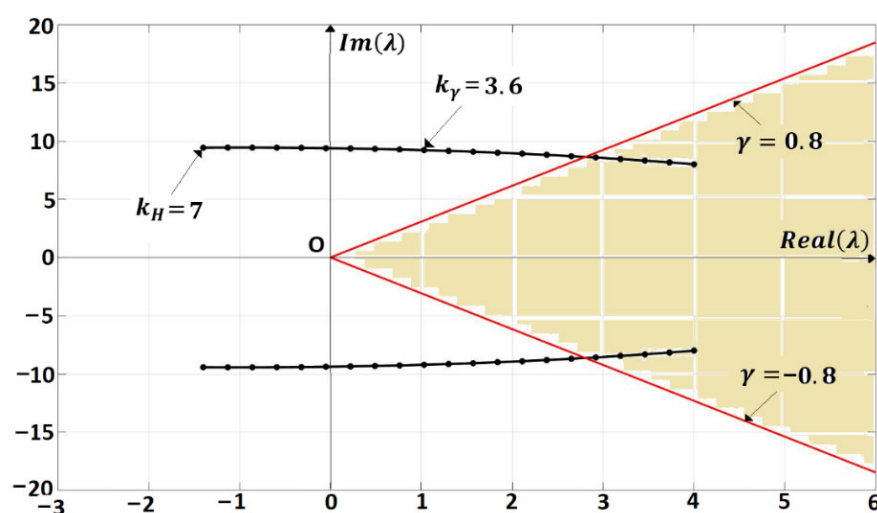


Figure 1. Eigenvalue migration by the control law (42), Example 1.

The conditions (47) become

$$\eta_1^* = \frac{\delta}{\lambda_{\max}(P)} - \frac{1}{\lambda_{\min}(P)} = 0.5368; \eta_2^* = \tau_{1\max}^2 \frac{\lambda_{\max}(P_1^2) \|A_{1d} A_{1d}^T\|}{\lambda_{\min}(P_1)} = 0.155; \quad (111)$$

$$\eta_1^* > \eta_1^* > 0, \tau \in \mathcal{I} \quad (112)$$

$$D^\gamma V(t) \leq -\eta_1^* V(t) + \eta_1^* V(t - \tau) \quad (113)$$

The first condition of Theorem 3 is verified. The second condition, the frequency condition, requires

$$\operatorname{Re}\left(c^T (j\omega I - A^H)^{-1} b\right) \leq \frac{1}{1 - \gamma} = 5 \quad (114)$$

We remark from Figure 2 that the polar plot of $G(j\omega)$ satisfies the inequality (113). The conditions of Theorem 3 are satisfied; the system (105)–(107) is asymptotically stable. In Figure 3 are shown the trajectories of the system for $k_{0.5} = 3.6$ and $\tau = 0.2$ s, ($x(t) = [0.2 : 0]$, $t \in [-0.2, 0]$).

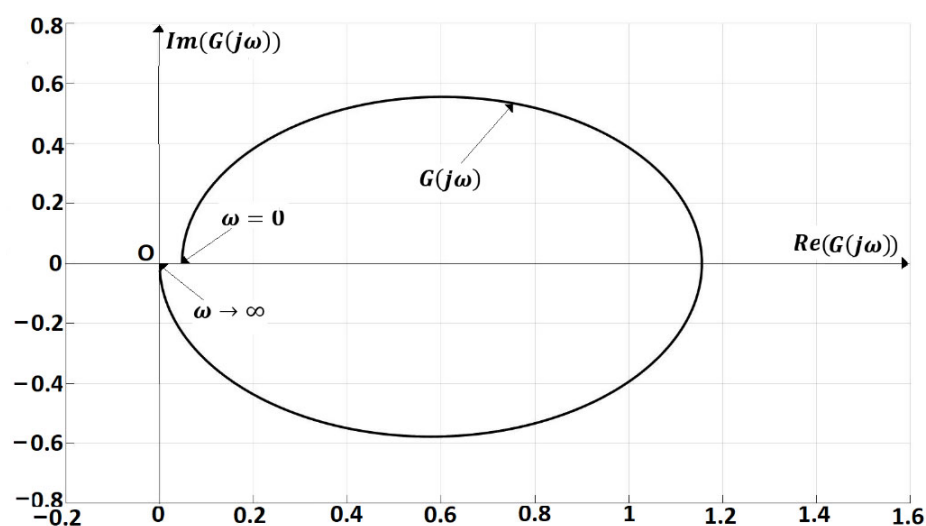


Figure 2. $G(j\omega)$ polar plot, Example 1.

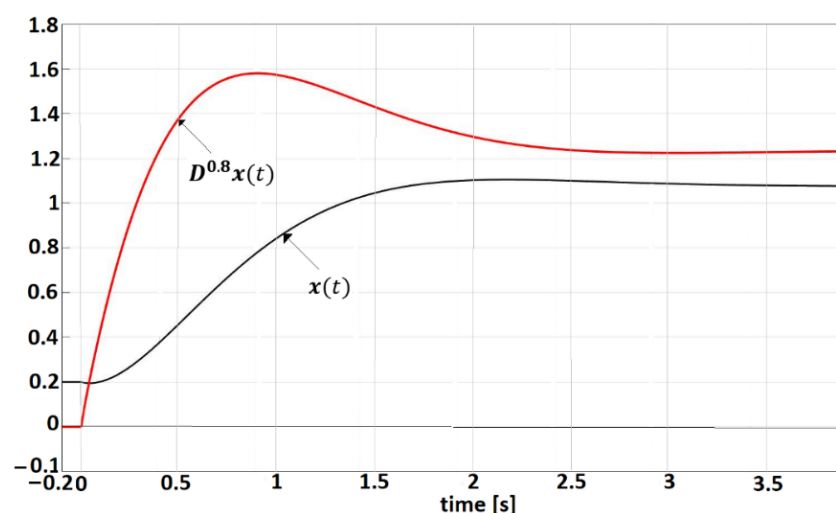


Figure 3. Trajectories of the system (105)–(107).

3.2. Numerical Example 2

Consider the system

$$D^{0.5}x_1(t) = A_{11}x_1(t) + A_{1d}x_1(t - \tau_1) + A_{12}x_2(t) - b_1u_1(t) \quad (115)$$

$$D^{0.7}x_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + A_{2d}x_2(t - \tau_2) - b_2u_2(t) \quad (116)$$

$$y_1(t) = c_1^T x_1(t); y_2(t) = c_2^T x_2(t) \quad (117)$$

where

$$A_{11} = \begin{bmatrix} 0 & 1 \\ -34 & 10 \end{bmatrix}; A_{1d} = \tau_1 \begin{bmatrix} 0.1 & 0 \\ 0 & 0.15 \end{bmatrix}; A_{12} = \begin{bmatrix} -0.017 & 0 \\ 0 & 0.011 \end{bmatrix}; b_1 = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}; c_1 = \begin{bmatrix} 2 \\ 2.2 \end{bmatrix}; \quad (118)$$

$$A_{22} = \begin{bmatrix} 0 & 1 \\ -25 & 6 \end{bmatrix}; A_{2d} = \tau_2 \begin{bmatrix} 0.011 & 0 \\ 0 & 0.015 \end{bmatrix}; A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; c_2 = \begin{bmatrix} 1.8 \\ 2.0 \end{bmatrix}; \quad (119)$$

$$\tau_1 \in \mathcal{J}_1 = [0.2 : 0.8]s; \tau_2 \in \mathcal{J}_2 = [0.2 : 1]s; \quad (120)$$

The matrix A_{11} is 0.5-fractional order unstable with $\lambda_{A_{11}} = 5 \pm j3$. The control law (67) ensures that matrix A_{11} is either 0.5-stable or Hurwitz stable. From Figure 4, it can be observed that for $k_{0.5} \in [1.4 : 5.5]$, A_{11} becomes 0.5-stable, whereas for $k_{H,1} \in [5.6:6]$, it becomes Hurwitz stable. According to (69) and (72), if we consider $k_{0.5} = 3$ and $k_{H,1} = 6$ yields $\mu_1 = 2$, $\mu_1^* = 3$ and the control law

$$u_1(t) = k_{\gamma 1} y_1(t) = k_{\gamma 1} c_1^T x_1(t) \quad (121)$$

The pairs (A_{11}^H, b_1) , (A_{11}^H, c_1) are completely controllable and observable, respectively, $(A_{11}^H = A_{11} - k_{H,1} b_1 c_1^T)$ and solving in MATLAB the system

$$A_{11}^{HT} P_1 + P_1 A_{11}^H = -q_1 q_1^T - \delta_1 I \quad (122)$$

$$-P_1 \mu_1^* \mu_1 b_1 - \frac{c_i}{2} = q_1 \quad (123)$$

for $\delta_1 = 25$, yields

$$P_1 = \begin{bmatrix} 3.49 & 0.746 \\ 0.746 & 0.746 \end{bmatrix}; q_1 = \begin{bmatrix} -2.5 \\ -2.34 \end{bmatrix}; \lambda_{P_1} = \begin{bmatrix} 0.55 \\ 3.67 \end{bmatrix}; \alpha_1 = 0.5; \quad (124)$$

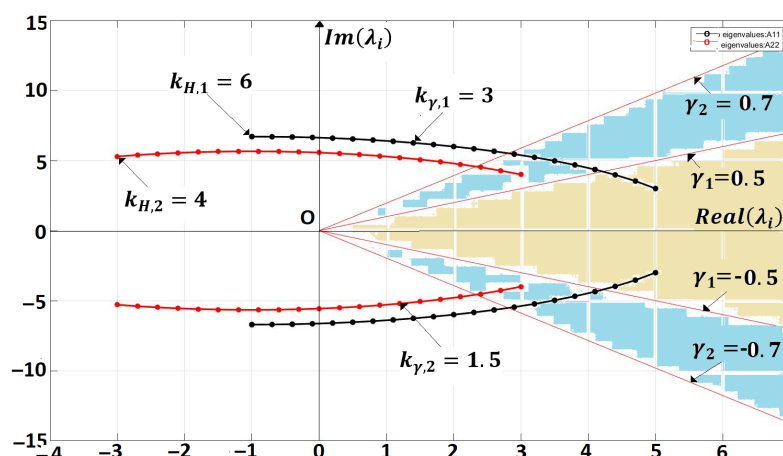


Figure 4. Eigenvalue migrations of A_{11} , A_{22} by the control law (42), Example 2.

Also, the matrix A_{22} is 0.7-fractional order unstable with $\lambda_{A_{11}} = 3 \pm j 4$. The control law (67) ensures that matrix A_{22} is either 0.7-stable or Hurwitz stable. From Figure 3, it can be observed that for $k_{0.7} \in [1.2 : 1.8]$ A_{11} becomes 0.7-stable, whereas for $k_{H,2} \in [1.9:4]$, it becomes Hurwitz stable. We consider $k_{0.7} = 1.5$ and $k_{H,2} = 4$ and obtain $\mu_2 = 2.67, \mu_2^* = 2.5$ and the control law

$$u_2(t) = k_{\gamma 2} y_2(t) = k_{\gamma 2} c_2^T x_2(t) \quad (125)$$

The pairs (A_{22}^H, b_2) , (A_{22}^H, c_2) are completely controllable and observable, respectively. $(A_{22}^H = A_{22} - k_{H,2} b_2 c_2^T)$ and solving the system in MATLAB

$$A_{22}^{HT} P_2 + P_2 A_{22}^H = -q_2 q_2^T - \delta_2 I \quad (126)$$

$$-P_2 \mu_2^* \mu_2 b_2 - \frac{c_2}{2} = \alpha q_2 \quad (127)$$

for $\delta_2 = 22$, yields

$$P_2 = \begin{bmatrix} 27.34 & 0.57 \\ 0.57 & 0.57 \end{bmatrix}; q_2 = \begin{bmatrix} -2.85 \\ -0.36 \end{bmatrix}; \lambda_{P_2} = \begin{bmatrix} 0.56 \\ 27.35 \end{bmatrix}; \alpha_2 = 0.5; \quad (128)$$

The inequalities (75) and (76) become

$$\eta_{11} = \frac{\delta_1 - 1}{\lambda_{\max}(P_1)} = 6.54; \eta_{21} = \tau_{1\max}^2 \frac{\lambda_{\max}(P_1^2) \|A_{1d} A_{1d}^T\|}{\lambda_{\min}(P_1)} = 0.352; \quad (129)$$

$$\eta_{12} = \frac{\delta_2 - 2}{\lambda_{\max}(P_2)} - \frac{\lambda_{\max}(P_1^2) \|A_{12} A_{12}^T\|}{\lambda_{\min}(P_2)} = 0.53; \eta_{22} = \tau_{2\max}^2 \frac{\lambda_{\max}(P_2^2) \|A_{2d} A_{2d}^T\|}{\lambda_{\min}(P_2)} = 0.016; \quad (130)$$

$$\eta_1^* = \min_i \eta_{1i} = 0.53; \eta_1^* = \max_i \eta_{2i} = 0.352 \quad (131)$$

These results can be summarized in

$$D^\gamma V(t) \leq -0.53 V(t) + 0.352 V(t - \tau); \tau \in \mathcal{J} \quad (132)$$

From (77), it follows that

$$\operatorname{Re} \left(c_i^T (j\omega I - A_i^H)^{-1} b_i \right) \leq \frac{\alpha_{\min}^2}{1 - \gamma_{\min}}, i = 1, 2 \quad (133)$$

$$\operatorname{Re}(G_i(j\omega)) \leq 0.5, i = 1, 2 \quad (134)$$

The inequality (77) is verified in Figure 5 by the polar plots of $G_i(j\omega), i = 1, 2$. All the conditions of Theorem 4 are satisfied, and we can conclude that the system (114)–(120) is asymptotically stable. The stability is also proved by the trajectories from Figure 6.

$$(x_1(t) = [0 : 0.1], x_2(t) = [0 : 0], t \in [-0.2, 0])$$

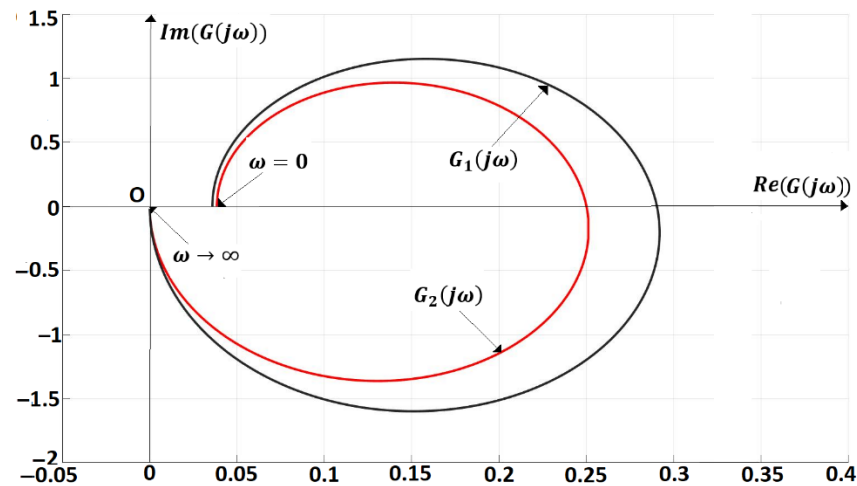


Figure 5. $G_i(j\omega)$, $i = 1, 2$, polar plots of Example 2.

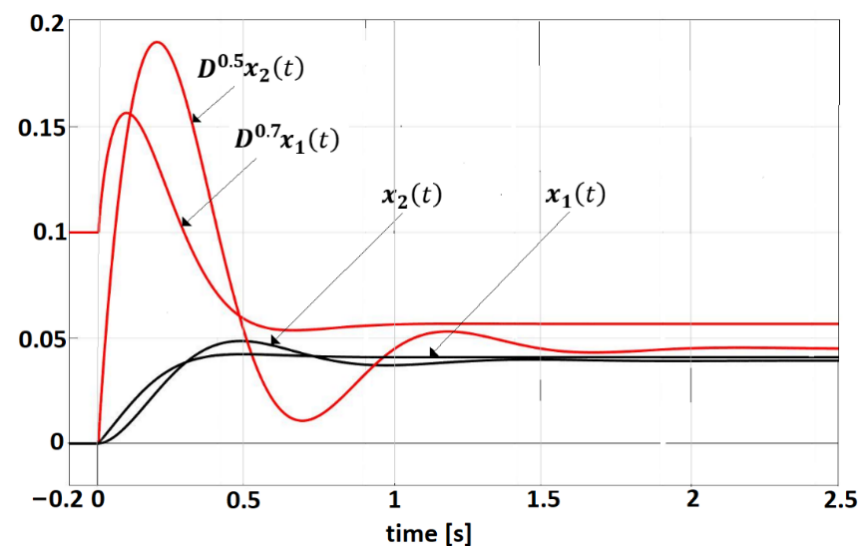


Figure 6. Trajectories of the system, Example 2.

3.3. Human–Machine System: Multi-Order Fractional System with Multiple State Delays

The human–machine system is modeled as a multi-fractional order system to capture its complex dynamics. The literature is quite extensive, and this is not the place to present a synthesis of these papers. The human–machine model discussed in this section is represented by the driver–vehicle model, which completely defines all the characteristics that allow for the determination of the conditions to achieve the desired performance. This model is based on the studies and publications synthesized in [26,27]. In this case, the human–machine system is a complex system where the human model is described by a fractional-order system, while the simplified dynamics of the vehicle's handling are described by second-order differential equations; see Figure 7.

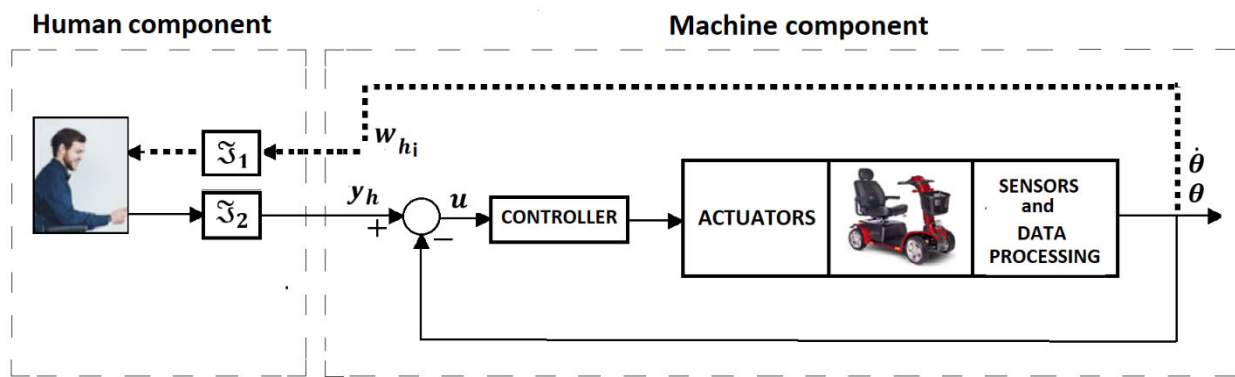


Figure 7. Driver–vehicle model.

The human model discussed herein is an extension of the one presented in [27]

$$D^{1.4}x_h(t) + a_{h1}D^{0.7}x_h(t) + a_{h2}x_h(t) = w_h(t) \quad (135)$$

$$y_h(t) = x_h(t) \quad (136)$$

where x_h defines the human state, y_h is the human operator's output, and the parameters a_{h1}, a_{h2} define the human driving characteristics.

The driver's ability to estimate the position (and eventually the velocity) of the vehicle is determined by the matrix D_h in which the coefficients d_{hi} , $i = 1, 2$, define the delay τ_1 ($\tau_1 \in \mathfrak{I}_1 = [\tau_{1min}, \tau_{1max}]$) through which these motion parameters are estimated

$$w = \begin{bmatrix} w_{h1} \\ w_{h2} \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}; D_h = \begin{bmatrix} d_{h1} & 0 \\ 0 & d_{h2} \end{bmatrix} \quad (137)$$

The driver's ability to enforce a desired motion trajectory is defined by y_h , a command generated with a delay τ_2 ($\tau_2 \in \mathfrak{I}_2 = [\tau_{2min}, \tau_{2max}]$) that corresponds to the time constant of the driver. The model (135), (136) becomes

$$D^{0.7}x_h(t) = A_h x_h(t) + D_h w_h(t) \quad (138)$$

$$y_h(t - \tau_2) = c_h^T x_h(t - \tau_2) \quad (139)$$

where $x_h(t) = [x_h(t), D^{0.7}x_h(t)]^T$

$$A_h = \begin{bmatrix} 0 & 1 \\ -a_{h2} & -a_{h1} \end{bmatrix}; c_h = 10^3 \begin{bmatrix} 1.1 \\ 0 \end{bmatrix} \quad (140)$$

Remark that A_h is Hurwitz for $a_{h1}, a_{h2} \in \mathfrak{D}_h$, this is expected, as a vehicle driver must necessarily be a stable person. The data defining this model correspond to a driver with normal driving skills, meaning an equivalent response time constant of approximately $\mathfrak{I}_1 = [0.2, 0.8]$ s and a good visual observation quality defined by $d_{h1} = 0.1, d_{h2} = 0.015$. The driver's ability to enforce a desired motion trajectory is defined by $y_h(t - \tau_2)$, a command generated with a delay $\mathfrak{I}_2 = [0.2, 1]$ s that corresponds to the time constant of the driver.

An electric vehicle with the front-wheel-steering control is considered [28].

$$J_s \ddot{\theta}(t) = -c_d \dot{\theta}(t) - c_s \theta(t) + b_s u(t) \quad (141)$$

where θ is the average angle of the two front wheel angles, inertial component $J_s = 0.008 \text{ kgm}^2$, c_s is the stiffness of steering system ($c_s = 2100 \text{ Nm/rad}$), damping $c_d = 0.136 \frac{\text{Nm s}}{\text{rad}}$, b_s is a

coefficient relative to steering ratio [28] and u is the driving torque. The model of (141) can be rewritten as $(\theta(t) = \theta_1(t))$

$$\dot{x}_\theta(t) = A_\theta x_\theta(t) + b_\theta u(t) \quad (142)$$

$$y(t) = c_\theta^T x_\theta(t) \quad (143)$$

$$A_\theta = \begin{bmatrix} 0 & 1 \\ -c_s/J_s & -c_d/J_s \end{bmatrix}; b_\theta = \begin{bmatrix} 0 \\ b_s/J_s \end{bmatrix}; c_\theta = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad (144)$$

where x_θ is the state vector, $x_\theta = [\theta_1 \ \theta_2]^T$, the system parameters are: $b_s = 10^{-3}$, $c_1 = 8$, $c_2 = 0.27$ and u is determined by two components

$$u(t) = y_h(t) - u_c(t) \quad (145)$$

The first component is imposed by human operator, it determines how the operator's state dictates the control command u applied to the vehicle and the second component represents the internal control that ensures the desired performance

$$u_c(t) = k_\theta y(t) \quad (146)$$

For $k_\theta = 8$, the matrix A_θ becomes Hurwitz and (142)–(144) can be rewritten as

$$\dot{x}_\theta(t) = A_\theta^H x_\theta(t) + b_\theta c_h^T x_h(t - \tau_2) \quad (147)$$

Consider the following Lyapunov generalized function for the human–machine system:

$$V(x) = x_h^T P_h x_h + x_\theta^T P_\theta x_\theta \quad (148)$$

By following the same procedure as described in the preceding examples, for the human model we chose and infer $k_h = 8$, $\mu_h = 3$, $\mu_h^* = 4$. Considering the controllability and observability of the pairs, (A_h^H, b_h) , (A_h^H, c_h) respectively, and solving in MATLAB the systems

$$A_h^{HT} P_h + P_h A_h^H = -q_h q_h^T - \delta_1 I \quad (149)$$

$$-P_h \mu_h^* \mu_h b_h - \frac{c_h}{2} = q_h \quad (150)$$

for $\delta_1 = 57$, yields

$$P_h = \begin{bmatrix} 23.19 & 0.491 \\ 0.491 & 0.491 \end{bmatrix}; q_h = \begin{bmatrix} -4.53 \\ -15.8 \end{bmatrix}; \lambda_{P_1} = \begin{bmatrix} 23.18 \\ 0.495 \end{bmatrix}; \alpha_1 = 1; \quad (151)$$

For the system (142), (143) ($\gamma = 1$), the lemma YKP's equations become Lyapunov equation

$$A_\theta^{HT} P_\theta + P_\theta A_\theta^H = -q_\theta q_\theta^T - \delta_2 I \quad (152)$$

For $\delta_2 = 47$ yields

$$P_\theta = \begin{bmatrix} 0.768 & 0.766 \\ 0.766 & 0.766 \end{bmatrix}; q_\theta = \begin{bmatrix} -34.46 \\ -12.51 \end{bmatrix}; \lambda_{P_1} = \begin{bmatrix} 0.783 \\ 0.783 \end{bmatrix}; \quad (153)$$

The conditions (75) and (76) of Theorem 4 become

$$\eta_{11} = \frac{\delta_1}{\lambda_{\max}(P_h)} - \frac{1}{\lambda_{\min}(P_h)} = 0.469; \eta_{21} = \tau_{1\max}^2 \frac{\lambda_{\max}(P_\theta^2) \|(b_\theta c_h^T)(b_\theta c_h^T)^T\|}{\lambda_{\min}(P_h)} = 0.198 \quad (154)$$

$$\eta_{21} = \frac{\delta_2}{\lambda_{\max}(P_\theta)} - \frac{1}{\lambda_{\min}(P_\theta)} = 45.69; \eta_{21} = \tau_{2\max}^2 \frac{\lambda_{\max}(P_h^2) \|D_h D_h^T\|}{\lambda_{\min}(P_\theta)} = 0.213 \quad (155)$$

$$\eta_1^* = 0.469; \eta_2^* = 0.213 \quad (156)$$

$$D^\gamma V(t) \leq -\eta_1^* V(t) + \eta_2^* V(t - \tau) \quad (157)$$

$$\eta_1^* > \eta_2^* \quad (158)$$

According to Remark 1 (Section 2.2.1) the frequency condition of the system (140) and (141), ($\gamma = 1$), is always satisfied. For the human model (138) and (139) yields, (Figure 8)

$$\operatorname{Re}\left(c_h^T (j\omega I - A_h^H)^{-1} b_h\right) \leq 3.3 \quad (159)$$

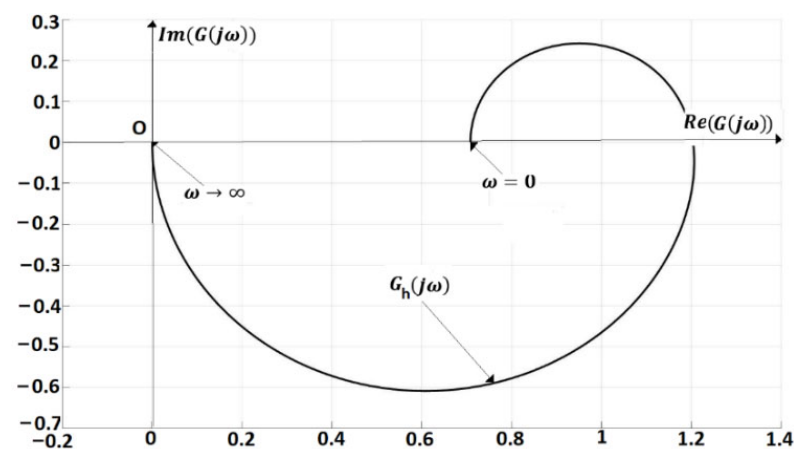


Figure 8. $G_h(j\omega)$ polar plot.

Since all conditions of Theorem 4 are verified, allow us to conclude that system (136), (137), (140), and (141) is asymptotically stable. The trajectories of the system for a desired control position $\theta^d = \pi/6$ and for the delays $\tau_1 = 0.2$ s, $\tau_2 = 0.3$ s are shown in Figure 9.

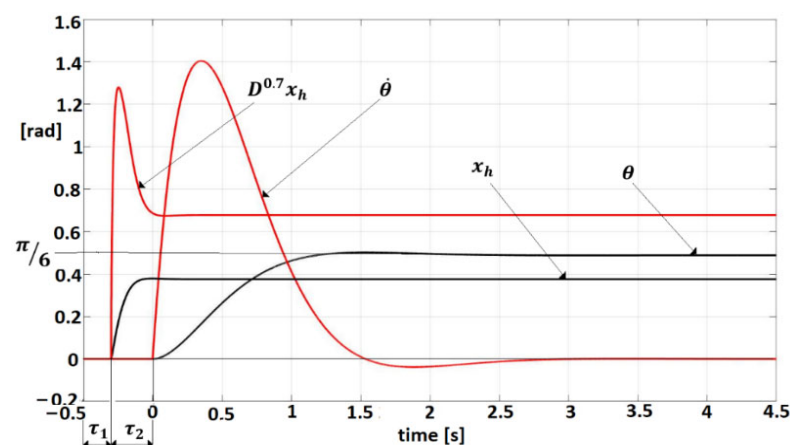


Figure 9. Trajectories of the human-machine system.

The driver needs 0.2 s to estimate the position and 0.3 s to generate new control. The asymptotic stability of the system is confirmed by the evolution of the trajectories presented in Figure 9.

4. Discussion

As discussed in Section 1, papers addressing the complex field of multi-fractional-order systems control with multiple delays are relatively scarce. Our theoretical results are based on papers [4–7], which offer algebraic criteria for these classes of systems. To the best of the authors' knowledge, none of these papers analyze these systems from a control perspective. For instance, in [4], the following model is studied:

$$\frac{d^\gamma x}{dt^\gamma} = y \quad (160)$$

$$\frac{d^\gamma y}{dt^\gamma} = -k_1 y + k_2 x(t - \tau) - y(t - \tau)^3 + B \cos t \quad (161)$$

The stability is studied by Laplace transform and by its characteristic equation

$$\begin{vmatrix} s^\gamma + k_1 & -1 \\ -e^{-\tau s} & s^\gamma + k_1 + k_2 \end{vmatrix} = 0 \quad (162)$$

Furthermore, these papers study positive fractional models, which restricts their scope of application. In [3] the following system is analyzed:

$$D^\alpha x(t) = Ax(t) + A_d x(t - \tau) + dw(t) \quad (163)$$

$$A = \begin{bmatrix} -2 & 0.6 & 0.3 \\ 0.3 & -3 & 1 \\ 0.2 & 0.5 & -4 \end{bmatrix}; A_d = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.1 \end{bmatrix} \quad (164)$$

where A is Metzler and $(A + A_d)$ is Hurwitz. The system is asymptotically stable ($w(t) = 1$). In [8] is studied the model is studied

$$D^{\alpha_i} x_i(t) = -c_i + f_i(x_i, x_i(t - \tau_i), \dots) \quad (165)$$

where $c_i \in R$, f_i are Lipschitz functions. The stability is verified by $(\Lambda + \sum A_d)$ in Lyapunov vector function

$$D^\alpha V \preceq \Lambda V + \sum A_d V(t - \tau) \quad (166)$$

In general, the analysis method in these papers is based on the condition that the matrix A in $D^\alpha x(t) = Ax(t)$ is a Metzler matrix. In physical processes, this condition is not always satisfied. For instance, in the human-machine system model from the previous section, the vehicle model is defined by:

$$A_\theta = \begin{bmatrix} 0 & 1 \\ -c_s/J_s & -c_d/J_s \end{bmatrix} \quad (167)$$

which is obviously not Metzler.

Our paper investigates the problem of establishing criteria that ensure, through the control system, the asymptotic stability for models defined by equations with multiple fractional exponents and multiple delays. In this regard, our paper extends these results to models derived from physical processes and proposes algebraic criteria associated with frequency-domain criteria.

5. Conclusions

The present work attempts to introduce several stability evaluation criteria for multiple fractional-order systems with multiple time delays. Furthermore, the presented

methods fall within the scope of techniques that serve as investigation and design tools by expressing these criteria in the form of polar plots, which are familiar in the design of automated control systems. These new stability criteria are presented as frequency criteria and represented graphically by polar plots similar to familiar Nyquist and Popov plots and provide advantages in control systems design. In Theorems 2–4 presented in our paper, the proposed control laws ensure γ -fractional-order stability; the formulated criteria encompass both simple algebraic criteria and frequency-domain representations via polar plots. It is worth noting that the critical point of these polar plots is not determined by nonlinearity or control coefficients—as is the case with Nyquist or Popov polar plots—but rather by the fractional exponent γ . We consider that, in many applications, these polar plots enable a much easier evaluation of system stability and can serve as a useful tool in control systems design. Throughout the paper, whenever possible, we have demonstrated the simplicity of the proposed methods and criteria in comparison to similar studies. We assume that establishing frequency-domain stability criteria and their associated graphical representations, similar to those obtained through the classical Nyquist and Popov criteria, constitutes an efficient tool for both the analysis and design of control systems for these classes of systems described by multiple fractional-order differential equations and multiple time delays.

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Appendix A

Proof of Theorem 1. Consider the generalized Lyapunov function $V(t)$ of MDFOS that satisfies (8)–(10).

Taking (7) and (8) into account, inequality (10) can be rewritten as

$$\begin{bmatrix} D^{\gamma_1} V_1(t) \\ D^{\gamma_2} V_2(t) \\ \vdots \\ D^{\gamma_N} V_N(t) \end{bmatrix} \leq -\eta_1 \begin{bmatrix} V_1(t) \\ V_2(t) \\ \vdots \\ V_N(t) \end{bmatrix} + \eta_2 \begin{bmatrix} V_1(t-\tau) \\ V_2(t-\tau) \\ \vdots \\ V_N(t-\tau) \end{bmatrix} \quad (\text{A1})$$

or

$$\begin{bmatrix} D^{\gamma_1} V_1(t) \\ D^{\gamma_2} V_2(t) \\ \vdots \\ D^{\gamma_N} V_N(t) \end{bmatrix} \leq -\mathbf{M}_1 \begin{bmatrix} V_1(t) \\ V_2(t) \\ \vdots \\ V_N(t) \end{bmatrix} + \mathbf{M}_2 \begin{bmatrix} V_1(t-\tau) \\ V_2(t-\tau) \\ \vdots \\ V_N(t-\tau) \end{bmatrix} \quad (\text{A2})$$

where $\mathbf{M}_1, \mathbf{M}_2 \in \mathbb{R}^{N \times N}$

$$\mathbf{M}_i = \eta_i \mathbf{I}_N, i = 1, 2 \quad (\text{A3})$$

The system (A2) is a fractional-order positive system where $-M_1$ is Metzler matrix and $(-M_1 + M_2)$ is Hurwitz matrix. According to ([8], Theorem 2), the system (MDFOS) is asymptotically stable independent of the magnitude of delays. $\square$

Appendix B

Consider the system

$$D^{\gamma_i} x(t) = Ax(t) \quad (A4)$$

Let $V(x) = x^T Px$ be the Lyapunov function.

The Caputo fractional derivative of $V(x)$ is [5,8–10]

$$D^{\gamma_i} (x^T Px) \leq 2x^T P(D^{\gamma_i} x) = 2x^T PAx \quad (A5)$$

and, using properties of the transpose, which yields:

$$x^T PAx = x^T A^T Px \quad (A6)$$

Consequently, (A5) can be rewritten as

$$D^{\gamma_i} (x^T Px) \leq x^T PAx + x^T A^T Px = x^T (PA + A^T P)x \quad (A7)$$

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