

Article

Exponent-Incidence Constraints for Tensor Eigenvectors of Multi-Hypergraphs

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Abstract

We study support-determined constraints for eigenvectors of nonnegative symmetric tensors whose support may contain repeated indices. Such tensors are naturally encoded by uniform multi-hypergraphs, where each multiedge is represented by an exponent vector $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$. Replacing the ordinary vertex-edge incidence matrix by the exponent-incidence matrix, we show that every nonzero-eigenvalue H -eigenvector satisfies linear incidence constraints in the transformed coordinates $y_i = x_i^k$. These constraints are invariant under positive scalar edge weights and reduce to the usual support-incidence constraints for ordinary squarefree hypergraphs. When the exponent-incidence matrix has a nontrivial left kernel, these relations provide nonzero support-level constraints in the transformed coordinates. We also describe the positive branch of the resulting constraint variety and prove a positive-weight realization criterion: a positive vector x can be realized as an eigenvector of some strictly positive edge-weighting of a fixed multi-hypergraph if and only if $x^{[k]}$ lies in the strictly positive coefficient cone generated by the exponent-incidence columns. Thus, the exponent-incidence constraint variety provides necessary algebraic constraints in general, while the strictly positive coefficient cone gives the exact positive-weight realization region on the positive branch.

Keywords: spectral hypergraph theory; tensor eigenvalues; adjacency tensor; nonlinear eigenvalue problems; hypergraphs; multi-hypergraphs

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1. Introduction

Eigenvectors of nonnegative tensors associated with uniform hypergraphs are a central object in spectral hypergraph theory. If $H = (V, E)$ is a k -uniform hypergraph, then its adjacency tensor is a symmetric nonnegative tensor of order k . For $m \in \mathbb{N}$, write

$$x^{[m]} = (x_1^m, \dots, x_n^m).$$

Throughout the paper, we use $\mathbb{N} = \{0, 1, 2, \dots\}$. An H -eigenpair of a real symmetric order- k tensor $\mathcal{A}$ is a pair

$$(\lambda, x) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$$

satisfying

$$\mathcal{A}x^{k-1} = \lambda x^{[k-1]}.$$

Tensor eigenvalue notions go back to the foundational work of Qi and Lim [1,2] and have since become standard in spectral hypergraph theory. Unlike the graph case $k = 2$, these equations are nonlinear polynomial equations, and even small supports can produce eigenvector varieties that are difficult to describe directly.

A useful way to obtain necessary constraints is to multiply the i -th eigenvalue equation by x_i and pass to transformed coordinates

$$y_i = x_i^k.$$

The adjacency tensor approach to uniform hypergraphs was developed in [3–5] and has since been used to study a range of spectral, structural, and computational problems for hypergraphs; see, for example, [6–9]. For the purposes of the present paper, the most relevant point is the incidence-based constraint method introduced in [10,11]. For squarefree k -uniform hypergraphs, the multiplied eigenvalue equations factor through the vertex-edge incidence matrix. Consequently, every eigenvector with nonzero eigenvalue must satisfy the linear relations in the left kernel of the incidence matrix after the substitution $y_i = x_i^k$.

The purpose of this paper is to extend this incidence-constraint framework from squarefree hypergraphs to multi-hypergraphs. In a multi-hypergraph, an edge may contain a vertex with multiplicity. Thus a k -uniform multiedge is naturally represented by an exponent vector

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| = \alpha_1 + \dots + \alpha_n = k.$$

The ordinary 0/1 incidence matrix is then replaced by the *exponent-incidence matrix*, whose column for the multiedge α is the vector α itself. This replacement is essential: for example, the multiedge 1, 1, 2 has exponent vector

$$(2, 1, 0, \dots, 0),$$

not the squarefree support vector $(1, 1, 0, \dots, 0)$.

The main result shows that the same support-level mechanism survives in this more general setting. After the substitution $y_i = x_i^k$, every nonzero-eigenvalue H -eigenvector satisfies the linear relations determined by the left kernel of the exponent-incidence matrix. Thus the constraint space depends only on the multi-hypergraph support and not on the positive edge weights. We also prove a positive-weight realization criterion: a positive vector can occur as an eigenvector for some strictly positive weighting of a fixed support precisely when its k -th power vector lies in the strictly positive coefficient cone generated by the exponent-incidence columns.

In tensor-based spectral clustering, cluster indicators or data rankings are often derived from a dominant nonnegative eigenvector of a nonnegative adjacency tensor [12]. The existence and positivity properties of such Perron eigenvectors are governed by Perron–Frobenius theory for nonnegative tensors [13], while their computation remains difficult because tensor eigenvalue equations are nonlinear polynomial systems [1,2]. When $\ker(M_{\mathcal{H}}^T) \neq 0$, these relations give nontrivial linear constraints in the transformed variables and may help reduce or parametrize the search space on the positive branch.

2. Exponent-Incidence Constraints

We recall the terminology of multi-hypergraphs as outlined in [10]. Let

$$V = \{1, \dots, n\}.$$

A k -uniform multi-hypergraph on V will be represented by a finite indexed set of multi-edges E . Each multiedge $e \in E$ has an associated exponent vector

$$\alpha(e) = (\alpha_1(e), \dots, \alpha_n(e)) \in \mathbb{N}^n, \quad |\alpha(e)| = k.$$

Here $\alpha_i(e)$ records the multiplicity with which the vertex i appears in the multiedge e . Ordinary k -uniform hypergraphs are the special case in which each $\alpha_i(e) \in \{0, 1\}$.

Throughout, the support of a multi-hypergraph means its collection of exponent vectors $\alpha(e)$; when E is indexed, repeated exponent vectors are allowed.

Definition 1 (Exponent-incidence matrix). *Let $\mathcal{H} = (V, E)$ be a k -uniform multi-hypergraph. The exponent-incidence matrix of $\mathcal{H}$ is the matrix*

$$M_{\mathcal{H}} = (M_{i,e})_{i \in V, e \in E}$$

defined by

$$M_{i,e} = \alpha_i(e).$$

Thus the e -th column of $M_{\mathcal{H}}$ is the exponent vector $\alpha(e)$.

Definition 2 (Exponent-incidence relation space). *The global exponent-incidence relation space of $\mathcal{H}$ is*

$$\mathcal{P}(\mathcal{H}) = \ker(M_{\mathcal{H}}^T) \subseteq \mathbb{R}^V.$$

Equivalently, $\ell = (\ell_1, \dots, \ell_n) \in \mathcal{P}(\mathcal{H})$ if and only if

$$\sum_{i=1}^n \alpha_i(e) \ell_i = 0 \quad \text{for every } e \in E.$$

The associated constraint ideal (see [11] for notation) is obtained by pulling these linear relations back under the coordinatewise k -th power map.

Definition 3 (Exponent-incidence constraint ideal). *The exponent-incidence constraint ideal of $\mathcal{H}$ is*

$$I_{\mathcal{P}} = \left\langle \ell_1 x_1^k + \dots + \ell_n x_n^k : \ell = (\ell_1, \dots, \ell_n) \in \mathcal{P}(\mathcal{H}) \right\rangle \subseteq \mathbb{R}[x_1, \dots, x_n].$$

For an ordinary squarefree hypergraph, $M_{\mathcal{H}}$ is the usual vertex-edge incidence matrix B . Hence $\mathcal{P}(\mathcal{H}) = \ker(B^T)$. If $\mathcal{L}(H)$ denotes a circuit-generated Pearson constraint space associated to an ordinary hypergraph H , then each circuit relation extends by zero to an element of $\ker(B^T)$. Thus

$$\mathcal{L}(H) \subseteq \mathcal{P}(H) = \ker(B^T).$$

The exponent-incidence relation space is therefore a natural global incidence version of the Pearson constraint space, and it is the form that extends most directly to multiedges.

Proposition 1 (Incidence constraints for eigenvectors). *Let $\mathcal{H} = (V, E)$ be a k -uniform multi-hypergraph with exponent-incidence matrix $M = M_{\mathcal{H}}$. Suppose a tensor eigenvalue problem supported on $\mathcal{H}$ has the property that, after multiplying the i -th equation by x_i , it can be written as*

$$Mz(x) = \lambda y(x), \quad y_i(x) = x_i^k.$$

If x is an eigenvector with $\lambda \neq 0$, then every polynomial in $I_{\mathcal{P}}$ vanishes at x .

Proof. Let $\ell \in \mathcal{P}(\mathcal{H}) = \ker(M^T)$. Multiplying

$$Mz(x) = \lambda y(x)$$

on the left by ℓ^T , we obtain

$$\ell^T Mz(x) = \lambda \ell^T y(x).$$

Since $\ell^T M = 0$, the left-hand side is zero. Therefore

$$\lambda \ell^T y(x) = 0.$$

Because $\lambda \neq 0$, we have

$$\ell^T y(x) = 0.$$

Substituting $y_i(x) = x_i^k$, this becomes

$$\sum_{i=1}^n \ell_i x_i^k = 0.$$

Thus every generator of $I_{\mathcal{P}}$ vanishes at x , and hence every polynomial in $I_{\mathcal{P}}$ vanishes at x . $\square$

3. Weighted Multi-Hypergraph Tensors and Support Invariance

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, write

$$\alpha! = \alpha_1! \cdots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Let $\mathbf{e}_i$ denote the i -th standard basis vector.

Definition 4 (Weighted multi-hypergraph adjacency tensor). Let $\mathcal{H} = (V, E, w)$ be a positive edge-weighted k -uniform multi-hypergraph, with weights $w_e > 0$. The associated symmetric order- k , dimension- n tensor $\mathcal{A}(\mathcal{H}, w) = (a_{i_1 \dots i_k})$ is defined as the sum of the edge tensors determined by the multiedges. More explicitly, if the ordered tuple $(i_1, \dots, i_k)$ has multiplicity vector $\beta \in \mathbb{N}^n$, then

$$a_{i_1 \dots i_k} = \sum_{e \in E: \alpha(e) = \beta} \frac{w_e \alpha(e)!}{(k-1)!}.$$

If no edge e has $\alpha(e) = \beta$, then $a_{i_1 \dots i_k} = 0$.

This normalization extends the standard squarefree hypergraph normalization. Indeed, if e is squarefree, then $\alpha(e)! = 1$, and the nonzero tensor entries contributed by e are $w_e / (k-1)!$.

Proposition 2 (Weighted multiedge eigenvalue equations). Let $\mathcal{A}(\mathcal{H}, w)$ be the weighted multi-hypergraph adjacency tensor above. Then the tensor eigenvalue equation

$$\mathcal{A}(\mathcal{H}, w)x^{k-1} = \lambda x^{[k-1]}$$

is equivalent to

$$\sum_{e \in E} w_e \alpha_i(e) x^{\alpha(e) - \mathbf{e}_i} = \lambda x_i^{k-1}, \quad i = 1, \dots, n,$$

where terms with $\alpha_i(e) = 0$ contribute zero.

Proof. Fix i and a multiedge e with $\alpha_i(e) > 0$. In the contraction $(\mathcal{A}x^{k-1})_i$, the remaining $k-1$ indices must have multiplicity vector $\alpha(e) - \mathbf{e}_i$. The number of such ordered tuples is

$$\frac{(k-1)!}{(\alpha_i(e)-1)! \prod_{j \neq i} \alpha_j(e)!}.$$

The tensor entry contributed by e is

$$\frac{w_e \alpha(e)!}{(k-1)!}.$$

Multiplying these two factors gives

$$\frac{w_e \alpha(e)!}{(k-1)!} \cdot \frac{(k-1)!}{(\alpha_i(e)-1)! \prod_{j \neq i} \alpha_j(e)!} = w_e \alpha_i(e).$$

Thus the contribution of e to the i -th equation is

$$w_e \alpha_i(e) x^{\alpha(e) - \mathbf{e}_i}.$$

Summing over all edges gives the stated equation. $\square$

Multiplying the equation in the preceding proposition by x_i gives

$$\sum_{e \in E} w_e \alpha_i(e) x^{\alpha(e)} = \lambda x_i^k.$$

Equivalently, with

$$M = M_{\mathcal{H}}, \quad D = \text{diag}(w_e : e \in E), \quad z_e(x) = x^{\alpha(e)}, \quad y_i(x) = x_i^k,$$

the multiplied equations have the matrix form

$$MDz(x) = \lambda y(x).$$

Theorem 1 (Support invariance under positive edge weights). *Let $\mathcal{H} = (V, E, w)$ be a positive edge-weighted k -uniform multi-hypergraph with exponent-incidence matrix $M = M_{\mathcal{H}}$. If*

$$\mathcal{A}(\mathcal{H}, w) x^{k-1} = \lambda x^{[k-1]}$$

with $\lambda \neq 0$, then for every $\ell \in \ker(M^T)$,

$$\sum_{i=1}^n \ell_i x_i^k = 0.$$

Consequently, the exponent-incidence constraint space and the associated constraint ideal depend only on the multi-hypergraph support, not on the particular positive edge weights.

Proof. By Proposition 1, it is enough to observe that the multiplied eigenvalue equations have the form

$$MDz(x) = \lambda y(x), \quad z_e(x) = x^{\alpha(e)}, \quad y_i(x) = x_i^k.$$

Equivalently, this is

$$Mz'(x) = \lambda y(x), \quad z'(x) = Dz(x).$$

Thus for every $\ell \in \ker(M^T)$, multiplication on the left by ℓ^T gives

$$0 = \ell^T Mz'(x) = \lambda \ell^T y(x).$$

Since $\lambda \neq 0$, we have $\ell^T y(x) = 0$, which is precisely

$$\sum_{i=1}^n \ell_i x_i^k = 0.$$

The relation space $\ker(M^T)$ depends only on the exponent-incidence matrix, hence only on the multi-hypergraph support. $\square$

Corollary 1 (Ordinary hypergraphs). *Let $H = (V, E, w)$ be a positive edge-weighted ordinary k -uniform hypergraph, and let B be its usual vertex-edge incidence matrix. If x is a nonzero-eigenvalue H -eigenvector, then every $\ell \in \ker(B^T)$ gives the constraint*

$$\sum_{i=1}^n \ell_i x_i^k = 0.$$

In particular, every circuit-generated Pearson relation $\ell \in \mathcal{L}(H) \subseteq \ker(B^T)$ gives such a constraint.

Proof. For an ordinary hypergraph, every exponent vector is squarefree, so $M_{\mathcal{H}} = B$. The conclusion follows immediately from Theorem 1. The final statement follows from the inclusion $\mathcal{L}(H) \subseteq \ker(B^T)$. $\square$

Corollary 2 (Symmetric tensor support determines exponent-incidence constraints). *Let $\mathcal{A} = (a_{i_1 \dots i_k})$ be a symmetric nonnegative order- k tensor. Let $\mathcal{H}_{\mathcal{A}}$ be the k -uniform multi-hypergraph whose edges are the exponent vectors $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ for which the corresponding symmetric support entries of $\mathcal{A}$ are positive. If*

$$\mathcal{A}x^{k-1} = \lambda x^{[k-1]}$$

with $\lambda \neq 0$, then for every $\ell \in \ker(M_{\mathcal{H}_{\mathcal{A}}}^T)$,

$$\sum_{i=1}^n \ell_i x_i^k = 0.$$

Proof. For each exponent vector α in the positive support of $\mathcal{A}$, let c_{α} be the common tensor entry on ordered tuples with multiplicity vector α . Since $\mathcal{A}$ is symmetric, this is well-defined; since α belongs to the positive support, $c_{\alpha} > 0$. Define

$$w_{\alpha} = \frac{(k-1)!}{\alpha!} c_{\alpha}.$$

Then $\mathcal{A}$ is the weighted adjacency tensor of the positive edge-weighted multi-hypergraph whose multiedges are the exponent vectors α and whose weights are w_{α} . The result follows from Theorem 1. $\square$

4. The Positive Branch Relevant to Perron Eigenvectors

In tensor-based spectral clustering, cluster indicators or data rankings are often derived from a dominant nonnegative eigenvector of a nonnegative adjacency tensor [12]. Finding these eigenvectors is notoriously difficult because tensor eigenvalue equations are highly nonlinear [1,2,13]. In many tensor-based spectral clustering settings, one is interested in a dominant nonnegative eigenvector, and under suitable irreducibility hypotheses this Perron vector is strictly positive and unique up to scaling.

The following shows that, on the positive branch, the nonlinear eigenvalue equations imply linear constraints after the coordinate transformation $y_i = x_i^k$. These constraints are support-level necessary conditions. They can reduce or parametrize the search space only when $\ker(M_{\mathcal{H}}^T)$ is nontrivial; if the exponent-incidence matrix has trivial left kernel, the incidence construction imposes no nonzero linear relations in the transformed coordinates.

Let

$$\phi_k : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \phi_k(x) = (x_1^k, \dots, x_n^k).$$

The exponent-incidence constraint variety is

$$V(I_{\mathcal{P}}) = \left\{ x \in \mathbb{R}^n : \ell_1 x_1^k + \dots + \ell_n x_n^k = 0 \text{ for all } \ell \in \mathcal{P}(\mathcal{H}) \right\}.$$

Since $\mathcal{P}(\mathcal{H}) = \ker(M_{\mathcal{H}}^T)$, we have

$$\mathcal{P}(\mathcal{H})^\perp = \text{im}(M_{\mathcal{H}}).$$

Therefore

$$V(I_{\mathcal{P}}) = \phi_k^{-1}(\text{im}(M_{\mathcal{H}})).$$

Thus every nonzero-eigenvalue H -eigenvector lies in the pullback of the exponent-incidence column space.

We define

$$I_{\mathcal{P}, \mathbb{C}} := I_{\mathcal{P}} \otimes_{\mathbb{R}} \mathbb{C} \subseteq \mathbb{C}[x_1, \dots, x_n].$$

Over $\mathbb{C}$, the complexified ideal $I_{\mathcal{P}, \mathbb{C}}$ need not be prime. It is the pullback of a linear ideal under the coordinatewise power map, and this pullback can split into several components reflecting different k -th-root branches. For Perron eigenvectors, however, the relevant geometry is simpler. On the positive orthant,

$$\phi_k : \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}^n$$

is a bijection. Hence the positive constraint branch is

$$V_+(I_{\mathcal{P}}) = V(I_{\mathcal{P}}) \cap \mathbb{R}_{>0}^n = \phi_k^{-1}(\text{im}(M_{\mathcal{H}}) \cap \mathbb{R}_{>0}^n).$$

This positive branch is the natural support-determined algebraic relaxation for positive eigenvectors. No converse is asserted at this level: membership in $V_+(I_{\mathcal{P}})$ gives necessary incidence constraints for nonzero-eigenvalue positive eigenvectors, but it does not by itself guarantee that a positive vector is realized by strictly positive edge weights.

5. Realization by Positive Edge Weights

In semi-supervised clustering applications, a target or ideal cluster assignment vector x may be known beforehand based on prior domain knowledge or partial labels [14].

The theorem below gives an exact criterion for deciding whether such a vector can be realized by a strictly positive weighting of a fixed multi-hypergraph support. When the condition is met, the proof gives an explicit formula for strictly positive edge weights for which the prescribed vector is an eigenvector. Under additional irreducibility hypotheses ensuring Perron uniqueness, this realized eigenvector is the Perron eigenvector for the corresponding weighted tensor.

The incidence variety above gives necessary constraints. The following theorem establishes the corresponding exact positive-weight realization condition on the positive branch: a positive vector x can be realized as an eigenvector for some strictly positive

edge weighting if and only if $x^{[k]}$ lies in the strictly positive coefficient cone generated by the columns of the exponent-incidence matrix M . This distinction is important because the strictly positive coefficient cone is generally smaller than the positive part of the incidence span.

Theorem 2 (Positive edge-weight realization). *Let $\mathcal{H} = (V, E)$ be a k -uniform multi-hypergraph with exponent-incidence matrix $M = M_{\mathcal{H}}$. Let $x \in \mathbb{R}_{>0}^V$, and set*

$$y_i = x_i^k.$$

Then there exist positive edge weights $w_e > 0$ and a scalar $\lambda > 0$ such that

$$\sum_{e \in E} w_e \alpha_i(e) x^{\alpha(e) - \mathbf{e}_i} = \lambda x_i^{k-1}, \quad i = 1, \dots, n,$$

if and only if there exists $t = (t_e)_{e \in E} \in \mathbb{R}_{>0}^E$ such that

$$Mt = y.$$

Equivalently,

$$x^{[k]} \in \text{cone}_{>0}(M),$$

where

$$\text{cone}_{>0}(M) = \{Mt : t \in \mathbb{R}_{>0}^E\}$$

is the strictly positive coefficient cone corresponding to strictly positive weights on every support edge. This is to be distinguished from the closed nonnegative cone

$$\text{cone}_{\geq 0}(M) = \{Mt : t \in \mathbb{R}_{\geq 0}^E\},$$

which allows zero coefficients and hence corresponds to deleting some support edges.

Proof. Suppose first that positive weights w_e and $\lambda > 0$ exist. After multiplying the i -th equation by x_i , we obtain

$$\sum_{e \in E} w_e \alpha_i(e) x^{\alpha(e)} = \lambda x_i^k.$$

Define

$$t_e = \frac{w_e}{\lambda} x^{\alpha(e)}.$$

Since $x \in \mathbb{R}_{>0}^V$, $w_e > 0$, and $\lambda > 0$, we have $t_e > 0$. The multiplied equations become

$$Mt = y.$$

Hence $y \in \text{cone}_{>0}(M)$.

Conversely, suppose there exists $t \in \mathbb{R}_{>0}^E$ such that $Mt = y$. Choose any $\lambda > 0$, and define

$$w_e = \frac{\lambda t_e}{x^{\alpha(e)}}.$$

Then $w_e > 0$, and

$$w_e x^{\alpha(e)} = \lambda t_e.$$

Therefore, for each i ,

$$\sum_{e \in E} w_e \alpha_i(e) x^{\alpha(e)} = \lambda \sum_{e \in E} \alpha_i(e) t_e = \lambda y_i = \lambda x_i^k.$$

Dividing by $x_i > 0$, we get

$$\sum_{e \in E} w_e \alpha_i(e) x^{\alpha(e) - \mathbf{e}_i} = \lambda x_i^{k-1}.$$

Thus x is an eigenvector for the positive edge-weighted multi-hypergraph. $\square$

The cone $\text{cone}_{>0}(M)$ is generally not the same object as the closed polyhedral cone $\text{cone}_{\geq 0}(M)$. The former requires every coefficient in the prescribed support to be strictly positive, while the latter permits zero coefficients and therefore allows some support edges to disappear. Thus Theorem 2 is a realization criterion for a fixed positive support, not merely for a sub-support.

If $\mathcal{H}$ satisfies a Perron–Frobenius irreducibility hypothesis ensuring that a positive eigenvector is unique up to scale, then a positive vector realized by Theorem 2 is the Perron eigenvector for the corresponding weighted tensor. Without such an irreducibility hypothesis, the theorem should be interpreted only as a positive-eigenvector realization result.

6. Examples

Example 1 (A genuine multiedge example). Let $k = 3$, $V = \{1, 2, 3\}$, and let $\mathcal{H}$ have multiedges

$$e_1 = 112, \quad e_2 = 233.$$

Equivalently,

$$\alpha(e_1) = (2, 1, 0), \quad \alpha(e_2) = (0, 1, 2).$$

The exponent-incidence matrix is

$$M = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Solving $M^T \ell = 0$ gives

$$2\ell_1 + \ell_2 = 0, \quad \ell_2 + 2\ell_3 = 0.$$

Thus

$$\ker(M^T) = \text{span}\{(1, -2, 1)\}.$$

The exponent-incidence constraint ideal is

$$I_{\mathcal{P}} = \langle x_1^3 - 2x_2^3 + x_3^3 \rangle.$$

This example shows why the exponent-incidence matrix is necessary for multiedges: the multiedge 112 contributes the coefficient $2\ell_1 + \ell_2$, not $\ell_1 + \ell_2$.

Example 2 (A squarefree two-edge hypergraph). Let $k = 3$, $V = \{1, 2, 3, 4\}$, and $E = \{123, 124\}$. The exponent-incidence matrix is the usual incidence matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The image is

$$\text{im}(M) = \{(a + b, a + b, a, b) : a, b \in \mathbb{R}\}.$$

Hence the transformed coordinates $y_i = x_i^3$ satisfy

$$y_1 = y_2, \quad y_1 = y_3 + y_4.$$

Equivalently,

$$\ker(M^T) = \text{span}\{(1, -1, 0, 0), (1, 0, -1, -1)\}.$$

Thus

$$I_{\mathcal{P}} = \langle x_1^3 - x_2^3, x_1^3 - x_3^3 - x_4^3 \rangle.$$

On the positive branch this becomes

$$x_1 = x_2, \quad x_1^3 = x_3^3 + x_4^3, \quad x_i > 0.$$

Example 3 (A strict cone-versus-span example). Let $k = 3$, $V = \{1, 2, 3, 4, 5\}$, and

$$E = \{123, 145, 125\}.$$

The exponent-incidence matrix is

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

with columns

$$b_1 = (1, 1, 1, 0, 0)^T, \quad b_2 = (1, 0, 0, 1, 1)^T, \quad b_3 = (1, 1, 0, 0, 1)^T.$$

Consider

$$y = \left(\frac{3}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}\right)^T.$$

Then

$$y = b_1 + b_2 - \frac{1}{2}b_3,$$

so $y \in \text{im}(M) \cap \mathbb{R}_{>0}^5$. Thus, if $x_i = y_i^{1/3}$, then x lies on the positive part of the exponent-incidence constraint variety

$$\phi_3^{-1}(\text{im}(M)).$$

However, $y \notin \text{cone}_{>0}(M)$. Indeed, if $Mt = y$, then the third and fourth coordinates force

$$t_1 = 1, \quad t_2 = 1.$$

The fifth coordinate then gives

$$t_2 + t_3 = \frac{1}{2},$$

so

$$t_3 = -\frac{1}{2}.$$

Thus the unique solution is

$$t = \left(1, 1, -\frac{1}{2}\right)^T,$$

which is not positive. Therefore, the positive part of the incidence span can be strictly larger than the positive incidence cone. Equivalently, the constraint variety is generally only an algebraic relaxation of the exact positive-weight realization region.

7. Conclusions

We have recast support-determined tensor eigenvector constraints in the language of exponent-incidence matrices. This replacement is necessary when one allows multiedges with repeated vertices: the correct column for a multiedge is its exponent vector, not merely its squarefree support. After multiplying the i -th H -eigenvalue equation by x_i , the transformed coordinates $y_i = x_i^k$ satisfy linear constraints determined by the left kernel of the exponent-incidence matrix.

The resulting constraint space is invariant under positive scalar edge weights. Thus, the associated constraint ideal depends only on the multi-hypergraph support. For ordinary squarefree hypergraphs this recovers the usual incidence constraints and contains the circuit-generated Pearson constraints as a subspace. For general symmetric nonnegative tensors, the same framework applies by encoding the positive support as exponent vectors.

For positive eigenvectors, the coordinatewise power map is injective on the positive orthant, so the positive branch of the constraint variety is the natural support-determined relaxation. The exact positive-weight realization region is smaller in general: it is the inverse image of the strictly positive coefficient cone generated by the exponent-incidence columns. The examples show that this cone condition can be strictly stronger than membership in the positive part of the incidence span.

This exponent-incidence viewpoint provides a unified framework for ordinary hypergraphs, multi-hypergraphs, and symmetric tensor supports with repeated indices. It also suggests further questions about how support geometry, strictly positive coefficient cones, and Perron–Frobenius theory interact for nonnegative tensor eigenvectors.

The results have several computational consequences. First, the exponent-incidence constraints provide support-level certificates for candidate tensor eigenvectors: any nonzero-eigenvalue eigenvector must satisfy the linear relations in $x^{[k]}$ -coordinates determined by $\ker(M_{\mathcal{H}}^T)$. Second, on the positive branch relevant to Perron-type eigenvectors, these constraints can reduce or parametrize the search space only when $\ker(M_{\mathcal{H}}^T)$ is non-trivial; otherwise, the exponent-incidence construction gives no nonzero linear relations in the transformed variables. Finally, the strictly positive coefficient cone criterion gives an inverse realization test: a positive vector x can occur as an eigenvector for some strictly positive weighting of a fixed multi-hypergraph support if and only if $x^{[k]}$ lies in the strictly positive coefficient cone generated by the exponent-incidence columns. Thus, the support determines necessary algebraic constraints in general and exact positive-weight realization obstructions on the positive branch.

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