

Article

Aperiodically Intermittent Control for Stochastic McKean–Vlasov Equations with Markovian Switching

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Abstract

To address the challenge of effectively stabilizing inherently unstable hybrid stochastic McKean–Vlasov equations (HMSDEs) while simultaneously minimizing control costs, this paper proposes a novel control strategy termed aperiodically intermittent control (AIC). Under the global Lipschitz condition, we first establish the existence and uniqueness theorem for the solutions to HMSDEs. Subsequently, we derive a generalized Itô formula for HMSDEs, based on which we construct a Lyapunov functional that explicitly incorporates both the law (distribution) of the solution and the underlying Markovian switching process. By employing the Lyapunov functional method, we rigorously construct AIC for the unstable HMSDEs and analyze the mean-square exponential stability of the controlled system. Furthermore, we demonstrate the applicability of the proposed AIC strategy through a mean-field stochastic Cohen–Grossberg–Hopfield neural network model. Finally, a numerical example is provided to illustrate the effectiveness and practical feasibility of the developed control approach.

Keywords: aperiodically intermittent control (AIC); stochastic McKean–Vlasov equations; Itô formula; Markov chain; stabilization

MSC: 60H10; 93D15; 60K35

1. Introduction

When studying a system, the stability should warrant more attention since the instability may lead to a state in which the system fails to operate continuously and reliably. Its key risks lie in disrupting normal operations, triggering wider problems, and leading to tangible or intangible losses [1–5]. The stabilization problem is one of the most important problems in control theory. This issue has attracted considerable scholarly attention from both the mathematical and control engineering communities, spurring the development of numerous stabilization strategies for dynamic systems. Among these approaches, feedback-based stabilization and intermittent control are widely regarded as the most prevalent and effective techniques. However, conventional feedback control requires continuous, real-time measurement of all system states and uninterrupted computation of control signals—conditions that often become impractical or prohibitively expensive when certain state variables are inaccessible or difficult to measure due to technological constraints. It is worth mentioning that even discrete-time observer-based feedback control requires the controller to remain active throughout the entire control horizon and still demands high-frequency, real-time feedback information, entailing significantly higher implementation costs and greater resource consumption. For example, wind power generation is always



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irregular; thus, requiring real-time high-frequency feedback information in a fixed duration for each wind time is usually unreasonable and impractical. Therefore, it is necessary to explore more general intermittent control technology to overcome these shortcomings. Aperiodically intermittent control (AIC), which means that the controller is active during the control intervals (or work time) and inactive during other time intervals (or the rest time) where the widths of control intervals and rest intervals are different, may satisfy these requirements.

In this paper, we consider an unstable McKean–Vlasov stochastic differential equation (MVSDE for short) with Markovian switching:

$$\begin{cases} dZ(t) = (b(Z(t), \mathcal{L}(Z(t)), r(t), t)dt + \iota(Z(t), \mathcal{L}(Z(t)), r(t), t)dW_t, & t \geq 0 \\ Z(0) = \xi \end{cases} \quad (1)$$

where $\xi \in \mathbb{R}^d$, $\mathcal{L}(Z(t))$ means the probability distribution of $Z(t)$, $r(t)$ is a Markov chain taking values in a finite space $\mathbb{S} = \{1, 2, \dots, N_0\}$, which will be defined in Section 2. $b : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$, $\iota : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}^+ \rightarrow \mathbb{R}^{d \times n}$. Stabilization problems of MVSDEs (1) without Markovian switching have been studied in Wu-Hu [6], Chen-Ren [7], and Shen et al. [8], where they establish the stabilization principle (Wu et al. [6]) via feedback control based on discrete-time state observation and aperiodically intermittent control, respectively, but neither of them investigates the case with Markovian switching. We all know that large particle systems with weak interactions are inevitably affected by common sudden environmental factors, such as stock prices in economics being influenced by government policies; therefore, large-particle systems driven by Markov chains with weak interactions are widely applied in fields such as optimal control, neural networks, machine learning, and physics; see [9–11] and the references therein. Inspired by these, the aim of this paper is to discuss how to design a periodically intermittent control to make the given unstable hybrid MVSDE system (1) become stable. The controlled system is shown as the form

$$\begin{cases} dZ(t) = (b(Z(t), \mathcal{L}(Z(t)), r(t), t) + U(Z(t), \mathcal{L}(Z(t)), r(t), t))dt \\ \quad + \iota(Z(t), \mathcal{L}(Z(t)), r(t), t)dW_t, & t \geq 0, \\ Z(0) = \xi, r(0) = r_0, \end{cases} \quad (2)$$

where $U(Z(t), \mathcal{L}(Z(t)), r(t), t)$ is an aperiodically intermittent control defined as follows:

$$U(Z(t), \mathcal{L}(Z(t)), r(t), t) = \begin{cases} -u(Z(t), \mathcal{L}(Z(t)), r(t), t), & t \in [t_k, s_k], \\ 0, & t \in (s_k, t_{k+1}], \end{cases} \quad k = 0, 1, 2, \dots, \quad (3)$$

where t_k, s_k are intermittent instants, $t_{k+1} - s_k$ is the rest width and $s_k - t_k$ is the work width. To the best of the authors' knowledge, it is the first time that periodically intermittent control has been used to stabilize a given unstable hybrid MVSDE. Let us outline the main points as follows:

- * In order to reduce the cost of continuous working time, the intermittent control strategy is used to stabilize the unstable MVSDE. It is the first study of stabilization for MVSDEs with Markovian switching via aperiodically intermittent control.
- * It is well known that Itô's formula plays a key role in the theory of stochastic analysis. We derive a multivariate version for processes depending on the solution's state, its distribution, and an underlying Markov chain without invoking an abstract probability space, enhancing applicability.

- * The long-time behavior, in the sense of mean square exponential stability for MVSDEs with Markovian switching is formulated via periodically intermittent control.

We end this part of the paper by describing how this article is organized. In Section 2, we introduce some necessary notations, definitions, and assumptions. In Section 3, we establish the Itô formula that jointly accounts for the system state, its law, and the Markovian switching signal—thereby fully capturing the tripartite dependence structure inherent in controlled regime-switching stochastic systems. In Section 4, we deal with the stabilization problem of the controlled HMVSDEs via aperiodically intermittent control. Finally, we present one example to explain the effectiveness of our proposed control strategy in Section 5.

2. Preliminaries

We will now define several notations and assumptions that will be employed throughout the following discussion. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^d$. Let A^T be the transpose of matrix $A \in \mathbb{R}^{d \times n}$ with $d, n \geq 1$, $\text{tr}(A)$ the trace of A and $|A| = \sqrt{\text{tr}(A^T A)}$ the trace norm of A . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be complete probability space equipped with the filtrations $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual condition. Let $\mathbb{E}$ be the expectation under $\mathbb{P}$. For a given random variable $Z : \Omega \rightarrow \mathbb{R}^d$, the mapping $\mathcal{L}(Z) : \Omega \ni \omega \mapsto \mathcal{L}(Z(\omega))$, is $\mathbb{P}$; it is almost surely well defined and is a function from $(\Omega, \mathcal{F}, \mathbb{P})$ into $\mathcal{P}^2(\mathbb{R}^d)$. $\mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ ($\mathbb{L}^p(\mathbb{R}^d)$ for short) denotes the family of $\mathbb{R}^d$ -valued random variables ξ with $\mathbb{E}|\xi|^p < \infty$.

Let $r(t)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N_0\}$. The generator of $r(t)$ is defined by $\Gamma = (\gamma_{ij})_{N_0 \times N_0}$ such that for a sufficiently small $\Delta > 0$,

$$\mathbb{P}(r(t + \Delta) = j | r(t) = i) = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j. \end{cases} \quad (4)$$

We say that γ_{ij} is transition rate from i to j satisfying $\gamma_{ij} \geq 0$ for $j \neq i$ and $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$ for all $i, j \in \mathbb{S}$. Moreover, we assume W_t and $r(t)$ are mutually independent.

Let $\mathcal{P}(\mathbb{R}^d)$ denote the space of all probability measures over $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ equipped with the weak topology, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ -field over $\mathbb{R}^d$, and let $\mathcal{P}^k(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^k \mu(dx) < \infty \right\}$ for any $k > 0$. For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and $q \geq 1$, the q -Wasserstein distance $W_q(\mu, \nu)$ is defined by

$$W_q(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^q \pi(dx, dy) \right)^{\frac{1}{q}} \right\},$$

where $\Pi(\mu, \nu)$ denotes all probability measurement couplings between μ and ν on $\mathbb{R}^d \times \mathbb{R}^d$.

Definition 1 ([12,13]). Let $h : \mathcal{P}^2(\mathbb{R}^d) \rightarrow \mathbb{R}$. It is said that $h \in C^1(\mathcal{P}^2(\mathbb{R}^d))$ if, there exists a function $\partial_\mu h : \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying the following properties:

1. *Measurability and integrability:* for every $\mu \in \mathcal{P}^2(\mathbb{R}^d)$, the map $x \mapsto \partial_\mu h(\mu)(x)$ is Borel-measurable, and $\int_{\mathbb{R}^d} |\partial_\mu h(\mu)(x)|^2 \mu(dx) < \infty$.
2. *Derivative property:* For any square-integrable random variables $X, Y \in L^2(\Omega; \mathbb{R}^d)$ defined on a rich enough probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mu = \mathcal{L}(X)$. Then, $h(\mathcal{L}(X + Y)) - h(\mu) = \mathbb{E}[\langle \partial_\mu h(\mu)(X), Y \rangle] + o(\|Y\|_{L^2})$, as $\|Y\|_{L^2} \rightarrow 0$.
3. *Continuity:* The mapping $(\mu, x) \mapsto \partial_\mu h(\mu)(x)$ is jointly continuous with respect to the product topology induced by the W_2 -Wasserstein metric on $\mathcal{P}^2(\mathbb{R}^d)$ and the Euclidean norm on $\mathbb{R}^d$. That is, for any sequences $\mu_n \rightarrow \mu \in W_2$ and $x_n \rightarrow x \in \mathbb{R}^d$, we have $\partial_{\mu_n} h(\mu_n)(x_n) \rightarrow \partial_\mu h(\mu)(x)$.

The function $\partial_\mu h$ is called the Wasserstein derivative (or L-derivative) of h . It is uniquely determined up to μ -null sets for each μ .

Definition 2 ([12,13]). It is said that $h \in C^2(\mathcal{P}^2(\mathbb{R}^d))$ if, for any $z \in \mathcal{P}^2(\mathbb{R}^d)$, $h \in C^1(\mathcal{P}^2(\mathbb{R}^d))$ and $\partial_\mu h(\mathbb{P}_\varepsilon)(\cdot)$ are differentiable, and its derivative $\partial_y \partial_\mu h : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{R}^+ \rightarrow \mathbb{R}^{d \times d}$ is continuous.

Definition 3 ([12,13]). Assume that $h : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times [0, +\infty) \rightarrow \mathbb{R}$, if $h(\cdot, \cdot, v)$ is in $C^{2,1}(\mathbb{R}^d \times [0, +\infty))$ for any z , and $h(s, y, \cdot) \in C^2(\mathcal{P}^2(\mathbb{R}^d))$ for every (s, y) , and all partial derivatives $\partial_1(y, v, s)$, $\partial_2(y, v, s)$, $\partial_2^2(y, v, s)$, $\partial_v(y, v, s)(x)$, $\partial_{\bar{v}, v}^2(y, v, s)(x)$ are jointly continuous with respect to (y, v, s, x) . Then, θ is said to be in the set $C^{2,2,1}(\mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{R}^+)$.

3. The Existence and Uniqueness and the Itô Formula

This section will state the existence and uniqueness of the solution of the controlled system (2) and establish the essential tool, Itô formula, for the asymptotic analysis. We impose the following condition.

Assumption 1. Assume that there exists a positive constant K such that for any $t_1, t_2 \in [0, T]$, $x, y \in \mathbb{R}^d$, $\mu, v \in \mathcal{P}^2(\mathbb{R}^d)$, and $i \in \mathbb{S}$,

$$|w(x, \mu, i, t_1) - w(y, v, i, t_2)|^2 \leq K(|x - y|^2 + \mathbb{W}_2^2(\mu, v)), \quad w = b, \iota, u.$$

Moreover, for $(v, i, t) \in \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{S} \times [0, +\infty)$, $w(0, v, i, t) = 0$.

The Lipschitz continuity of b , ι and u in Assumption 1 ensures the well-posedness of MVSDE (2). More details can see [3,14,15].

Definition 4. Define a function $\Theta : \mathbb{S} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Theta(i, z) = \sum_{j \in \mathbb{S}} (j - i) \mathbf{1}_{\mathcal{T}_{ij}}(z), \quad i \in \mathbb{S}, \quad z \in [0, \infty), \quad (5)$$

where $\{\mathcal{T}_{ij} : i, j \in \mathbb{S}\}$ is a sequence of intervals on the real line defined in the manner

$$\begin{aligned} \mathcal{T}_{12} &= [0, \gamma_{12}), \quad \mathcal{T}_{13} = [\gamma_{12}, \gamma_{12} + \gamma_{13}), \dots, \\ \mathcal{T}_{1N_0} &= \left[\sum_{j=2}^{N_0-1} \gamma_{1j}, q_1 \right), \quad \mathcal{T}_{21} = [q_1, q_1 + \gamma_{21}), \\ \mathcal{T}_{23} &= [q_1 + \gamma_{21}, q_1 + \gamma_{21} + \gamma_{23}), \end{aligned}$$

and so on, where $q_i = -\gamma_{ii}$. For convenience of notation, $\mathcal{T}_{ij} = \emptyset$ if $\gamma_{ij} = 0$, and $\mathcal{T}_{ii} = \emptyset$. Then, we have

$$dr(t) = \int_{\mathbb{R}} \Theta(r(t^-), z) p(dt, dz), \quad (6)$$

where $p(dt, dz)$ is a Poisson random measure with intensity $dt \times \tilde{m}(dz)$, and $\tilde{m}$ is the Lebesgue measure on $\mathbb{R}$.

Theorem 1. (Generalized Itô formula) Let $V(z, \mu, i, t)$ belong to $C^{2,2,1}(\mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}^+; \mathbb{R}^+)$. Then $V(Z(t), \mathcal{L}(Z(t)), r(t), t)$ is an Itô process of the form

$$V(Z(t), \mathcal{L}(Z(t)), r(t), t) = V(Z(0), \mathcal{L}(Z(0)), r(0), 0) + \int_0^t \left[L^\mu V(Z(s), \mathcal{L}(Z(s)), r(s), s) \right] ds + M(t), \quad (7)$$

where

$$\begin{aligned} & L^\mu V(Z(t), \mathcal{L}(Z(t)), r(t), t) \\ = & \partial_t V(Z(t), \mathcal{L}(Z(t)), r(t), t) + \partial_z V(Z(t), \mathcal{L}(Z(t)), r(t), t) b(Z(t), \mathcal{L}(Z(t)), r(t), t) \\ & + \int_{\mathbb{R}^d} \partial_\mu V(Z(t), \mathcal{L}(Z(t)), r(t), t) (y) b(y, \mathcal{L}(Z(t)), r(t), t) \mu_t(dy) \\ & + \frac{1}{2} \text{tr}(\partial_{zz} V(Z(t), \mathcal{L}(Z(t)), r(t), t) \iota(Z(t), \mathcal{L}(Z(t)), r(t), t) (\iota(Z(t), \mathcal{L}(Z(t)), r(t), t))^T) \\ & + \frac{1}{2} \int_{\mathbb{R}^d} \text{tr}(\partial_y \partial_\mu V(Z(t), \mathcal{L}(Z(t)), r(t), t) (y) \iota(y, \mathcal{L}(Z(t)), r(t), t) \iota^T(y, \mathcal{L}(Z(t)), r(t), t)) \mu_t(dy) \\ & + \sum_{j \in \mathbb{S}} \gamma_{r(t)j} V(Z(t), \mathcal{L}(Z(t)), j, t) \end{aligned} \quad (8)$$

and

$$\begin{aligned} M(t) = & \int_0^t \partial_z V(Z(s), \mathcal{L}(Z(s)), r(s), s) \iota(Z(s), \mathcal{L}(Z(s)), r(s), s) dW_s \\ & + \int_0^t \int_{\mathbb{R}} \left[V(Z(s), \mathcal{L}(Z(s)), r(0) + \Theta(r(s-), z), s) \right. \\ & \left. - V(Z(s), \mathcal{L}(Z(s)), r(s), s) \right] p(ds, dz). \end{aligned} \quad (9)$$

Because its proof is too long, we place it in the Appendix A so as to make the context more compact.

Definition 5. The system (1) is said to be p -th moment exponentially stable for any initial value $Z(0)$, if there exist positive constants α and λ such that

$$\mathbb{E} \|Z(t)\|^p \leq \lambda \|Z(0)\|^p e^{-\alpha(t-t_0)},$$

when $p = 2$, it is called exponentially stable in the mean square.

Theorem 2. Assume that Assumption 1 holds and there is a function $V(z, \mu, i, t) \in C^{2,1,1}(\mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}^+; \mathbb{R})$, and a_1, a_2, a_3, p are positive numbers such that all $t \geq 0$

$$\begin{aligned} a_1 \int_{\mathbb{R}^d} |z|^p \mu(dx) & \leq \int_{\mathbb{R}^d} V(z, \mu, i, t) \mu(dz), \\ \int_{\mathbb{R}^d} V(z, \mu, i, t) \mu(dz) & \leq a_2 \int_{\mathbb{R}^d} |z|^p \mu(dz), \\ \int_{\mathbb{R}^d} L^\mu V(z, \mu, i, t) \mu(dz) & \geq a_3 \int_{\mathbb{R}^d} |z|^p \mu(dz), \end{aligned} \quad (10)$$

then the trivial solution of (1) is p th-moment exponentially unstable; that is,

$$\mathbb{E} |Z(t)|^p \geq \frac{a_1}{a_2} |Z(0)|^p e^{a_3(t-t_0)}.$$

Proof. Without losing generality, we only prove the case of $t_0 = 0$. Define the stopping time $\tau_N = \inf\{t \geq t_0 : |Z(t)| \geq N\}$. Employing the Itô formula to $e^{-a_3(t \wedge \tau_N - t_0)} V(Z(t \wedge \tau_N), \mu(t \wedge \tau_N), r(t \wedge \tau_N), t \wedge \tau_N)$, we obtain

$$\begin{aligned} & \mathbb{E}[e^{-a_3(t \wedge \tau_N - t_0)} V(Z(t \wedge \tau_N), \mu(t \wedge \tau_N), r(t \wedge \tau_N), t \wedge \tau_N)] \\ &= \mathbb{E}V(Z(t_0), \mu(t_0), r(t_0), t_0) + \mathbb{E} \int_{t_0}^{t \wedge \tau_N} e^{-a_3(s-t_0)} [L^\mu V(Z(s), \mu(s), r(s), s) - a_3 V(Z(s), \mu(s), r(s), s)] ds. \end{aligned} \quad (11)$$

By condition (10), we get

$$\begin{aligned} a_1 \mathbb{E}|Z(t_0)|^p &\leq \mathbb{E}V(Z(t_0), \mu(t_0), r(t_0), t_0) \\ &\leq \mathbb{E}[e^{-a_3(t \wedge \tau_N - t_0)} V(Z(t \wedge \tau_N), \mu(t \wedge \tau_N), r(t \wedge \tau_N), t \wedge \tau_N)] \\ &\leq a_2 \mathbb{E}|Z(t \wedge \tau_N)|^p e^{-a_3(t \wedge \tau_N - t_0)}. \end{aligned}$$

Letting $N \rightarrow \infty$, it holds that

$$\mathbb{E}|Z(t \wedge \tau_N)|^p \geq \frac{a_1}{a_2} \mathbb{E}|Z(t_0)|^p e^{a_3(t-t_0)}.$$

□

4. The Main Results

The primary purpose of this section is to study the stabilization issue for system (1) using the aperiodically intermittent control. For our stabilization purpose related to the controlled system (2), we impose an extra condition.

Assumption 2. The lower bound of control intervals (i.e., $\inf_m (s_m - t_m)$) is a , which can be very small and the upper bound of control periods (i.e., $\sup_m (t_{m+1} - t_m)$) is S , potentially reaching a significant magnitude. Additionally, $a \leq S$.

To illustrate the mechanism of aperiodically intermittent control, Figure 1 is presented. The average control rate is given by $(s_k - t_k + s_{k+1} - t_{k+1} + s_{k+2} - t_{k+2}) / (t_{k+3} - t_k)$, providing a comprehensive understanding of the control dynamics.

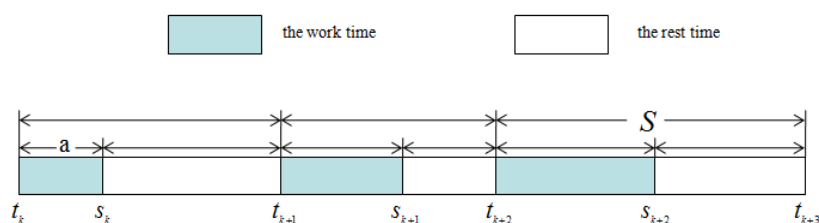


Figure 1. Simple aperiodically intermittent control.

Theorem 3. Suppose that the width $t_{k+1} - s_k$ for all $k = 0, 1, 2, \dots$ is bounded. And suppose we have function $V : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and positive constants c_1, b_1, b_2, γ, M such that

$$c_1 \int_{\mathbb{R}^d} |z|^2 \mu(dz) \leq \int_{\mathbb{R}^d} V(z, \mu, i, t) \mu(dz), \quad \forall (z, \mu, i, t) \in \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}^+; \quad (12)$$

$$\int_{\mathbb{R}^d} L^\mu V(z, \mu, i, t) \mu(dz) \leq -b_1 \int_{\mathbb{R}^d} V(z, \mu, i, t) \mu(dz), \quad t \in (t_k, s_k] \quad (13)$$

and

$$\int_{\mathbb{R}^d} L^\mu V(z, \mu, i, t) \mu(dz) \leq b_2 \int_{\mathbb{R}^d} V(z, \mu, i, t) \mu(dz), \quad t \in (s_k, t_{k+1}]. \quad (14)$$

If

$$e^{(\lambda+b_2)\sum_{k=0}^i(t_{k+1}-s_k)} < e^{(\lambda-\gamma)\sum_{k=0}^{i-1}(t_{k+1}-t_k)}, \quad i = 1, 2, \dots, \quad (15)$$

then $\mathbb{E}|Z(t)|^2 < Me^{-\gamma t}$, where $\lambda > \gamma > 0$ and $\lambda = b_1$. That is, the trivial solution of the controlled system (2) is exponentially stable through the aperiodically intermittent control design.

Proof. By employing the generalized Itô formula to $e^{\lambda t}V(Z(t), \mathcal{L}(Z(t)), r(t), t)$, we get

$$\begin{aligned} d(e^{\lambda t}V(Z(t), \mathcal{L}(Z(t)), r(t), t)) &= e^{\lambda t}[\lambda V(Z(t), \mathcal{L}(Z(t)), r(t), t) + V_t(Z(t), \mathcal{L}(Z(t)), r(t), t) \\ &+ V_z(Z(t), \mathcal{L}(Z(t)), r(t), t)[b(Z(t), \mathcal{L}(Z(t)), r(t), t) + U(Z(t), \mathcal{L}(Z(t)), r(t), t))] \\ &+ \frac{1}{2}e^{\lambda t}\text{tr}\left(V_{zz}(Z(t), \mathcal{L}(Z(t)), r(t), t)u^T(Z(t), \mathcal{L}(Z(t)), r(t), t)\right)dt \\ &+ \frac{1}{2}\int_{\mathbb{R}^d}\text{tr}(\partial_y\partial_\mu V(Z(t), \mathcal{L}(Z(t)), r(t), t)(y)\iota(y, \mathcal{L}(Z(t)), r(t), t)^T(y, \mathcal{L}(Z(t)), r(t), t))\mu_t(dy) \\ &+ e^{\lambda t}V_z(Z(t), \mathcal{L}(Z(t)), r(t), t)\iota(Z(t), \mathcal{L}(Z(t)), r(t), t)dW_t \\ &+ \int_{\mathbb{R}}\left[V(Z(t), \mathcal{L}(Z(t)), r(0) + \Theta(r(t-), z), t) - V(Z(t), \mathcal{L}(Z(t)), r(t), t)\right]p(dt, dz). \end{aligned}$$

Hence, for any $t \in (t_0, s_0]$, we have

$$\begin{aligned} e^{\lambda t}V(Z(t), \mathcal{L}(Z(t)), r(t), t) &= e^{\lambda t_0}V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0) + M_t^{t_0} \\ &+ \int_{t_0}^t e^{\lambda s}(\lambda V(Z(s), \mathcal{L}(Z(s)), r(s), s) + L^\mu V(Z(s), \mathcal{L}(Z(s)), r(s), s))ds \end{aligned} \quad (16)$$

where

$$\begin{aligned} M_t^{t_0} &= \int_{t_0}^t \int_{\mathbb{R}} e^{\lambda s} \left[V(Z(s), \mathcal{L}(Z(s)), r(t_0) + \Theta(r(s-), z), s) - V(Z(s), \mathcal{L}(Z(s)), r(s), s) \right] p(ds, dz) \\ &+ \int_{t_0}^t e^{\lambda s} V_z(Z(s), \mathcal{L}(Z(s)), r(s), s) \iota(Z(s), \mathcal{L}(Z(s)), r(s), s) dW_s. \end{aligned}$$

Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the standard augmented filtration generated by the Brownian motion W_t , Poisson random measure $p(ds, dz)$ and Markov switching process $r(t)$. Under mild predictability and integrability assumptions on the integrands, the Brownian stochastic integral satisfies $\mathbb{E}\left[\int_u^t \phi(s) dW_s \mid \mathcal{F}_u\right] = 0$ for any $0 \leq u \leq t$. For the Poisson integral, we recall the compensated Poisson measure $\tilde{p}(ds, dz) = p(ds, dz) - ds\tilde{m}(dz)$, where the deterministic drift part is incorporated into the infinitesimal generator L^μ , and the remaining compensated Poisson integral has zero conditional expectation over $[u, t]$. Taking the conditional expectation of $M_t^{t_0}$ with respect to $\mathcal{F}_u$, the expectations of both stochastic integrals on the interval $[u, t]$ equal zero, which gives $\mathbb{E}[M_t^{t_0} \mid \mathcal{F}_u] = M_u^{t_0}$. Hence, $M_t^{t_0}$ is a martingale adapted to the filtration $\{\mathcal{F}_t\}$ (see [16]).

It follows from (13) and (16) that

$$e^{\lambda t}V(Z(t), \mathcal{L}(Z(t)), r(t), t) \leq e^{\lambda t_0}V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0) + M_t^{t_0}.$$

Taking expectation on the both sides, it gives

$$\mathbb{E}[e^{\lambda t}V(Z(t), \mathcal{L}(Z(t)), r(t), t)] \leq \mathbb{E}[e^{\lambda t_0}V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)].$$

In particular,

$$\mathbb{E}[e^{\lambda s_0}V(Z(s_0), \mathcal{L}(Z(s_0)), r(s_0), s_0)] \leq \mathbb{E}[e^{\lambda t_0}V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)], \quad (17)$$

since $s_0 > t_0$. Similarly, for any $t \in (s_0, t_1]$, we can obtain

$$\begin{aligned} e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t) &= e^{\lambda s_0} V(Z(s_0), \mathcal{L}(Z(s_0)), r(s_0), s_0) + M_t^{s_0} \\ &\quad + \int_{s_0}^t e^{\lambda s} (\lambda V(Z(s), \mathcal{L}(Z(s)), r(s), s) + L^H V(Z(s), \mathcal{L}(Z(s)), r(s), s)) ds \\ &\leq e^{\lambda s_0} V(Z(s_0), \mathcal{L}(Z(s_0)), r(s_0), s_0) + M_t^{s_0} \\ &\quad + \int_{s_0}^t e^{\lambda s} (\lambda + b_2) V(Z(s), \mathcal{L}(Z(s)), r(s), s) ds. \end{aligned}$$

Taking expectation on both sides, then we have

$$\begin{aligned} \mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &= \mathbb{E}[e^{\lambda s_0} V(Z(s_0), \mathcal{L}(Z(s_0)), r(s_0), s_0)] \\ &\quad + (\lambda + b_2) \int_{s_0}^t \mathbb{E}[e^{\lambda s} V(Z(s), \mathcal{L}(Z(s)), r(s), s)] ds. \end{aligned}$$

Applying the Gronwall inequality to give

$$\mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] \leq \mathbb{E}[e^{\lambda s_0} V(Z(s_0), \mathcal{L}(Z(s_0)), r(s_0), s_0)] e^{(\lambda + b_2)(t - s_0)},$$

then by inequality (17), we have

$$\mathbb{E}[e^{\lambda s_0} V(Z(s_0), \mathcal{L}(Z(s_0)), r(s_0), s_0)] e^{(\lambda + b_2)(t - s_0)} \leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda + b_2)(t - s_0)}.$$

So we can get, for $t \in (s_0, t_1]$,

$$\begin{aligned} \mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &\leq \mathbb{E}[e^{\lambda s_0} V(Z(s_0), \mathcal{L}(Z(s_0)), r(s_0), s_0)] e^{(\lambda + b_2)(t - s_0)} \\ &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda + b_2)(t - s_0)}. \end{aligned} \quad (18)$$

Similarly, for any $t \in (t_1, s_1]$, we have

$$\begin{aligned} \mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &\leq \mathbb{E}[e^{\lambda t_1} V(Z(t_1), \mathcal{L}(Z(t_1)), r(t_1), t_1)] \\ &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda + b_2)(t_1 - s_0)}, \end{aligned}$$

and for any $t \in (s_1, t_2]$, we have

$$\begin{aligned} \mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &\leq \mathbb{E}[e^{\lambda s_1} V(Z(s_1), \mathcal{L}(Z(s_1)), r(s_1), s_1)] e^{(\lambda + b_2)(t - s_1)} \\ &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda + b_2)(t_1 - s_0) + (\lambda + b_2)(t - s_1)}. \end{aligned}$$

Repeating a similar process, for any $t \in (t_2, s_2]$, we have

$$\begin{aligned} \mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &\leq \mathbb{E}[e^{\lambda t_2} V(Z(t_2), \mathcal{L}(Z(t_2)), r(t_2), t_2)] \\ &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda + b_2)(t_1 - s_0) + (\lambda + b_2)(t_2 - s_1)}, \end{aligned}$$

and for any $t \in (s_2, t_3]$, we have

$$\begin{aligned} \mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &\leq \mathbb{E}[e^{\lambda s_2} V(Z(s_2), \mathcal{L}(Z(s_2)), r(s_2), s_2)] e^{(\lambda + b_2)(t - s_2)} \\ &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda + b_2)(t_1 - s_0) + (\lambda + b_2)(t_2 - s_1) + (\lambda + b_2)(t - s_2)}. \end{aligned}$$

For any $t \in (t_i, s_i], i = 1, 2, \dots$, we get

$$\begin{aligned} \mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &\leq \mathbb{E}[e^{\lambda s_1} V(Z(s_1), \mathcal{L}(Z(s_1)), r(s_1), s_1)] e^{(\lambda + b_2)(t - s_1)} \\ &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda + b_2) \sum_{k=0}^{i-1} (t_{k+1} - s_k)}, \end{aligned}$$

and for any $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots$, we have

$$\mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] \leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda+b_2) \sum_{k=0}^{i-1} (t_{k+1}-s_k) + (\lambda+b_2)(t-s_i)}.$$

Hence, we get

$$\mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] \leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)}.$$

Moreover, we can get

$$\begin{aligned} \mathbb{E}V(Z(t), \mathcal{L}(Z(t)), r(t), t) &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)} e^{-\gamma t + (\gamma-\lambda)t} \\ &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)} e^{-\gamma t + (\gamma-\lambda) \sum_{k=0}^{i-1} (t_{k+1}-t_k)}. \end{aligned}$$

Therefore, by (12), we have

$$\mathbb{E}|Z(t)|^2 \leq \frac{1}{c} \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] \frac{e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)}}{e^{(\lambda-\gamma) \sum_{k=0}^{i-1} (t_{k+1}-t_k)}} e^{-\gamma t}.$$

Then, $\mathbb{E}|Z(t)|^2 \leq Me^{-\gamma t}$, where $M = \frac{1}{c} \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] \frac{e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)}}{e^{(\lambda-\gamma) \sum_{k=0}^{i-1} (t_{k+1}-t_k)}}$.

This proof is completed. $\square$

Remark 1. It should be mentioned that when $s_i - t_i = a$ and $t_{i+1} - t_i = S$, then the control policy is described as periodically intermittent. Then, we can deduce the obtained result as follows:

Corollary 1. If conditions (12)–(14) and

$$e^{(\lambda+b_2)(S-a)} < e^{(\lambda-\gamma)S}$$

hold, then Equation (1) is exponentially stable in the mean square through the periodically intermittent control.

5. An Application

In this section, we consider the mean-field stochastic Cohen–Grossberg–Hopfield neural network in the following form

$$\begin{cases} dZ(t) = -D(t, r(t)) [C(Z(t)) - QG(Z(t))] dt + H(Z(t), \mathcal{L}(Z(t)), r(t)) dW(t), & t \geq 0, \\ Z(0) = z_0, \end{cases} \quad (19)$$

where $Z(t) = (Z_1(t), Z_2(t), \dots, Z_n(t))^T$, $Z_i(t)$ denotes the state variable of the i th neuron at time t , $D(t, r(t)) = \text{diag}\{d_1(t, r(t)), \dots, d_n(t, r(t))\}$ represents an amplification, $C(Z(t)) = (c_1(Z_1(t)), c_2(Z_2(t)), \dots, c_n(Z_n(t)))^T$ is an appropriate behaviour function so that system (19) keeps bounded, $G(Z(t)) = (g_1(Z_1(t)), g_2(Z_2(t)), \dots, g_n(Z_n(t)))^T$ represents the activation function of the i th neuron at time t , $H(Z(t)) = (h_1(Z_1(t), \mathcal{L}(Z_1(t)), r_1(t)), h_2(Z_2(t), \mathcal{L}(Z_2(t)), r_2(t)), \dots, h_n(Z_n(t), \mathcal{L}(Z_n(t)), r_n(t)))^T$ is the diffusion function, $Q = q(ij)_{m \times m}$ is the connection weight matrix.

For the unstable system (19), we choose an aperiodically intermittent control in the following form

$$U(t) = \begin{cases} -u(t)Z(t), & t \in (t_i, s_i] \\ 0, & t \in (s_i, t_{i+1}] \end{cases} \quad i = 1, 2, \dots$$

where $u(t) = \text{diag}\{u_1(t), u_2(t), \dots, u_n(t)\}$, $u_i(0) > 0, i = 1, 2, \dots, n$ denote initial values, and the update law is set to be

$$\dot{u}_i(t) = q_i e^{-\lambda t} |Z_i(t)|^2$$

q_i and λ are positive constants. Then, the controlled Hopfield neural network model has the following form

$$\begin{cases} dZ(t) = (-D(t, r(t))[C(Z(t)) - QG(Z(t))] + U(t))dt + H(Z(t), \mathcal{L}(Z(t)), r(t))dW(t), & t \geq 0, \\ Z(0) = z_0, \end{cases} \quad (20)$$

Let $J = -D(t, r(t))[C(Z(t)) - QG(Z(t))] + U(t)$, and assume that J and H satisfy both Lipschitz and linear growth conditions to guarantee the existence and uniqueness of the solutions for $t \geq t_0$. Assume that $J(0, \mu_0, i, t) = 0$ and $H(0, \mu_0, i, t) = 0$, then the system (19) has a trivial solution $Z(0, \mu_0, i, t) = 0$ with $z_0 = 0$. In what follows, let $Z(t) = Z(t, \mu_t, r(t), t_0, z_0, \mu_0, r_0)$ be the solution of the system (20) such that $Z(t_0) = z_0$.

Assumption 3. (A1) For each $i \in \{1, 2, \dots, n\}$, $g_i(z)$ is Lipschitz-continuous and there exist positive constants G_i such that $|g_i(z)| \leq G_i|z|$, $\forall z \in \mathbb{R}$.

(A2) Suppose that there exists a positive constant u_i such that $u_i(t)$ satisfies $0 \leq u_i \leq u_i(t)$.

(A3) Suppose that there exist positive constants $\underline{d}_i$ and $\bar{d}_i$ such that $d_i(t, r(t))$ satisfy $\underline{d}_i \leq d_i(t, r(t)) \leq \bar{d}_i$.

(A4) Suppose that there exists a positive constant ζ_i such that $\zeta_i z^2 \leq z c_i(z)$.

(A5) There exist constants ϵ_i such that trace $[H(Z(t), \mathcal{L}(Z(t)), r(t))^T V_{zz}(Z(t), \mathcal{L}(Z(t)), r(t)) H(Z(t), \mathcal{L}(Z(t)), r(t))] \leq \sum_{i=1}^n \epsilon_i (|Z_i(t)|^2 + \mathbb{W}_2^2(\mathcal{L}(Z_i(t)), \delta_0))$.

The following theorem suggests that the controlled Hopfield neural network model with control function (20) is exponentially stable through an aperiodically intermittent control design.

Theorem 4. Under Assumption 3, let

$$b_1 = 2 \min_{1 \leq i \leq n} \{\underline{d}_i \zeta_i - \epsilon_i + \rho\} - 2 \max_{1 \leq i \leq n} \{\bar{d}_i\} |Q| |G| \quad (21)$$

$$b_2 = -2 \min_{1 \leq i \leq n} \{\underline{d}_i \zeta_i - \epsilon_i\} + 2 \max_{1 \leq i \leq n} \{\bar{d}_i\} |Q| |G|. \quad (22)$$

If $b_1 \geq 0$, $b_2 \geq 0$ and $e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)} \leq e^{(\lambda-\gamma) \sum_{k=0}^{i-1} (t_{k+1}-t_k)}$, $i = 1, 2, \dots$, then $\mathbb{E}|Z(t)|^2 \leq M e^{-\gamma t}$, where $G = \max_{1 \leq i \leq n} G_i$, $\lambda = b_1$, $\lambda > \gamma > 0$, and ρ is a positive constant provided $\rho > \lambda + K$. Then, the controlled Hopfield neural network model (20) is exponentially stable via aperiodically intermittent control.

Proof. Define

$$V(Z(t), \mathcal{L}(Z(t)), r(t), t) = Z^T(t)Z(t) + \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))^2}{q_i}.$$

When $t \in (t_i, s_i]$, it follows from Assumption 3 that

$$\begin{aligned} L^\mu V(Z(t), \mathcal{L}(Z(t)), r(t), t) = & 2Z^T(t) (-D(t, r(t))[C(Z(t)) - QG(Z(t))] - u(t)Z(t)) \\ & + [H^T(Z(t), \mathcal{L}(Z(t)), r(t))H(Z(t), \mathcal{L}(Z(t)), r(t))] \end{aligned}$$

$$\begin{aligned}
& -\lambda \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))^2}{q_i} - 2 \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))}{q_i} \dot{u}_i \\
& \leq 2Z^T(t) (-D(t, r(t)) [C(Z(t)) - QG(Z(t))]) \\
& \quad + [H^T(Z(t), \mathcal{L}(Z(t)), r(t)) H(Z(t), \mathcal{L}(Z(t)), r(t))] \\
& \quad - \lambda \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))^2}{q_i} - 2\rho \sum_{i=1}^n |Z_i(t)|^2 \\
& \leq -2 \sum_{i=1}^n d_i(t, r(t)) c_i(Z_i(t)) Z_i(t) + 2 \max_{1 \leq i \leq n} \bar{d}_i |Z(t)| |Q| |G(Z(t))| \\
& \quad + \sum_{i=1}^n \epsilon_i (|Z_i(t)|^2 + \mathbb{W}_2^2(\mathcal{L}(Z_i(t)), \delta_0)) - 2\rho \sum_{i=1}^n |Z_i(t)|^2 \\
& \quad - \lambda \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))^2}{q_i} \\
& \leq -2 \sum_{i=1}^n (d_i \zeta_i - \frac{1}{2} \epsilon_i + \rho) |Z_i(t)|^2 + 2 \max_{1 \leq i \leq n} \{\bar{d}_i\} |Z(t)| |Q| |G(Z(t))| \\
& \quad + \sum_{i=1}^n \epsilon_i \mathbb{W}_2^2(\mathcal{L}(Z_i(t)), \delta_0) - \lambda \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))^2}{q_i} \\
& \leq -\left(2 \min_{1 \leq i \leq n} \{d_i \zeta_i - \frac{1}{2} \epsilon_i + \rho\} - 2 \max_{1 \leq i \leq n} \{\bar{d}_i\} |Q| |G|\right) |Z(t)|^2 \\
& \quad + \sum_{i=1}^n \epsilon_i \mathbb{W}_2^2(\mathcal{L}(Z_i(t)), \delta_0) - \lambda \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))^2}{q_i} \\
& \leq -\left(2 \min_{1 \leq i \leq n} \{d_i \zeta_i - \frac{1}{2} \epsilon_i + \rho\} - 2 \max_{1 \leq i \leq n} \{\bar{d}_i\} |Q| |G|\right) |Z(t)|^2 \\
& \quad + \sum_{i=1}^n \epsilon_i \mathbb{W}_2^2(\mathcal{L}(Z_i(t)), \delta_0). \tag{23}
\end{aligned}$$

Taking expectation on both sides of (23) and considering

$$\begin{aligned}
\mathbb{W}_2^2(\mu, \nu) &= \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dZ, dy) \right\} \\
&\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d(\mathbb{P}_{(X, Y)}(x, y)) = \mathbb{E}|X - Y|^2, \tag{24}
\end{aligned}$$

then we have

$$\mathbb{E}[L^\mu V(Z(t), \mathcal{L}(Z(t)), r(t), t)] \leq -b_1 \mathbb{E}V(Z(t), \mathcal{L}(Z(t)), r(t), t).$$

In a similar way, when $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned}
L^\mu V(Z(t), \mathcal{L}(Z(t)), r(t), t) &\leq 2Z^T(t) (-D(t, r(t)) [C(Z(t)) - QG(Z(t))]) \\
&\quad + [H^T(Z(t), \mathcal{L}(Z(t)), r(t)) H(Z(t), \mathcal{L}(Z(t)), r(t))] \\
&\leq -2 \sum_{i=1}^n d_i(t, r(t)) c_i(Z_i(t)) Z_i(t) + 2 \max_{1 \leq i \leq n} \bar{d}_i |Z(t)| |Q| |G(Z(t))| \\
&\quad + \sum_{i=1}^n \epsilon_i (|Z_i(t)|^2 + \mathbb{W}_2^2(\mathcal{L}(Z_i(t)), \delta_0)) \\
&\leq \left(-2 \min_{1 \leq i \leq n} \{d_i \zeta_i - \frac{1}{2} \epsilon_i\} + 2 \max_{1 \leq i \leq n} \{\bar{d}_i\} |Q| |G|\right) |Z(t)|^2 \\
&\quad + \sum_{i=1}^n \epsilon_i \mathbb{W}_2^2(\mathcal{L}(Z_i(t)), \delta_0), \tag{25}
\end{aligned}$$

and

$$\mathbb{E}[L^\mu V(Z(t), \mathcal{L}(Z(t)), r(t), t)] \leq b_2 \mathbb{E}V(Z(t), \mathcal{L}(Z(t)), r(t), t).$$

Applying the Itô formula to $e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t) = e^{\lambda t} (Z^T(t)Z(t) + \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))^2}{q_i})$, we can get

$$\begin{aligned} d[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &= e^{\lambda t} \{ \lambda V(Z(t), \mathcal{L}(Z(t)), r(t), t) + V_t(Z(t), \mathcal{L}(Z(t)), r(t), t) \\ &\quad + V_z(Z(t), \mathcal{L}(Z(t)), r(t), t) [(-D(t, r(t))(C(Z(t)) - QG(Z(t))) \\ &\quad + U(t))] + 2H^T H(Z(t), \mathcal{L}(Z(t)), r(t)) \} dt \\ &\quad + e^{\lambda t} V_z(Z(t), \mathcal{L}(Z(t)), r(t), t) H(Z(t), \mathcal{L}(Z(t)), r(t)) dW(t) \\ &\leq e^{\lambda t} \left\{ \lambda \left(Z^T(t)Z(t) + \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))^2}{q_i} \right) \right. \\ &\quad - \lambda \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))^2}{q_i} - 2 \sum_{i=1}^n e^{-\lambda t} \frac{(\rho - u_i(t))}{q_i} \dot{u}_i \\ &\quad + 2Z^T(t) [(-D(t, r(t))[C(Z(t)) - QG(Z(t))] \\ &\quad + U(t))] + 2H^T H(Z(t), \mathcal{L}(Z(t)), r(t)) \} dt \\ &\quad + 2e^{\lambda t} Z^T(t) H(Z(t), \mathcal{L}(Z(t)), r(t)) dW(t). \end{aligned}$$

So, for any $t \in (t_0, s_0]$, we have

$$\begin{aligned} e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t) &= e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0) + \int_{t_0}^t e^{\lambda s} \{ \lambda V(Z(s), \mathcal{L}(Z(s)), r(s), s) \\ &\quad + L^\mu V(Z(s), \mathcal{L}(Z(s)), r(s), s) \} ds + 2 \int_{t_0}^t e^{\lambda s} Z^T(s) H(Z(s), \mathcal{L}(Z(s)), r(s)) dW(s). \end{aligned}$$

Taking expectation on the both sides of the above equality, we obtain

$$\mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] \leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)].$$

When $t \in (s_0, t_1]$, repeating the previous steps, we have

$$\begin{aligned} e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t) &= e^{\lambda s_0} V(Z(s_0), \mathcal{L}(Z(s_0)), r(s_0), s_0) + \int_{s_0}^t e^{\lambda s} \{ \lambda V(Z(s), \mathcal{L}(Z(s)), r(s), s) \\ &\quad + L^\mu V(Z(s), \mathcal{L}(Z(s)), r(s), s) \} ds + 2 \int_{s_0}^t e^{\lambda s} Z^T(s) H(Z(s), \mathcal{L}(Z(s)), r(s)) dW(s). \end{aligned}$$

Based on an expectation of both sides of the above equality, we obtain

$$\begin{aligned} \mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &\leq \mathbb{E}[e^{\lambda s_0} V(Z(s_0), \mathcal{L}(Z(s_0)), r(s_0), s_0)] \\ &\quad + (\lambda + b_2) \int_{s_0}^t \mathbb{E}[e^{\lambda s} V(Z(s), \mathcal{L}(Z(s)), r(s), s)] ds \\ &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] \\ &\quad + (\lambda + b_2) \int_{s_0}^t \mathbb{E}[e^{\lambda s} V(Z(s), \mathcal{L}(Z(s)), r(s), s)] ds. \end{aligned}$$

Apply the Gronwall inequality to give

$$\mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] \leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0) e^{(\lambda + b_2)(t - s_0)}].$$

From Theorem 3, we can easily obtain

$$\mathbb{E}[e^{\lambda t} V(Z(t), \mathcal{L}(Z(t)), r(t), t)] \leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0) e^{(\lambda + b_2) \sum_{k=0}^i (t_{k+1} - s_k)}].$$

Then, we have

$$\begin{aligned}\mathbb{E}[V(Z(t), \mathcal{L}(Z(t)), r(t), t)] &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0) e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)} e^{-\gamma t + (-\lambda+\gamma)t}] \\ &\leq \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0) e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)} e^{-\gamma t + (-\lambda+\gamma) \sum_{k=0}^{i-1} (t_{k+1}-t_k)}].\end{aligned}$$

Therefore, by (12) we have

$$\mathbb{E}|Z(t)|^2 \leq \frac{1}{c} \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] \frac{e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)}}{e^{(\lambda-\gamma) \sum_{k=0}^{i-1} (t_{k+1}-t_k)}} e^{-\gamma t}.$$

Then,

$$\mathbb{E}|Z(t)|^2 \leq M e^{-\gamma t},$$

where $M = \frac{1}{c} \mathbb{E}[e^{\lambda t_0} V(Z(t_0), \mathcal{L}(Z(t_0)), r(t_0), t_0)] \frac{e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)}}{e^{(\lambda-\gamma) \sum_{k=0}^{i-1} (t_{k+1}-t_k)}}$. That is, the control system (20) is exponentially stable in the mean square. This proof is completed. $\square$

6. Numerical Example

We present the following mean-field stochastic Coen–Grossberg–Hopfield neural network model with Markov switching to illustrate the effectiveness of the designed aperiodically intermittent controller:

$$dZ(t) = -D(t, r(t))[C(Z(t)) - QG(Z(t))]dt + H(Z(t), \mathcal{L}(Z(t)), r(t))dW(t), \quad 0 \leq t \leq T, \quad (26)$$

where $Z(t) = (Z_1(t), Z_2(t))^T$ denotes the state vector of the neurons at time t , $W(t)$ is a two-dimensional standard Brownian motion, and $r(t)$ is a right-continuous Markov chain taking values in $\mathbb{S} = \{1, 2\}$ with generator

$$\Gamma = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}.$$

The system parameters are chosen as follows:

$$C(Z(t)) = \begin{bmatrix} Z_1(t) \\ Z_2(t) \end{bmatrix}, \quad G(Z(t)) = \begin{bmatrix} \tanh Z_1(t) \\ \tanh Z_2(t) \end{bmatrix}, \quad Q = \begin{bmatrix} \frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{12} & \frac{3}{4} \end{bmatrix},$$

$$D(t, 1) = \begin{bmatrix} -0.54 - 0.1 \sin t & 0 \\ 0 & -0.95 - 0.1 \sin t \end{bmatrix}, \quad D(t, 2) = \begin{bmatrix} -0.35 - 0.1 \sin t & 0 \\ 0 & -0.35 - 0.1 \sin t \end{bmatrix}.$$

The diffusion coefficient matrices are given by

$$\begin{aligned}H(Z(t), \mathcal{L}(Z(t)), 1) &= \begin{bmatrix} |0.2Z_1(t)| - |0.1Z_2(t)| & 0 \\ 0 & |0.2Z_2(t)| - |0.1Z_1(t)| \end{bmatrix} + 0.5 \int_{\mathbb{R}} z \mu_t(dz), \\ H(Z(t), \mathcal{L}(Z(t)), 2) &= \begin{bmatrix} |0.1Z_1(t)| - |0.2Z_2(t)| & 0 \\ 0 & |0.1Z_2(t)| - |0.2Z_1(t)| \end{bmatrix} + 0.5 \int_{\mathbb{R}} z \mu_t(dz).\end{aligned}$$

6.1. Stability Analysis of the Uncontrolled System

To analyze the stability of system (26), we take the Lyapunov function $V(z, \mu, i, t) = z_1^2 + z_2^2$ for $i = 1, 2$. For $r(t) = 1$, the weak generator yields

$$\begin{aligned}
L^\mu V(Z(t), \mathcal{L}(Z(t)), 1, t) &= 2Z_1(t) \left((0.54 + 0.1 \sin t)Z_1(t) - \frac{1}{3} \tanh Z_1(t) + \frac{1}{4} \tanh Z_2(t) \right) \\
&\quad + 2Z_2(t) \left((0.95 + 0.1 \sin t)Z_2(t) + \frac{1}{12} \tanh Z_1(t) - \frac{3}{4} \tanh Z_2(t) \right) \\
&\quad + (|0.2Z_1(t)| - |0.1Z_2(t)|)^2 + (|0.2Z_2(t)| - |0.1Z_1(t)|)^2 \\
&\geq 0.18|Z_1(t)|^2 + 0.18|Z_2(t)|^2 = 0.18V(Z(t), \mathcal{L}(Z(t)), 1, t).
\end{aligned}$$

For $r(t) = 2$, similarly,

$$\begin{aligned}
L^\mu V(Z(t), \mathcal{L}(Z(t)), 2, t) &= 2Z_1(t) \left((0.35 + 0.1 \sin t)Z_1(t) - \frac{1}{3} \tanh Z_1(t) + \frac{1}{4} \tanh Z_2(t) \right) \\
&\quad + 2Z_2(t) \left((0.35 + 0.1 \sin t)Z_2(t) + \frac{1}{12} \tanh Z_1(t) - \frac{3}{4} \tanh Z_2(t) \right) \\
&\quad + (|0.1Z_1(t)| - |0.2Z_2(t)|)^2 + (|0.1Z_2(t)| - |0.2Z_1(t)|)^2 \\
&\geq 0.18|Z_1(t)|^2 + 0.18|Z_2(t)|^2 = 0.18V(Z(t), \mathcal{L}(Z(t)), 2, t).
\end{aligned}$$

According to Theorem 2, the solution of system (26) is unstable in the mean-square sense.

6.2. Design of the Aperiodically Intermittent Controller

We now design the aperiodically intermittent controller in the form

$$U(t) = \begin{cases} -u(t)Z(t), & t \in (t_i, s_i], \\ 0, & t \in (s_i, t_{i+1}], \end{cases} \quad i = 0, 1, 2, \dots, \quad (27)$$

where $u(t) = 1.6 + 0.1 \sin t$. The controlled system is then given by

$$dZ(t) = (-D(t, r(t))[C(Z(t)) - QG(Z(t))] + U(t))dt + H(Z(t), \mathcal{L}(Z(t)), r(t))dW(t), \quad t \geq 0. \quad (28)$$

It can be verified that

$$H^T H(Z(t), \mathcal{L}(Z(t)), r(t)) \leq 0.5(|Z_1(t)|^2 + |Z_2(t)|^2) + 0.5(\mathbb{W}_2^2(\mathcal{L}(Z_1(t)), \delta_0) + \mathbb{W}_2^2(\mathcal{L}(Z_2(t)), \delta_0)),$$

$$b_1 = 2 \min_{1 \leq j \leq 2} \{d_j \zeta_j - e_j\} - 2 \max_{1 \leq j \leq 2} \{\bar{d}_j\} \|Q\| \|G\| = 1.403,$$

$$b_2 = -2 \min_{1 \leq j \leq 2} \{d_j \zeta_j - e_j\} + 2 \max_{1 \leq j \leq 2} \{\bar{d}_j\} \|Q\| \|G\| = 2.696,$$

where $\rho = 1.5$, $d_j = 0.25$, $\bar{d}_j = 1.05$, $e_j = 0.5$, $u_j = 1.4$ for $i, j = 1, 2$, $\|Q\| = 0.2291$ and $\|G\| = 2$. Choosing $\gamma = 0.1$, for any initial data $\xi \in \mathbb{R}$ and $i_0 \in \{1, 2\}$, system (28) admits a unique global solution for $t \in \mathbb{R}^+$. Moreover, the average dwell-time condition

$$e^{(\lambda+b_2) \sum_{k=0}^i (t_{k+1}-s_k)} < e^{(\lambda-\gamma) \sum_{k=0}^i (s_k-t_k)}, \quad i = 1, 2, \dots$$

is satisfied. By Theorem 4, the controlled system (28) is exponentially stable in the mean square.

6.3. Numerical Simulation Results

To verify the theoretical results, we perform numerical simulations using the Euler–Maruyama method with a constant time step $\Delta t = 10^{-3}$. The number of sample paths for Monte Carlo averaging is set to $N = 1000$. The initial condition is $Z(0) = (1, -1)^T$ and the initial mode of the Markov chain is $r(0) = 1$.

From Figure 2, we observe that the uncontrolled system is unstable as $\mathbb{E}|Z(t)|^2$ grows unbounded. In contrast, the controlled system shows clear exponential decay of $\mathbb{E}|Z(t)|^2$, confirming the effectiveness of the proposed aperiodically intermittent controller. The sample trajectory further demonstrates that the designed controller successfully stabilizes the system states to zero.

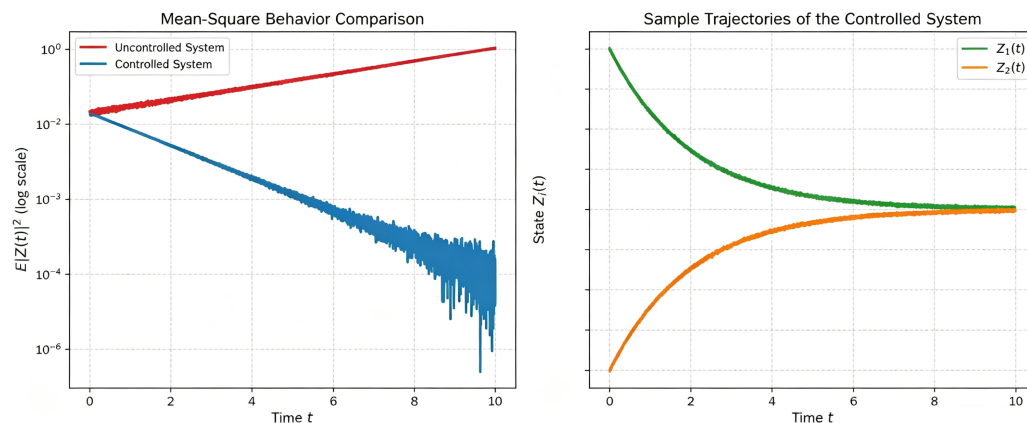


Figure 2. (Left) Evolution of the empirical mean-square state $\mathbb{E}|Z(t)|^2$ for the uncontrolled system (26) (red line) and the controlled system (28) (blue line). The uncontrolled system diverges exponentially, while the controlled system exhibits exponential mean-square stability. (Right) A sample trajectory of the state variables $Z_1(t)$ and $Z_2(t)$ under the aperiodically intermittent controller, showing that the state converges to the origin.

7. Conclusions

In this paper, we have focused on a type of unstable hybrid stochastic McKean–Vlasov equations (MVSDEs). We have proved the well-posedness of the MVSDEs with Markovian switching first and then established the multivariate Itô formula for this type of hybrid MVSDEs. Based on these, we have designed aperiodically intermittent control for an unstable MVSDE with Markovian switching. With it, we have studied the mean square exponential stability for the controlled system. Furthermore, we have given the mean-field stochastic Cohen–Grossberg–Hopfield neural network to elaborate on the specific application of the control method. Finally, we presented the Hopfield neural network model to check the results of our theory.

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Appendix A

Proof of Theorem 1. Let $0 < \tau_1 < \tau_2 \cdots < \tau_v < t$ represent all the times when $r(s)$ has a jump. Moreover, we emphasize that in the Markov jump process, when a jump occurs at time t , $r(t)$ and $r(t^-) = \lim_{s \uparrow t} r(s)$ are distinguishable.

Let $\mu_t = \mathcal{L}(Z(t))$, then $h(Z(t), \mathcal{L}(Z(t)), r, t) = h(Z(t), \mu_t, r, t)$, where $h = V, b, u, \iota$, and r represents some positive number taking value in the state space $S = \{1, 2, \dots, N_0\}$. Define $\bar{V}(t, z) = V(z, \mu_t, r, t)$, and then $\bar{V}(t, Z(t)) = V(Z(t), \mu_t, r, t)$. Moreover, based on $V \in C^{2,2,1}(\mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}^+)$, it holds that $\bar{V}$ is C^2 in x . However, we do not know the differentiability of $\bar{V}$ in t . Note that the differentiability of $\bar{V}$ in t comes from two parts; $V(z, \mu, r, t)$ in t for fixed z, μ and $V(z, \mu_t, r, s)$ in t for fixed z, s . Therefore, to apply the classical Itô formula to $\bar{V}(t, Z(t))$, we only need to consider the second part by $V \in C^{2,2,1}(\mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times \mathbb{S} \times \mathbb{R}^+)$.

Step 1. For any positive integer N , let $((Z^l(t))_{t \geq 0})_{1 \leq l \leq N}$ be a sequence of independent and identically distributed (i.i.d.) copies of $(Z(t))_{t \geq 0}$. That is, for any $l \geq 1$,

$$dZ^l(t) = b(Z^l(t), \mathcal{L}(Z^l(t)), r, t)dt + \iota(Z^l(t), \mathcal{L}(Z^l(t)), r, t)dW_t^l,$$

where $W^l, l = 1, 2, \dots, N$ are mutually independent and have the same distribution to that of W . Set $H(\mu_t) := V(x, \mu_t, r, t)$ and

$$H^N(x^1, x^2, \dots, x^N) := H\left(\frac{1}{N} \sum_{l=1}^N \delta_{x^l}\right), \quad (\text{A1})$$

then,

$$\partial_{x^i} H^N(x^1, x^2, \dots, x^N) = \frac{1}{N} \partial_{\mu} H\left(\frac{1}{N} \sum_{l=1}^N \delta_{x^l}\right)(x^i), \quad (\text{A2})$$

$$\partial_{x^i x^j}^2 H^N(x^1, x^2, \dots, x^N) = \frac{1}{N} \partial_y \partial_{\mu} H\left(\frac{1}{N} \sum_{l=1}^N \delta_{x^l}\right)(x^i) \delta_{i,j} + \frac{1}{N^2} \partial_{\mu}^2 H\left(\frac{1}{N} \sum_{l=1}^N \delta_{x^l}\right)(x^i, x^j). \quad (\text{A3})$$

Applying the Itô formula to $H^K(Z^1(t), Z^2(t), \dots, Z^N(t))$ and taking the expectation on two sides, we obtain that for $0 \leq t < t+h \leq T$,

$$\begin{aligned} \mathbb{E}H^N(Z^1(t+h), Z^2(t+h), \dots, Z^N(t+h)) &= \mathbb{E}H^N(Z^1(t), Z^2(t), \dots, Z^N(t)) \\ &+ \sum_{i=1}^N \left(\int_t^{t+h} \mathbb{E} \partial_{z^i} H^N(Z^1(s), Z^2(s), \dots, Z^N(s)) b(Z^i(s), \mathcal{L}^1(Z^i(s)), r, s) ds \right. \\ &+ \frac{1}{2} \int_t^{t+h} \mathbb{E} \partial_{z^i z^i}^2 H^N(Z^1(s), Z^2(s), \dots, Z^N(s)) u^T(Z^i(s), \mathcal{L}^1(Z^i(s)), r, s) ds \\ &= \mathbb{E}H^N(Z^1(t), Z^2(t), \dots, Z^N(t)) \\ &+ N \int_t^{t+h} \mathbb{E} \partial_{z^1} H^N(Z^1(s), Z^2(s), \dots, Z^N(s)) b(Z^1(s), \mathcal{L}^1(Z^1(s)), r, s) ds \\ &+ \frac{N}{2} \int_t^{t+h} \mathbb{E} \partial_{z^1 z^1}^2 H^N(Z^1(s), Z^2(s), \dots, Z^N(s)) u^T(Z^1(s), \mathcal{L}^1(Z^1(s)), r, s) ds. \end{aligned} \quad (\text{A4})$$

Inserting (A2) and (A3) in the above equality, we get

$$\begin{aligned} \mathbb{E}H\left(\frac{1}{N} \sum_{l=1}^N \delta_{Z^l(t+h)}\right) &= \mathbb{E}H\left(\frac{1}{N} \sum_{l=1}^N \delta_{Z^l(t)}\right) + N \int_t^{t+h} \mathbb{E} \partial_{\mu} H\left(\frac{1}{N} \sum_{l=1}^N \delta_{Z^l(s)}\right)(Z^1(s)) b(Z^1(s), \mathcal{L}^1(Z^1(s)), r, s) ds \\ &+ \frac{1}{2} \int_t^{t+h} \mathbb{E} \partial_y \partial_{\mu} H\left(\frac{1}{N} \sum_{l=1}^N \delta_{Z^l(s)}\right)(Z^1(s)) u^T(Z^1(s), \mathcal{L}^1(Z^1(s)), r, s) ds \\ &+ \frac{1}{2N} \int_t^{t+h} \mathbb{E} \partial_{\mu}^2 H\left(\frac{1}{N} \sum_{l=1}^N \delta_{Z^l(s)}\right)(Z^1(s), Z^1(s)) u^T(Z^1(s), \mathcal{L}^1(Z^1(s)), r, s) ds. \end{aligned}$$

Noting that

$$\lim_{N \rightarrow \infty} \mathbf{E} \left[\sup_{0 \leq s \leq t} W_2^2 \left(\frac{1}{N} \sum_{l=1}^N \delta_{Z^l(s)}, \mu_s \right) \right] = 0. \quad (\text{A5})$$

It follows from the continuity and boundedness of H , $\partial_\mu H$, $\partial_y \partial_\mu H$, and boundedness of $\partial_\mu^2 H$, b , ι , and the dominated convergence theorem that

$$\begin{aligned} H(\mu_{t+h}) &= H(\mu_t) + \int_t^{t+h} \mathbb{E} \partial_\mu H(\mu_s)(Z^1(s)) b(Z^1(s), \mathcal{L}^1(Z^1(s)), r, s) ds \\ &\quad + \frac{1}{2} \int_t^{t+h} \mathbb{E} \partial_y \partial_\mu H(\mu_s)(Z^1(s)) u^T(Z^1(s), \mathcal{L}^1(Z^1(s)), r, s) ds. \end{aligned}$$

By some calculus, we determine that

$$\partial_t H(\mu_t) = \int_{\mathbb{R}^d} \partial_\mu H(\mu_t)(y) b(y, \mu_t, r, t) \mu_t(dy) + \frac{1}{2} \int_{\mathbb{R}^d} \partial_y \partial_\mu H(\mu_t)(y) u^T(y, \mu_t, r, t) \mu_t(dy).$$

Thus, we know that $\bar{V}(t, x)$ is C^1 in t and C^2 in x .

Step 2. We can use the classical Itô formula to obtain

$$\begin{aligned} dV(Z(t), \mu_t, r, t) &= d\bar{V}(t, Z(t)) \\ &= \partial_t \bar{V}(t, Z(t)) dt + \partial_z \bar{V}(t, Z(t)) b(Z(t), \mu_t, r, t) dt + \partial_z \bar{V}(t, Z(t)) \iota(Z(t), \mu_t, r, t) dW_t \\ &= \partial_t V(Z(t), \mu_t, r, t) dt + \partial_t V(z, \mu_t, r, s)|_{z=Z(t), s=t} dt \\ &\quad + \partial_z V(Z(t), \mu_t, r, t) b(Z(t), \mu_t, r, t) dt + \partial_z V(Z(t), \mu_t, r, t) \iota(Z(t), \mu_t, r, t) dW_t \\ &\quad + \frac{1}{2} \partial_z^2 V(Z(t), \mu_t, r, t) u^T(Z(t), \mu_t, r, t) dt \\ &= \partial_t V(Z(t), \mu_t, r, t) dt + \int_{\mathbb{R}^d} \partial_\mu V(Z(t), \mu_t, r, t)(y) b(y, \mu_t, r, t) \mu_t(dy) dt \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} \partial_y \partial_\mu V(Z(t), \mu_t, r, t)(y) u^T(y, \mu_t, r, t) \mu_t(dy) dt \\ &\quad + \partial_z V(Z(t), \mu_t, r, t) b(Z(t), \mu_t, r, t) dt + \partial_z V(Z(t), \mu_t, r, t) \iota(Z(t), \mu_t, r, t) dW_t \\ &\quad + \frac{1}{2} \partial_z^2 V(Z(t), \mu_t, r, t) u^T(Z(t), \mu_t, r, t) dt. \end{aligned} \quad (\text{A6})$$

Step 3. Let

$$\begin{aligned} &LV(Z(t), \mu_t, r, t) \\ &= \partial_t V(Z(t), \mu_t, r, t) + \partial_z V(Z(t), \mu_t, r, t) b(Z(t), \mu_t, r, t) + \frac{1}{2} \partial_z^2 V(Z(t), \mu_t, r, t) u^T(Z(t), \mu_t, r, t) \\ &\quad + \int_{\mathbb{R}^d} \partial_\mu V(Z(t), \mu_t, r, t)(y) b(y, \mu_t, r, t) \mu_t(dy) + \frac{1}{2} \int_{\mathbb{R}^d} \partial_y \partial_\mu V(Z(t), \mu_t, r, t)(y) u^T(y, \mu_t, r, t) \mu_t(dy). \end{aligned}$$

On the intervals $[0, \tau_1)$, $[\tau_1, \tau_2)$, $\dots$, $[\tau_v, t]$, by (A6) we have

$$\begin{aligned} &V(Z(\tau_1), \mu_{\tau_1}, r(0), \tau_1) - V(Z(0), \mu_0, r(0), 0) \\ &= \int_0^{\tau_1} LV(Z(s), \mu_s, r(0), s) ds + \int_0^{\tau_1} \partial_z V(Z(s), \mu_s, r(0), s) \iota(Z(s), \mu_s, r(0), s) dW_s, \\ &V(Z(\tau_{k+1}), \mu_{\tau_{k+1}}, r(\tau_k), \tau_{k+1}) - V(Z(\tau_k), \mu_{\tau_k}, r(\tau_k), \tau_k) \\ &= \int_{\tau_k}^{\tau_{k+1}} LV(Z(s), \mu_s, r(\tau_k), s) ds + \int_{\tau_k}^{\tau_{k+1}} \partial_z V(Z(s), \mu_s, r(\tau_k), s) \iota(Z(s), \mu_s, r(\tau_k), s) dW_s, \end{aligned}$$

$$\begin{aligned}
& V(Z(t), \mu_t, r(\tau_v), t) - V(Z(\tau_v), \mu_{\tau_v}, r(\tau_v), \tau_v) \\
&= \int_{\tau_v}^t LV(Z(s), \mu_s, r(\tau_s), s) ds + \int_{\tau_v}^t \partial_z V(Z(s), \mu_s, r(\tau_s), s) \iota(Z(s), \mu_s, r(\tau_s), s) dW_s.
\end{aligned}$$

Adding them over k from 1 to $v - 1$, we get

$$\begin{aligned}
& V(Z(t), \mathcal{L}^1(Z(t)), r(t), t) - V(Z(0), \mathcal{L}(Z(0)), r(0), 0) \\
&= \int_0^t LV(Z(s), \mathcal{L}(Z(s)), r(s), s) ds + \int_0^t \partial_z V(Z(s), \mathcal{L}(Z(s)), r(s), s) \iota(Z(s), \mathcal{L}(Z(s)), r(s), s) dW_s \\
&\quad + \sum_{k=1}^v [V(Z(\tau_k), \mathcal{L}(Z(\tau_k)), r(\tau_k), \tau_k) - V(Z(\tau_k), \mathcal{L}(Z(\tau_k)), r(\tau_k-), \tau_k)] \\
&= \int_0^t LV(Z(s), \mathcal{L}(Z(s)), r(s), s) ds + \int_0^t \partial_z V(Z(s), \mathcal{L}(Z(s)), r) \iota(Z(s), \mathcal{L}(Z(s)), r(s), s) dW_s \\
&\quad + \int_0^t \int_{\mathbb{R}} [V(Z(s), \mathcal{L}(Z(s)), r(s) + \Theta(r(s-), z), s) - V(Z(s), \mathcal{L}(Z(s)), r(s), s)] ds \times \tilde{m}(dz).
\end{aligned}$$

Considering

$$\begin{aligned}
& \int_{\mathbb{R}} [V(Z(s), \mathcal{L}(Z(s)), r(s) + \Theta(r(s-), z), s) - V(Z(s), \mathcal{L}(Z(s)), r(s), s)] \tilde{m}(dz) \\
&= \sum_{j \in S} \gamma_{rj} V(Z(s), \mathcal{L}(Z(s)), j, s),
\end{aligned}$$

the assertion (7) is obtained. The proof is completed. $\square$

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