

Article

A Grothendieck-Type Compactness Principle for Super Weakly Compact Sets

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Abstract

The classical Grothendieck compactness principle states that every norm-compact subset of a Banach space lies in the closed convex hull of a norm null sequence. Replacing the norm topology by the weak topology yields a characterization of Banach spaces with the Schur property. In the present paper, we establish an extension of this principle to the framework of super weakly compact sets. Specifically, we prove that for a Banach space X with the weak Banach–Saks property, every super weakly compact subset of X is contained in the closed convex hull of a uniformly weakly null sequence if and only if X has the Schur property. In addition, we demonstrate that every Banach space failing the Schur property contains a weakly null sequence which is not uniformly weakly null.

Keywords: super weakly compact set; Schur property; Grothendieck principle; uniformly weakly null sequence

MSC: 46B50; 46B20; 46B03

1. Introduction

Over the past decade, Grothendieck’s theorem in various topological senses has been studied by many researchers, and numerous interesting results have been obtained, which have played an important role in the generalization of Grothendieck’s theorem. The classical Grothendieck compactness principle, originating from Grothendieck’s foundational work on topological tensor products, states that every norm-compact subset of a Banach space is contained in the closed convex hull of a norm null sequence [1], p. 112. This seemingly elementary result has proven to be remarkably influential, underpinning subsequent developments in the theory of Banach spaces, including the approximation property and the structure of compact operators.

Much of this recent activity has focused on the extent to which the norm topology can be replaced by other topologies. A key result in this direction, due to Dowling et al. [2], concerns the weak topology: a Banach space has the Schur property, meaning that weak and norm convergence of sequences coincide, if and only if every weakly compact set is contained in the closed convex hull of a weakly null sequence. This characterization, now referred to as the weak Grothendieck compactness principle, is highly nontrivial; its sufficiency relies on deep basic-sequence arguments and shows that spaces satisfying the principle are quite limited. Among classical sequence spaces, ℓ_1 is the standard example of a space with the Schur property. Subsequent work has extended this theme



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in several directions. For instance, Dowling and Mupasiri [3] treated the Mackey topology, while Botelho, Luiz, and Miranda [4] established an analogue for the absolute weak topology, showing that a Banach lattice has the positive Schur property precisely when every absolutely weakly compact set lies in the closed convex hull of an absolutely weakly null sequence. In a different development, ref. [5] introduced a family of topologies τ_ξ ($0 \leq \xi \leq \omega_1$) via a repeated averages hierarchy and proved a corresponding version of Grothendieck's theorem. Further extensions to symmetric sequence spaces and Fréchet spaces are discussed in [6,7], while ref. [8] offers an operator-theoretic treatment of the weak Grothendieck compactness principle.

Super weakly compact sets, as a local characterization of super-reflexive spaces, are analogous to weakly compact sets which serve as a local characterization of reflexive spaces. This concept was proposed by L.X. Cheng and coauthors [9] through the construction of finite representability of sets, which is a localized version of the finite representability of Banach spaces. A subset $A \subset X$ is relatively super weakly compact if every set finitely representable in A is relatively weakly compact. Super weak compactness appears more frequently than one might expect, and it often underlies many classical results in Banach space theory. As an illustration, bounded closed convex super weakly compact sets are equivalent to sets having the finite dentable property, uniform convexifiability, the finite tree property, and so on (see [10]). Super weakly compact sets exist widely in general spaces. Examples include bounded subsets of super-reflexive spaces, which are relatively super weakly compact; moreover, X is super-reflexive iff B_X is super weakly compact. Also, every weakly compact subset of $L_1(\mu)$ is super weakly compact (see [11], Proposition 6.1). The canonical basis $(e_j)_{j=1}^\infty$ of c_0 is uniformly weakly null and therefore relatively super weakly compact. We direct the interested reader to [12–16] for more detailed studies on super weakly compact sets.

Despite the substantial progress on Grothendieck's theorem in various topological settings, the corresponding principle for super weakly compact sets—a class of sets that parallels weakly compact sets in the theory of super-reflexive spaces—has remained largely unexplored. The absence of a canonical topology for super weak compactness presents a fundamental obstacle: unlike the weak topology, which is intrinsically tied to the dual pair (X, X^*) , super weak compactness is defined via finite representability and therefore lacks a direct topological description. This gap is not merely technical; bridging it would deepen our understanding of the interplay between local convexity, super-reflexivity, and compactness phenomena. In this paper, we circumvent this difficulty by employing the weak Banach–Saks property and adopting the sequential approach from [2], proving that super weakly compact sets admit a characterization analogous to the weak Grothendieck theorem. We further establish that every Banach space failing the Schur property contains a weakly null sequence which is not uniformly weakly null.

2. Preliminaries

Throughout this paper, we adopt standard notation (see [17]). For basic facts on super weakly compact sets, see [9,10].

Definition 1 ([9]). *A (weakly closed, resp.) subset A of a Banach space X is said to be relatively super weakly compact (super weakly compact, resp.) provided that every subset B of a Banach space Y which is finitely representable in A is relatively weakly compact.*

We shall simply write SWC for super weakly compact. SWC sets can also be characterized by ultrapower and finitely dentable property (see [14,16]).

Definition 2 ([18]). A sequence $(x_n)_{n=1}^{\infty} \subset X$ is said to be uniformly weakly convergent to $x \in X$ provided that $(x_n - x)_{n=1}^{\infty}$ is uniformly weak null, i.e., for every $\varepsilon > 0$ there exists $n_{\varepsilon} \in \mathbb{N}$ such that

$$\#\{n \in \mathbb{N} : |\langle x^*, x_n - x \rangle| \geq \varepsilon\} \leq n_{\varepsilon} \quad \text{for all } x^* \in B_{X^*}, \quad (1)$$

where B_{X^*} is the closed unit ball of X^* and $n_{\varepsilon} \in \mathbb{N}$ is independent of x^* .

It follows from the definition that every uniformly weakly convergent sequence is weakly convergent. Note that a uniformly weakly null set becomes SWC by adding $\{0\}$ [10]. Uniformly weakly null sets are the most prototypical examples of SWC sets. The following definitions and theorems can be found in [19].

A sequence $(x_m)_{m=1}^{\infty}$ in a Banach space X is called Cesàro-summable whenever its arithmetic means

$$\left(\frac{1}{m} \sum_{i=1}^m x_i \right)_{m=1}^{\infty}$$

converge in the norm of X .

Definition 3 ([19]). A Banach space X is said to have the Banach–Saks property if for every bounded sequence $(x_n)_{n=1}^{\infty} \subset X$, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that the Cesàro means

$$\frac{1}{m} \sum_{k=1}^m x_{n_k}$$

converge in norm to some $x \in X$ as $m \rightarrow \infty$.

Definition 4 ([19]). A Banach space X is said to have the weak Banach–Saks property if for every weakly null sequence $(x_n)_{n=1}^{\infty} \subset X$ (i.e., $x_n \xrightarrow{w} 0$), there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that

$$\left\| \frac{1}{m} \sum_{k=1}^m x_{n_k} \right\| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Theorem 1 ([19]). Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence of vectors in a Banach space X . Then the following are equivalent.

- (i) There exists $x \in X$ such that $x_n \rightarrow x$ uniformly weakly in X .
- (ii) There exists $x \in X$ such that

$$\lim_{n \rightarrow \infty} \sup_{k_1 < k_2 < \dots < k_n} \left\| \frac{1}{n} \sum_{i=1}^n x_{k_i} - x \right\| = 0.$$

- (iii) Every subsequence of $(x_n)_{n=1}^{\infty}$ is Cesàro-summable in X , and each to the same limit x .
- (iv) Every subsequence of $(x_n)_{n=1}^{\infty}$ is Cesàro-summable in X .

Thus, uniform weak convergence is characterized by the property that every subsequence is Cesàro-summable and all such Cesàro means converge to a common limit.

Theorem 2 ([19]). Let X be a Banach space. Then X has the Banach–Saks property if and only if every bounded sequence in X has a subsequence which converges uniformly weakly to some $x \in X$.

This theorem can be compared with the Eberlein–Šmulian theorem, which states that a Banach space X is reflexive if and only if every bounded sequence in X has a weakly convergent subsequence. It follows that a space with the Banach–Saks property is reflexive, but the converse is not necessarily true.

Theorem 3 ([19]). *Let X be a Banach space. Then X has the weak Banach-Saks property if and only if every weakly null sequence in X has a subsequence which converges uniformly weakly to zero in X .*

Consequently, the Banach-Saks property implies the weak Banach-Saks property. In particular, $L_1[0, 1]$ enjoys the weak Banach-Saks property (see [18], p. 112).

For the following properties of Schur spaces, see [17], p. 37.

Definition 5 ([17]). *A Banach space X has the Schur property (or X is a Schur space) if weak and norm sequential convergence coincide in X , i.e., a sequence $(x_n)_{n=1}^{\infty}$ in X converges weakly to 0 if and only if $(x_n)_{n=1}^{\infty}$ converges to 0 in norm.*

Example 1. *Neither the spaces ℓ_p for $1 < p < \infty$ nor c_0 have the Schur property, since the canonical basis is weakly null but does not converge to 0 in norm.*

The space ℓ_1 has the Schur property. Schur spaces are ℓ_1 -saturated.

Theorem 4 ([17]). *Let X be a Banach space with the Schur property. Then a subset W of X is weakly compact if and only if W is norm-compact.*

3. Main Results

We will need some properties of SWC sets before proving the theorem. These can be found in [10].

Proposition 1 ([10]). *Let $(x_n)_{n=1}^{\infty}$ be a uniformly weak null sequence in a Banach space X , and let $K = \overline{\text{co}}\{\pm x_n : n \in \mathbb{N}\}$. Then K is a SWC set.*

Proposition 2 ([10]). *The union of two relatively super weakly compact sets in a Banach space is again relatively super weakly compact.*

The following is the Grothendieck theorem that is valid for super weakly compact sets.

Proposition 3 ([10]). *A subset A of a Banach space X is relatively super weakly compact if (and only if) for every $\varepsilon > 0$ there exists a relatively super weakly compact set B such that $A \subset B + \varepsilon B_X$.*

The classical Grothendieck compactness principle is as follows [1].

Theorem 5 ([1]). *Every norm-compact subset of a Banach space is contained in the closed convex hull of a norm null sequence.*

Theorem 6 ([2]). *Every weakly compact subset of a Banach space X is contained in the closed convex hull of a weakly null sequence if and only if X has the Schur property.*

This extends the classical Grothendieck theorem, which establishes the analogous property for weakly compact sets. We shall establish a version of Grothendieck's theorem for super weakly compact sets, following the proof method of Dowling et al. [2] based on bimonotone bases.

Definition 6 ([17]). *A Schauder basis $(e_i)_{i=1}^{\infty}$ in a Banach space is bimonotone if for every $n, m \in \mathbb{N}$ with $n < m$, the projection $P_{[n,m]}$ defined by $P_{[n,m]}(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=n}^{m-1} a_i e_i$ satisfies $\|P_{[n,m]}\| = 1$. When $n = 1$, the basis is said to be monotone.*

Remark 1. A Banach space X with a basis $(e_i)_{i=1}^\infty$ can be renormed with an equivalent bimonotone norm as follows:

$$\left\| \sum_{i=1}^{\infty} a_i e_i \right\| = \sup_{n < m} \left\| P_{[n,m)} \left(\sum_{i=1}^{\infty} a_i e_i \right) \right\|.$$

As every separable Banach space embeds isometrically into $C[0, 1]$, each separable Banach space is isometric to a closed subspace of a space possessing a normalized bimonotone basis. In particular, $Y \equiv \overline{\text{span}}\{y_n\}$ is separable, and hence such a space contains an isometric copy of Y .

Our principal result is stated as follows.

Theorem 7. Let X be a Banach space with the weak Banach–Saks property. Then every super weakly compact subset of X contained in the closed convex hull of a uniformly weakly null sequence X has the Schur property.

Proof. The sufficiency follows immediately from Theorems 5 and 6. We now prove the necessity.

Suppose that X does not have the Schur property. Then there exists a sequence $(x_n)_{n=1}^\infty$ that converges weakly to null but does not converge in norm. Without loss of generality, we may assume $\|x_n\| = 1$ for all $n \in \mathbb{N}$.

Since X has the weak Banach–Saks property, there exists a subsequence $(x'_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ which is uniformly weakly null. For simplicity, we still denote this subsequence by $(x_n)_{n=1}^\infty$. Define

$$K_n = \{n \cdot x : x \in \overline{\text{co}}\{x_i : i \in \mathbb{N}\}\} \cap \frac{1}{n} B_X.$$

and

$$K = \bigcup_{n=1}^{\infty} K_n.$$

where B_X is the closed unit ball of X . We notice that $\bigcup_{j \geq n} K_j \subset \frac{1}{n} B_X$, then $K \subset \bigcup_{j=1}^{n-1} K_j \cup \frac{1}{n} B_X$. Moreover, $0 \in K_j$ for all $j \in \mathbb{N}$ since the sequence $(x_n)_{n=1}^\infty$ is uniformly weakly null. It follows that

$$K \subset \bigcup_{j=1}^{n-1} K_j + \frac{1}{n} B_X.$$

The following shows that K is a super weakly compact set. Since $(x_n)_{n=1}^\infty$ is uniformly weakly null, by Propositions 1 and 2, both $\overline{\text{co}}(x_n)_{n=1}^\infty$ and $\bigcup_{j=1}^{n-1} K_j$ are super weakly compact. In particular, K_n is super weakly compact. Then, by Proposition 3 (which establishes Grothendieck's theorem for super weakly compact sets), we conclude that K is super weakly compact.

By the given condition, there exists a sequence $(y_n)_{n=1}^\infty$ which is uniformly weakly null such that $K \subset \overline{\text{co}}(y_n)_{n=1}^\infty$. Let $Y \equiv \overline{\text{span}}(y_n)_{n=1}^\infty$. Then Y is a separable space containing K and $\overline{\text{co}}(y_n)_{n=1}^\infty$. By Remark 1, Y can be regarded as a subspace of a Banach space with a normalized bimonotone basis $(e_i)_{i=1}^\infty$. Fix an integer N , $0 < \varepsilon < 1$, and a sequence $(\varepsilon_i)_{i=1}^\infty$ decreasing to 0 with $\sum_{i=1}^\infty \varepsilon_i < \varepsilon$. Since $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are uniformly weakly null sequences, there exist increasing sequences $(m_i)_{i=1}^\infty$, $(u_i)_{i=1}^\infty$ and $(v_i)_{i=1}^\infty$ in $\mathbb{N}$ satisfying the following properties:

1. $\|x_{m_j} - P_{[v_{2j-1}, v_{2j})}(x_{m_j})\| < \varepsilon_j^2$, for all $j \in \mathbb{N}$.
2. $\|P_{[v_{2j-1}, v_{2j})}(x_{m_i})\| < \varepsilon_i \varepsilon_j$, for all $i \neq j$.
3. $\|P_{[v_{2j-1}, v_{2j})}(y_i)\| < \varepsilon_j$, for all $i \notin [u_j, u_{j+1})$.

Since $(x_{m_i})_{i=1}^\infty$ is uniformly weakly null, $0 \in \overline{\text{co}}(x_{m_i})_{i=1}^\infty$. We can scale it by N . After scaling, it follows that there exists $D \in \mathbb{N}$ and $(\lambda_i)_{i=1}^D \subseteq [0, 1]$ such that

$$\left\| \sum_{i=1}^D \lambda_i x_{m_i} \right\| \leq \frac{1}{N} \quad \text{and} \quad \sum_{i=1}^D \lambda_i = N.$$

(Indeed, choose N convex combinations of the x_{m_i} 's, in blocks with disjoint support, such that the norms of the convex combinations are each less than $1/N^2$ and sum them). This is equivalent to

$$\sum_{i=1}^D \lambda_i x_{m_i} \in K_N.$$

Since $K_N \subset K$ and $K \subset \overline{\text{co}}(y_n)_{n=1}^\infty$, if we now fix $\delta = \frac{\varepsilon}{D}$, there exists a sequence $(\alpha_i)_{i=1}^\infty$ in $[0, 1]$ such that $\sum_{i=1}^\infty \alpha_i = 1$ and

$$\left\| \sum_{i=1}^D \lambda_i x_{m_i} - \sum_{i=1}^\infty \alpha_i y_i \right\| < \delta.$$

For each $1 \leq j \leq D$, since $(e_i)_{i=1}^\infty$ is bimonotone, we obtain the following estimate:

$$\begin{aligned} \delta &> \left\| \sum_{i=1}^D \lambda_i x_{m_i} - \sum_{i=1}^\infty \alpha_i y_i \right\| \\ &> \left\| P_{[v_{2j-1}, v_{2j}]} \left(\sum_{i=1}^D \lambda_i x_{m_i} - \sum_{i=1}^\infty \alpha_i y_i \right) \right\| \\ &\geq \left\| \sum_{i=1}^D \lambda_i P_{[v_{2j-1}, v_{2j}]}(x_{m_i}) \right\| - \sum_{i=1}^\infty \alpha_i \|P_{[v_{2j-1}, v_{2j}]}(y_i)\| \\ &\geq \left\| \sum_{i=1}^D \lambda_i P_{[v_{2j-1}, v_{2j}]}(x_{m_i}) \right\| - \sum_{i \in [u_j, u_{j+1}]} \alpha_i \|y_i\| - \sum_{i \notin [u_j, u_{j+1}]} \alpha_i \|P_{[v_{2j-1}, v_{2j}]}(y_i)\| \\ &\geq \left\| \sum_{i=1}^D \lambda_i P_{[v_{2j-1}, v_{2j}]}(x_{m_i}) \right\| - \sum_{i \in [u_j, u_{j+1}]} \alpha_i \|y_i\| - \varepsilon_j \quad (\text{by 3}) \\ &\geq \lambda_j \|x_{m_j}\| - \lambda_j \|x_{m_j} - P_{[v_{2j-1}, v_{2j}]}(x_{m_j})\| - \sum_{\substack{i \leq D \\ i \neq j}} \lambda_i \|P_{[v_{2j-1}, v_{2j}]}(x_{m_i})\| \\ &\quad - \sum_{i \in [u_j, u_{j+1}]} \alpha_i \|y_i\| - \varepsilon_j \\ &> \lambda_j - \lambda_j \varepsilon_j^2 - \sum_{\substack{i \leq D \\ i \neq j}} \lambda_i \varepsilon_i \varepsilon_j - \sum_{i \in [u_j, u_{j+1}]} \alpha_i \|y_i\| - \varepsilon_j \quad (\text{by 1 and 2}) \\ &= \lambda_j - \varepsilon_j \left(\lambda_j \varepsilon_j + \sum_{\substack{i \leq D \\ i \neq j}} \lambda_i \varepsilon_i \right) - \sum_{i \in [u_j, u_{j+1}]} \alpha_i \|y_i\| - \varepsilon_j \\ &= \lambda_j - \varepsilon_j \sum_{i=1}^D \lambda_i \varepsilon_i - \sum_{i \in [u_j, u_{j+1}]} \alpha_i \|y_i\| - \varepsilon_j \\ &> \lambda_j - \sum_{i \in [u_j, u_{j+1}]} \alpha_i \|y_i\| - 2\varepsilon_j. \end{aligned}$$

Summing these inequalities over $1 \leq j \leq D$ yields

$$D\delta > \sum_{j=1}^D \lambda_j - \sum_{i=1}^\infty \alpha_i \|y_i\| - 2\varepsilon \geq N - \sup_{i \in \mathbb{N}} \|y_i\| - 2\varepsilon,$$

so that

$$\sup_{i \in \mathbb{N}} \|y_i\| > N - D\delta - 2\varepsilon = N - 3\varepsilon.$$

However, since $\varepsilon > 0$ can be chosen arbitrarily small, $\sup_{i \in \mathbb{N}} \|y_i\| \geq N$. Then, since N is an arbitrary positive integer, $\sup_{i \in \mathbb{N}} \|y_i\| = \infty$. But any uniformly weakly null sequence is bounded; indeed, it is weakly null and therefore norm-bounded by the uniform boundedness principle. Hence $\sup_i \|y_i\| < \infty$, a contradiction. Therefore X must have the Schur property. $\square$

Since norm convergence and weak convergence are equivalent for sequences in a Schur space, if a Banach space X does not have the Schur property, then there exists a sequence in X that is weakly convergent but not norm convergent. We will show that this sequence is not uniformly weakly convergent either.

Lemma 1 ([20]). *Every infinite-dimensional Banach space contains a basic sequence.*

This lemma was proved by S. Mazur [20], p. 4. As a special case, we have the following result.

Lemma 2 ([20]). *Let $(x_n)_{n=1}^\infty$ be a sequence in a Banach space X . If $\liminf_n \|x_n\| > 0$ and $(x_n)_{n=1}^\infty$ converges weakly to 0, then $(x_n)_{n=1}^\infty$ has a subsequence $(x_{n_k})_{k=1}^\infty$ which is a basic sequence of X .*

Theorem 8. *Let X be a Banach space without the Schur property. Then there exists a weakly null sequence in X which is not uniformly weakly null.*

Proof. Let $(x_n)_{n=1}^\infty$ be a weakly null sequence that is not norm convergent to zero, so that $\liminf_n \|x_n\| > 0$. Normalizing if necessary, we assume $\|x_n\| = 1$ for every n . We now show that this sequence is not uniformly weakly null. Suppose first that it is not Cesàro-summable; then Theorem 1 implies that it cannot be uniformly weakly null.

Otherwise, we construct a new sequence $(y_n)_{n=1}^\infty$. Take a sequence $(\sigma_i)_{i=1}^\infty$ with $\sigma_i \in \{-1, 1\}$.

$$\begin{aligned} y_i &= \sigma_i x_i, & \text{where } i &= 1, 2, \dots, n_1, \\ y_{n_1+i} &= \sigma_{n_1} x_{n_1}, & \text{where } i &= 1, 2, \dots, n_2 - n_1, \\ y_{n_2+i} &= \sigma_{n_2+i} x_{n_2+i}, & \text{where } i &= 1, 2, \dots, n_3 - n_2, \\ y_{n_3+i} &= \sigma_{n_3} x_{n_3}, & \text{where } i &= 1, 2, \dots, n_4 - n_3, \\ y_{n_4+i} &= \sigma_{n_4+i} x_{n_4+i}, & \text{where } i &= 1, 2, \dots, n_5 - n_4, \\ &\dots \end{aligned}$$

Repeating this process, we obtain a sequence $(y_n)_{n=1}^\infty$. By choosing appropriate numbers n_j , the following conditions are satisfied:

$$\begin{aligned} \left\| \frac{1}{n_1} \sum_{i=1}^{n_1} y_i \right\| &= \left\| \frac{1}{n_1} \sum_{i=1}^{n_1} \sigma_i x_i \right\| < \frac{1}{2}, \\ \left\| \frac{1}{n_2} \sum_{i=1}^{n_2} y_i \right\| &\geq 1 - \frac{1}{2^2}, \\ \left\| \frac{1}{n_3} \sum_{i=1}^{n_3} y_i \right\| &< \frac{1}{2^3}, \\ &\dots \end{aligned}$$

Proceeding in this pattern, we observe that:

1. When j is odd, the arithmetic mean of the sequence $(y_n)_{n=1}^\infty$ satisfies

$$\left\| \frac{1}{n_j} \sum_{i=1}^{n_j} y_i \right\| \leq \frac{1}{2^j}.$$

2. When j is even, the arithmetic mean of the sequence $(y_n)_{n=1}^\infty$ satisfies

$$\left\| \frac{1}{n_j} \sum_{i=1}^{n_j} y_i \right\| \geq 1 - \frac{1}{2^j}.$$

We now explain that Case 1 can be guaranteed. When j is odd, $y_{n_j} = \sigma_{n_j} x_{n_j}$. Then

$$\lim_{n \rightarrow \infty} \inf_{\sigma_i \in \{-1, 1\}} \left\| \frac{1}{n} \sum_{i=1}^n \sigma_i x_i \right\| = 0.$$

Otherwise,

$$\lim_{n \rightarrow \infty} \inf_{\sigma_i \in \{-1, 1\}} \left\| \frac{1}{n} \sum_{i=1}^n \sigma_i x_i \right\| = C \neq 0,$$

then

$$C \left\| \sum_{i=1}^n \sigma_i x_i \right\|_1 \leq \left\| \sum_{i=1}^n \sigma_i x_i \right\| \leq \left\| \sum_{i=1}^n \sigma_i x_i \right\|_1,$$

which implies that $(x_n)_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_1 , contradicting the fact that $(x_n)_{n=1}^\infty$ is not norm null.

Consequently, as $n \rightarrow \infty$ (equivalently, as $j \rightarrow \infty$), the norm of the Cesàro mean of $(y_n)_{n=1}^\infty$ approaches either 0 or 1. Therefore, $(y_n)_{n=1}^\infty$ fails to be Cesàro-summable.

We claim that $(y_n)_{n=1}^\infty$ is weakly null. Indeed, for any $x^* \in X^*$ and $\varepsilon > 0$, weak nullity of $(x_n)_{n=1}^\infty$ yields some $N_\varepsilon > 0$ such that for all $m > N_\varepsilon$,

$$|\langle x^*, x_m \rangle| < \varepsilon.$$

By the construction of $(y_n)_{n=1}^\infty$, for this $x^* \in X^*$ and $\varepsilon > 0$, there exists $M_\varepsilon > N_\varepsilon$ such that for all $n > M_\varepsilon$,

$$|\langle x^*, y_n \rangle| < \varepsilon.$$

Hence $(y_n)_{n=1}^\infty$ is weakly null.

In summary, the sequence $(y_n)_{n=1}^\infty$ converges weakly to 0, but it is not Cesàro-summable. Therefore, by the definition of uniformly weakly null sequences, $(y_n)_{n=1}^\infty$ is not uniformly weakly null. This completes the proof. $\square$

4. Conclusions and Outlook

In this paper, we establish a Grothendieck-type compactness principle for super weakly compact sets. Our main result (Theorem 7) states that for a Banach space X with the weak Banach–Saks property, every super weakly compact subset is contained in the closed convex hull of a uniformly weakly null sequence if and only if X has the Schur property. This extends the classical Grothendieck principle from norm-compact and weakly compact sets to the setting of super weak compactness.

Our results generalize the work of Dowling et al., who characterized Schur spaces via the weak Grothendieck principle, by employing the weak Banach–Saks property as a bridge to super weak compactness—a connection not previously observed in earlier

studies focusing on other topologies, such as the Mackey dual topology or the absolute weak topology.

The implications are twofold. First, we provide a new characterization of the Schur property in terms of super weakly compact sets and uniformly weakly null sequences. Second, we offer a partial answer to the open question of whether a topological characterization of super weak compactness exists. Although we do not define such a topology, our sequential approach via the weak Banach–Saks property circumvents this difficulty and yields a meaningful Grothendieck-type result.

Our construction of a weakly null sequence that is not uniformly weakly null in every non-Schur space reinforces the necessity of the Schur property in the characterization. This complements the classical fact that Schur spaces are exactly those in which weak and norm convergence coincide, and highlights the subtlety that uniform weak nullity is strictly stronger than mere weak convergence.

Future research directions may include the following: (1) weakening the weak Banach–Saks property in the characterization; (2) developing a topology for super weak compactness; and (3) exploring potential extensions to the setting of ξ -Schur spaces.

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