

Article

A Heavy-Tailed QLindley Distribution for Modelling Skewed Lifetime Data

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Abstract

Lifetime data arising in engineering reliability, survival analysis, actuarial science, and environmental studies often exhibit substantial right-skewness, extreme observations, and heterogeneous hazard-rate structures. Classical lifetime distributions may not adequately capture these characteristics, thereby affecting risk assessment, reliability evaluation, and predictive performance. In this paper, we introduce the Heavy-Tailed QLindley (HTQL) distribution, a new two-parameter heavy-tailed extension of the QLindley model obtained through the New Family of Heavy-Tailed (NFHT) transformation. The proposed distribution provides greater flexibility for modelling positively skewed and heavy-tailed data while preserving analytical tractability. The HTQL model accommodates increasing, decreasing, bathtub-shaped, unimodal, and nearly constant hazard rate functions, making it suitable for applications in reliability analysis, survival studies, actuarial science, and environmental modelling. Several mathematical and statistical properties of the HTQL distribution are derived, including explicit expressions for the quantile function, ordinary and incomplete moments, order statistics, and reliability measures. Important tail-based risk measures such as Value-at-Risk, Tail Value-at-Risk, Tail Variance, Tail Variance Premium, and Expected Shortfall are also obtained. Parameter estimation is investigated using maximum likelihood, ordinary least squares, Cramér–von Mises, and Bayesian approaches, together with bootstrap confidence intervals. A Monte Carlo simulation study is conducted to evaluate the finite-sample performance of the proposed estimators. The practical usefulness of the HTQL distribution is illustrated using three real-world datasets from pharmacokinetics, engineering reliability, and environmental studies. The empirical results show that the HTQL distribution provides highly competitive fits compared with several classical, Lindley-type, and heavy-tailed distributions. Overall, the proposed model constitutes a flexible and parsimonious alternative for modelling positive heavy-tailed data.



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1. Introduction

Modelling positively skewed and heavy-tailed data remains an important challenge in several areas of applied statistics, including reliability engineering, survival analysis, actuarial science, environmental studies, and biomedical research. In many practical situations, classical lifetime distributions fail to adequately capture extreme observations, asymmetric behaviour, and heterogeneous hazard-rate structures. Ignoring these characteristics may lead to poor predictive performance and underestimation of risk, particularly in applications where rare but influential events play a significant role.

Among the numerous lifetime distributions proposed in the literature, the QLindley distribution introduced by [1] has received considerable attention due to its analytical simplicity and improved fitting performance relative to classical Lindley-type models. By extending the classical Lindley distribution, the QLindley model offers greater flexibility for modelling positively skewed lifetime data while retaining a relatively simple mathematical structure. These characteristics make it an attractive alternative in reliability and survival studies.

Despite its flexibility, the QLindley distribution may exhibit limitations when modelling datasets exhibiting pronounced right-skewness, tail sensitivity, extreme observations, and complex hazard-rate shapes. Such features frequently arise in engineering reliability, actuarial science, environmental studies, and risk analysis, motivating the search for more flexible heavy-tailed extensions.

To address these limitations, considerable research has focused on developing flexible probability models that accommodate varying tail behaviours. Several important contributions have appeared in recent years. For example, ref. [2] introduced the Type-I heavy-tailed family, while [3] proposed the heavy-tailed exponential distribution for actuarial applications. Additional contributions include the heavy-tailed log-logistic distribution of [4], the Gleser-type heavy-tailed model of [5], and the generalised heavy-tailed Weibull-G and Topp–Leone-G families proposed by [6,7]. Moreover, ref. [8] developed a new heavy-tailed class for financial applications. These developments highlight the growing interest in constructing parsimonious distributions that accommodate diverse tail behaviours while preserving analytical tractability.

Throughout this paper, analytical tractability refers to the ability to derive explicit or semi-explicit expressions for key distributional characteristics, including the cumulative distribution function, quantile function, moments, reliability measures, and risk measures, thereby avoiding extensive numerical integration and facilitating practical implementation.

A particularly relevant contribution was introduced by [9] through the New Family of Heavy-Tailed (NFHT) distributions. This family provides a simple and effective mechanism for generating flexible extensions of existing baseline models via an additional shape parameter that controls tail behaviour. Using the T-X transformation framework, the NFHT family extends several classical lifetime distributions while maintaining mathematical tractability.

Motivated by these considerations, this paper introduces a new two-parameter model called the Heavy-Tailed QLindley (HTQL) distribution, obtained by applying the NFHT transformation to the QLindley baseline distribution. The proposed model extends the flexibility of the classical QLindley distribution while maintaining a relatively simple mathematical structure.

The proposed extension offers several advantages over existing lifetime models. First, it preserves the parsimonious two-parameter structure while substantially increasing flexibility in the tail region through an additional shape parameter. Second, the model can accommodate a wider range of hazard-rate behaviours and degrees of skewness than the baseline QLindley distribution. Third, despite this increased flexibility, important distributional quantities remain available in explicit or semi-explicit form, facilitating both theoretical analysis and practical implementation. These characteristics make the HTQL distribution particularly attractive for applications involving extreme observations, heterogeneous failure mechanisms, and datasets exhibiting sensitivity to tail behaviour and allow the proposed model to capture complex lifetime behaviours while maintaining a parsimonious parameterisation.

The HTQL distribution possesses several attractive features. It accommodates different degrees of skewness and tail behaviour, generates increasing, decreasing, bathtub-shaped, unimodal, and nearly constant hazard-rate functions, and admits explicit analytical expressions for several statistical and reliability properties. Moreover, the proposed model provides a parsimonious alternative to more complex lifetime distributions with enhanced tail flexibility.

The practical usefulness of the HTQL distribution is illustrated through applications involving pharmacokinetic measurements, engineering reliability data, and environmental observations, where pronounced skewness, extreme values, and heterogeneous ageing patterns are frequently encountered.

The main contributions of this paper can be summarised as follows:

1. Introduction of a new extension of the QLindley distribution with enhanced tail flexibility;
2. Derivation of important structural, statistical, and reliability properties, including moments, quantile function, order statistics, and reliability measures;
3. Development of several inferential procedures based on maximum likelihood, least squares, Cramér–von Mises, and Bayesian estimation;
4. Derivation of actuarial risk measures such as Value-at-Risk and Tail Value-at-Risk;
5. Assessment of the finite-sample performance of the estimators through Monte Carlo simulations;
6. Illustration of the flexibility and applicability of the proposed model using real-world datasets.

Additional mathematical derivations, extended simulation results, MCMC diagnostic plots, bootstrap interval analyses, and supplementary graphical results are provided in the Supplementary Material.

The remainder of this paper is organised as follows. Section 2 introduces the HTQL distribution and presents its main structural properties. Section 3 investigates several statistical and reliability characteristics. Section 4 discusses actuarial risk measures associated with the proposed model. Section 5 presents the estimation procedures. Section 6 contains the Monte Carlo simulation study, while Section 7 illustrates the performance of the HTQL distribution using real datasets. Finally, Section 8 concludes the paper with some final remarks and possible directions for future research.

2. Heavy-Tailed QLindley Distribution and Its Main Structural Functions

In this section, we introduce the Heavy-Tailed QLindley (HTQL) distribution obtained by applying the New Family of Heavy-Tailed (NFHT) transformation to the QLindley baseline distribution. The main structural functions of the proposed model are derived, including the probability density function (PDF), cumulative distribution function (CDF), survival function, hazard rate function, and quantile function.

Let $G(x)$ and $g(x)$ denote the CDF and PDF of a baseline distribution, respectively. According to the NFHT family proposed by [9], the generated distribution is defined through the CDF

$$F(x) = \frac{\beta G(x)}{\beta - 1 + G(x)}, \quad x > 0, \quad \beta > 1. \quad (1)$$

Differentiating (1) with respect to x , the corresponding PDF is obtained as

$$f(x) = \frac{\beta(\beta - 1)g(x)}{[\beta - 1 + G(x)]^2}.$$

Now consider the QLindley distribution with parameter $\alpha > 0$, whose CDF and PDF are respectively given by

$$G(x; \alpha) = 1 - \left(1 + \frac{\alpha^2 x}{1 + \alpha}\right) e^{-\alpha x}, \quad x > 0, \quad (2)$$

and

$$g(x; \alpha) = \frac{\alpha^2}{1 + \alpha} (1 + \alpha x) e^{-\alpha x}, \quad x > 0.$$

Substituting (2) into (1), the CDF of the HTQL distribution becomes

$$F(x; \alpha, \beta) = \frac{\beta \left[1 - \left(1 + \frac{\alpha^2 x}{1 + \alpha}\right) e^{-\alpha x}\right]}{\beta - \left(1 + \frac{\alpha^2 x}{1 + \alpha}\right) e^{-\alpha x}}, \quad x > 0, \quad (3)$$

where $\alpha > 0$ and $\beta > 1$.

Differentiating (3), the corresponding PDF is obtained as

$$f(x; \alpha, \beta) = \frac{\alpha \beta (\beta - 1) (\alpha^2 x + 1) e^{-\alpha x}}{(1 + \alpha) \left[\beta - \left(1 + \frac{\alpha^2 x}{1 + \alpha}\right) e^{-\alpha x}\right]^2}, \quad x > 0. \quad (4)$$

The survival function of the HTQL distribution is therefore given by

$$S(x; \alpha, \beta) = 1 - F(x; \alpha, \beta) = \frac{(\beta - 1) \left(1 + \frac{\alpha^2 x}{1 + \alpha}\right) e^{-\alpha x}}{\beta - \left(1 + \frac{\alpha^2 x}{1 + \alpha}\right) e^{-\alpha x}}.$$

Consequently, the hazard rate function takes the form

$$h(x; \alpha, \beta) = \frac{f(x; \alpha, \beta)}{S(x; \alpha, \beta)} = \frac{\alpha \beta (\alpha^2 x + 1)}{(1 + \alpha) \left[\beta - \left(1 + \frac{\alpha^2 x}{1 + \alpha}\right) e^{-\alpha x}\right] \left(1 + \frac{\alpha^2 x}{1 + \alpha}\right)}.$$

The quantile function can be obtained explicitly in terms of the Lambert-W function. Let $u = F(x)$ with $0 < u < 1$. After some algebraic manipulations, the quantile function of the HTQL distribution is given by

$$Q(u) = \frac{-\alpha W\left(-\frac{(1+\alpha)}{\alpha} \cdot \frac{\beta(u-1)}{u-\beta} \cdot e^{-(1+\alpha)/\alpha}\right) - (1+\alpha)}{\alpha^2}, \quad (5)$$

where $W(\cdot)$ denotes the Lambert-W function. A detailed derivation of the quantile function is provided in Supplementary Material S2.1.

The explicit representation in (5) facilitates random number generation, Monte Carlo simulation studies, and the computation of tail-based risk measures.

Figure 1 presents selected shapes of the PDF and hazard rate function of the HTQL distribution for different parameter combinations. The PDF exhibits substantial flexibility, accommodating right-skewed and tail-sensitive behaviours, whereas the hazard rate function can assume increasing, decreasing, bathtub-shaped, and unimodal forms. These features highlight the adaptability of the proposed model for reliability and survival analysis applications. Figure 1 also illustrates the effect of the parameters on the density and hazard-rate functions. In particular, increasing β leads to noticeable changes in tail behaviour and hazard-rate shape, whereas α primarily affects the concentration and decay rate of the distribution.

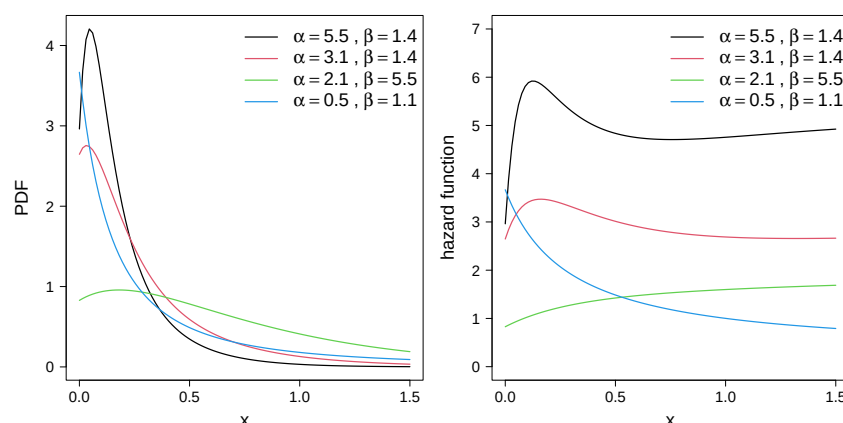


Figure 1. PDF and hazard rate functions of the HTQL distribution for selected parameter values.

Although the parameter space is restricted to $\beta > 1$, values of β close to one produce behaviour increasingly similar to that of the baseline QLindley distribution, while larger values yield greater departures from the baseline model through enhanced tail flexibility.

3. Statistical and Reliability Properties

This section investigates several important statistical and reliability properties of the proposed HTQL distribution. Explicit expressions are derived for the linear representation, ordinary moments, incomplete moments, order statistics, and reliability functions. These results highlight the analytical tractability of the proposed model and facilitate its use in reliability analysis, actuarial science, and risk modelling.

3.1. Linear Representation

A useful representation of the PDF for the HTQL model can be obtained via a power series expansion. Using the identity

$$(1 - z)^{-2} = \sum_{k=0}^{\infty} (k+1)z^k, \quad |z| < 1,$$

the denominator of (4) can be expanded as

$$\left[\beta - \left(1 + \frac{\alpha^2 x}{1 + \alpha} \right) e^{-\alpha x} \right]^{-2} = \frac{1}{\beta^2} \sum_{k=0}^{\infty} \frac{k+1}{\beta^k} \left[\left(1 + \frac{\alpha^2 x}{1 + \alpha} \right) e^{-\alpha x} \right]^k.$$

Substituting this expansion into (4), we obtain

$$f(x) = \frac{\alpha(\beta - 1)}{(1 + \alpha)\beta} (\alpha^2 x + 1) e^{-\alpha x} \sum_{k=0}^{\infty} \frac{k+1}{\beta^k} \left(1 + \frac{\alpha^2 x}{1 + \alpha} \right)^k e^{-k\alpha x}.$$

Applying the binomial expansion

$$\left(1 + \frac{\alpha^2 x}{1 + \alpha}\right)^k = \sum_{j=0}^k \binom{k}{j} \left(\frac{\alpha^2}{1 + \alpha}\right)^j x^j,$$

the PDF can be expressed as

$$f(x) = \frac{\alpha(\beta - 1)}{(1 + \alpha)\beta} \sum_{k=0}^{\infty} \frac{k+1}{\beta^k} \sum_{j=0}^k \binom{k}{j} \left(\frac{\alpha^2}{1 + \alpha}\right)^j (\alpha^2 x^{j+1} + x^j) e^{-(k+1)\alpha x}. \quad (6)$$

Additional algebraic details of the series expansion are presented in Supplementary Material S2.2. The representation in (6) is useful for deriving several mathematical properties of the HTQL distribution.

3.2. Ordinary Moments

Let $X \sim \text{HTQL}(\alpha, \beta)$. The r -th ordinary moment is given by

$$\mu'_r = \frac{\alpha(\beta - 1)}{(1 + \alpha)\beta} \sum_{k=0}^{\infty} \frac{k+1}{\beta^k} \sum_{j=0}^k \binom{k}{j} \left(\frac{\alpha^2}{1 + \alpha}\right)^j \left[\alpha^2 \frac{\Gamma(r + j + 2)}{[(k + 1)\alpha]^{r+j+2}} + \frac{\Gamma(r + j + 1)}{[(k + 1)\alpha]^{r+j+1}} \right].$$

Consequently, the mean and variance are respectively given by

$$\mathbb{E}(X) = \mu'_1, \quad \text{Var}(X) = \mu'_2 - (\mu'_1)^2.$$

The derivational details for the ordinary moments are provided in Supplementary Material S2.3. The coefficients of skewness and kurtosis can also be computed from the ordinary moments using standard relationships. Numerical values of selected descriptive measures for different parameter combinations are also reported in the referred Supplementary Material.

3.3. Incomplete Moments

The r -th incomplete moment of the HTQL distribution is defined as

$$\mu'_r(t) = \int_0^t x^r f(x) dx, \quad t > 0.$$

Substituting the PDF representation in (6), we obtain

$$\mu'_r(t) = \frac{\alpha(\beta - 1)}{(1 + \alpha)\beta} \sum_{k=0}^{\infty} \frac{k+1}{\beta^k} \sum_{j=0}^k \binom{k}{j} \left(\frac{\alpha^2}{1 + \alpha}\right)^j J_{j,k}(t),$$

where

$$J_{j,k}(t) = \int_0^t x^r (\alpha^2 x^{j+1} + x^j) e^{-(k+1)\alpha x} dx.$$

Using the lower incomplete gamma function

$$\gamma(s, t) = \int_0^t u^{s-1} e^{-u} du,$$

we obtain

$$\mu'_r(t) = \frac{\alpha(\beta - 1)}{(1 + \alpha)\beta} \sum_{k=0}^{\infty} \frac{k+1}{\beta^k} \sum_{j=0}^k \binom{k}{j} \left(\frac{\alpha^2}{1 + \alpha}\right)^j \left[\alpha^2 \frac{\gamma(r + j + 2, (k + 1)\alpha t)}{[(k + 1)\alpha]^{r+j+2}} + \frac{\gamma(r + j + 1, (k + 1)\alpha t)}{[(k + 1)\alpha]^{r+j+1}} \right]. \quad (7)$$

Additional derivational steps for the incomplete moments are given in Supplementary Material S2.4. Incomplete moments are useful for deriving actuarial quantities, reliability measures, and tail-based risk measures discussed in subsequent sections.

3.4. Order Statistics

Let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ denote the order statistics of a random sample from the HTQL distribution. The PDF of the i -th order statistic is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1-F(x)]^{n-i} f(x),$$

where $f(x)$ and $F(x)$ are respectively given by (4) and (3). Substituting the corresponding expressions yields

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \left[\frac{\beta \left(1 - \left(1 + \frac{\alpha^2 x}{1+\alpha} \right) e^{-\alpha x} \right)}{\beta - \left(1 + \frac{\alpha^2 x}{1+\alpha} \right) e^{-\alpha x}} \right]^{i-1} \times \left[\frac{(\beta-1) \left(1 + \frac{\alpha^2 x}{1+\alpha} \right) e^{-\alpha x}}{\beta - \left(1 + \frac{\alpha^2 x}{1+\alpha} \right) e^{-\alpha x}} \right]^{n-i} f(x).$$

Order statistics are useful in reliability analysis, life-testing experiments, and inferential procedures involving censored samples.

3.5. Mean Residual Life and Mean Inactivity Time

The mean residual life (MRL) function is an important reliability measure defined by

$$m(t) = \mathbb{E}(X - t \mid X > t) = \frac{1}{S(t)} \int_t^\infty S(x) dx, \quad t > 0,$$

where $S(t)$ is the survival function. Using incomplete moments, the MRL function can be expressed as

$$m(t) = \frac{\mu'_1 - \mu'_1(t)}{S(t)} - t. \quad (8)$$

Similarly, the mean inactivity time (MIT) function is defined as

$$M(t) = \mathbb{E}(t - X \mid X \leq t) = t - \frac{1}{F(t)} \int_0^t x f(x) dx.$$

Hence,

$$M(t) = t - \frac{\mu'_1(t)}{F(t)}. \quad (9)$$

The explicit forms of (8) and (9) follow directly from the incomplete moments in (7). These functions provide insights into the ageing behaviour and reliability structure of the HTQL distribution. Additional reliability properties and numerical illustrations are presented in Supplementary Material S3.

4. Risk Analysis

In this section, we derive several important tail-based risk measures associated with the HTQL distribution. These measures are widely used in actuarial science, insurance, finance, and reliability analysis to quantify potential extreme losses and tail exposure. Owing to the explicit form of the CDF and quantile functions, the proposed model admits explicit expressions for the Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), Tail Variance

(TV), Tail Variance Premium (TVP), and Expected Shortfall (ES). Additional derivations for the tail-based risk measures are provided in Supplementary Material S4.

4.1. Value-at-Risk and Tail Value-at-Risk

Let X be a random variable following the HTQL distribution. The Value-at-Risk (VaR) at confidence level $u \in (0, 1)$ is defined as the u th quantile of the distribution:

$$\text{VaR}_u = Q(u) = F^{-1}(u).$$

Using the quantile function given in Equation (5), we obtain

$$\text{VaR}_u = \frac{-\alpha W\left(-\frac{(1+\alpha)}{\alpha} \frac{\beta(u-1)}{u-\beta} e^{-(1+\alpha)/\alpha}\right) - (1+\alpha)}{\alpha^2}, \quad 0 < u < 1,$$

where $W(\cdot)$ denotes the Lambert-W function.

The Tail Value-at-Risk (TVaR), also known as Conditional Tail Expectation, is defined as

$$\text{TVaR}_u = \mathbb{E}(X \mid X > \text{VaR}_u) = \frac{1}{1-u} \int_u^1 Q(p) dp.$$

Let $q_u = \text{VaR}_u$. Using the first incomplete moment given in Equation (7), we obtain

$$\text{TVaR}_u = \frac{1}{1-u} [\mu - \mu'_1(q_u)], \quad (10)$$

where $\mu = \mathbb{E}(X)$ and $\mu'_1(t)$ denotes the first incomplete moment. Therefore, the TVaR of the HTQL distribution can be evaluated directly using the incomplete moments derived in Section 3.

4.2. Tail Variance and Tail Variance Premium

The Tail Variance (TV) measures the variability of losses beyond the Value-at-Risk threshold and is defined by

$$\text{TV}(u) = \text{Var}(X \mid X > q_u),$$

where $q_u = \text{VaR}_u$. Using conditional moments, we obtain

$$\text{TV}(u) = \mathbb{E}(X^2 \mid X > q_u) - [\text{TVaR}_u]^2.$$

Since

$$\mathbb{E}(X^2 \mid X > q_u) = \frac{1}{1-u} \int_{q_u}^{\infty} x^2 f(x) dx = \frac{\mu'_2 - \mu'_2(q_u)}{1-u},$$

the TV becomes

$$\text{TV}(u) = \frac{\mu'_2 - \mu'_2(q_u)}{1-u} - \left(\frac{\mu - \mu'_1(q_u)}{1-u} \right)^2, \quad (11)$$

where $\mu'_2 = \mathbb{E}(X^2)$ is the second ordinary moment and $\mu'_2(t)$ denotes the second incomplete moment.

The Tail Variance Premium (TVP) combines the expected tail loss and its variability and is defined as

$$\text{TVP}_u = \text{TVaR}_u + \eta \text{TV}(u),$$

where $\eta > 0$ is a loading parameter.

Substituting Equations (10) and (11), we obtain

$$\text{TVP}_u = \frac{\mu - \mu'_1(q_u)}{1-u} + \eta \left[\frac{\mu'_2 - \mu'_2(q_u)}{1-u} - \left(\frac{\mu - \mu'_1(q_u)}{1-u} \right)^2 \right].$$

4.3. Expected Shortfall

The Expected Shortfall (ES) at confidence level $u \in (0, 1)$ is defined as

$$\text{ES}_u = \frac{1}{u} \int_0^u \text{VaR}_t dt.$$

Using the relationship between ES and the incomplete first moment, we obtain

$$\text{ES}_u = \frac{1}{u} \mu'_1(q_u).$$

Hence, the Expected Shortfall can be directly evaluated using the closed-form expression of the incomplete moment derived previously. Table 1 presents numerical values of the principal risk measures for selected parameter combinations and confidence levels. All tail-based risk measures increase with the confidence level u , reflecting the progressive concentration of probability mass in the upper tail region. In particular, the VaR and TVaR values increase substantially as u approaches 1, indicating greater exposure to extreme losses.

The results also show that larger values of β yield systematically higher VaR, TVaR, and TVP, confirming that the parameter β plays an important role in controlling tail heaviness and overall tail risk. This effect is especially pronounced for high confidence levels, where the differences between parameter configurations become considerably larger. Furthermore, the TV values remain relatively stable across confidence levels in some scenarios, suggesting that the HTQL distribution can accommodate both moderate and highly volatile tail behaviours, depending on the parameter configuration.

Comparing the two values of α , larger values of α lead to smaller risk measures overall, reflecting lighter effective tail behaviour and shorter upper-tail support. These numerical results illustrate the flexibility of the HTQL distribution in modelling distinct risk profiles and reinforce its potential usefulness in actuarial science, finance, reliability analysis, and environmental risk assessment.

Table 1. Numerical values of VaR_u , TVaR_u , TV_u , TVP_u , and ES_u for the HTQL distribution.

u	$\alpha = 0.5, \beta = 1.2$					$\alpha = 0.5, \beta = 4.0$				
	VaR_u	TVaR_u	TV_u	TVP_u	ES_u	VaR_u	TVaR_u	TV_u	TVP_u	ES_u
0.75	1.1661	2.8456	3.8662	4.7787	0.3703	3.2156	5.5553	5.3370	8.2238	1.2351
0.80	1.4562	3.2312	4.0876	5.2750	0.4286	3.7420	6.0769	5.3052	8.7295	1.3748
0.85	1.8734	3.7580	4.3354	5.9257	0.5005	4.4205	6.7475	5.2619	9.3785	1.5330
0.90	2.5382	4.5488	4.6088	6.8532	0.5936	5.3746	7.6885	5.1993	10.2881	1.7182
0.95	3.8445	5.9964	4.8878	8.4403	0.7256	6.9947	9.2837	5.0952	11.8313	1.9484
0.99	7.3193	9.5645	4.9900	12.0595	0.9025	10.6874	12.9234	4.8933	15.3701	2.2080
u	$\alpha = 2.6, \beta = 1.2$					$\alpha = 2.6, \beta = 4.0$				
	VaR_u	TVaR_u	TV_u	TVP_u	ES_u	VaR_u	TVaR_u	TV_u	TVP_u	ES_u
0.75	0.3504	0.7183	0.1662	0.8014	0.1272	0.8086	1.2845	0.2124	1.3907	0.3511
0.80	0.4202	0.8019	0.1726	0.8882	0.1433	0.9188	1.3902	0.2094	1.4949	0.3831
0.85	0.5165	0.9141	0.1796	1.0039	0.1622	1.0588	1.5251	0.2059	1.6281	0.4185
0.90	0.6637	1.0791	0.1868	1.1725	0.1857	1.2529	1.7128	0.2016	1.8136	0.4591
0.95	0.9401	1.3744	0.1930	1.4708	0.2171	1.5774	2.0285	0.1954	2.1262	0.5085
0.99	1.6420	2.0838	0.1910	2.1793	0.2567	2.3054	2.7414	0.1850	2.8339	0.5627

5. Estimation Strategies

In this section, we discuss the estimation of the unknown parameters of the HTQL distribution under both classical and Bayesian paradigms. To provide a comprehensive inferential framework, we consider four estimation procedures: Maximum Likelihood Estimation (MLE), Ordinary Least Squares Estimation (OLS), Cramér–von Mises Estimation (CvM), and Bayesian Estimation. In addition, bootstrap confidence intervals are employed to assess the variability and precision of the estimators.

Given the nonlinear structure of the objective functions and the variety of estimation procedures considered in this work, bootstrap resampling was adopted as a unified approach for assessing estimator variability and constructing confidence intervals. This strategy avoids the need for method-specific derivations of analytical variance, facilitates direct comparisons among competing estimation procedures, and provides a practical inferential framework applicable to all estimation methods examined in this study.

All estimation procedures considered in this study require numerical optimisation due to the nonlinear structure of the corresponding objective functions. The parameter space is restricted to $\alpha > 0$ and $\beta > 1$, ensuring that the optimisation is performed over the model's admissible region. Furthermore, the objective functions are continuously differentiable with respect to the model parameters over the admissible parameter space, which supports the numerical implementation of the estimation procedures. In our simulation experiments and real-data applications, stable convergence was consistently achieved, and no numerical difficulties were observed.

5.1. Maximum Likelihood Estimation

Let $x_1, x_2, \dots, x_n$ be a random sample from the HTQL distribution with PDF given in Equation (4). The log-likelihood function is

$$\begin{aligned} \ell(\alpha, \beta) = & n \log \alpha + n \log \beta + n \log(\beta - 1) - n \log(1 + \alpha) + \sum_{i=1}^n \log(\alpha^2 x_i + 1) \\ & - \alpha \sum_{i=1}^n x_i - 2 \sum_{i=1}^n \log \left[\beta - \left(1 + \frac{\alpha^2 x_i}{1 + \alpha} \right) e^{-\alpha x_i} \right]. \end{aligned} \quad (12)$$

Differentiating $\ell(\alpha, \beta)$ with respect to the parameters yields the score equations

$$\frac{\partial \ell}{\partial \alpha} = 0, \quad \frac{\partial \ell}{\partial \beta} = 0.$$

Since these equations do not admit closed-form solutions, the maximum likelihood estimators (MLEs) of α and β are obtained numerically using iterative optimisation procedures such as the Newton–Raphson or BFGS algorithms. Under standard regularity conditions, the MLEs are consistent, asymptotically normal, and asymptotically efficient.

5.2. Ordinary Least Squares Estimation

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ denote the ordered sample. The Ordinary Least Squares (OLS) estimators are obtained by minimising the squared distance between the fitted CDF and the empirical distribution function [10]. Specifically, the OLS estimator $\hat{\theta}_{\text{OLS}} = (\hat{\alpha}_{\text{OLS}}, \hat{\beta}_{\text{OLS}})$ is defined as the minimiser of

$$Q(\alpha, \beta) = \sum_{i=1}^n \left[F(x_{(i)}; \alpha, \beta) - \frac{i}{n+1} \right]^2,$$

where $F(x; \alpha, \beta)$ denotes the CDF for the HTQL model given in Equation (3).

Substituting the CDF for the HTQL model yields

$$Q(\alpha, \beta) = \sum_{i=1}^n \left[\frac{\beta \left[1 - \left(1 + \frac{\alpha^2 x_{(i)}}{1+\alpha} \right) e^{-\alpha x_{(i)}} \right]}{\beta - \left(1 + \frac{\alpha^2 x_{(i)}}{1+\alpha} \right) e^{-\alpha x_{(i)}}} - \frac{i}{n+1} \right]^2.$$

The OLS estimates are obtained numerically using standard optimisation routines.

5.3. Cramér–Von Mises Estimation

The Cramér–von Mises (CvM) estimation method is a minimum-distance approach based on the discrepancy between the empirical and fitted distribution functions. The CvM estimator $\hat{\theta}_{\text{CvM}} = (\hat{\alpha}_{\text{CvM}}, \hat{\beta}_{\text{CvM}})$ minimises

$$W(\alpha, \beta) = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{(i)}; \alpha, \beta) - \frac{2i-1}{2n} \right]^2.$$

For the HTQL distribution,

$$W(\alpha, \beta) = \frac{1}{12n} + \sum_{i=1}^n \left[\frac{\beta \left[1 - \left(1 + \frac{\alpha^2 x_{(i)}}{1+\alpha} \right) e^{-\alpha x_{(i)}} \right]}{\beta - \left(1 + \frac{\alpha^2 x_{(i)}}{1+\alpha} \right) e^{-\alpha x_{(i)}}} - \frac{2i-1}{2n} \right]^2.$$

The minimisation of $W(\alpha, \beta)$ is performed numerically using optimisation algorithms such as Nelder–Mead or BFGS.

5.4. Bayesian Estimation

Bayesian inference combines prior information with the likelihood function to obtain posterior distributions for the model parameters [11,12]. Let $\theta = (\alpha, \beta)$ denote the parameter vector of the HTQL distribution.

5.4.1. Prior and Posterior Distributions

To ensure positivity and posterior propriety, we assume independent gamma priors:

$$\alpha \sim \text{Gamma}(a_0, b_0), \quad \beta - 1 \sim \text{Gamma}(c_0, d_0),$$

where $\text{Gamma}(r, s)$ denotes the gamma distribution with shape r and rate s . The corresponding hyperparameters were selected to represent weakly informative prior beliefs, ensuring that posterior inference was primarily driven by the observed data while preserving the positivity constraints of the parameters. Specifically, we adopted $a_0 = b_0 = c_0 = d_0 = 0.001$.

Given an i.i.d. sample $\mathbf{x} = (x_1, \dots, x_n)$, the posterior PDF is

$$\pi(\alpha, \beta | \mathbf{x}) \propto L(\alpha, \beta; \mathbf{x}) \pi(\alpha, \beta),$$

where $L(\alpha, \beta; \mathbf{x})$ is the likelihood function associated with Equation (12). Since the posterior distribution does not belong to a standard parametric family, Markov chain Monte Carlo (MCMC) methods are employed for posterior sampling.

5.4.2. MCMC Implementation

Posterior samples are generated using a Metropolis–Hastings algorithm with componentwise updates. At iteration t , candidate values are proposed through log-normal random walks:

$$\alpha^* = \alpha^{(t-1)} \exp(\sigma_\alpha \varepsilon_\alpha), \quad \beta^* = \beta^{(t-1)} \exp(\sigma_\beta \varepsilon_\beta),$$

where $\varepsilon_\alpha, \varepsilon_\beta \sim N(0, 1)$. The proposed values are accepted according to the standard Metropolis–Hastings acceptance probabilities. Posterior summaries are then computed from the retained MCMC samples after discarding an initial burn-in period.

Point estimates are obtained using posterior means, while uncertainty is quantified through equal-tail credible intervals and highest posterior density (HPD) intervals.

5.5. Bootstrap Confidence Intervals

To assess the variability of the estimators, bootstrap confidence intervals are constructed following the resampling approach of [13]. Let $\hat{\Xi} = (\hat{\alpha}, \hat{\beta})$ denote an estimator obtained from the original sample.

For each bootstrap replication $s = 1, \dots, S$:

1. Generate a bootstrap sample $\mathbf{x}^{*(s)} = (x_1^{*(s)}, \dots, x_n^{*(s)})$ by sampling with replacement from the observed data.
2. Compute the bootstrap estimate $\hat{\Xi}^{*(s)}$.

The empirical distribution of the bootstrap replicates $\hat{\Xi}^{*(1)}, \hat{\Xi}^{*(2)}, \dots, \hat{\Xi}^{*(S)}$ is then used to estimate standard errors and construct confidence intervals.

In this work, we consider four bootstrap interval procedures: Normal intervals, Basic intervals, Percentile intervals, Bias-corrected and accelerated (BCa) intervals. These approaches provide a comprehensive assessment of the stability and reliability of the proposed estimators.

All estimation procedures considered in this study require numerical optimisation due to the nonlinear structure of the objective functions. The parameter space is constrained to $\alpha > 0$ and $\beta > 1$, which facilitates stable optimisation by restricting the search to admissible regions. In our numerical implementation, convergence was consistently achieved across the simulation scenarios and real-data applications considered in this paper. Although closed-form solutions are not available, the objective functions were found to be smooth and well-behaved over the parameter ranges investigated, with no evidence of numerical instability or convergence failures.

Finally, although formal asymptotic properties are not derived in this paper, the likelihood-based and minimum-distance estimators considered here can be viewed within the general framework of M-estimation under appropriate regularity conditions. The development of rigorous results on consistency and asymptotic normality for the HTQL model is an interesting topic for future research.

6. Comparative Simulation Study of Estimation Methods

In this section, we evaluate the finite-sample performance of the estimation procedures introduced in Section 5 through Monte Carlo simulation experiments. The comparison includes the Maximum Likelihood Estimator (MLE), Ordinary Least Squares Estimator (OLS), Cramér–von Mises Estimator (CvM), and Bayesian Estimator for the HTQL distribution.

This study focuses on the empirical bias and mean squared error (MSE) of the estimators under different parameter settings and sample sizes.

6.1. Simulation Design

Random samples were generated from the HTQL distribution using the quantile function given in Equation (5). Three parameter configurations were considered to evaluate the finite-sample performance of the estimators under different distributional shapes and tail behaviours. The selected scenarios represent distinct combinations of the model parameters and allow us to assess the robustness of the proposed estimation procedures across a variety of settings: $(\alpha, \beta) = (1.5, 1.2)$, $(\alpha, \beta) = (0.8, 2.0)$, and $(\alpha, \beta) = (0.8, 0.25)$. The sample sizes considered were $n \in \{20, 50, 80, 100, 200, 300\}$. For each configuration, $N = 1000$ Monte Carlo replications were generated. For every simulated sample, the parameters were estimated using the four inferential procedures described in Section 5.

The performance of the estimators was assessed through the empirical bias and mean squared error defined as

$$\text{Bias}(\hat{\theta}) = \frac{1}{N} \sum_{r=1}^N (\hat{\theta}_r - \theta), \quad \text{and} \quad \text{MSE}(\hat{\theta}) = \frac{1}{N} \sum_{r=1}^N (\hat{\theta}_r - \theta)^2,$$

where θ denotes the true parameter value and $\hat{\theta}_r$ is the estimate obtained from the r th replication.

Bootstrap percentile confidence intervals based on $B = 1000$ resamples were also constructed for the MLEs in order to evaluate empirical coverage probabilities and interval lengths.

6.2. Simulation Results

Table 2 summarises the empirical bias, MSE, coverage probability (CP), and average confidence length (ACL) for the estimators of α and β under one representative parameter configuration. The remaining simulation scenarios are reported in the Supplementary Material S5.

Table 2. Simulation results for the HTQL distribution: Bias (MSE), coverage probability (CP), and average confidence length (ACL) for $(\alpha, \beta) = (1.5, 1.2)$.

n	Method	$\hat{\alpha}$	$\hat{\beta}$	CP (α)	CP (β)	ACL (α)	ACL (β)
20	MLE	0.045 (0.082)	−0.031 (0.067)	0.938	0.932	0.462	0.392
	OLS	0.062 (0.095)	−0.048 (0.081)	0.926	0.918	0.498	0.425
	CvM	0.058 (0.090)	−0.041 (0.075)	0.931	0.925	0.480	0.408
	Bayes	0.043 (0.078)	−0.030 (0.065)	0.942	0.931	0.452	0.381
50	MLE	0.021 (0.041)	−0.015 (0.033)	0.944	0.939	0.295	0.252
	OLS	0.030 (0.050)	−0.022 (0.041)	0.932	0.925	0.318	0.275
	CvM	0.026 (0.046)	−0.018 (0.037)	0.938	0.931	0.305	0.262
	Bayes	0.020 (0.038)	−0.014 (0.032)	0.948	0.937	0.284	0.242
80	MLE	0.015 (0.028)	−0.010 (0.021)	0.949	0.946	0.228	0.198
	OLS	0.022 (0.034)	−0.014 (0.026)	0.937	0.931	0.248	0.215
	CvM	0.019 (0.031)	−0.012 (0.024)	0.943	0.938	0.236	0.205
	Bayes	0.014 (0.025)	−0.009 (0.020)	0.951	0.948	0.221	0.192
100	MLE	0.012 (0.022)	−0.008 (0.018)	0.951	0.948	0.205	0.178
	OLS	0.018 (0.027)	−0.011 (0.022)	0.939	0.934	0.223	0.194
	CvM	0.015 (0.025)	−0.009 (0.020)	0.946	0.942	0.212	0.185
	Bayes	0.011 (0.020)	−0.007 (0.017)	0.952	0.949	0.198	0.171
200	MLE	0.007 (0.012)	−0.005 (0.009)	0.953	0.950	0.146	0.128
	OLS	0.010 (0.015)	−0.007 (0.012)	0.941	0.938	0.162	0.141
	CvM	0.009 (0.014)	−0.006 (0.011)	0.947	0.944	0.153	0.133
	Bayes	0.006 (0.010)	−0.004 (0.008)	0.948	0.947	0.141	0.124
300	MLE	0.005 (0.008)	−0.003 (0.006)	0.954	0.952	0.118	0.098
	OLS	0.007 (0.010)	−0.004 (0.008)	0.943	0.940	0.132	0.112
	CvM	0.006 (0.009)	−0.004 (0.007)	0.949	0.946	0.124	0.105
	Bayes	0.004 (0.007)	−0.003 (0.006)	0.951	0.950	0.112	0.093

Overall, the results indicate that all estimation methods improve as the sample size increases, with both bias and MSE decreasing towards zero. The MLE and Bayesian estimators generally provide the best finite-sample performance, particularly for moderate and large sample sizes. The OLS and CvM estimators also produce satisfactory estimates, although with slightly larger variability for smaller sample sizes.

The simulation results further indicate that the HTQL distribution can be reliably estimated even in pronounced heavy-tailed scenarios. In particular, the Bayesian estimator tends to exhibit greater numerical stability in small samples due to the regularising effect of the prior distributions.

Regarding interval estimation, the bootstrap confidence intervals achieved empirical coverage probabilities close to the nominal confidence level, while interval lengths decreased as the sample size increased.

Additional simulation configurations are reported in Supplementary Tables S1 and S2.

7. Applications to Real-World Data: Model Fitting and Comparison

To illustrate the practical usefulness and flexibility of the proposed HTQL distribution, we analyse three real-world datasets arising from distinct applied areas: pharmacokinetics, engineering reliability, and environmental science. These datasets exhibit pronounced right-skewness, heterogeneous variability, and extreme observations, which constitute the primary motivation for introducing the HTQL model.

To benchmark the performance of the proposed distribution, we compare it with several well-established lifetime models from the literature, namely the HTED [3], HTLLD [4], XL [14], NXL [15], PXL, PL [16], TPLD [17], GXL [18], Weibull, Lindley (L), and QLindley [1] distributions. These competing models represent a broad class of one- and two-parameter lifetime distributions, thereby providing a comprehensive basis for comparison.

Model adequacy is assessed using the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov–Smirnov (KS) statistic, and Anderson–Darling (AD) statistic, together with their associated p -values. In general, smaller AIC, BIC, KS, and AD values combined with larger p -values indicate superior model performance. When interpreting information criteria such as AIC and BIC, it is important to note that small differences between competing models may not necessarily indicate meaningful differences in model performance. In particular, differences of less than approximately two units are often regarded as providing only weak evidence in favour of one model over another. Therefore, the model comparisons reported in this section should be interpreted jointly with goodness-of-fit statistics, graphical diagnostics, parameter estimates, and the overall adequacy of the fitted distributions.

The datasets considered in this study are described below.

- **Dataset I:** Plasma concentrations of indomethacin (mcg/mL), consisting of $n = 66$ observations, previously analysed by [19]. These observations are commonly used in pharmacokinetic studies to investigate drug absorption and elimination behaviour.
- **Dataset II:** Time between failures (in thousands of hours) of secondary reactor pumps, consisting of $n = 23$ observations, reported by [19,20]. This dataset is frequently employed in reliability and lifetime modelling studies.
- **Dataset III:** Vinyl chloride concentrations (mg/L) from clean-up gradient monitoring wells, consisting of $n = 34$ observations, reported by [21]. The data exhibit substantial positive skewness and heavy-tailed behaviour, making them suitable for environmental risk modelling.

Table 3 summarises the principal descriptive statistics for the three datasets. In all cases, the datasets exhibit positive skewness and moderate-to-high kurtosis, confirming asymmetric, heavy-tailed behaviour.

Table 3. Descriptive statistics for the three datasets.

Dataset	Mean	Median	SD	Skewness	Kurtosis	Min.	Max.
Dataset I	0.5918	0.3400	0.6326	1.4476	4.4709	0.0500	2.7200
Dataset II	1.5779	0.6140	1.9307	1.3643	3.5445	0.0620	6.5600
Dataset III	1.8794	1.1500	1.9526	1.6037	5.0054	0.1000	8.0000

Figure 2 presents several exploratory graphical analyses for the three datasets, including total time on test (TTT) plots and histograms. The plots consistently reveal substantial right-skewness, heavy or moderately heavy tails, and extreme observations. These empirical characteristics justify the use of flexible heavy-tailed models such as the HTQL distribution.

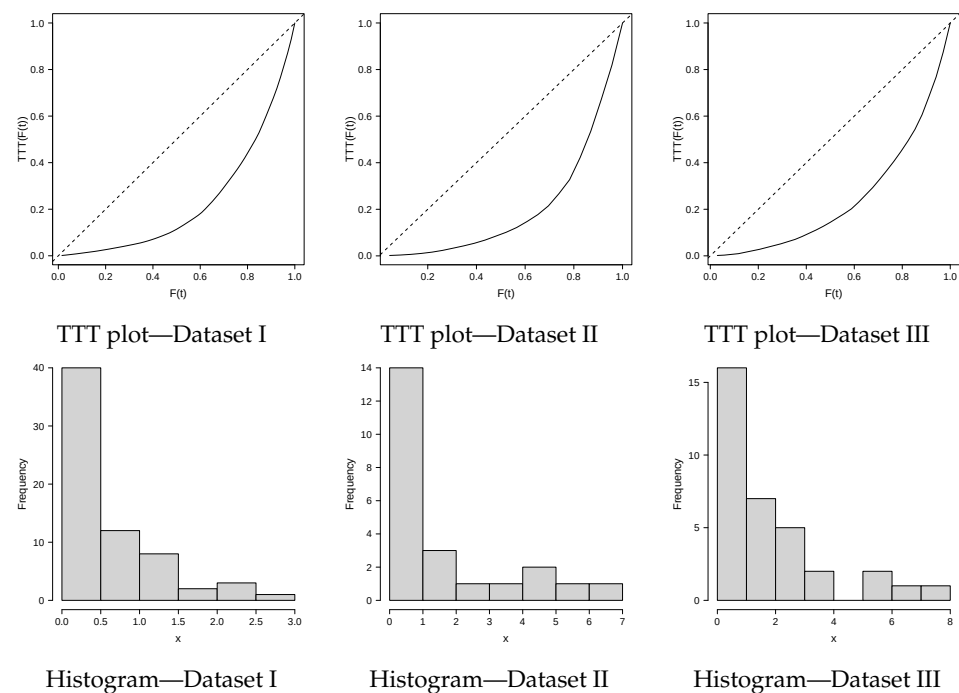
**Figure 2.** Exploratory plots for the three datasets.

Figure 3 displays the fitted PDFs of the HTQL model together with the competing distributions. Visual inspection indicates that the HTQL distribution provides an excellent fit across all datasets, particularly in the upper-tail regions.

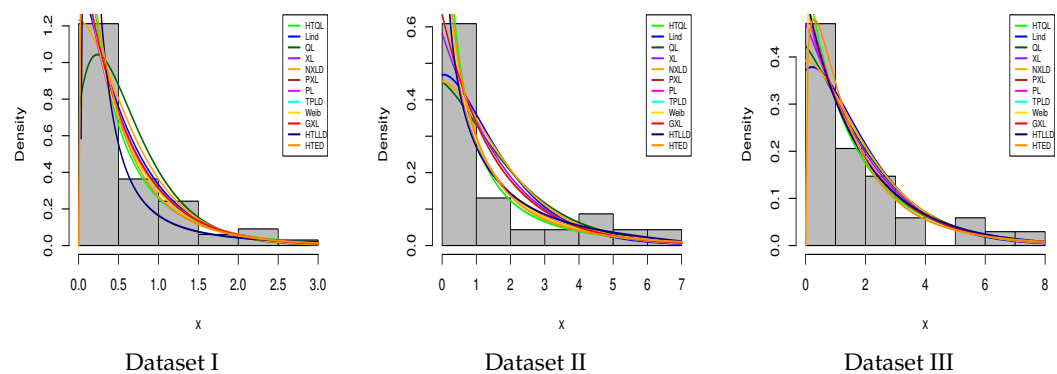
**Figure 3.** Fitted PDFs of the HTQL distribution and competing models for the three datasets.

Table 4 reports the parameter estimates (with standard errors), AIC, BIC, KS, and AD statistics together with their corresponding p -values for all fitted models across the three datasets.

For Dataset I, several candidate distributions provide very similar fits according to the information criteria and goodness-of-fit measures. In particular, although the XL distribution attains the smallest AIC value, the difference relative to the HTQL model is less than one unit, indicating that both models provide essentially comparable empirical fits. It is also worth noting that some competing models exhibit parameter estimates close to the boundary of the parameter space, accompanied by relatively large standard errors, suggesting weak parameter identifiability. In contrast, the HTQL parameter estimates remain within the interior of the admissible parameter space and are associated with moderate standard errors, suggesting satisfactory numerical behaviour of the estimation procedure for this dataset.

Table 4. Summary of parameter estimates (with standard errors) and goodness-of-fit measures for the three datasets.

Model	Parameters (Estimate (SE))	AIC	BIC	AD	<i>p</i> -Value	KS	<i>p</i> -Value
Dataset I							
HTQL	$\alpha = 1.4810$ (0.4196), $\beta = 1.4061$ (0.1932)	65.3343	68.0136	1.2391	0.930	0.1163	0.755
Lind	$\theta = 2.2152$ (0.2208)	66.2739	68.9636	2.2041	0.585	0.1654	0.495
QL	$\theta = 2.6833$ (0.3190)	76.8744	79.0640	6.1224	0.515	0.2333	0.585
XL	$\theta = 1.8933$ (0.2020)	64.9222	69.1118	1.8206	0.720	0.1533	0.595
NXLD	$\theta = 2.4615$ (0.2691)	69.4544	71.6440	3.2451	0.595	0.1908	0.575
PXL	$\alpha = 0.9474$ (0.0898), $\theta = 1.8878$ (0.2018)	66.5887	70.9680	1.4742	0.765	0.1343	0.540
PL	$\alpha = 0.9082$ (0.0874), $\theta = 2.2031$ (0.2201)	67.2221	71.6014	1.4918	0.780	0.1328	0.595
TPLD	$\alpha = 0.0010$ (0.5881), $\theta = 1.6907$ (0.6225)	66.7586	71.1379	1.7635	0.695	0.1513	0.510
Weibull	$k = 0.9546$ (0.0904), $\lambda = 0.5787$ (0.0791)	66.5121	70.8914	1.4719	0.780	0.1348	0.580
GXL	$\alpha = 1.6897$ (0.2251), $\theta = 0.0010$ (72.7707)	66.7586	71.1379	1.7644	0.690	0.1513	0.540
HTED	$a = 1.3131$ (0.4072), $c = 1.1448$ (0.1741)	66.6718	73.2408	1.4884	0.785	0.1412	0.635
	$\alpha = 0.2395$ (0.2631)						
HTLLD	$a = 1.1162$ (0.1371), $b = 0.0214$ (0.0256)	65.8803	74.0389	1.4544	0.815	0.1245	0.715
	$c = 10.0000$ (11.4065), $\alpha = 1.8937$ (2.0755)						
Dataset II							
HTQL	$\alpha = 0.4025$ (0.2547), $\beta = 1.3350$ (0.3059)	67.8526	70.1236	0.2734	0.990	0.0838	0.950
Lind	$\theta = 0.9575$ (0.1505)	72.6108	73.7463	2.2992	0.650	0.2441	0.655
QL	$\theta = 0.8161$ (0.1834)	71.7925	72.9280	2.5380	0.580	0.2640	0.585
XL	$\theta = 0.8290$ (0.1350)	70.1360	71.2715	1.6137	0.690	0.2143	0.725
NXLD	$\theta = 0.9001$ (0.1682)	72.3578	73.4933	2.4546	0.685	0.2562	0.710
PXL	$\alpha = 0.7791$ (0.1215), $\theta = 0.9855$ (0.1839)	69.1771	71.4481	0.4153	0.945	0.1186	0.855
PL	$\alpha = 0.7253$ (0.1130), $\theta = 1.1952$ (0.2121)	69.4952	71.7662	0.4287	0.915	0.1190	0.835
TPLD	$\alpha = 0.0010$ (0.2022), $\theta = 0.6348$ (0.2405)	70.9795	73.2505	1.3051	0.710	0.1994	0.690
Weibull	$k = 0.8077$ (0.1298), $\lambda = 1.3915$ (0.3805)	69.0278	71.2988	0.4042	0.965	0.1184	0.790
GXL	$\alpha = 0.6338$ (0.2060), $\theta = 0.0010$ (50.3276)	70.9795	73.2505	1.3112	0.685	0.2000	0.690
HTED	$a = 0.3848$ (0.2212), $c = 0.9021$ (0.2225)	70.5477	73.9542	0.3677	0.925	0.1081	0.865
	$\alpha = 0.1886$ (0.3461)						
HTLLD	$a = 10.0000$ (12.1215), $b = 7.3706$ (2.0733)	72.1789	76.7209	0.4475	0.925	0.1226	0.815
	$c = 0.0744$ (0.0936), $\alpha = 0.0477$ (0.0852)						
Dataset III							
HTQL	$\alpha = 0.5762$ (0.2050), $\beta = 2.6592$ (2.1222)	114.7444	117.7971	0.2416	0.995	0.0741	0.970
Lind	$\theta = 0.8238$ (0.1054)	114.9073	118.1336	0.6872	0.850	0.1326	0.780
QL	$\theta = 0.7335$ (0.1361)	115.7758	118.3021	0.5532	0.845	0.1291	0.670
XL	$\theta = 0.7145$ (0.0944)	115.4008	118.9271	0.3902	0.960	0.1081	0.845
NXLD	$\theta = 0.7846$ (0.1191)	114.2263	115.7526	0.6377	0.790	0.1328	0.685
PXL	$\alpha = 0.9516$ (0.1175), $\theta = 0.7446$ (0.1231)	115.2340	118.2867	0.3004	0.830	0.0935	0.895
PL	$\alpha = 0.8832$ (0.1095), $\theta = 0.9139$ (0.1452)	115.5198	118.5726	0.3119	0.910	0.0944	0.915
TPLD	$\alpha = 0.0010$ (0.4159), $\theta = 0.5331$ (0.4228)	114.9052	117.9579	0.2720	0.900	0.0890	0.900
Weibull	$k = 1.0102$ (0.1327), $\lambda = 1.8879$ (0.3390)	114.8992	117.9520	0.2826	0.700	0.0918	0.885
GXL	$\alpha = 0.5321$ (0.3225), $\theta = 0.0010$ (82.7451)	114.9052	117.9579	0.2742	0.985	0.0902	0.910
HTED	$a = 0.4336$ (0.2093), $c = 1.2466$ (0.2756)	116.0526	120.6316	0.2556	0.975	0.0842	0.945
	$\alpha = 0.2365$ (0.4381)						
HTLLD	$a = 10.0000$ (19.9491), $b = 9.5237$ (4.9515)	118.7050	124.8104	0.2955	0.915	0.0938	0.885
	$c = 0.0984$ (0.2015), $\alpha = 0.0083$ (0.0297)						

For Dataset II, the HTQL distribution achieves the lowest values of the information criteria among the candidate models. Nevertheless, the differences relative to the Weibull, PXL, and PL distributions are smaller than two units, indicating that these models provide broadly similar descriptions of the data. It should be noted that the standard error of the estimate of α is relatively large compared with the estimated value, suggesting a moderate degree of uncertainty in estimating this parameter. Similar behaviour, however, is observed for several competing models, particularly those involving parameters close to the boundary of the admissible parameter space. Overall, the estimation results exhibit satisfactory numerical behaviour and provide an adequate description of the data.

For Dataset III, the NXLD distribution achieves slightly lower AIC and BIC values than the HTQL model while using fewer parameters. However, the differences in the information criteria are less than one unit, indicating that both distributions provide very similar fits to the observed data. The estimate of the parameter β in the HTQL model has a relatively large standard error, indicating moderate uncertainty in its estimation. Nevertheless, the estimation procedure exhibited satisfactory numerical behaviour and achieved goodness-of-fit statistics comparable to those of the leading competing distributions. These results suggest that the HTQL distribution constitutes a competitive alternative for modelling the dataset while providing additional flexibility through its heavy-tailed construction.

Taken together, the results from the three datasets indicate that the HTQL model exhibits competitive performance relative to the candidate distributions considered in this study. Across the three datasets, the proposed model generally attains favourable values of the information criteria and goodness-of-fit measures, while yielding associated p -values comparable to those of the best competing models. The HTQL distribution is among the models that adequately describe the data according to the goodness-of-fit tests, providing a flexible and practically useful alternative for modelling positively skewed and heavy-tailed data.

Supplementary Material S6.2 presents the profile log-likelihood curves and joint contour plots for the parameters α and β of the HTQL model across the three datasets. In all cases, the profile log-likelihood functions exhibit smooth and well-defined maxima, while the contour plots reveal regular likelihood surfaces around the estimated parameter values. These graphical results provide empirical evidence of stable numerical behaviour and support the implementation of the maximum likelihood estimation procedure adopted in this work.

To reduce the length of the main manuscript and improve readability, additional inferential results are reported in Supplementary Material S6.2–S6.5. These include: (i) detailed comparisons among OLS, CvM, and Bayesian estimation procedures; (ii) bootstrap confidence intervals obtained using Normal, Basic, Percentile, and BCa methods; (iii) trace plots and Gelman–Rubin convergence diagnostics for the MCMC algorithms; and (iv) additional posterior summaries and convergence analyses.

The supplementary analyses provide additional evidence regarding the numerical behaviour of the estimators and the performance of the Bayesian estimation procedure across all datasets. In particular, the trace plots and Gelman–Rubin diagnostics indicate satisfactory convergence of the MCMC chains, while the bootstrap and posterior summaries suggest reasonably stable parameter estimates under the scenarios considered.

8. Conclusions

In this paper, we introduced the Heavy-Tailed QLindley (HTQL) distribution, a new heavy-tailed extension of the QLindley model designed to provide greater flexibility for modelling positively skewed and heavy-tailed lifetime data. The proposed distribution is particularly suitable for applications arising in reliability analysis, actuarial science, survival

studies, environmental modelling, and risk management, where classical lifetime models may fail to adequately capture tail behaviour and heterogeneous hazard-rate structures.

Several important mathematical and statistical properties of the HTQL distribution were derived, including explicit expressions for the quantile function, ordinary and incomplete moments, order statistics, and reliability measures. The proposed model also exhibited considerable flexibility in the shape of the hazard rate function, allowing for increasing, decreasing, bathtub-shaped, and unimodal behaviours commonly encountered in practical applications.

A comprehensive inferential framework was developed based on maximum likelihood, ordinary least squares, Cramér–von Mises, and Bayesian estimation methods. Bootstrap confidence intervals were additionally considered to evaluate the variability and stability of the estimators. The Monte Carlo simulation study showed that all estimation procedures perform satisfactorily as the sample size increases, with the maximum likelihood and Bayesian approaches generally providing the best finite-sample performance.

The practical applicability of the HTQL distribution was illustrated through three real-world datasets from pharmacokinetics, engineering reliability, and environmental studies. The empirical analyses demonstrated that the proposed model provides highly competitive fits compared with several classical and generalised lifetime distributions. In particular, the HTQL distribution showed substantial flexibility in capturing skewness, tail heaviness, and diverse distributional shapes observed in the analysed datasets.

Overall, the HTQL distribution constitutes a flexible, parsimonious, and competitive alternative for modelling positive heavy-tailed data. Its analytical tractability through the availability of explicit expressions for important distributional, reliability, and risk-related quantities, flexible hazard-rate behaviour, and satisfactory inferential performance make it a useful addition to the growing class of heavy-tailed lifetime distributions.

Supplementary Materials: The following supporting information can be downloaded at: <https://www.mdpi.com/article/10.3390/math14132395/s1>, Supplementary S1: Overview of the Supplementary Material; Supplementary S2: Additional Mathematical Derivations; Supplementary S3: Additional Reliability Properties; Supplementary S4: Additional Risk Measure Derivations; Supplementary S5: Additional Simulation Results; Supplementary S6: Additional Results for the Real Data Applications.

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