

Article

Evaluation of q -Truncated-Exponential-Based Hahn–Appell Polynomials in the Framework of Quantum Calculus

Waseem Ahmad Khan ¹, Oğuz Yağcı ², Khidir Shaib Mohamed ^{3,*} and Osman Osman ^{3,*}¹ Department of Electrical Engineering, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia; wkhan1@pmu.edu.sa² Department of Mathematics, Kırıkkale University, Kırıkkale 71450, Türkiye; 1588151031@kku.edu.tr³ Department of Mathematics, College of Science, Qassim University, Buraidah 51452, Saudi Arabia

* Correspondence: k.idris@qu.edu.sa (K.S.M.); o.osman@qu.edu.sa (O.O.)

Abstract

This paper develops a two-variable truncated-exponential-based Hahn–Appell polynomial family of order r in quantum q -calculus, generated by $\mathcal{A}(t)(1 - \eta t^r)^{-1} e_q^{wt}(\zeta t)$, where $\mathcal{A}(t)$ is a normalized determining function and e_q^{wt} denotes the Hahn-type q -exponential. From this relation, we derive connections with the underlying Hahn–Appell basis, the finite expansion induced by the truncation factor, the Hahn lowering property, operational resolvents, step- r recurrences, determinant formulas, and Hahn-factorial representations. We also obtain origin values, normalized monicity, parameter-connection and parameter-differentiation identities, quasi-monomial operators, and the associated Hahn-grid difference equation. Limiting and special cases recover known q -truncated-exponential and classical truncated-exponential Appell constructions. For the unit-Appell specialization $\mathcal{A}(t) = 1$, numerical computations for $q = 1/2$, $r = 2$, $\eta = 1$, and $w = 1$ provide explicit polynomials, zero tables, diagnostics, and plots that illustrate parameter sensitivity, Hahn-grid deformation, conjugate symmetry of non-real zeros, origin zeros in odd degrees, and the observed layered rightward spread of zero clouds.

Keywords: Hahn–Appell polynomials; truncated-exponential polynomials; quantum q -calculus; Hahn difference operator; generating functions; quasi-monomiality; differential equations; zero distribution

MSC: 05A30; 11B83; 33E20; 33E30; 39A13



Academic Editors: Shiqing Zhang and Janusz Brzdęk

Received: 4 June 2026

Revised: 23 June 2026

Accepted: 2 July 2026

Published: 5 July 2026

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1. Introduction

Generalized two-variable polynomial families play an important role in the analysis of differential, difference, and q -difference equations arising in mathematical physics. In particular, truncated-exponential constructions provide a convenient setting for producing hybrid families that combine Appell-type structures with multivariable generating functions. The classical two-variable truncated polynomials serve as a basic model in this direction.

The 2-variable truncated polynomials (2VTP), denoted by $e_n^{(r)}(\zeta, \eta)$, are defined by the generating relation [1]

$$\frac{e^{\zeta t}}{1 - \eta t^r} = \sum_{n=0}^{\infty} e_n^{(r)}(\zeta, \eta) \frac{t^n}{n!}, \quad (1)$$

where

$$e_n^{(r)}(\zeta, \eta) = n! \sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} \frac{\eta^k \zeta^{n-rk}}{(n-rk)!}. \quad (2)$$

The representation (1) and (2) shows how the parameter η truncates the exponential-type expansion through powers of t^r . This mechanism has appeared in a number of recent generalizations of Appell, Hermite, and Gould–Hopper type polynomials. Recent q -truncated and truncated-exponential Appell-type extensions, including q -truncated exponential polynomial, truncated-exponential-based general-Appell, and q -Appell convolution constructions, have also been studied in [2–4].

The present work is carried out in the setting of quantum q -calculus. Over the last decades, q -analogues have become a standard tool in combinatorics, special functions, approximation theory, and mathematical physics. They provide natural deformations of classical formulas and often reveal additional algebraic structure. Throughout the paper we assume $q \in \mathbb{C}$ with $|q| < 1$ and use the standard notation of Andrews, Askey, and Roy [5]. We also refer to recent work on multivariable q -Hermite, q -Appell, and q -Laguerre–Appell type polynomial families; see, for example, [6–12]. In this setting, the q -Pochhammer symbol, q -numbers, q -factorials, and q -exponential functions form the basic analytic ingredients needed for the Hahn–Appell construction developed below.

In this paper, we adopt the following key definitions for q -calculus, using the standard conventions for basic hypergeometric notation and quantum calculus [5,13,14]. Assume $0 < q < 1$. The q -shifted factorial $(a; q)_n$ is defined as:

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - q^k a), \quad n \in \mathbb{N}, \quad (a; q)_0 = 1, \quad |q| < 1. \quad (3)$$

This notation plays a central role in q -calculus, encapsulating the discrete nature of q -analogues and enabling various identities and transformations essential to the theory.

The q -analogue of a number d is given by [5,13]

$$[d]_q = \frac{1 - q^d}{1 - q}, \quad 0 < |q| < 1, \quad n \geq 1, \quad d \in \mathbb{C}. \quad (4)$$

The q -factorial is given by [5,13]

$$[n]_q! = \begin{cases} \prod_{k=1}^n [k]_q, & 0 < q < 1, \quad n \geq 1 \\ 1, & n = 0. \end{cases} \quad (5)$$

Gauss’s q -binomial formula is given by [5,13]

$$(\zeta \pm a)_q^n = \sum_{k=0}^n \binom{n}{k}_q q^{\binom{n-k}{2}} \zeta^k (\pm a)^{n-k}. \quad (6)$$

The two standard q -exponential functions are given by [5,8,13,15,16]

$$e_q(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{[n]_q!}, \quad 0 < |q| < 1, \quad (7)$$

and

$$E_q(\zeta) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} \zeta^n}{[n]_q!}, \quad 0 < |q| < 1, \quad (8)$$

Their product identity is given by [5,13]

$$e_q(\zeta)E_q(\eta) = \sum_{n=0}^{\infty} \frac{(\zeta + \eta)_q^n}{[n]_q!}. \quad (9)$$

Consequently

$$e_q(\zeta)E_q(-\zeta) = 1. \quad (10)$$

The Jackson q -derivative of a function $f(\zeta)$ is defined as [5,13,14]

$$\hat{D}_{q,\zeta}f(\zeta) = \frac{f(q\zeta) - f(\zeta)}{\zeta(1-q)}, \quad 0 < |q| < 1, \quad \zeta \neq 0. \quad (11)$$

In particular, we have

$$\hat{D}_{q,\zeta}\zeta^n = [n]_q\zeta^{n-1}, \quad (12)$$

$$\hat{D}_{q,\zeta}e_q(\alpha\zeta) = \alpha e_q(\alpha\zeta), \quad \alpha \in \mathbb{C}, \quad (13)$$

and

$$\hat{D}_{q,\zeta}^k e_q(\alpha\zeta) = \alpha^k e_q(\alpha\zeta), \quad k \in \mathbb{N}, \quad \alpha \in \mathbb{C}, \quad (14)$$

where $\hat{D}_{q,\zeta}^k$ denotes the k th-order q -derivative with respect to ζ .

The q -derivative of the product of two functions $f(\zeta)$ and $g(\zeta)$ is given by [8,17]

$$\hat{D}_{q,\zeta}(f(\zeta)g(\zeta)) = f(\zeta)\hat{D}_{q,\zeta}g(\zeta) + g(q\zeta)\hat{D}_{q,\zeta}f(\zeta), \quad (15)$$

and

$$\hat{D}_{q,\zeta}\left(\frac{f(\zeta)}{g(\zeta)}\right) = \frac{g(q\zeta)\hat{D}_{q,\zeta}f(\zeta) - f(q\zeta)\hat{D}_{q,\zeta}g(\zeta)}{g(\zeta)g(q\zeta)}. \quad (16)$$

The Hahn-type q -exponential used in the sequel is defined by the Hahn factorial basis [3,13,18,19]:

$$e_q^{wt}(\zeta t) := \sum_{n=0}^{\infty} (\zeta)_q^n \frac{t^n}{[n]_q!} = \frac{(-wt; q)_{\infty}}{(((1-q)\zeta - w)t; q)_{\infty}}. \quad (17)$$

Here, $0 < q < 1$ in the analytic statements, $w > 0$ is the Hahn step used in the real numerical examples, and $\zeta, t \in \mathbb{C}$. The infinite products $(a; q)_{\infty}$ converge absolutely for every fixed complex a when $|q| < 1$. The quotient in (17) is therefore meromorphic as a function of t and is analytic away from the zeros of the denominator, namely away from

$$(((1-q)\zeta - w)t) = q^{-m}, \quad m = 0, 1, 2, \dots$$

The power-series expansion in t is valid in the disk

$$|((1-q)\zeta - w)t| < 1,$$

with the usual interpretation that the radius is infinite when $((1-q)\zeta - w) = 0$. In the algebraic parts of the paper, including coefficient comparisons, determinant formulas, and operator identities, (17) is used as a formal power series; in that setting no analytic convergence assumption is required. If desired, the parameter w may be complex in the formal theory, but the numerical and graphical sections below keep $w > 0$. In particular,

$$e_q^{0t}(\zeta t) = e_q(\zeta t), \quad \lim_{q \rightarrow 1^-} e_q^{wt}(\zeta t) = e^{\zeta t}$$

in the corresponding classical limit. A second Hahn exponential is sometimes introduced through the dual Hahn factorials by

$$E_q^{wt}(\zeta t) := \sum_{n=0}^{\infty} q^{\binom{n}{2}} [\zeta]_{q,w}^n \frac{t^n}{[n]_q!}. \quad (18)$$

Only the first exponential (17) is used in the present construction. The polynomial bases of Hahn calculus are $\{(t)_{q,w}^n\}_{n \geq 0}$ and $\{[t]_{q,w}^n\}_{n \geq 0}$, following the usual Hahn factorial notation [3,13,18], which are given by

$$(t)_{q,w}^n = t(t-w)(t-[2]_q w)(t-[3]_q w) \cdots (t-[n-1]_q w) \quad (19)$$

$$[t]_{q,w}^n = t(qt+w)(q^2t+[2]_q w) \cdots (q^{n-1}t+[n-1]_q w). \quad (20)$$

Applying the Hahn difference operator [13,18] to (19) and (20), we obtain

$$\hat{\mathcal{D}}_{q,w}(t)_{q,w}^n = [n]_q [t]_{q,w}^{n-1},$$

$$\hat{\mathcal{D}}_{q,w}[t]_{q,w}^n = [n]_q [qt+w]_{q,w}^{n-1}$$

where $w > 0$, $q \in (0, 1)$ and $n \geq 1$.

The q -dilatation operator T_z , which acts on any function of the complex variable z , in the following manner [8,20]

$$T_z^k f(z) = f(q^k z), \quad k \in \mathbb{R}, \quad (21)$$

satisfies the property

$$T_z^{-1} T_z^1 f(z) = f(z). \quad (22)$$

The q -derivative of the q -exponential function $e_q(\eta t^m)$ is given as [21]

$$\hat{D}_{q,t} e_q(\eta t^m) = \eta t^{m-1} T_{(\eta;m)} e_q(\eta t^m), \quad (23)$$

where

$$T_{(\eta;m)} = \frac{1 - q^m T_\eta^m}{1 - q T_\eta} = 1 + q T_\eta + \cdots + q^{m-1} T_\eta^{m-1}.$$

Al-Salam introduced the generating function of q -Appell polynomials $\mathcal{A}_{n,q}(\zeta)$ in the form [22–24]

$$\mathcal{A}_q(t) e_q(\zeta t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q}(\zeta) \frac{t^n}{[n]_q!}, \quad (24)$$

$$\mathcal{A}_q(t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q} \frac{t^n}{[n]_q!}, \quad \mathcal{A}_q(t) \neq 0, \quad \mathcal{A}_{0,q} = 1. \quad (25)$$

Important subclasses of the q -Appell family are obtained by the following choices of $\mathcal{A}_q(t)$:

Remark 1. The Table 1 records standard conventions from the q -Appell literature. In the normalized Appell setting used in the main body of this paper, the determining series must be invertible and, when monicity and the determinant formula are stated in their unit form, it is normalized by $\mathcal{A}(0) = 1$. Thus the Bernoulli factor is compatible with this convention, whereas the Euler factor $[2]_q / (e_q(t) + 1)$ has constant term $[2]_q / 2$ and is not normalized unless $q = 1$. A normalized Euler companion is $2 / (e_q(t) + 1)$. The Genocchi factor $[2]_q t / (e_q(t) + 1)$ has zero constant term, so it defines a shifted Appell-type sequence but does not satisfy the invertibility assumptions needed for the determinant, inverse-transfer, and monicity statements below without modification.

Table 1. Several q -Appell polynomial families.

No.	q -Appell Polynomials	Generating Function	$\mathcal{A}_q(t)$
I	q -Bernoulli polynomials [13,25]	$\frac{t}{e_q(t) - 1} e_q(xt) = \sum_{n=0}^{\infty} \mathcal{B}_{n,q}(x) \frac{t^n}{[n]_q!}$	$\frac{t}{e_q(t) - 1}$
II	q -Euler polynomials [13,25]	$\frac{[2]_q}{e_q(t) + 1} e_q(xt) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{[n]_q!}$	$\frac{[2]_q}{e_q(t) + 1}$
III	q -Genocchi polynomials [13,25]	$\frac{[2]_q t}{e_q(t) + 1} e_q(xt) = \sum_{n=0}^{\infty} \mathcal{G}_{n,q}(x) \frac{t^n}{[n]_q!}$	$\frac{[2]_q t}{e_q(t) + 1}$

For $x = 0$, the q -Appell polynomials $\mathcal{A}_{n,q}(x)$ reduce to the associated q -numbers $\mathcal{A}_{n,q}$. Table 2 records the first few values of the q -Bernoulli numbers $\mathcal{B}_{n,q}$, the q -Euler numbers $\mathcal{E}_{n,q}$, and the q -Genocchi numbers $\mathcal{G}_{n,q}$; see [13,25].

Table 2. The first five q -numbers $\mathcal{B}_{n,q}$, $\mathcal{E}_{n,q}$, and $\mathcal{G}_{n,q}$.

n	0	1	2	3	4
$\mathcal{B}_{n,q}$	1	$-(1+q)^{-1}$	$q^2([3]_q!)^{-1}$	$(1-q)q^3([2]_q)^{-1}([4]_q)^{-1}$	$q^4(1-q^2-2q^3-q^4+q^6)([2]_q! [3]_q [5]_q)^{-1}$
$\mathcal{E}_{n,q}$	1	$-\frac{1}{2}$	$\frac{1}{4}(-1+q)$	$\frac{1}{8}(-1+2q+q^2-q^3)$	$\frac{1}{16}(q-1)[3]_q!(q^2-4q+1)$
$\mathcal{G}_{n,q}$	0	$\frac{2-q}{1+q}$	$\frac{q(q-5)}{(1+q)^2}$	$-\frac{3q^2(q-5)}{(1+q)^3} - \frac{3q(2-q)}{(1+q)^2} - \frac{q}{1+q}$	$\frac{-3q}{1+q} \left(\frac{3q^3+10q^2-28q+7}{(1+q)^3} \right)$

Remark 2. Table 2 is included only as background on common q -number conventions. In particular, entries normalized by $\mathcal{E}_{0,q} = 1$ correspond to a normalized Euler convention and should not be mixed with the unnormalized factor $[2]_q/(e_q(t) + 1)$ when $q \neq 1$. None of the proofs in the sequel depends on the numerical entries of Table 2.

The monomiality, or quasi-monomiality, principle is a useful device in the study of polynomial sequences and their operational properties [20,21,26]. In its modern form, it provides a systematic way to characterize polynomial families through a pair of raising and lowering operators. This point of view has proved especially effective for hybrid and multivariable families, including several recent q -analogues; see [20,21,26]. In the present setting, it offers a natural language for describing the Hahn–Appell structure generated in Section 2.

For a q -polynomial set $p_{n,q}(\zeta)$ ($n \in \mathbb{N}$, $\zeta \in \mathbb{C}$), the corresponding q -multiplicative and q -derivative operators $\widehat{M}_q$ and $\widehat{P}_q$ are realized in the standard quasi-monomial form [21,26]

$$\widehat{M}_q \{p_{n,q}(\zeta)\} = p_{n+1,q}(\zeta), \quad (26)$$

and

$$\widehat{P}_q \{p_{n,q}(\zeta)\} = [n]_q p_{n-1,q}(\zeta). \quad (27)$$

The operators $\widehat{M}_q$ and $\widehat{P}_q$ satisfy the following commutation relation:

$$[\widehat{M}_q, \widehat{P}_q] = \widehat{P}_q \widehat{M}_q - \widehat{M}_q \widehat{P}_q. \quad (28)$$

The structural properties of the polynomials $p_{n,q}(\zeta)$ can be read from the operator pair $(\widehat{M}_q, \widehat{P}_q)$. In particular, if these operators admit a differential realization, then $p_{n,q}(\zeta)$ satisfies

$$\widehat{M}_q \widehat{P}_q \{p_{n,q}(\zeta)\} = [n]_q p_{n,q}(\zeta), \quad (29)$$

and

$$\widehat{P}_q \widehat{M}_q \{p_{n,q}(\zeta)\} = [n+1]_q p_{n,q}(\zeta). \quad (30)$$

In view of (26) and (27), the commutator relation is to be understood through its action on the basis elements:

$$[\widehat{M}_q, \widehat{P}_q] \{p_{n,q}(\zeta)\} = ([n+1]_q - [n]_q) p_{n,q}(\zeta) = q^n p_{n,q}(\zeta). \quad (31)$$

Moreover, repeated use of the raising relation (26) gives

$$\widehat{M}_q^r \{p_{n,q}\} = p_{n+r,q}(\zeta). \quad (32)$$

In particular, we have

$$p_{n,q}(\zeta) = \widehat{M}_q^n \{p_{0,q}\} = \widehat{M}_q^n \{1\}, \quad (33)$$

where the initial element of the polynomial sequence $\{p_{n,q}(\zeta)\}_{n \geq 0}$ is $p_{0,q}(\zeta) = 1$.

Moreover, the generating function of $p_{n,q}(\zeta)$ can be written as

$$e_q(\widehat{M}_q t) \{1\} = \sum_{n=0}^{\infty} p_{n,q}(\zeta) \frac{t^n}{[n]_q!}. \quad (34)$$

Motivated by these developments, we now turn to the two-variable truncated-exponential-based Hahn–Appell family generated by a Hahn-type exponential basis. The central point of the paper is that the truncation factor $(1 - \eta t^r)^{-1}$ can be added without losing the Appell-type Hahn operator structure.

The present family is distinguished from three closely related frameworks. Classical truncated-exponential Appell polynomials attach the factor $(1 - \eta t^r)^{-1}$ to the ordinary exponential kernel; q -truncated-exponential polynomials replace the exponential by the Jackson q -exponential but usually keep a fixed unit determining factor; and general q -Appell constructions allow an arbitrary determining function but do not include the Hahn shift in the factorial basis. The construction in (37) combines all three mechanisms in a single normalized framework: an arbitrary invertible Appell factor $\mathcal{A}(t)$, the finite truncation mechanism determined by (r, η) , and the Hahn factorial deformation governed by (q, w) . This combination is not merely a change of variables: the Hahn operator in (46) lowers the shifted factorial basis rather than the ordinary monomial basis, so the determinant, finite-resolvent, and Hahn-grid identities have algebraic content that differs from the corresponding Jackson or classical formulas.

Relative to the recent q -truncated-exponential setting, especially the unit-Appell construction in [3], the main new contributions of this work are as follows:

- a normalized determining factor $\mathcal{A}(t)$ is incorporated, giving an arbitrary Hahn–Appell layer over the q -truncated-exponential polynomial system;
- the ordinary Jackson q -exponential basis is replaced by the Hahn factorial basis $e_q^{wt}(\zeta t)$, so that the deformation depends simultaneously on q and on the Hahn shift w ;
- the Hahn lowering operator $\mathcal{H}_{q,w,\zeta}$ is used to derive a finite operational resolvent, step- r recurrence, determinant representation, Hahn-factorial expansion, and Hahn-grid difference equation;
- normalized special subclasses are separated from conventional non-normalized Euler and Genocchi factors, clarifying when monicity and inverse-determinant formulas are valid;
- reproducible numerical benchmarks and zero-distribution plots are supplied for the unit-Appell specialization, with the exact origin-zero and conjugate-symmetry mechanisms distinguished from empirical zero-cloud observations.

A concise comparison of the related Appell-type frameworks and the distinguishing features of the present Hahn–Appell construction is provided in Table 3.

Table 3. Comparison of related Appell-type frameworks with the present Hahn–Appell construction.

Framework	Typical Generating Kernel	Main Deformation	Distinction of the Present Paper
Classical truncated-exponential Appell systems [1]	$\mathcal{A}(t)e^{\zeta t}/(1-\eta t^r)$	finite step- r correction	Replaced by the Hahn factorial kernel $e_q^{wt}(\zeta t)$ and a Hahn difference operator.
q -truncated-exponential polynomials [3]	$e_q(\zeta t)/(1-\eta t^r)$ or related unit-Appell forms	Jackson q -deformation and truncation	Extended by an arbitrary normalized Appell factor $\mathcal{A}(t)$ and by the Hahn shift w .
General and q -Appell families [4,23,24]	$\mathcal{A}(t)e_q(\zeta t)$ or related q -kernels	determining function $\mathcal{A}(t)$	The finite truncation factor and the Hahn-grid operator are incorporated simultaneously.
Present Hahn–Appell family	$\mathcal{A}(t)e_q^{wt}(\zeta t)/(1-\eta t^r)$	Appell, truncation, and Hahn-grid deformation	Produces finite-resolvent, determinant, parameter-connection, and Hahn-lattice identities in one framework.

The results below are organized to make the logical dependence on the generating function explicit. The new family is introduced through a single Hahn–Appell generating relation and is connected to the untruncated Hahn–Appell basis by an exact finite expansion. The Hahn operator $\mathcal{H}_{q,w,\zeta}$ is then shown to be the lowering operator for both the base sequence and the truncated sequence. The truncation factor is converted into an operational resolvent, a step- r recurrence, and an origin-value formula, while determinant, Hahn-factorial, and quasi-monomial representations are derived for computation and structural analysis. Additional parameter identities are also obtained, including connection formulas between two truncation parameters, η -derivative relations, monicity, and a Hahn-grid difference equation. Finally, the unit-Appell specialization $\mathcal{A}(t) = 1$ is used to verify the first explicit polynomials, illustrate parameter-sensitive applications, and summarize the main zero-distribution features.

The logical progression and dependence structure of the principal results are summarized in Table 4.

Table 4. Logical roadmap of the main structural results.

Stage	Main Result	Depends on	Role in the Paper
Definition and comparison	Generating function (37) and convolution (42)	Hahn exponential and normalized $\mathcal{A}(t)$	Fixes the family and connects it to existing Hahn and q -Appell bases.
Hahn lowering	Lemma 1 and Theorem 3	Hahn factorial identity (47)	Establishes Appell-type structure with respect to $\mathcal{H}_{q,w,\zeta}$.
Truncation mechanism	finite expansion (51), resolvent (52), and step recurrence (58)	coefficient comparison in (37)	Shows that the truncation produces a terminating step- r correction.
Algebraic representations	determinant (56), Hahn-factorial expansion (63), and operational transfer (74)	inverse series of $\mathcal{A}(t)$ and $\mathfrak{B}_\eta(t)$	Provides finite formulas suitable for symbolic and numerical computation.
Parameter and grid identities	(69), (70) and (73)	finite expansion and Hahn lowering	Describes sensitivity with respect to η and evaluation on the Hahn lattice.
Numerical and zero analysis	Benchmark validation and zero-distribution analysis, together with Proposition 1	unit-Appell specialization of the preceding identities	Verifies selected identities and separates exact zero properties from empirical observations.

For ease of reference, the principal symbols, parameters, polynomial families, and operators used throughout the paper are collected in Table 5.

Table 5. Summary of principal symbols and operators used throughout the paper.

Symbol	Meaning
q	Quantum deformation parameter, usually $0 < q < 1$ in analytic and numerical statements.
w	Hahn shift or grid parameter; the numerical illustrations use $w = 1$.
r	Positive integer truncation order in the factor $(1 - \eta t^r)^{-1}$.
η	Truncation parameter controlling the step- r correction terms.
$[n]_q, [n]_q!, \binom{n}{k}_q$	Standard q -number, q -factorial, and Gaussian binomial coefficient.
$(\zeta)_{q,w}^n$	Hahn shifted factorial basis $\prod_{j=0}^{n-1} (\zeta - [j]_q w)$.
$e_q^{wt}(\zeta t)$	Hahn-type exponential $\sum_{n \geq 0} (\zeta)_{q,w}^n t^n / [n]_q!$.
$\mathcal{A}(t)$	Normalized determining series $\sum_{n \geq 0} \mathcal{A}_n t^n / [n]_q!$ with $\mathcal{A}(0) = 1$.
$\mathcal{P}_n(\zeta; w)$	Untruncated Hahn–Appell basis generated by $\mathcal{A}(t)e_q^{wt}(\zeta t)$.
${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w)$	Truncated-exponential-based Hahn–Appell polynomial generated by (37).
$e_n^{(r)}(\zeta, \eta; w)$	Unit-Appell specialization obtained from ${}_{e(r)}\mathcal{P}_n$ by taking $\mathcal{A}(t) = 1$.
$\mathcal{H}_{q,w,\zeta}$	Hahn lowering operator in (46).
$\mathfrak{B}_\eta(t)$	Transfer series $\mathcal{A}(t) / (1 - \eta t^r)$ with coefficients $b_n^{(r)}(\eta)$.
β_n	Inverse coefficients of $1/\mathcal{A}(t)$ used in the determinant representation.

The first quantitative benchmark checks of the principal identities are reported in Table 6.

Table 6. Quantitative validation of selected identities for the benchmark unit-Appell specialization.

Identity Tested	Benchmark Parameters	Common Value	Residual
Step-2 recurrence (58)	$n = 6, \zeta = 1$	18.774261474609	0
Hahn lowering (50)	$n = 6, \zeta = 1$	18.774261474609	0
Origin value (68)	$n = 6, \zeta = 0$	18.774261474609	0
Parameter connection (69)	$n = 6, \zeta = 1, \theta = \frac{1}{2}$	18.774261474609	0

The next section introduces the family in a form compatible with the Appell formalism and derives the recurrence, operator, and determinant identities that follow directly from the generating function.

2. Truncated-Exponential-Based Hahn–Appell Polynomials

In this section, we introduce the two-variable truncated-exponential-based Hahn–Appell polynomials ${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w)$ and collect the identities needed later. The construction combines the q -Appell formulation of Al-Salam with the Hahn calculus and q -truncated-exponential setting [3,18,23]. Unless explicitly stated otherwise, the determining function is the normalized formal q -power series

$$\mathcal{A}(t) = \sum_{n=0}^{\infty} \mathcal{A}_n \frac{t^n}{[n]_q!}, \quad \mathcal{A}_0 = \mathcal{A}(0) = 1. \quad (35)$$

This normalization is the assumption under which the monicity, determinant, inverse-series, and quasi-monomial inverse formulas below are stated. As the underlying Hahn–Appell basis, we write

$$\mathcal{A}(t)e_q^{wt}(\zeta t) = \sum_{n=0}^{\infty} \mathcal{P}_n(\zeta; w) \frac{t^n}{[n]_q!}. \quad (36)$$

Theorem 1. Following the truncated-exponential Appell convention [1,3], let $r \in \mathbb{N}$ with $r \geq 1$. The two-variable truncated-exponential-based Hahn–Appell polynomials of order r , denoted by ${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w)$, are defined by the generating function

$$\frac{\mathcal{A}(t)}{1 - \eta t^r} e_q^{wt}(\zeta t) = \sum_{n=0}^{\infty} {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) \frac{t^n}{[n]_q!}. \quad (37)$$

Proof. Using (36) together with the geometric expansion

$$\frac{1}{1 - \eta t^r} = \sum_{j=0}^{\infty} \eta^j t^{rj},$$

we obtain

$$\frac{\mathcal{A}(t)}{1 - \eta t^r} e_q^{wt}(\zeta t) = \left(\sum_{j=0}^{\infty} \eta^j t^{rj} \right) \left(\sum_{n=0}^{\infty} \mathcal{P}_n(\zeta; w) \frac{t^n}{[n]_q!} \right).$$

Collecting coefficients of like powers of t defines the sequence ${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w)$ and yields (37). $\square$

Remark 3. The symbol η is retained for the truncation parameter in order to remain consistent with the truncated-exponential Appell convention used in [1,3]. It is independent of the Hahn shift w : the parameter η occurs only in the denominator $1 - \eta t^r$ and controls the step- r correction terms, whereas w occurs in the Hahn factorials and in the Hahn operator. Thus no identification between η and w is intended.

Remark 4. Setting $\eta = 1$ in Theorem 37 yields the following specialization.

Corollary 1. The one-parameter specialization ${}_{e(r)}\mathcal{P}_n(\zeta; w)$ is defined by

$$\frac{\mathcal{A}(t)}{1 - t^r} e_q^{wt}(\zeta t) = \sum_{n=0}^{\infty} {}_{e(r)}\mathcal{P}_n(\zeta; w) \frac{t^n}{[n]_q!}. \quad (38)$$

Remark 5. Taking $\mathcal{A}(t) = 1$ in (37) reduces the Hahn–Appell family to the two-variable truncated-exponential-based Hahn polynomials $e_n^{(r)}(\zeta, \eta; w)$, generated by [3]

$$\frac{1}{1 - \eta t^r} e_q^{wt}(\zeta t) = \sum_{n=0}^{\infty} e_n^{(r)}(\zeta, \eta; w) \frac{t^n}{[n]_q!}. \quad (39)$$

Remark 6. Taking $\mathcal{A}(t) = 1$ and $w \rightarrow 0^+$ in (37) reduces the Hahn–Appell family to the two-variable q -truncated-exponential polynomials $e_{n,q}^{(r)}(\zeta, \eta)$, generated by [3]

$$\frac{1}{1 - \eta t^r} e_q(\zeta t) = \sum_{n=0}^{\infty} e_{n,q}^{(r)}(\zeta, \eta) \frac{t^n}{[n]_q!}. \quad (40)$$

Remark 7. As $w \rightarrow 0^+$, Equation (37) reduces to the two-variable q -truncated-exponential-based Appell polynomials of order r , denoted by ${}_{e(r)}\mathcal{A}_n(\zeta, \eta)$, with generating function [1,3]

$$\frac{\mathcal{A}(t)}{1 - \eta t^r} e_q(\zeta t) = \sum_{n=0}^{\infty} {}_{e(r)}\mathcal{A}_n(\zeta, \eta) \frac{t^n}{[n]_q!}. \quad (41)$$

Remark 8. As $w \rightarrow 0^+$ and $q \rightarrow 1^-$, Equation (37) reduces to the corresponding classical truncated-exponential-based Appell polynomials ${}_{e(r)}\mathcal{A}_n(\zeta, \eta)$ (see [1,3]).

Theorem 2. The truncated-exponential-based Hahn–Appell polynomials satisfy the convolution formula

$$e^{(r)}\mathcal{P}_n(\zeta, \eta; w) = \sum_{k=0}^n \binom{n}{k}_q \mathcal{A}_k e_{n-k}^{(r)}(\zeta, \eta; w), \quad (42)$$

where the coefficients $\mathcal{A}_k$ are determined by the normalized Maclaurin expansion (35).

Proof. From Equations (37) and (39), we obtain

$$\sum_{n=0}^{\infty} e^{(r)}\mathcal{P}_n(\zeta, \eta; w) \frac{t^n}{[n]_q!} = \mathcal{A}(t) \sum_{n=0}^{\infty} e_n^{(r)}(\zeta, \eta; w) \frac{t^n}{[n]_q!}. \quad (43)$$

Substituting the expansion (35) for $\mathcal{A}(t)$ into the right-hand side of (43) and comparing coefficients of like powers of t yields (42). $\square$

Hahn-Type Operator and Determinant Formulas

To derive Hahn-side identities directly from the generating function, we use the shifted factorial basis associated with Hahn calculus [3,13,18]

$$(\zeta)_{q,w}^0 = 1, \quad (\zeta)_{q,w}^n = \prod_{j=0}^{n-1} (\zeta - [j]_q w), \quad n \geq 1, \quad (44)$$

for which

$$e_q^{wt}(\zeta t) = \sum_{n=0}^{\infty} \frac{(\zeta)_{q,w}^n}{[n]_q!} t^n. \quad (45)$$

We also introduce the Hahn operator acting on the ζ -variable [13,18]:

$$(\mathcal{H}_{q,w,\zeta} f)(\zeta) = \frac{f(q\zeta + w) - f(\zeta)}{(q-1)\zeta + w}. \quad (46)$$

Lemma 1. For every integer $n \geq 1$, the Hahn operator satisfies

$$\mathcal{H}_{q,w,\zeta} \{(\zeta)_{q,w}^n\} = [n]_q (\zeta)_{q,w}^{n-1}. \quad (47)$$

Consequently,

$$\mathcal{H}_{q,w,\zeta} e_q^{wt}(\zeta t) = t e_q^{wt}(\zeta t). \quad (48)$$

Proof. By (44),

$$(q\zeta + w)_{q,w}^n = \prod_{j=0}^{n-1} (q\zeta + w - [j]_q w) = q^{n-1} (q\zeta + w) \prod_{j=0}^{n-2} (\zeta - [j]_q w).$$

Hence

$$(q\zeta + w)_{q,w}^n - (\zeta)_{q,w}^n = \left(q^{n-1} (q\zeta + w) - (\zeta - [n-1]_q w) \right) (\zeta)_{q,w}^{n-1}.$$

Using $[n]_q = 1 + q[n-1]_q$, the factor in parentheses becomes

$$(q^n - 1)\zeta + [n]_q w = [n]_q ((q-1)\zeta + w),$$

which proves (47). Applying (47) termwise to (45) gives

$$\mathcal{H}_{q,w,\zeta} e_q^{wt}(\zeta t) = \sum_{n=1}^{\infty} \frac{[n]_q (\zeta)_{q,w}^{n-1}}{[n]_q!} t^n = t \sum_{m=0}^{\infty} \frac{(\zeta)_{q,w}^m}{[m]_q!} t^m,$$

which is exactly (48). $\square$

Theorem 3. The Hahn–Appell basis and its truncated-exponential extension satisfy the following lowering relations

$$\mathcal{H}_{q,w,\zeta} \mathcal{P}_n(\zeta; w) = [n]_q \mathcal{P}_{n-1}(\zeta; w), \quad n \geq 1, \quad (49)$$

and

$$\mathcal{H}_{q,w,\zeta} e^{(r)} \mathcal{P}_n(\zeta, \eta; w) = [n]_{q^{(r)}} \mathcal{P}_{n-1}(\zeta, \eta; w), \quad n \geq 1. \quad (50)$$

Proof. Applying $\mathcal{H}_{q,w,\zeta}$ to (36) and using (48), we obtain

$$\sum_{n=0}^{\infty} \mathcal{H}_{q,w,\zeta} \mathcal{P}_n(\zeta; w) \frac{t^n}{[n]_q!} = t \sum_{n=0}^{\infty} \mathcal{P}_n(\zeta; w) \frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty} [n]_q \mathcal{P}_{n-1}(\zeta; w) \frac{t^n}{[n]_q!},$$

and coefficient comparison yields (49). The proof of (50) is identical after replacing (36) by (37). $\square$

Theorem 4. For every $n \geq 0$, the truncated-exponential-based Hahn–Appell polynomials admit the finite expansion

$$e^{(r)} \mathcal{P}_n(\zeta, \eta; w) = \sum_{k=0}^{\lfloor n/r \rfloor} \eta^k \frac{[n]_q!}{[n-rk]_q!} \mathcal{P}_{n-rk}(\zeta; w). \quad (51)$$

Equivalently,

$$e^{(r)} \mathcal{P}_n(\zeta, \eta; w) = \left(I - \eta \mathcal{H}_{q,w,\zeta}^r \right)^{-1} \mathcal{P}_n(\zeta; w), \quad (52)$$

where the inverse is not a bounded-operator inverse on an infinite-dimensional function space. It is the formal Neumann inverse on the finite-dimensional polynomial space Π_n of polynomials of degree at most n . Since $\mathcal{H}_{q,w,\zeta}$ lowers degree by one, $\mathcal{H}_{q,w,\zeta}^{rk} \mathcal{P}_n = 0$ whenever $rk > n$, and hence

$$\left(I - \eta \mathcal{H}_{q,w,\zeta}^r \right)^{-1} \mathcal{P}_n(\zeta; w) = \sum_{k=0}^{\lfloor n/r \rfloor} \eta^k \mathcal{H}_{q,w,\zeta}^{rk} \mathcal{P}_n(\zeta; w).$$

Proof. Multiplying (36) by $(1 - \eta t^r)^{-1} = \sum_{k \geq 0} \eta^k t^{rk}$ gives

$$\sum_{n=0}^{\infty} e^{(r)} \mathcal{P}_n(\zeta, \eta; w) \frac{t^n}{[n]_q!} = \left(\sum_{k=0}^{\infty} \eta^k t^{rk} \right) \left(\sum_{m=0}^{\infty} \mathcal{P}_m(\zeta; w) \frac{t^m}{[m]_q!} \right).$$

Comparing coefficients of t^n gives (51). On the other hand, repeated use of (49) yields

$$\mathcal{H}_{q,w,\zeta}^{rk} \mathcal{P}_n(\zeta; w) = \frac{[n]_q!}{[n-rk]_q!} \mathcal{P}_{n-rk}(\zeta; w), \quad 0 \leq rk \leq n,$$

while $\mathcal{H}_{q,w,\zeta}^{rk} \mathcal{P}_n(\zeta; w) = 0$ for $rk > n$. Thus the Neumann series terminates after $\lfloor n/r \rfloor$ terms on Π_n , and substitution into the finite series gives (52). $\square$

The determinant formula below is a triangular inversion formula, in the spirit of determinantal treatments of Appell-type polynomial systems [20,27]. It is useful because it converts the generating-function definition into an explicit finite algebraic representation in terms of the base truncated Hahn polynomials. Thus the arbitrary Appell factor $\mathcal{A}(t)$ enters only through the inverse coefficients β_n , while the dependence on (ζ, η, w) remains concentrated in $e_n^{(r)}(\zeta, \eta; w)$.

Theorem 5. Under the normalization (35), let

$$\frac{1}{\mathcal{A}(t)} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]_q!}, \quad \beta_0 = 1. \quad (53)$$

Then the base truncated Hahn polynomials $e_n^{(r)}(\zeta, \eta; w)$ and the Hahn–Appell family are related by

$$e_n^{(r)}(\zeta, \eta; w) = \sum_{k=0}^n \binom{n}{k}_q \beta_k {}_{e(r)}\mathcal{P}_{n-k}(\zeta, \eta; w). \quad (54)$$

Consequently,

$${}_{e(r)}\mathcal{P}_0(\zeta, \eta; w) = 1, \quad {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = e_n^{(r)}(\zeta, \eta; w) - \sum_{k=1}^n \binom{n}{k}_q \beta_k {}_{e(r)}\mathcal{P}_{n-k}(\zeta, \eta; w), \quad (55)$$

and the determinant representation

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \begin{vmatrix} \beta_0 & 0 & 0 & \cdots & 0 & e_0^{(r)}(\zeta, \eta; w) \\ \beta_1 & \binom{1}{1}_q \beta_0 & 0 & \cdots & 0 & e_1^{(r)}(\zeta, \eta; w) \\ \beta_2 & \binom{2}{1}_q \beta_1 & \binom{2}{2}_q \beta_0 & \cdots & 0 & e_2^{(r)}(\zeta, \eta; w) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_n & \binom{n}{1}_q \beta_{n-1} & \binom{n}{2}_q \beta_{n-2} & \cdots & \binom{n}{n-1}_q \beta_1 & e_n^{(r)}(\zeta, \eta; w) \end{vmatrix}. \quad (56)$$

Proof. By (53), the reciprocal of $\mathcal{A}(t)$ is represented by the formal q -exponential series

$$\frac{1}{\mathcal{A}(t)} = \sum_{k=0}^{\infty} \beta_k \frac{t^k}{[k]_q!}, \quad \beta_0 = 1.$$

Multiplying both sides of (37) by this inverse series gives

$$\frac{1}{1 - \eta t^r} e_q^{wt}(\zeta t) = \left(\sum_{k=0}^{\infty} \beta_k \frac{t^k}{[k]_q!} \right) \left(\sum_{m=0}^{\infty} {}_{e(r)}\mathcal{P}_m(\zeta, \eta; w) \frac{t^m}{[m]_q!} \right).$$

Using (39) on the left-hand side and multiplying the two q -factorial series on the right-hand side, we obtain

$$\sum_{n=0}^{\infty} e_n^{(r)}(\zeta, \eta; w) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k}_q \beta_k {}_{e(r)}\mathcal{P}_{n-k}(\zeta, \eta; w) \right) \frac{t^n}{[n]_q!}.$$

Comparison of coefficients of $t^n / [n]_q!$ proves (54).

Since $\beta_0 = 1$, Equation (54) is a lower triangular system with unit diagonal. For $n = 0$ it gives ${}_{e(r)}\mathcal{P}_0(\zeta, \eta; w) = e_0^{(r)}(\zeta, \eta; w) = 1$. For $n \geq 1$, isolating the term corresponding to $k = 0$ in (54) gives

$$e_n^{(r)}(\zeta, \eta; w) = {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) + \sum_{k=1}^n \binom{n}{k}_q \beta_k {}_{e(r)}\mathcal{P}_{n-k}(\zeta, \eta; w),$$

which is equivalent to the recursive identity (55).

To obtain the determinant formula, write the first $n + 1$ equations in (54) as the linear system

$$\sum_{j=0}^m \binom{m}{j}_q \beta_{m-j} {}_{e(r)}\mathcal{P}_j(\zeta, \eta; w) = e_m^{(r)}(\zeta, \eta; w), \quad m = 0, 1, \dots, n.$$

The coefficient matrix is lower triangular and all its diagonal entries are $\beta_0 = 1$; hence its determinant is one. Applying Cramer's rule to the unknown ${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w)$ amounts to replacing the last column of this triangular coefficient matrix by the right-hand side vector $(e_0^{(r)}, e_1^{(r)}, \dots, e_n^{(r)})^T$. The resulting determinant is precisely (56). This completes the proof. $\square$

Example 1. The first non-trivial instance of (56) is

$${}_{e(r)}\mathcal{P}_2(\zeta, \eta; w) = \begin{vmatrix} \beta_0 & 0 & e_0^{(r)}(\zeta, \eta; w) \\ \beta_1 & \binom{1}{1}_q \beta_0 & e_1^{(r)}(\zeta, \eta; w) \\ \beta_2 & \binom{2}{1}_q \beta_1 & e_2^{(r)}(\zeta, \eta; w) \end{vmatrix}.$$

Under the standing normalization $\beta_0 = 1$, this gives

$${}_{e(r)}\mathcal{P}_2(\zeta, \eta; w) = e_2^{(r)}(\zeta, \eta; w) - [2]_q \beta_1 e_1^{(r)}(\zeta, \eta; w) + ([2]_q \beta_1^2 - \beta_2) e_0^{(r)}(\zeta, \eta; w),$$

which is the same expression obtained recursively from (55). This example also fixes the notation in (56): products such as $\binom{m}{j}_q \beta_{m-j}$ are scalar coefficients multiplying the corresponding unknown polynomial, and the final column contains only the base polynomials $e_m^{(r)}(\zeta, \eta; w)$.

Remark 9. If the determining factor is invertible but not normalized, say $\mathcal{A}(0) = a_0 \neq 1$, then $\beta_0 = 1/a_0$ and the triangular inversion above acquires the corresponding diagonal factor. The unit-diagonal determinant (56) is therefore the normalized form used throughout the main theory.

Remark 10. The operational resolvent (52) and the determinant representation (56) serve different but complementary purposes. The resolvent explains why the truncation is finite on each degree: $\mathcal{H}_{q,w,\zeta}^r$ lowers the degree by r , so the Neumann series terminates after $\lfloor n/r \rfloor$ terms. It is therefore a degree-wise algebraic identity, not an analytic inversion problem. This is the main advantage over a purely generating-function description, because it immediately yields the step- r recurrence, origin values, and finite computation scheme.

The determinant formula, by contrast, isolates the contribution of the Appell determining function. Once the inverse coefficients β_k of $1/\mathcal{A}(t)$ are known, (56) reconstructs ${}_{e(r)}\mathcal{P}_n$ from the unit-Appell truncated Hahn polynomials $e_m^{(r)}(\zeta, \eta; w)$ through a triangular finite system. This makes explicit which parts of the family come from the arbitrary Appell layer and which parts come from the Hahn-truncated kernel. In computational terms, (52) is best suited for repeated evaluation at fixed n , while (56) is useful for comparing different choices of $\mathcal{A}(t)$ within the same Hahn-truncated background.

Corollary 2. For $0 \leq n < r$, the truncation parameter does not affect the first r members:

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \mathcal{P}_n(\zeta; w), \quad 0 \leq n < r. \quad (57)$$

For $n \geq r$, the same sequence satisfies the finite-step recurrence

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \mathcal{P}_n(\zeta; w) + \eta \frac{[n]_q!}{[n-r]_q!} {}_{e(r)}\mathcal{P}_{n-r}(\zeta, \eta; w). \quad (58)$$

Equivalently,

$$\left(I - \eta \mathcal{H}_{q,w,\zeta}^r \right) {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \mathcal{P}_n(\zeta; w). \quad (59)$$

Proof. Equation (58) follows by multiplying (37) by $1 - \eta t^r$ and comparing coefficients of t^n . Formula (57) is the case in which no term of degree t^{n-r} exists. Finally, (59) follows from the repeated lowering law (50). $\square$

Remark 11. Corollary 2 separates the Appell contribution from the truncation contribution. In particular, the order r produces a ladder relation with step length r ; this is the algebraic source of the parity phenomena observed in the numerical examples when $r = 2$.

3. Recurrence Relations, Summation Formulas, and the Monomiality Principle

In this section, we record additional structural consequences of the generating function (37), using standard Appell-type operational and quasi-monomial techniques [20,21,24,26]. These results are useful for computation because they express the family in the Hahn factorial basis, give exact finite recurrences, describe the dependence on the truncation parameter, and identify the monomiality operators of the sequence.

The coefficient series $\mathfrak{B}_\eta(t)$ gives a compact way to state the Hahn factorial expansion, the origin value, the operational transfer formula, and the quasi-monomiality principle. In particular, the arbitrary Appell factor $\mathcal{A}(t)$ enters these formulas only through the coefficients $b_n^{(r)}(\eta)$.

Set, with notation consistent with the coefficient conventions for q -Appell-type generating functions [23,24],

$$\mathfrak{B}_\eta(t) := \frac{\mathcal{A}(t)}{1 - \eta t^r} = \sum_{n=0}^{\infty} b_n^{(r)}(\eta) \frac{t^n}{[n]_q!}. \quad (60)$$

Then, coefficient comparison gives

$$b_n^{(r)}(\eta) = \sum_{j=0}^{\lfloor n/r \rfloor} \eta^j \frac{[n]_q!}{[n-rj]_q!} \mathcal{A}_{n-rj}. \quad (61)$$

Lemma 2. For $0 \leq m \leq n$,

$$\mathcal{H}_{q,w,\zeta}^m \{ {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) \} = \frac{[n]_q!}{[n-m]_q!} {}_{e(r)}\mathcal{P}_{n-m}(\zeta, \eta; w), \quad (62)$$

while $\mathcal{H}_{q,w,\zeta}^m \{ {}_{e(r)}\mathcal{P}_n \} = 0$ for $m > n$.

Proof. The case $m = 1$ is exactly (50). Repeated application of (50) gives (62); if $m > n$, the iteration reaches the constant polynomial and then gives zero. $\square$

Theorem 6. The truncated-exponential-based Hahn–Appell polynomials possess the finite Hahn factorial expansion

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \sum_{k=0}^n \binom{n}{k}_q b_k^{(r)}(\eta) (\zeta)_{q,w}^{n-k}, \quad (63)$$

where $b_k^{(r)}(\eta)$ is given by (61). Equivalently,

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \sum_{k=0}^n \binom{n}{k}_q \left(\sum_{j=0}^{\lfloor k/r \rfloor} \eta^j \frac{[k]_q!}{[k-rj]_q!} \mathcal{A}_{k-rj} \right) (\zeta)_{q,w}^{n-k}. \quad (64)$$

Proof. Using (60) and (45), the generating function (37) can be written as

$$\left(\sum_{k=0}^{\infty} b_k^{(r)}(\eta) \frac{t^k}{[k]_q!} \right) \left(\sum_{m=0}^{\infty} (\zeta)_{q,w}^m \frac{t^m}{[m]_q!} \right).$$

The Cauchy product for q -factorial series gives the coefficient of $t^n / [n]_q!$ as the right-hand side of (63). Substituting (61) into (63) gives (64). $\square$

Corollary 3. Under the normalization (35), for every $n \geq 0$, ${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w)$ is a monic polynomial of exact degree n in the variable ζ . More precisely,

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = (\zeta)_{q,w}^n + \text{terms of degree at most } n-1. \quad (65)$$

Consequently,

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \zeta^n + O(\zeta^{n-1}) \quad (\zeta \rightarrow \infty). \quad (66)$$

Proof. In the Hahn factorial expansion (63), the term corresponding to $k = 0$ is

$$\binom{n}{0}_q b_0^{(r)}(\eta) (\zeta)_{q,w}^n = (\zeta)_{q,w}^n,$$

because $b_0^{(r)}(\eta) = \mathcal{A}_0 = 1$. Every term with $k \geq 1$ contains $(\zeta)_{q,w}^{n-k}$ and therefore has degree at most $n-1$. Since $(\zeta)_{q,w}^n$ is monic as an ordinary polynomial in ζ , Equations (65) and (66) follow. $\square$

Remark 12. For a non-normalized but invertible determining series with $\mathcal{A}(0) = a_0 \neq 0$, the same argument gives leading term $a_0(\zeta)_{q,w}^n$. Hence monicity in the ordinary sense is equivalent to the normalization $a_0 = 1$.

Corollary 4. The value at $\zeta = 0$ is determined solely by the coefficient of $\mathfrak{B}_\eta(t)$:

$${}_{e(r)}\mathcal{P}_n(0, \eta; w) = b_n^{(r)}(\eta) = \sum_{j=0}^{\lfloor n/r \rfloor} \eta^j \frac{[n]_q!}{[n-rj]_q!} \mathcal{A}_{n-rj}. \quad (67)$$

In the unit-Appell specialization $\mathcal{A}(t) = 1$,

$$e_n^{(r)}(0, \eta; w) = \begin{cases} \eta^{n/r} [n]_q!, & r \mid n, \\ 0, & r \nmid n. \end{cases} \quad (68)$$

Proof. Since $(0)_{q,w}^0 = 1$ and $(0)_{q,w}^m = 0$ for every $m \geq 1$, only the term $k = n$ remains in (63). This proves (67). If $\mathcal{A}(t) = 1$, then $\mathcal{A}_0 = 1$ and $\mathcal{A}_m = 0$ for $m \geq 1$, and (68) follows immediately. $\square$

Remark 13. Formula (68) gives an analytic explanation of the zero at the origin in the odd-degree unit-Appell polynomials when $r = 2$. Hence this feature is not merely a numerical artifact of the zero plots.

The next identities make explicit how the sequence changes when the truncation parameter is varied and how the Hahn lowering relation becomes a first-order difference equation on the Hahn lattice, following the standard Hahn-difference notation [13,18].

Theorem 7. Let η and θ be two complex parameters. Then, for every $n \geq 0$,

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = {}_{e(r)}\mathcal{P}_n(\zeta, \theta; w) + (\eta - \theta) \sum_{j=1}^{\lfloor n/r \rfloor} \eta^{j-1} \frac{[n]_q!}{[n-rj]_q!} {}_{e(r)}\mathcal{P}_{n-rj}(\zeta, \theta; w). \quad (69)$$

In particular, the specialization $\theta = 0$ gives the finite expansion (51).

Proof. Denote the generating function in (37) by $F_\eta(t)$. For the parameter θ we have

$$F_\theta(t) = \frac{\mathcal{A}(t)}{1 - \theta t^r} e_q^{wt}(\zeta t).$$

Therefore

$$F_\eta(t) = \frac{1 - \theta t^r}{1 - \eta t^r} F_\theta(t) = \left(1 + (\eta - \theta) \frac{t^r}{1 - \eta t^r} \right) F_\theta(t).$$

Since

$$\frac{t^r}{1 - \eta t^r} = \sum_{j=1}^{\infty} \eta^{j-1} t^{rj},$$

we obtain

$$\sum_{n=0}^{\infty} {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) \frac{t^n}{[n]_q!} = \left(1 + (\eta - \theta) \sum_{j=1}^{\infty} \eta^{j-1} t^{rj} \right) \sum_{m=0}^{\infty} {}_{e(r)}\mathcal{P}_m(\zeta, \theta; w) \frac{t^m}{[m]_q!}.$$

Comparison of the coefficients of t^n gives (69). When $\theta = 0$, Equation (37) reduces to the untruncated Hahn–Appell generating function (36), so that ${}_{e(r)}\mathcal{P}_m(\zeta, 0; w) = \mathcal{P}_m(\zeta; w)$; hence (69) becomes (51). $\square$

Theorem 8. As a polynomial in η , ${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w)$ has degree at most $\lfloor n/r \rfloor$. Moreover, for every integer $s \geq 0$,

$$\frac{\partial^s}{\partial \eta^s} {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \sum_{j=s}^{\lfloor n/r \rfloor} \frac{j!}{(j-s)!} \eta^{j-s} \frac{[n]_q!}{[n-rj]_q!} \mathcal{P}_{n-rj}(\zeta; w), \quad (70)$$

where the sum is understood to be zero when $s > \lfloor n/r \rfloor$. In particular, for $n \geq r$,

$$\frac{\partial}{\partial \eta} {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \frac{[n]_q!}{[n-r]_q!} {}_{e(r)}\mathcal{P}_{n-r}(\zeta, \eta; w) + \eta \frac{[n]_q!}{[n-r]_q!} \frac{\partial}{\partial \eta} {}_{e(r)}\mathcal{P}_{n-r}(\zeta, \eta; w). \quad (71)$$

Proof. The finite expansion (51) writes ${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w)$ as a finite polynomial in η :

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \sum_{j=0}^{\lfloor n/r \rfloor} \eta^j \frac{[n]_q!}{[n-rj]_q!} \mathcal{P}_{n-rj}(\zeta; w).$$

Taking the s th ordinary derivative with respect to η term by term gives (70). The degree assertion and the vanishing for $s > \lfloor n/r \rfloor$ follow from the same finite representation. Finally, differentiating the step- r recurrence (58) with respect to η gives (71). $\square$

Theorem 9. For every $n \geq 1$, the truncated-exponential-based Hahn–Appell polynomials satisfy

$${}_{e(r)}\mathcal{P}_n(q\zeta + w, \eta; w) - {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = ((q-1)\zeta + w)[n]_{q^{e(r)}}\mathcal{P}_{n-1}(\zeta, \eta; w). \quad (72)$$

Equivalently, on the Hahn lattice $\zeta = [m]_q w$ one has

$${}_{e(r)}\mathcal{P}_n([m+1]_q w, \eta; w) - {}_{e(r)}\mathcal{P}_n([m]_q w, \eta; w) = q^m w [n]_{q^{e(r)}}\mathcal{P}_{n-1}([m]_q w, \eta; w), \quad (73)$$

for $m = 0, 1, 2, \dots$

Proof. By the definition of the Hahn operator (46) and by the lowering law (50),

$$\frac{{}_{e(r)}\mathcal{P}_n(q\zeta + w, \eta; w) - {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w)}{(q-1)\zeta + w} = [n]_{q^{e(r)}}\mathcal{P}_{n-1}(\zeta, \eta; w).$$

Multiplication by $(q-1)\zeta + w$ gives (72) as a polynomial identity. If $\zeta = [m]_q w$, then $q\zeta + w = [m+1]_q w$ and

$$(q-1)[m]_q w + w = q^m w.$$

Substituting these two identities into (72) gives (73). $\square$

Remark 14. Equation (73) is included because it gives a direct recursive evaluation scheme on the Hahn lattice. Starting from the origin values

$${}_{e(r)}\mathcal{P}_n(0, \eta; w) = b_n^{(r)}(\eta),$$

one obtains successively

$${}_{e(r)}\mathcal{P}_n([m+1]_q w, \eta; w) = {}_{e(r)}\mathcal{P}_n([m]_q w, \eta; w) + q^m w [n]_{q^{e(r)}}\mathcal{P}_{n-1}([m]_q w, \eta; w).$$

Thus values of degree n on the grid can be computed from values of degree $n-1$ and the known initial value at $m=0$. This explains the computational role of the Hahn-grid relation in the numerical sections.

Theorem 10. Let

$$\mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta}) = \sum_{k=0}^{\infty} b_k^{(r)}(\eta) \frac{\mathcal{H}_{q,w,\zeta}^k}{[k]_q!}.$$

Then, for every $n \geq 0$,

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta})(\zeta)_{q,w}^n. \quad (74)$$

Moreover, if $\mathfrak{B}_\eta(t)$ is invertible as a formal power series, then

$$\mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta})^{-1} {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = (\zeta)_{q,w}^n. \quad (75)$$

Proof. Using (47),

$$\mathcal{H}_{q,w,\zeta}^k (\zeta)_{q,w}^n = \frac{[n]_q!}{[n-k]_q!} (\zeta)_{q,w}^{n-k}, \quad 0 \leq k \leq n,$$

and the result is zero for $k > n$. Hence

$$\mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta})(\zeta)_{q,w}^n = \sum_{k=0}^n \binom{n}{k}_q b_k^{(r)}(\eta) (\zeta)_{q,w}^{n-k},$$

which is identical with (63). Since $\mathfrak{B}_\eta(0) = \mathcal{A}(0) = 1$ by the standing normalization, the inverse formal series exists, and (75) follows by applying it to both sides of (74). $\square$

Theorem 11. Let $\mathcal{R}_{q,w}$ denote the Hahn raising operator on the factorial basis, defined in accordance with the monomiality formalism [21,26] by

$$\mathcal{R}_{q,w}\{(\zeta)_{q,w}^n\} = (\zeta)_{q,w}^{n+1}, \quad n \geq 0. \quad (76)$$

Define

$$\hat{P}_{q,w} := \mathcal{H}_{q,w,\zeta}, \quad \hat{M}_{q,w}^{(r)} := \mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta})\mathcal{R}_{q,w}\mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta})^{-1}. \quad (77)$$

Then the truncated-exponential-based Hahn–Appell sequence is quasi-monomial with respect to $\hat{M}_{q,w}^{(r)}$ and $\hat{P}_{q,w}$:

$$\hat{M}_{q,w}^{(r)}\{ {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) \} = {}_{e(r)}\mathcal{P}_{n+1}(\zeta, \eta; w), \quad (78)$$

and

$$\hat{P}_{q,w}\{ {}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) \} = [n]_{q_{e(r)}}\mathcal{P}_{n-1}(\zeta, \eta; w). \quad (79)$$

Consequently,

$$\hat{M}_{q,w}^{(r)}\hat{P}_{q,w}\{ {}_{e(r)}\mathcal{P}_n \} = [n]_{q_{e(r)}}\mathcal{P}_n, \quad \hat{P}_{q,w}\hat{M}_{q,w}^{(r)}\{ {}_{e(r)}\mathcal{P}_n \} = [n+1]_{q_{e(r)}}\mathcal{P}_n. \quad (80)$$

Proof. By the operational transfer formula (74),

$${}_{e(r)}\mathcal{P}_n(\zeta, \eta; w) = \mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta})(\zeta)_{q,w}^n.$$

Applying $\hat{M}_{q,w}^{(r)}$ and using (76), we get

$$\hat{M}_{q,w}^{(r)}{}_{e(r)}\mathcal{P}_n = \mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta})\mathcal{R}_{q,w}(\zeta)_{q,w}^n = \mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta})(\zeta)_{q,w}^{n+1} = {}_{e(r)}\mathcal{P}_{n+1},$$

which proves (78). Formula (79) is the lowering relation (50). Finally, applying (78) and (79) successively gives (80). $\square$

Remark 15. The operator $\hat{M}_{q,w}^{(r)}$ in (77) is independent of the degree index n ; it is a single operator on the polynomial space $\Pi = \bigcup_{N \geq 0} \Pi_N$. To make its action explicit, let

$$\mathfrak{B}_\eta(t)^{-1} = \sum_{j=0}^{\infty} c_j^{(r)}(\eta) \frac{t^j}{[j]_q!}, \quad c_0^{(r)}(\eta) = 1,$$

and let $f \in \Pi_N$. Then

$$g(\zeta) := \mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta})^{-1}f(\zeta) = \sum_{j=0}^N c_j^{(r)}(\eta) \frac{\mathcal{H}_{q,w,\zeta}^j f(\zeta)}{[j]_q!}$$

is a finite sum. Writing g in the Hahn factorial basis as

$$g(\zeta) = \sum_{m=0}^N g_m(\zeta)_{q,w}^m, \quad g_m = \frac{1}{[m]_q!} \left(\mathcal{H}_{q,w,\zeta}^m g \right)(0),$$

the raising operator and the monomiality operator act by

$$\mathcal{R}_{q,w}g = \sum_{m=0}^N g_m(\zeta)_{q,w}^{m+1}, \quad \hat{M}_{q,w}^{(r)}f = \mathfrak{B}_\eta(\mathcal{H}_{q,w,\zeta}) \left(\sum_{m=0}^N g_m(\zeta)_{q,w}^{m+1} \right).$$

All sums terminate because $\mathcal{H}_{q,w,\zeta}$ is degree lowering. The raising and lowering operators do not commute; with the commutator convention used in (28),

$$[\widehat{M}_{q,w}^{(r)}, \widehat{P}_{q,w}] \{ {}_{e(r)}\mathcal{P}_n \} = \left(\widehat{P}_{q,w} \widehat{M}_{q,w}^{(r)} - \widehat{M}_{q,w}^{(r)} \widehat{P}_{q,w} \right) \{ {}_{e(r)}\mathcal{P}_n \} = q^n {}_{e(r)}\mathcal{P}_n.$$

This is the Hahn–Appell analogue of the usual quasi-monomial commutation law.

Remark 16. If $\mathcal{N}_{q,w}$ is the degree operator determined by $\mathcal{N}_{q,w}(\zeta)_{q,w}^n = n(\zeta)_{q,w}^n$, then the abstract raising operator in (76) can be written on the Hahn factorial basis as

$$\mathcal{R}_{q,w} = \zeta - w[\mathcal{N}_{q,w}]_q.$$

Thus the monomiality operator (77) is a conjugated Hahn-factorial raising operator, and the factor $1 - \eta t^r$ enters only through the transfer series $\mathfrak{B}_\eta$.

4. Special Cases

Particular choices of the determining function $\mathcal{A}(t)$ in (37) produce standard Hahn-based subclasses. The results of Sections 2 and 3 are stated under the normalized condition $\mathcal{A}(0) = 1$; consequently, the examples below distinguish normalized choices from conventional unnormalized factors used in parts of the q -Appell literature [13,25].

Example 2. The q -Bernoulli determining function

$$\mathcal{A}(t) = \frac{t}{e_q(t) - 1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,q} \frac{t^n}{[n]_q!}$$

has constant term 1 and therefore satisfies the standing normalization. The corresponding truncated-exponential-based Hahn–Bernoulli polynomials ${}_{e(r)}\mathcal{B}_n(\zeta, \eta; w)$ are generated by

$$\frac{t}{e_q(t) - 1} \frac{e_q^{wt}(\zeta t)}{1 - \eta t^r} = \sum_{n=0}^{\infty} {}_{e(r)}\mathcal{B}_n(\zeta, \eta; w) \frac{t^n}{[n]_q!}. \quad (81)$$

Consequently,

$${}_{e(r)}\mathcal{B}_n(\zeta, \eta; w) = \sum_{k=0}^n \binom{n}{k}_q \mathcal{B}_{k,q} e_{n-k}^{(r)}(\zeta, \eta; w). \quad (82)$$

Example 3. The conventional Euler factor $[2]_q / (e_q(t) + 1)$ has constant term $[2]_q / 2$. To remain inside the normalized setting, we use the companion factor

$$\widetilde{\mathcal{A}}_E(t) = \frac{2}{e_q(t) + 1} = \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q} \frac{t^n}{[n]_q!}, \quad \widetilde{\mathcal{E}}_{0,q} = 1.$$

The normalized truncated-exponential-based Hahn–Euler polynomials ${}_{e(r)}\widetilde{\mathcal{E}}_n(\zeta, \eta; w)$ are generated by

$$\frac{2}{e_q(t) + 1} \frac{e_q^{wt}(\zeta t)}{1 - \eta t^r} = \sum_{n=0}^{\infty} {}_{e(r)}\widetilde{\mathcal{E}}_n(\zeta, \eta; w) \frac{t^n}{[n]_q!}. \quad (83)$$

Hence,

$${}_{e(r)}\widetilde{\mathcal{E}}_n(\zeta, \eta; w) = \sum_{k=0}^n \binom{n}{k}_q \widetilde{\mathcal{E}}_{k,q} e_{n-k}^{(r)}(\zeta, \eta; w). \quad (84)$$

If one instead uses the conventional factor $[2]_q / (e_q(t) + 1)$, all generated polynomials in (83) and (84) are multiplied by $[2]_q / 2$ and are not monic for $q \neq 1$.

Example 4. The conventional q -Genocchi factor

$$\mathcal{A}_G(t) = \frac{[2]_q t}{e_q(t) + 1} = \sum_{n=0}^{\infty} \mathcal{G}_{n,q} \frac{t^n}{[n]_q!}$$

has $\mathcal{A}_G(0) = 0$. Therefore it is not invertible as a formal power series and lies outside the normalized Appell class used for the determinant, inverse-transfer, and monicity results. It still defines a shifted Hahn–Genocchi-type sequence by

$$\frac{[2]_q t}{e_q(t) + 1} \frac{e_q^{wt}(\zeta t)}{1 - \eta t^r} = \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{G}_n(\zeta, \eta; w) \frac{t^n}{[n]_q!}. \quad (85)$$

For this shifted sequence, coefficient comparison gives the formal convolution

$${}_{e^{(r)}}\mathcal{G}_n(\zeta, \eta; w) = \sum_{k=0}^n \binom{n}{k}_q \mathcal{G}_{k,q} e_{n-k}^{(r)}(\zeta, \eta; w), \quad (86)$$

but the normalized unit-diagonal determinant and monicity assertions should not be applied to (85) without first passing to an invertible normalized companion factor.

5. Application-Oriented Interpretation and Numerical Illustrations

The preceding sections provide exact identities for the truncated-exponential-based Hahn–Appell family. This section adds application-oriented interpretation and computable illustrations that can be used as small benchmark cases without changing the theoretical flow of the paper. We continue with the unit-Appell specialization $\mathcal{A}(t) = 1$, because in this case all quantities follow directly from the finite Hahn-factorial expansion and no additional Appell coefficients are needed. Thus

$$e_n^{(r)}(\zeta, \eta; w) = [n]_q! \sum_{j=0}^{\lfloor n/r \rfloor} \frac{\eta^j}{[n - rj]_q!} (\zeta)_{q,w}^{n-rj}. \quad (87)$$

Formula (87) gives a finite evaluation algorithm, and the same expression also makes the roles of the truncation parameter η and the Hahn-grid parameter q visible. In the examples below, all parameters not explicitly varied are kept fixed. More precisely, the truncation-parameter test uses $q = \frac{1}{2}$, $r = 2$, and $w = 1$; the Hahn-grid test uses $r = 2$, $\eta = 1$, and $w = 1$; and the response-surface test uses the fixed values stated in each figure caption. We use the sixth-degree member because $n = 6$ contains four truncation levels for $r = 2$ and is still small enough for direct verification.

5.1. Potential Application Contexts

The family is potentially useful in problems where a finite correction of an Appell-type basis is needed on a Hahn lattice. In q -difference equations, the lowering law (50) and the lattice identity (73) provide a natural basis for constructing polynomial trial solutions and finite recursive schemes. In approximation theory, the monic Hahn-factorial expansion (63) can be used to build degree-preserving polynomial bases adapted to nonuniform q -grids. In special-function theory, the normalized determining function $\mathcal{A}(t)$ makes it possible to generate Bernoulli-, Euler-, and shifted Genocchi-type subclasses from the same kernel, while the truncation parameter η introduces finite lower-degree corrections. In mathematical physics, such deformed Appell systems can serve as symbolic models for discretized oscillator-type or ladder-operator problems in which the underlying grid is not equally spaced. The numerical examples below are therefore not proposed as a single

applied model; rather, they demonstrate how the formulas can be used as reproducible computational modules in these settings.

5.2. Computational Protocol

All numerical tables and figures in Sections 4 and 5 were generated from the finite Formula (87); no infinite series was truncated during the numerical stage. The coefficients were formed in the Hahn factorial basis and then expanded in the ordinary power basis only for root computation and plotting. The calculations were carried out in Python 3 using symbolic rational arithmetic where possible and high-precision floating-point arithmetic otherwise. For the root tables, the working precision was set to 80 decimal digits. The zeros were computed from the ordinary power-basis coefficient vector by a companion-matrix/eigenvalue routine and were then checked by direct substitution. The accepted residual tolerance was

$$\max_j |e_n^{(2)}(\zeta_j, 1; 1)| < 10^{-25}$$

for the listed roots before rounding. The displayed numerical values are rounded to three decimal places. The EPS graphics were produced from the same coefficient data, using the plotted parameter grids described in the captions. These computations are intended to provide reproducible benchmarks; zero-location claims not proved above are therefore stated only as empirical observations for the indicated parameter values.

5.3. Quantitative Validation of Selected Identities

To connect the numerical part with the preceding theory, Table 6 verifies representative identities using the unit-Appell model with $q = \frac{1}{2}$, $r = 2$, $\eta = 1$, $w = 1$, and $\zeta = 1$ unless another parameter is explicitly stated. The calculations were performed with exact rational arithmetic and then displayed in decimal form. The zero residuals therefore confirm the algebraic identities for these benchmark data before graphical visualization is considered.

5.4. Application 1: Truncation-Parameter Tuning

For fixed $q = \frac{1}{2}$, $r = 2$, and $w = 1$, the parameter η controls the strength of the correction terms in (87). Table 7 records benchmark values for $e_6^{(2)}(\zeta, \eta; 1)$ and the associated numerical zero spread. The row $\eta = 0$ gives the untruncated Hahn factorial $(\zeta)_{q,1}^6$, while positive values of η add the lower-degree Hahn-factorial components predicted by the step-2 recurrence.

Table 7. Truncation-parameter sensitivity of $e_6^{(2)}(\zeta, \eta; 1)$ for $q = \frac{1}{2}$, $r = 2$, and $w = 1$.

η	$e_6^{(2)}(0, \eta; 1)$	$e_6^{(2)}(1, \eta; 1)$	$e_6^{(2)}(2, \eta; 1)$	Real Zeros	$\max \Re(\zeta_j)$	$\max \Im(\zeta_j) $
0.0	0.000	0.000	0.002	6	1.938	0.000
0.5	2.347	2.347	9.084	0	2.459	1.216
1.0	18.774	18.774	44.762	0	2.750	1.663
1.5	63.363	63.363	121.118	0	2.976	2.001

Figure 1 gives the corresponding two-dimensional profiles. The deformation is not merely a vertical shift: the lower-degree components induced by η change the curvature and remove the six real Hahn-grid zeros that occur at $\eta = 0$.

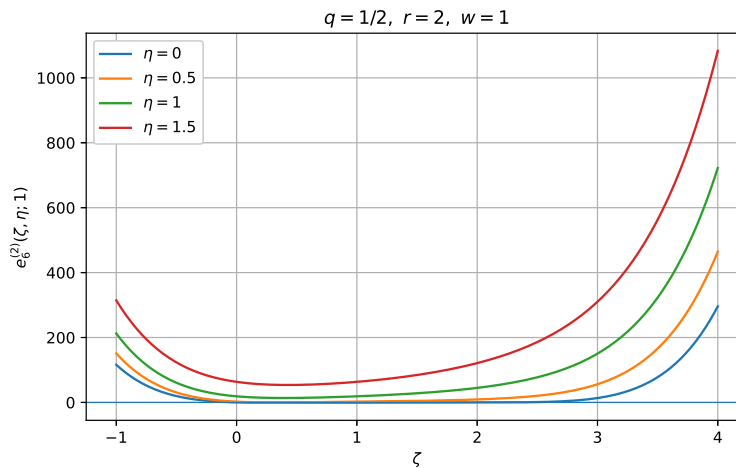


Figure 1. Two-dimensional truncation-parameter profiles of $e_6^{(2)}(\zeta, \eta; 1)$ for $q = \frac{1}{2}$, $r = 2$, $w = 1$, and $\eta \in \{0, 0.5, 1, 1.5\}$. The comparison shows how the finite lower-degree correction induced by η changes both scale and curvature relative to the untruncated Hahn-factorial case.

5.5. Application 2: Hahn-Grid Deformation

With $r = 2$, $\eta = 1$, and $w = 1$ fixed, the parameter q determines the Hahn-grid nodes $[j]_q w$ and the scaling factors $[n]_q!$. Hence it is useful to record how the same degree responds to different q -levels. Table 8 gives a compact benchmark for $e_6^{(2)}(\zeta, 1; 1)$ when $r = 2$ and $w = 1$.

Table 8. Hahn-grid sensitivity of $e_6^{(2)}(\zeta, 1; 1)$ for $r = 2$, $\eta = 1$, and $w = 1$.

q	$[6]_q!$	$e_6^{(2)}(0, 1; 1)$	$e_6^{(2)}(2, 1; 1)$	$\max \Re(\zeta_j)$	$\max \Im(\zeta_j) $
0.25	3.869	3.869	12.353	2.069	1.247
0.50	18.774	18.774	44.762	2.750	1.663
0.75	111.002	111.002	236.174	3.799	2.792

The two-dimensional comparison in Figure 2 illustrates the same effect on the real axis. For this fixed degree and parameter set, larger q increases the scale of the polynomial and moves the observed zero spread farther to the right.

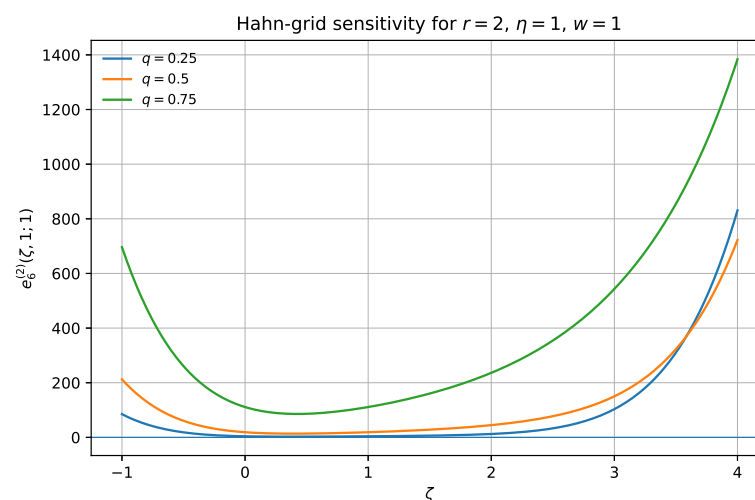


Figure 2. Two-dimensional Hahn-grid profiles of $e_6^{(2)}(\zeta, 1; 1)$ for $q \in \{0.25, 0.5, 0.75\}$, $r = 2$, $\eta = 1$, and $w = 1$. Larger q changes the Hahn nodes $[j]_q$ and the factorial weights, producing the visible scale increase.

5.6. Application 3: Response Surfaces

The finite form (87) is also suitable for two-parameter visualization. Figure 3a varies (ζ, η) with $q = \frac{1}{2}$, $r = 2$, and $w = 1$ fixed. Figure 3b varies (q, η) at the sample point $\zeta = 2$ with $r = 2$ and $w = 1$ fixed. These surfaces are included as numerical illustrations of the operational formulas, not as asymptotic claims.

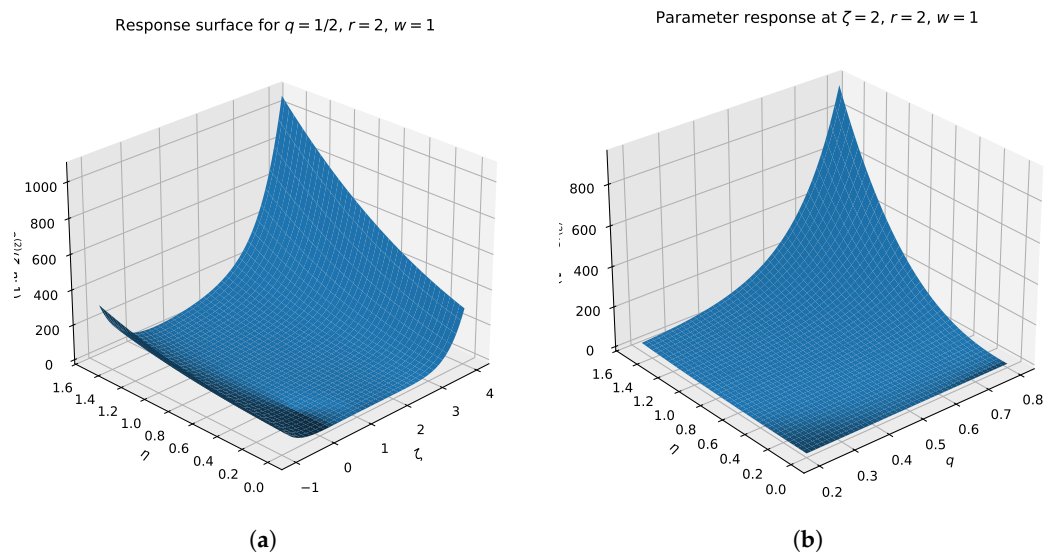


Figure 3. Organized two-panel view of the three-dimensional response surfaces. Panel (a) visualizes the finite truncation response summarized in Table 7; panel (b) shows the joint effect of Hahn-grid deformation and truncation strength. (a) Response over (ζ, η) for fixed $q = \frac{1}{2}$, $r = 2$, and $w = 1$. (b) Response over (q, η) at the fixed point $\zeta = 2$, with $r = 2$ and $w = 1$.

The computations in this section provide reproducible test cases for the finite-step recurrence, the origin-value formula, and the Hahn-factorial expansion. They also complement the zero-distribution plots below by showing how this unit-Appell specialization changes under controlled parameter variation.

6. Zero Distribution and Graphical Representations

Since the determining function $\mathcal{A}(t)$ in (37) is arbitrary, the numerical discussion in this section is carried out for the unit-Appell specialization $\mathcal{A}(t) = 1$, namely for the truncated-exponential-based Hahn polynomials $e_n^{(r)}(\zeta, \eta; w)$ defined by (39) and motivated by the q -truncated-exponential construction in [3]. Throughout this section, unless explicitly stated otherwise, the same parameter set is used:

$$q = \frac{1}{2}, \quad r = 2, \quad \eta = 1, \quad w = 1.$$

The computations below serve four purposes. First, they verify the explicit coefficient formula. Second, they record the exact origin-zero mechanism predicted by Corollary 4. Third, they display the real-axis profiles of the first few members. Fourth, they show the two- and three-dimensional geometry of the zeros.

Using the Hahn exponential basis [3,13,18]

$$e_q^{wt}(\zeta t) = \sum_{m=0}^{\infty} \frac{(\zeta)_{q,w}^m}{[m]_q!} t^m \quad \text{with} \quad (\zeta)_{q,w}^m = \prod_{j=0}^{m-1} (\zeta - [j]_q w),$$

Equation (39) yields the explicit numerical model

$$e_n^{(2)}(\zeta, 1; 1) = [n]_q! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{[n-2k]_q!} \prod_{j=0}^{n-2k-1} (\zeta - [j]_q), \quad (88)$$

$$[j]_q = \frac{1-q^j}{1-q}, \quad q = \frac{1}{2}.$$

The first few members generated by (88) are listed in Table 9.

Table 9. The first few polynomials $e_n^{(2)}(\zeta, 1; 1)$ for $q = \frac{1}{2}$, $r = 2$, $\eta = 1$, and $w = 1$.

n	$e_n^{(2)}(\zeta, 1; 1)$
0	1
1	ζ
2	$\zeta^2 - \zeta + \frac{3}{2}$
3	$\zeta^3 - \frac{5\zeta^2}{2} + \frac{33\zeta}{8}$
4	$\zeta^4 - \frac{17\zeta^3}{4} + \frac{293\zeta^2}{32} - \frac{189\zeta}{32} + \frac{315}{64}$
5	$\zeta^5 - \frac{49\zeta^4}{8} + \frac{2237\zeta^3}{128} - \frac{5817\zeta^2}{256} + \frac{20385\zeta}{1024}$

Proposition 1. For the unit-Appell family $e_n^{(r)}(\zeta, \eta; w)$ with real q , η , and w , all coefficients are real; hence non-real zeros occur in conjugate pairs. Moreover, if $\mathcal{A}(t) = 1$ and $r \nmid n$, then $e_n^{(r)}(0, \eta; w) = 0$. In particular, for $r = 2$ every odd-degree unit-Appell polynomial has the zero $\zeta = 0$.

Proof. The real-coefficient assertion follows directly from the finite expansion in (88), since the Hahn factorial factors and q -factorials are real under the stated assumptions. The value at the origin is already given by (68); it vanishes whenever r does not divide n . For $r = 2$, this is precisely the odd-degree case. $\square$

Remark 17. Proposition 1 turns two visual features of the plots into exact statements. The remaining geometric behavior of the zero cloud, such as the rightward spreading and layered clustering, is numerical for the chosen parameter set and is therefore reported as an observed tendency.

The zeros were computed numerically from the coefficient vectors in (88). Because the coefficients are real, the non-real zeros occur in conjugate pairs. Moreover, every odd-degree member in the present parameter set possesses the trivial zero $\zeta = 0$. Table 10 lists the real root when it exists and the zeros in the upper half-plane for $2 \leq n \leq 8$; the conjugate zeros are understood.

Table 10. Numerically computed zeros of $e_n^{(2)}(\zeta, 1; 1)$ for $2 \leq n \leq 8$ when $q = \frac{1}{2}$, $r = 2$, $\eta = 1$, and $w = 1$.

n	Real Zero (s)	Zero(s) with $\Im(\zeta) > 0$
2	none	$0.500 + 1.118i$
3	0	$1.250 + 1.601i$
4	none	$0.234 + 0.828i$, $1.891 + 1.753i$
5	0	$0.679 + 1.350i$, $2.384 + 1.743i$
6	none	$0.137 + 0.658i$, $1.145 + 1.648i$, $2.750 + 1.663i$
7	0	$0.429 + 1.135i$, $1.567 + 1.798i$, $3.020 + 1.555i$
8	none	$0.090 + 0.545i$, $0.768 + 1.461i$, $1.928 + 1.854i$, $3.221 + 1.441i$

Table 11 gives an additional numerical check. The column $e_n^{(2)}(0, 1; 1)$ confirms the exact origin-value law, while the columns $e_n^{(2)}(1, 1; 1)$ and $e_n^{(2)}(2, 1; 1)$ provide reproducible benchmark values for the graphs.

Table 11. Numerical diagnostics for $e_n^{(2)}(\zeta, 1; 1)$ when $q = \frac{1}{2}$, $r = 2$, $\eta = 1$, and $w = 1$.

n	$e_n^{(2)}(0, 1; 1)$	$e_n^{(2)}(1, 1; 1)$	$e_n^{(2)}(2, 1; 1)$	Real Zeros	$\max \Re(\zeta_j)$
2	1.500	1.500	3.500	0	0.500
3	0	2.625	6.250	1	1.250
4	4.922	4.922	11.734	0	1.891
5	0	9.536	22.736	1	2.384
6	18.774	18.774	44.762	0	2.750
7	0	37.255	88.825	1	3.020
8	74.219	74.219	176.956	0	3.221

Analytical Interpretation of the Observed Zero Patterns

The zero data can be interpreted through the finite expansion rather than only as a graphical phenomenon. When $\eta = 0$, the unit-Appell polynomial reduces to the Hahn factorial $(\zeta)_{q,w}^n$, so its zeros are exactly the Hahn-grid points

$$0, [1]_q w, [2]_q w, \dots, [n-1]_q w.$$

For $\eta \neq 0$, the terms with degrees $n - r, n - 2r, \dots$ perturb this grid polynomial by lower-degree Hahn-factorial components. Since the coefficients depend polynomially on η and rationally on q away from singular parameter values, the zeros vary continuously with these parameters except at multiple-root collisions. This explains why the profiles in Section 4 show a gradual deformation of the zero cloud rather than unrelated root sets.

Two features are exact. First, real parameter values give real coefficients, hence conjugate symmetry. Second, the origin value in (68) forces a zero at $\zeta = 0$ whenever $r \nmid n$; for $r = 2$ this is exactly the odd-degree case. For even degrees with $r = 2$, the constant term is nonzero and equals $\eta^{n/2} [n]_q!$, so no origin zero is forced. The observed rightward spread is consistent with the fact that the unperturbed Hahn nodes $[j]_q w$ are non-negative and increase with j , while the truncation adds lower-degree components without changing monicity. Nevertheless, the data in Table 10 do not constitute a proof of global zero bounds or asymptotic location. They should be read as finite-degree evidence suggesting that rigorous zero-location theory is a natural next problem.

For the fixed parameter set $q = \frac{1}{2}$, $r = 2$, $\eta = 1$, and $w = 1$, Figure 4a displays the real profiles of the first few polynomials on the interval $-1 \leq \zeta \leq 4$. Figure 5a shows the corresponding two-dimensional zero cloud in the complex plane for $2 \leq n \leq 8$. Figure 5b gives the stacked three-dimensional distribution $(\Re \zeta, \Im \zeta, n)$. Finally, Figure 4b gives a three-dimensional degree-profile representation over the real ζ -axis.

Proposition 1 gives the exact explanation for two of the plotted features: conjugate symmetry and the origin zero in odd degree. The remaining geometry is empirical for the selected parameter set. The data in Tables 10 and 11 and Figures 4b and 5a show that the zeros are symmetric with respect to the real axis and that odd-degree polynomials contain the zero $\zeta = 0$. For the specific values $q = \frac{1}{2}$, $r = 2$, $\eta = 1$, and $w = 1$, they also indicate that, as n increases from 2 to 8, the zero set spreads to the right half-plane and forms layered conjugate clusters. No theorem on asymptotic zero distribution, zero bounds, or global zero location is claimed here. These patterns are numerical observations for the chosen values of q , r , η , and w , and analytical zero-geometry results remain open for future work.

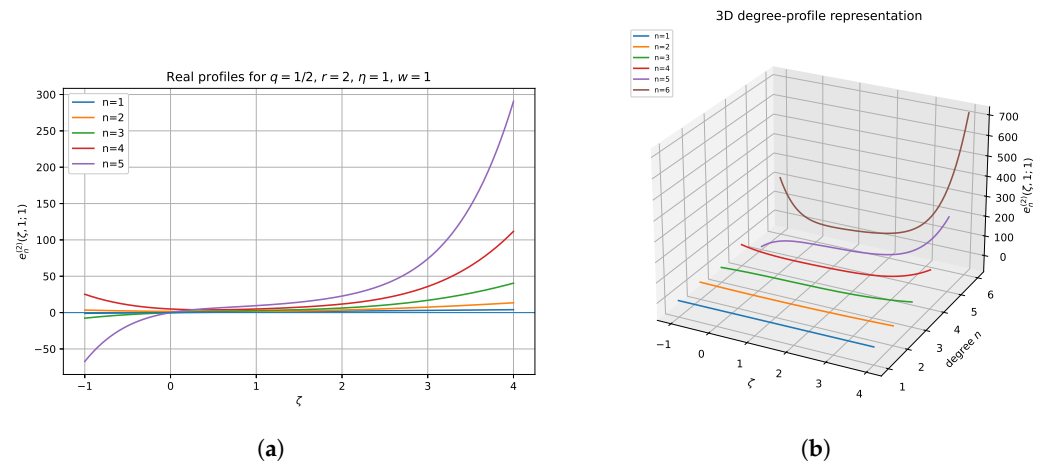


Figure 4. Organized profile plots of $e_n^{(2)}(\zeta, 1; 1)$ for $q = \frac{1}{2}, r = 2, \eta = 1$, and $w = 1$. Panel (a) shows the two-dimensional real profiles on $-1 \leq \zeta \leq 4$, while panel (b) compares degree growth on the real axis with the finite values reported in Table 11. (a) Real profiles for $1 \leq n \leq 5$. (b) Degree-profile representation for $1 \leq n \leq 6$.

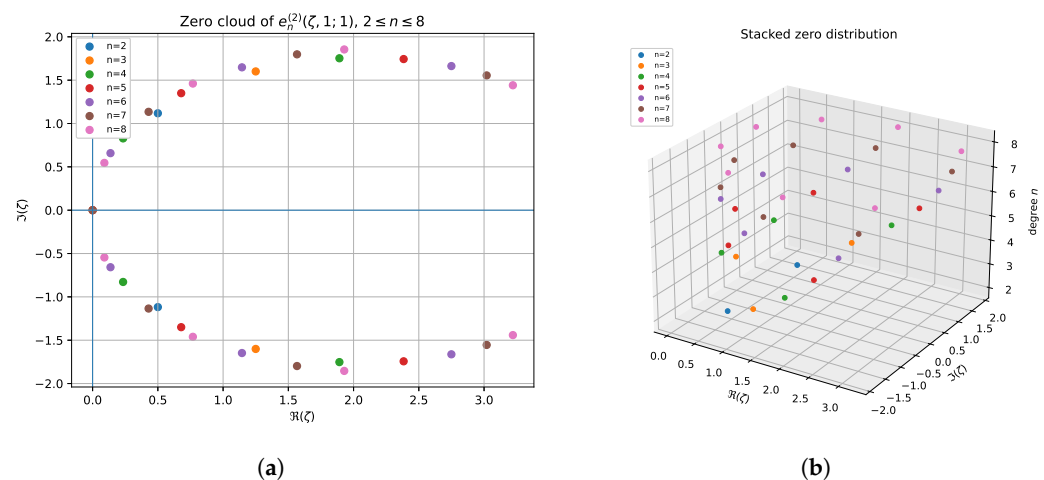


Figure 5. Organized zero-distribution plots of $e_n^{(2)}(\zeta, 1; 1)$ for $2 \leq n \leq 8$ when $q = \frac{1}{2}, r = 2, \eta = 1$, and $w = 1$. Conjugate symmetry is exact by Proposition 1; the rightward spread is an empirical feature of this parameter set. (a) Two-dimensional zero cloud for $2 \leq n \leq 8$. (b) Stacked zero distribution in $(\Re\zeta, \Im\zeta, n)$ -space.

7. Limitations and Future Research Directions

The preceding results establish a finite algebraic and operational theory, but several questions remain outside the scope of the present paper. First, the analytical convergence of the Hahn exponential is stated only to justify the generating function where needed; most coefficient identities are formal polynomial identities. A fuller analytic treatment could examine domains of convergence, parameter-dependent singularities, and limiting regimes in which $q \rightarrow 1^-$, $w \rightarrow 0^+$, or $n \rightarrow \infty$ simultaneously. Second, although normalized monicity is proved, no orthogonality measure is constructed for the general family. Determining whether special choices of $\mathcal{A}(t)$, η , and w lead to orthogonal or biorthogonal subclasses is an open problem.

Third, the quasi-monomial operators identify a ladder structure, but a spectral interpretation of the corresponding Hahn-grid difference equation has not been developed. Such a study would require boundary conditions and an appropriate functional space on the Hahn lattice. Fourth, the zero-distribution results in Section 5 are deliberately fi-

nite and computational. Exact conjugate symmetry and the origin-zero mechanism are proved, but global zero bounds, interlacing properties, asymptotic zero measures, and root-transition thresholds as functions of (q, η, w, r) remain open. Finally, possible applications to q -difference equations, approximation schemes on nonuniform grids, and deformed special-function models require problem-specific analysis beyond the benchmark examples presented here.

These limitations point to several natural directions for future work: asymptotic expansions for large degree, rigorous zero-location theory, orthogonality or moment problems for special normalized subclasses, spectral analysis of the Hahn-grid difference operator, and numerical algorithms that exploit the finite step- r recurrence for stable high-degree computation.

8. Conclusions

This paper has been organized consistently around the two-variable truncated-exponential-based Hahn–Appell family generated by (37). The main results show that the truncation factor $(1 - \eta t^r)^{-1}$ preserves the Hahn–Appell lowering structure while adding a finite step- r correction mechanism.

The principal structural conclusions can be stated in a unified form. The family is connected to the underlying Hahn–Appell basis by the finite expansion (51), and the Hahn operator $\mathcal{H}_{q,w,\zeta}$ lowers the sequence exactly as in an Appell system, as stated in (50). The truncation factor is represented operationally by $(I - \eta \mathcal{H}_{q,w,\zeta}^r)^{-1}$ and equivalently by the step- r recurrence (58). In addition, the determinant, Hahn-factorial, origin-value, normalized monicity, parameter-connection, parameter-differentiation, Hahn-grid difference, and quasi-monomial representations provide computable forms of the sequence.

For the parameter choice

$$q = \frac{1}{2}, \quad r = 2, \quad \eta = 1, \quad w = 1,$$

the explicit expressions listed in Table 9 and the zeros reported in Table 10 agree with direct symbolic and numerical computation from (88). In particular, the symmetry of the zero set with respect to the real axis follows from real coefficients, while the root $\zeta = 0$ for odd degrees follows from the origin-value formula. The application-oriented computations further show how the finite Hahn-factorial formula can be used for truncation-parameter tuning, Hahn-grid sensitivity tests, and two-parameter response visualization. The observed rightward spreading and layered clustering of the non-real zeros remain numerical phenomena for this unit-Appell specialization; analytical bounds and asymptotic zero-distribution results are not asserted in the present paper.

Together with the notation table, theorem roadmap, validation checks, and limitations section added above, these results provide a coherent starting point for normalized special subclasses obtained from different choices of $\mathcal{A}(t)$ and for future analytical work on zero geometry, asymptotic behavior, orthogonality, spectral interpretation, sensitivity with respect to the truncation parameter, and parameter-dependent transitions in Hahn–Appell-type polynomial systems.

Author Contributions: Conceptualization, O.O.; methodology, O.O.; writing—original draft, O.Y.; writing—review and editing, W.A.K. and K.S.M.; supervision, O.O. All authors have read and agreed to the published version of the manuscript.

Funding: The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2026).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2026).

Conflicts of Interest: The authors declare no conflicts of interest.

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