

Article

Existence of Solutions for Generalized Mixed Weak Vector Variational–Hemivariational Inequalities

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Abstract

In this paper, we consider the classes of generalized Stampacchia mixed weak vector variational–hemivariational inequalities (in short, GSMWVVHIs) and generalized Minty mixed weak vector variational–hemivariational inequalities (in short, GMMWVVHIs) formulated in terms of bifunctions. By employing the Knaster–Kuratowski–Mazurkiewicz–Fan (in short, KKM–Fan) lemma, we deduce an existence result for the solutions of GSMWVVHIs without imposing any monotonicity assumptions on bifunctions and relaxed compactness hypotheses. Moreover, under a generalized, stable pseudomonotonicity hypothesis on bifunctions, we derive an existence theorem for the solutions of GMMWVVHIs and establish an equivalence relation between the solutions of the considered vector variational–hemivariational inequalities. Furthermore, uniqueness results for the solutions of the considered vector variational–hemivariational inequalities are established under suitable assumptions. In addition, we establish the stability of the solution of GSMWVVHIs under some appropriate conditions. We furnish several illustrative examples to demonstrate the significance of the results established in this paper. Finally, we demonstrate that the sustainable electricity generation planning problem can be naturally formulated as a GSMWVVHI.

Keywords: vector variational–hemivariational inequalities; bifunctions; existence results; uniqueness results; KKM–Fan lemma; stability analysis

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1. Introduction

The notion of variational inequality (in short, VI) was introduced by Hartman and Stampacchia [1]. Since then, VIs have become fundamental tools for addressing various problems emerging in science, economics, and engineering, including free boundary problems [2], optimization problems [3,4], complementarity problems [5], equilibrium problems [6], and electricity production models [7]. Giannessi [8,9] proposed the vector-valued versions of classical Stampacchia [10] and Minty [11] VIs in the Euclidean space setting. Vector variational inequalities (in short, VVIs) play a crucial role in modeling and analyzing a wide range of problems, such as vector traffic equilibrium problems [12], vector network equilibrium problems (see [13–15]), vector migration equilibrium models [16], and multiobjective optimization problems (see [17,18]). As a result, various researchers

have extensively investigated VVIs in different settings (see [19–26] and the references cited therein).

Panagiotopoulos [27] introduced the concept of hemivariational inequality for analysis of boundary value problems encountered in the study of partial differential equations. Hemivariational inequality is a generalization of variational inequality that incorporates nonconvex potential functions. Over the past few decades, hemivariational inequalities have served as effective tools for studying problems arising in science, engineering, and technology (see [28,29] and the references cited therein). The fusion of VIs and hemivariational inequalities gives rise to a broader class of inequalities—namely, variational–hemivariational inequalities, which involve both convex and nonconvex functions. The concept of variational–hemivariational inequality was first proposed by Motreanu and Rădulescu [30] in the setting of reflexive Banach spaces. Several real-life problems arising in mechanics and engineering, such as unilateral contact problems [31], nonsmooth transportation problems [32], and frictional contact problems [33], can be effectively modeled as variational–hemivariational inequalities. For further insights into hemivariational and variational–hemivariational inequalities, we refer to [34–37] and the references cited therein.

VVIs and hemivariational inequalities defined in terms of bifunctions have emerged as an interesting area of research over the last few decades. For instance, Bianchi and Schaible [38] investigated existence and uniqueness results for equilibrium problems under quasimonotonicity and pseudomonotonicity assumptions on bifunctions. Lalitha and Mehta [39] established existence results for Stampacchia and Minty VVIs formulated in terms of bifunctions by employing the KKM-Fan lemma under weak cone-pseudomonotonicity assumptions. Moreover, using the KKM-Fan lemma, Guu and Li [40] investigated existence results for Stampacchia and Minty VVIs involving generalized bifunctions and established relationships among the considered VVIs by utilizing weak cone-pseudomonotonicity hypotheses. Ansari and Lee [41] deduced existence results for VVIs defined in terms of bifunctions by using the KKM-Fan lemma and established relationships between the solutions of the considered VVIs and nonsmooth vector optimization problems. Furthermore, by employing the KKM-Fan lemma, Ansari and Rezaei [42] established existence results for Stampacchia- and Minty-type VVIs in terms of bifunctions and developed the relationships between their solutions under pseudomonotonicity and hemicontinuity hypotheses in topological vector spaces. Noor et al. [43] developed iterative algorithms for bifunction hemivariational inequalities in the Hilbert-space setting. Existence theorems for solutions of generalized set-valued variational inequalities were established by Plubtieng and Thammathiwat [44] using the KKM-Fan lemma in Banach spaces. Recently, Mehta et al. [45] studied existence results for mixed VIs involving bifunctions and analyzed the Levitin–Polyak well-posedness of the considered variational inequalities.

In view of the above discussion, it is important to emphasize that the pseudomonotonicity of bifunctions plays a vital role in establishing existence results for various classes of VVIs (see, for instance, [39,40,42]). Moreover, several researchers have introduced generalized notions of pseudomonotonicity for investigation of existence results and stability analysis for the solution of VIs, mixed VIs, and variational–hemivariational inequalities involving set-valued mappings. For instance, He [46] conducted a stability analysis of set-valued VIs in reflexive Banach spaces under stable pseudomonotonicity assumptions on set-valued mappings. Using Φ -pseudomonotonicity of set-valued mappings, Zhong and Huang [47] established stability results for mixed VIs in reflexive Banach spaces. Furthermore, Tang and Huang [48] introduced the notions of stably ϕ -pseudomonotone and quasimonotone set-valued mappings and derived existence results for variational–hemivariational inequalities in the setting of reflexive Banach spaces. Despite these developments, existence, uniqueness, and stability results for more general classes of vec-

tor variational–hemivariational inequalities formulated in terms of bifunctions, such as GSMWVHIs and GMMWVHIs, have not yet been investigated under the assumption of generalized stable pseudomonotonicity of the involved bifunctions. In this paper, our objective is to address these research gaps by establishing existence and uniqueness results for solutions of GSMWVHIs and GMMWVHIs, as well as conducting stability analysis for the solution of GSMWVHIs.

Motivated by the works reported in [39,41,45], in this paper, we consider generalized Stampacchia mixed weak vector variational–hemivariational inequalities and generalized Minty mixed weak vector variational–hemivariational inequalities in terms of bifunctions—namely, GSMWVHIs and GMMWVHIs. We introduce the notions of generalized monotone, strongly monotone, stable pseudomonotone, and stable strongly pseudomonotone bifunctions. By employing the KKM-Fan lemma, we establish an existence result for solutions of GSMWVHIs without relying on the monotonicity of bifunctions and under relaxed compactness assumptions. Based on the generalized stable pseudomonotonicity assumptions on bifunctions, we deduce an existence theorem for solutions of GMMWVHIs and establish a relationship between the solutions of GSMWVHIs and GMMWVHIs. Using appropriate assumptions, we establish uniqueness results for the solutions of GSMWVHIs and GMMWVHIs. The stability analysis for the solution of GSMWVHIs is established under some suitable conditions.

The novelty and contributions of this paper are discussed as follows:

- (a) The results established in this paper generalize the corresponding results derived by Lee [17] from VVIs to broader classes of vector variational–hemivariational inequalities involving bifunctions—namely, GSMWVHIs and GMMWVHIs. Moreover, the results with respect to the existence of solutions of GSMWVHIs and GMMWVHIs generalize the corresponding results established by Mehta et al. [45] from mixed VIs to generalized mixed-vector variational–hemivariational inequalities.
- (b) It is significant to note that the inner product satisfies the linearity property, whereas bifunctions are not necessarily linear (see [41,43]). Moreover, the results established in this paper rely on weaker assumptions than monotonicity—namely, generalized stable pseudomonotonicity. As a consequence, the results established in this paper are applicable to the study of the broader classes of VVIs as compared to the results available in the existing literature (see [8,17]).
- (c) We consider the sustainable electricity generation planning problem (in short, SEGPP). Moreover, we demonstrate that SEGPP can be naturally formulated as a GSMWVHI and can be effectively solved without imposing monotonicity assumptions on the corresponding bifunctions and the relaxed compactness hypothesis.

The rest of the paper is structured as follows: In Section 2, we provide the basic preliminaries used for the subsequent analysis. In Section 3, we establish existence results for the solutions of GSMWVHIs by utilizing the KKM-Fan lemma. Moreover, an existence theorem for solutions of GMMWVHIs is established under generalized stable pseudomonotonicity hypotheses on bifunctions, and the relationship between the solutions of GSMWVHIs and GMMWVHIs is examined. Moreover, by using generalized stable, strong pseudomonotonicity assumptions on the involved bifunctions, uniqueness results for the solutions of GSMWVHIs and GMMWVHIs are derived under some appropriate conditions. In Section 4, we establish the stability of the solution of GSMWVHIs. In Section 5, we present a practical application to the sustainable electricity generation planning problem. Section 6 concludes the paper and highlights several potential directions for future research.

2. Notations and Mathematical Preliminaries

The set of natural numbers is denoted by the symbol $\mathbb{N}$. For some $n \in \mathbb{N}$, the notations $\mathbb{R}^n$ and $\mathbb{R}_+^n$ are used to denote the n -dimensional Euclidean space and the non-negative orthant of $\mathbb{R}^n$, respectively. Mathematically, $\mathbb{R}_+^n$ is defined as follows:

$$\mathbb{R}_+^n := \{\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n : z_i \geq 0, i = 1, 2, \dots, n\}.$$

Moreover, $\text{int } \mathbb{R}_+^n$ denotes the interior of $\mathbb{R}_+^n$. The inner product and its corresponding norm in $\mathbb{R}^n$ are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. The $\emptyset$ symbol is employed to denote the empty set. For $s \in \mathbb{N}$, let $\mathbf{w} := (w_1, w_2, \dots, w_s) \in \mathbb{R}^s$. The following inequalities play a key role in the subsequent analysis:

$$\begin{aligned} \mathbf{w} < 0 &\Leftrightarrow w_j < 0, \text{ for all } j \in \{1, 2, \dots, s\}, \\ \mathbf{w} > 0 &\Leftrightarrow w_j > 0, \text{ for all } j \in \{1, 2, \dots, s\}. \end{aligned}$$

The $\mathbf{w} \not< 0$ symbol (or $\mathbf{w} \not> 0$) is employed to represent the negation of $\mathbf{w} < 0$ (or $\mathbf{w} > 0$). Let $\emptyset \neq \mathcal{S} (\subseteq \mathbb{R}^n)$ and $\Phi : \mathcal{S} \rightarrow \mathbb{R}$ be a real-valued function. The Φ function is said to be locally Lipschitz at $\mathbf{z} \in \mathcal{S}$ with rank K_0 ($K_0 \in \mathbb{R}$, $K_0 > 0$) if there exists a neighborhood (U) of $\mathbf{z}$ and some constant (K_0) such that

$$|\Phi(\mathbf{x}) - \Phi(\mathbf{y})| \leq K_0 \|\mathbf{x} - \mathbf{y}\|, \text{ for all } \mathbf{x}, \mathbf{y} \in U. \quad (1)$$

The Φ function is said to be locally Lipschitz on $\mathcal{S}$ with rank K_0 if (1) is satisfied for every $\mathbf{z} \in \mathcal{S}$. Here, constant K_0 is the same for all $\mathbf{z} \in \mathcal{S}$.

The definitions of a convex set, convex function, and convex hull of a set are presented in the following definitions from Hiriart-Urruty and Lemaréchal [49].

Definition 1. Let $\emptyset \neq \mathcal{S} (\subseteq \mathbb{R}^n)$. Then:

(i) $\mathcal{S}$ is a convex set if, for any $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, the following inclusion holds:

$$(1-t)\mathbf{z} + t\mathbf{y} \in \mathcal{S}, \text{ for all } t \in [0, 1].$$

(ii) A function ($\Phi : \mathcal{S} \rightarrow \mathbb{R}$) is a convex function on convex set $\mathcal{S}$, provided that for any arbitrary $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, the following inequality holds:

$$\Phi((1-t)\mathbf{z} + t\mathbf{y}) \leq (1-t)\Phi(\mathbf{z}) + t\Phi(\mathbf{y}), \text{ for all } t \in [0, 1].$$

Definition 2. Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a non-empty set. The convex hull of $\mathcal{S}$, denoted by $\text{co } \mathcal{S}$, is defined as follows:

$$\text{co } \mathcal{S} := \left\{ \sum_{i=1}^m \alpha_i \mathbf{z}^i : \mathbf{z}^i \in \mathcal{S}, \alpha_i \geq 0, i = 1, 2, \dots, m, \sum_{i=1}^m \alpha_i = 1 \right\}.$$

The notions of a Clarke directional derivative and Clarke subdifferential are from Clarke [50].

Definition 3. Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz function on $\mathbb{R}^n$. Then:

(i) The Clarke directional derivative of Φ at $\mathbf{z} \in \mathbb{R}^n$ in direction $\mathbf{v} \in \mathbb{R}^n$, denoted by $\Phi^\circ(\mathbf{z}; \mathbf{v})$, is defined as follows:

$$\Phi^\circ(\mathbf{z}; \mathbf{v}) := \limsup_{\mathbf{y} \rightarrow \mathbf{z}, t \downarrow 0} \frac{\Phi(\mathbf{y} + t\mathbf{v}) - \Phi(\mathbf{y})}{t}.$$

- (ii) The Clarke subdifferential of the Φ function at $\mathbf{z} \in \mathbb{R}^n$, denoted by $\partial_c \Phi(\mathbf{z})$, is defined as follows:

$$\partial_c \Phi(\mathbf{z}) := \{\eta \in \mathbb{R}^n : \Phi^\circ(\mathbf{z}; \mathbf{v}) \geq \langle \eta, \mathbf{v} \rangle, \text{ for all } \mathbf{v} \in \mathbb{R}^n\}.$$

The following lemma from Clarke [50] presents various properties of the Clarke directional derivative and Clarke subdifferential of a locally Lipschitz function.

Lemma 1. Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz function on $\mathbb{R}^n$ with rank K_0 . Then:

- (i) The function expressed as $\mathbf{v} \mapsto \Phi^\circ(\mathbf{z}; \mathbf{v})$ is finite, positively homogeneous, and subadditive on $\mathbb{R}^n$.
- (ii) $\Phi^\circ(\mathbf{z}; \mathbf{v})$ is upper semicontinuous as a function of $(\mathbf{z}, \mathbf{v})$ and, as a function of $\mathbf{v}$ alone, is Lipschitz with rank K_0 on $\mathbb{R}^n$.
- (iii) For every $\mathbf{z} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$, we have

$$\Phi^\circ(\mathbf{z}; \mathbf{v}) := \max\{\langle \zeta, \mathbf{v} \rangle : \zeta \in \partial_c \Phi(\mathbf{z})\}.$$

- (iv) For every $\mathbf{z} \in \mathbb{R}^n$, the Clarke subdifferential $(\partial_c \Phi(\mathbf{z}))$ is a compact set.

The notion of a upper Kuratowski semicontinuous set-valued map from Singer [51] is presented in the following definition.

Definition 4. Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map. Then, we say that T is

- (i) Upper Kuratowski semicontinuous at $\mathbf{z} \in \mathbb{R}^n$ if, for any sequences, $\{\mathbf{z}_k\}_{k=1}^\infty \subseteq \mathbb{R}^n$ and $\{\mathbf{v}_k\}_{k=1}^\infty \subseteq \mathbb{R}^n$, with each $\mathbf{v}_k \in T(\mathbf{z}_k)$ ($k \in \mathbb{N}$) satisfying

$$\lim_{k \rightarrow \infty} \mathbf{z}_k = \mathbf{z} \text{ and } \lim_{k \rightarrow \infty} \mathbf{v}_k = \mathbf{v},$$

we have $\mathbf{v} \in T(\mathbf{z})$.

- (ii) Upper Kuratowski semicontinuous on $\mathbb{R}^n$ if T is upper Kuratowski semicontinuous at every $\mathbf{z} \in \mathbb{R}^n$.

Now, we recall the concept of a proper subodd bifunction (see, for instance, [45]).

Definition 5. A bifunction $(F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R})$ is called proper subodd if, for every $\mathbf{v}_j \in \mathbb{R}^n$ ($j = 1, 2, \dots, m$) with $\sum_{j=1}^m \mathbf{v}_j = 0$, the following inequality is satisfied:

$$F(\mathbf{z}; \mathbf{v}_1) + F(\mathbf{z}; \mathbf{v}_2) + \dots + F(\mathbf{z}; \mathbf{v}_m) \geq 0, \text{ for all } \mathbf{z} \in \mathbb{R}^n.$$

In the following definition, we introduce the concept of a generalized hemicontinuous bifunction.

Definition 6. Let $\emptyset \neq \mathcal{S} (\subseteq \mathbb{R}^n)$ be a convex set, and let $T : \mathcal{S} \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. A bifunction $(F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R})$ is said to be generalized hemicontinuous on $T(\mathcal{S})$ if for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$ and for each $\mathbf{w}_t \in T(t\mathbf{y} + (1-t)\mathbf{z})$ ($t \in (0, 1)$) such that $\mathbf{w}_t \rightarrow \bar{\mathbf{w}} \in T(\mathbf{z})$, as $t \downarrow 0$, the function expressed as $t \mapsto F(\mathbf{w}_t, \mathbf{y} - \mathbf{z})$ is continuous at 0^+ .

Remark 1. If $T(\mathbf{z}) = \{\mathbf{z}\}$, for every $\mathbf{z} \in \mathcal{S}$, then Definition 6 reduces to Definition 2.2 presented by Lalitha and Mehta [39].

In the following definition, we introduce the notions of generalized monotone and generalized stable pseudomonotone bifunctions. For further insights into monotonicity and stable pseudomonotonicity, we refer to [38,43,48].

Definition 7. Let $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a bifunction such that $F(\mathbf{v}, 0) = 0$, for all $\mathbf{v} \in \mathbb{R}^n$. For any $\mathcal{O} \neq \mathcal{S} (\subseteq \mathbb{R}^n)$, we assume that $T : \mathcal{S} \rightrightarrows \mathbb{R}^n$ is a set-valued map such that $\psi : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ and $g : \mathcal{S} \rightarrow \mathbb{R}$ are continuous and lower semicontinuous on $\mathcal{S} \times \mathcal{S}$ and $\mathcal{S}$, respectively. Then, F is said to be

(i) T -monotone if, for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, the following inequality holds for all $\mathbf{v} \in T(\mathbf{z}), \mathbf{w} \in T(\mathbf{y})$:

$$F(\mathbf{v}, \mathbf{y} - \mathbf{z}) + F(\mathbf{w}, \mathbf{z} - \mathbf{y}) \leq 0.$$

(ii) Strongly T -monotone if there exists α ($\alpha \in \mathbb{R}, \alpha > 0$) such that, for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$ and for all $\mathbf{v} \in T(\mathbf{z}), \mathbf{w} \in T(\mathbf{y})$, the following inequality holds:

$$F(\mathbf{v}, \mathbf{y} - \mathbf{z}) + F(\mathbf{w}, \mathbf{z} - \mathbf{y}) \leq -\alpha \|\mathbf{y} - \mathbf{z}\|^2.$$

(iii) Stably T -stably (g, ψ) -pseudomonotone with respect to sets $V \subseteq \mathbb{R}^n$ and $\Theta \in \mathbb{R}^n$ if, for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, the following inequality holds for all $\mathbf{v} \in T(\mathbf{z}), \mathbf{w} \in T(\mathbf{y}), \zeta \in V$:

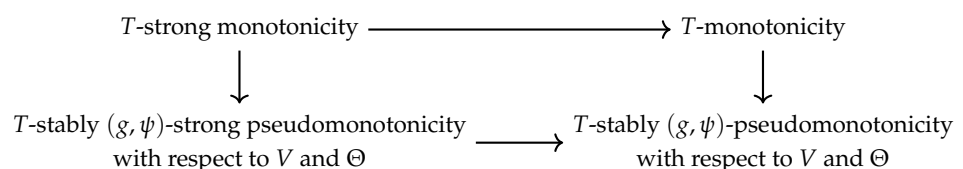
$$\begin{aligned} & F(\mathbf{v}, \mathbf{y} - \mathbf{z}) - \langle \Theta, \mathbf{y} - \mathbf{z} \rangle + g(\mathbf{y}) + \psi(\mathbf{z}, \mathbf{y}) - g(\mathbf{z}) - \psi(\mathbf{z}, \mathbf{z}) - \langle \zeta, \mathbf{y} - \mathbf{z} \rangle \geq 0 \\ \implies & F(\mathbf{w}, \mathbf{z} - \mathbf{y}) - \langle \Theta, \mathbf{z} - \mathbf{y} \rangle - g(\mathbf{y}) - \psi(\mathbf{z}, \mathbf{y}) + g(\mathbf{z}) + \psi(\mathbf{z}, \mathbf{z}) + \langle \zeta, \mathbf{y} - \mathbf{z} \rangle \leq 0. \end{aligned}$$

(iv) T -stably, (g, ψ) -strongly pseudomonotone with respect to sets $V \subseteq \mathbb{R}^n$ and $\Theta \in \mathbb{R}^n$ if there exists β ($\beta \in \mathbb{R}, \beta > 0$) such that, for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, the following inequality holds for all $\mathbf{v} \in T(\mathbf{z}), \mathbf{w} \in T(\mathbf{y}), \zeta \in V$

$$\begin{aligned} & F(\mathbf{v}, \mathbf{y} - \mathbf{z}) - \langle \Theta, \mathbf{y} - \mathbf{z} \rangle + g(\mathbf{y}) + \psi(\mathbf{z}, \mathbf{y}) - g(\mathbf{z}) - \psi(\mathbf{z}, \mathbf{z}) - \langle \zeta, \mathbf{y} - \mathbf{z} \rangle \geq 0 \\ \implies & F(\mathbf{w}, \mathbf{z} - \mathbf{y}) - \langle \Theta, \mathbf{z} - \mathbf{y} \rangle - g(\mathbf{y}) - \psi(\mathbf{z}, \mathbf{y}) + g(\mathbf{z}) + \psi(\mathbf{z}, \mathbf{z}) + \langle \zeta, \mathbf{y} - \mathbf{z} \rangle \\ & \leq -\beta \|\mathbf{y} - \mathbf{z}\|^2. \end{aligned}$$

Remark 2. 1. If $\Theta = 0, V = \{0\}$ and $T(\mathbf{z}) = \{\mathbf{z}\}$ for every $\mathbf{z} \in \mathcal{S}$ and $g(\mathbf{z}) = 0 = \psi(\mathbf{z}, \mathbf{y})$ for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, then T -stably (g, ψ) -pseudomonotonicity with respect to sets V and Θ reduces to pseudomonotonicity of F as introduced by Komlósi [52] in Definition 3.2.
2. If $T(\mathbf{z}) = \{\mathbf{z}\}$ for every $\mathbf{z} \in \mathcal{S}$, then T -strong monotonicity of F reduces to strong monotonicity of F as presented by Mehta et al. [45] in Definition 3.2.

The following diagram represents the relationships between generalized monotonicity and generalized stable pseudomonotonicity of bifunctions.



Remark 3. It is worth noting that T -stably (g, ψ) -pseudomonotone (or strongly pseudomonotone) bifunctions need not be T -monotone (or strongly monotone) bifunctions. Moreover, the T -stably (g, ψ) -pseudomonotonicity of a bifunction (F) with respect to V and Θ does not, in general, imply T -stably, (g, ψ) -strong pseudomonotonicity with respect to V and Θ . To illustrate this fact, we furnish the following example.

Example 1. We define $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows:

$$F(\mathbf{v}, \mathbf{y}) := (v_2 - 1)y_2, \text{ for all } \mathbf{y} = (y_1, y_2), \mathbf{v} = (v_1, v_2) \in \mathbb{R}^2.$$

Let $\mathcal{S} := \{\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2 : z_1 = 0, 3 \leq z_2 \leq 5\}$ and $T : \mathcal{S} \rightrightarrows \mathbb{R}^2$ be defined as follows:

$$T(\mathbf{z}) := \{\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2 : v_1 = 0, v_2 = -\frac{z_2}{3} + 1\}, \text{ for all } \mathbf{z} \in \mathcal{S}.$$

We define $g : \mathcal{S} \rightarrow \mathbb{R}$ and $\psi : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ as follows:

$$g(\mathbf{z}) := z_2, \psi(\mathbf{z}, \mathbf{y}) := z_2 + (y_2)^2, \text{ for all } \mathbf{z}, \mathbf{y} \in \mathcal{S}.$$

Let $V := \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \in \mathbb{R}, \xi_2 = 0\}$ and $\Theta = (1, 0) \in \mathbb{R}^2$. For any $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, $\xi \in V$, and $\mathbf{v} \in T(\mathbf{z})$, we consider the following:

$$F(\mathbf{v}, \mathbf{y} - \mathbf{z}) - \langle \Theta, \mathbf{y} - \mathbf{z} \rangle + g(\mathbf{y}) + \psi(\mathbf{z}, \mathbf{y}) - g(\mathbf{z}) - \psi(\mathbf{z}, \mathbf{z}) - \langle \xi, \mathbf{y} - \mathbf{z} \rangle \geq 0.$$

This implies that

$$(y_2 - z_2) \left(1 + y_2 + \frac{2z_2}{3} \right) \geq 0. \quad (2)$$

In view of the fact that $3 \leq y_2, z_2 \leq 5$, according to (2), we have

$$y_2 - z_2 \geq 0. \quad (3)$$

Now, for any $\mathbf{w} \in T(\mathbf{y})$ and $\xi \in V$, we have the following:

$$\begin{aligned} & F(\mathbf{w}, \mathbf{z} - \mathbf{y}) - \langle \Theta, \mathbf{z} - \mathbf{y} \rangle - g(\mathbf{y}) - \psi(\mathbf{z}, \mathbf{y}) + g(\mathbf{z}) + \psi(\mathbf{z}, \mathbf{z}) + \langle \xi, \mathbf{y} - \mathbf{z} \rangle \\ &= (z_2 - y_2) \left(1 + z_2 + \frac{2y_2}{3} \right). \end{aligned} \quad (4)$$

Upon the combination of (3) and (4), the following inequality holds:

$$F(\mathbf{w}, \mathbf{z} - \mathbf{y}) - \langle \Theta, \mathbf{z} - \mathbf{y} \rangle - g(\mathbf{y}) - \psi(\mathbf{z}, \mathbf{y}) + g(\mathbf{z}) + \psi(\mathbf{z}, \mathbf{z}) + \langle \xi, \mathbf{y} - \mathbf{z} \rangle \leq 0.$$

As a result, F is T -stably (g, ψ) -pseudomonotone with respect to V and Θ . However, F is not T -stably, (g, ψ) -strongly pseudomonotone with respect to V and Θ .

Now, we verify that F is not T -monotone. Let $\mathbf{z}, \mathbf{y} \in \mathcal{S}$ and $\mathbf{v} \in T(\mathbf{z})$, $\mathbf{w} \in T(\mathbf{y})$. Then,

$$\begin{aligned} & F(\mathbf{v}, \mathbf{y} - \mathbf{z}) + F(\mathbf{w}, \mathbf{z} - \mathbf{y}) = \left(-\frac{z_2}{3} \right) (y_2 - z_2) + \left(-\frac{y_2}{3} \right) (z_2 - y_2) \\ &= \frac{1}{3} (y_2 - z_2)^2. \end{aligned} \quad (5)$$

Let us consider $\mathbf{z} = (0, z_2)$ and $\mathbf{y} = (0, y_2)$ such that $\mathbf{z}, \mathbf{y} \in \mathcal{S}$ and $z_2 \neq y_2$. Then, it follows from (5) that

$$F(\mathbf{v}, \mathbf{y} - \mathbf{z}) + F(\mathbf{w}, \mathbf{z} - \mathbf{y}) > 0.$$

Hence, F is not T -monotone.

The following KKM map and KKM-Fan lemma play a key role in establishing existence results for the solutions of GSMWVVHIs and GMMWVVHIs (see [53,54]).

Definition 8. Let $\mathcal{S} (\subseteq \mathbb{R}^n)$ be a non-empty closed set and $\mathcal{G} : \mathcal{S} \rightrightarrows \mathcal{S}$ be a set-valued map $\mathcal{G}$ is referred to as a KKM map if, for any set $\{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^p\} \subseteq \mathcal{S}$, the following inclusion holds:

$$\text{co}\{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^p\} \subseteq \bigcup_{j=1}^p \mathcal{G}(\mathbf{y}^j).$$

Lemma 2. Let $\emptyset \neq \mathcal{S} (\subseteq \mathbb{R}^n)$ and $\mathcal{G} : \mathcal{S} \rightrightarrows \mathcal{S}$ be a KKM map. We assume that the following conditions are satisfied:

- (i) For every $\mathbf{z} \in \mathcal{S}$, $\mathcal{G}(\mathbf{z})$ is a closed set.
- (ii) $\mathcal{G}(\mathbf{z})$ is a compact set for some $\mathbf{z} \in \mathcal{S}$.

Then,

$$\bigcap_{\mathbf{z} \in \mathcal{S}} \mathcal{G}(\mathbf{z}) \neq \emptyset.$$

3. Existence and Uniqueness Results for GSMWVVHs and GMMWVVHs

In this section, by employing the KKM-Fan lemma, an existence theorem for solutions of GSMWVVHs is established without relying on any monotonicity assumptions on bifunctions under a relaxed compactness hypothesis. Under generalized stable pseudomonotonicity assumptions on bifunctions, we establish that GMMWVVHs admit at least one solution. Subsequently, we deduce an equivalence relation between the solutions of GSMWVVHs and GMMWVVHs under the generalized hemicontinuity and generalized stable pseudomonotonicity hypotheses on the involved bifunctions. Moreover, the uniqueness results for the solutions of GSMWVVHs and GMMWVVHs are established under a generalized stable, strong pseudomonotonicity hypothesis on bifunctions and some appropriate conditions.

In the remainder of the paper, the following conditions are assumed to be satisfied:

- (B1) Let $\emptyset \neq \mathcal{S}$ be a convex subset of $\mathbb{R}^n$ and $\Theta \in \mathbb{R}^n$.
- (B2) For each $j \in I := \{1, \dots, p\}$, let $T_j : \mathcal{S} \rightrightarrows \mathbb{R}^n$ be a set-valued map and $F_j : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a bifunction such that $F_j(\mathbf{v}, 0) = 0$, for all $\mathbf{v} \in \mathbb{R}^n$. Furthermore, we assume that $P_j : \mathbb{R}^n \rightarrow \mathbb{R}$ is a locally Lipschitz function on $\mathbb{R}^n$ for every $j \in I$.
- (B3) For every $j \in I$, let $\psi_j : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ and $g_j : \mathcal{S} \rightarrow \mathbb{R}$ be continuous and lower semicontinuous functions on $\mathcal{S} \times \mathcal{S}$ and $\mathcal{S}$, respectively.

Consider the following generalized Stampacchia and Minty mixed weak vector variational–hemivariational inequalities formulated in terms of bifunctions.

Generalized Stampacchia mixed weak vector variational–hemivariational inequality (GSMWVVHI): Find $\hat{\mathbf{z}} \in \mathcal{S}$ such that there exists $\hat{\mathbf{v}}_j \in T_j(\hat{\mathbf{z}})$ ($j \in I$) satisfying

$$\begin{aligned} & (F_1(\hat{\mathbf{v}}_1, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) \\ & + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\hat{\mathbf{v}}_p, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_p(\hat{\mathbf{z}}, \mathbf{y}) \\ & + g_p(\mathbf{y}) - \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_p(\hat{\mathbf{z}}) + P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0, \end{aligned}$$

for every $\mathbf{y} \in \mathcal{S}$.

Generalized Minty mixed weak vector variational–hemivariational inequality (MMWVVHI): Find $\hat{\mathbf{z}} \in \mathcal{S}$ such that for every $\mathbf{y} \in \mathcal{S}$ and $\mathbf{w}_j \in T_j(\mathbf{y})$ ($j \in I$),

$$\begin{aligned} & (F_1(\mathbf{w}_1, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_1(\hat{\mathbf{z}}, \mathbf{y}) - g_1(\mathbf{y}) + \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_1(\hat{\mathbf{z}}) \\ & - P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\mathbf{w}_p, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_p(\hat{\mathbf{z}}, \mathbf{y}) - g_p(\mathbf{y}) \\ & + \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_p(\hat{\mathbf{z}}) - P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\geq 0. \end{aligned}$$

Remark 4. 1. If $I = \{1\}$ and $\Theta = 0$, T_1 is a single-valued map, $g_1(\mathbf{z}) = 0 = \psi_1(\mathbf{z}, \mathbf{y})$ for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, and $F_1(\mathbf{w}_1, \mathbf{y} - \mathbf{z}) = \langle T_1(\mathbf{z}), \mathbf{y} - \mathbf{z} \rangle$, then the GSMWVVHI reduces to a VI of the following form: Find $\mathbf{z} \in \mathcal{S}$ such that

$$\langle T(\mathbf{z}), \mathbf{y} - \mathbf{z} \rangle \geq 0, \text{ for all } \mathbf{y} \in \mathcal{S},$$

as considered in [1,3].

2. For every $j \in I$, let $T_j : \mathcal{S} \rightarrow \mathbb{R}^n$ be a continuous function and $F_j(\mathbf{w}_j, \mathbf{y} - \mathbf{z}) = \langle T_j(\mathbf{z}), \mathbf{y} - \mathbf{z} \rangle$. If $\Theta = 0$ and, for every $j \in I$, $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, $\psi_j(\mathbf{z}, \mathbf{y}) = 0 = g_j(\mathbf{z})$, P_j is a constant function on $\mathbb{R}^n$, then GSMWVVHIs and GMMWVVHIs reduce to Stampacchia- and Minty-type weak vector variational inequalities, as studied by Giannessi [8].
3. If $I = \{1\}$ and $T_1(\mathbf{z}) = \{\mathbf{z}\}$ for every $\mathbf{z} \in \mathcal{S}$, $\Theta = 0$ and $\psi_1(\mathbf{z}, \mathbf{y}) = 0 = g_1(\mathbf{z})$ for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, P_1 is a constant function on $\mathbb{R}^n$, and $F_1(\mathbf{w}_1, \mathbf{y} - \mathbf{z}) = \langle T_1(\mathbf{z}), \mathbf{y} - \mathbf{z} \rangle$; then, $P_1^\circ(\mathbf{z}; \mathbf{y} - \mathbf{z}) = 0$ for every $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, and GSMWVVHIs and GMMWVVHIs reduce to SVIs and MVIs, respectively, as studied by Komlósi [55].
4. Let $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$ ($j \in I$) be a locally Lipschitz function at $\mathbf{z} \in \mathcal{S}$. If $\Theta = 0$, $T_j(\mathbf{z}) = \partial_c f_j(\mathbf{z})$ and $F_j(\mathbf{w}_j, \mathbf{y} - \mathbf{z}) = \langle \xi_j, \mathbf{y} - \mathbf{z} \rangle$ for every $j \in I$, $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, $\psi_j(\mathbf{z}, \mathbf{y}) = 0 = g_j(\mathbf{z})$ for every $j \in I$, $\mathbf{y}, \mathbf{z} \in \mathcal{S}$, and P_j is a constant function on $\mathbb{R}^n$ for every $j \in I$, then GSMWVVHIs and GMMWVVHIs reduce to WSVVIs and WMVVIs, respectively, as follows:

(WSVVI) : Find $\mathbf{z} \in \mathcal{S}$, such that for any $\mathbf{y} \in \mathcal{S}$, there exists $\xi_j \in \partial_c f_j(\mathbf{z})$ ($j \in I$)

$$(\langle \xi_1, \mathbf{y} - \mathbf{z} \rangle, \langle \xi_2, \mathbf{y} - \mathbf{z} \rangle, \dots, \langle \xi_p, \mathbf{y} - \mathbf{z} \rangle) \notin \text{int } \mathbb{R}_+^p,$$

(WMVVI) : Find $\mathbf{z} \in \mathcal{S}$, such that for any $\mathbf{y} \in \mathcal{S}$, and any $\xi_j \in \partial_c f_j(\mathbf{y})$ ($j \in I$)

$$(\langle \xi_1, \mathbf{z} - \mathbf{y} \rangle, \langle \xi_2, \mathbf{z} - \mathbf{y} \rangle, \dots, \langle \xi_p, \mathbf{z} - \mathbf{y} \rangle) \notin \text{int } \mathbb{R}_+^p,$$

as considered by Lee [17].

5. If $\Theta = 0$, $T_j(\mathbf{z}) = \{\mathbf{z}\}$, and $\psi_j(\mathbf{z}, \mathbf{y}) = 0 = g_j(\mathbf{z})$ for every $j \in I$, $\mathbf{y}, \mathbf{z} \in \mathcal{S}$ and P_j is a constant function on $\mathbb{R}^n$ for every $j \in I$, then GMMWVVHI reduces to a weak Minty VVI of the following form: Find $\mathbf{z} \in \mathcal{S}$ such that for every $\mathbf{y} \in \mathcal{S}$

$$(F_1(\mathbf{y}, \mathbf{z} - \mathbf{y}), \dots, F_p(\mathbf{y}, \mathbf{z} - \mathbf{y})) \not\geq 0,$$

as considered by Ansari and Lee [41].

6. If $I = \{1\}$ and $T_1(\mathbf{z}) = \{\mathbf{z}\}$ for every $\mathbf{z} \in \mathcal{S}$, $\Theta = 0$, $\psi_1(\mathbf{z}, \mathbf{y}) = 0$ for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, and P_1 is a constant function on $\mathbb{R}^n$, then $P_1^\circ(\mathbf{z}; \mathbf{y} - \mathbf{z}) = 0$ for every $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, and GSMWVVHIs and GMMWVVHIs reduce to the corresponding Stampacchia and Minty mixed VIs involving bifunctions, as studied by Mehta et al. [45].

By employing the KKM-Fan lemma, an existence result for solutions of GSMWVVHIs is established in the following theorem without imposing any monotonicity assumptions on bifunctions under a relaxed compactness assumption.

Theorem 1. Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^n$, such that for every $j \in I$ and $\mathbf{z} \in \mathcal{S}$, $\psi_j(\mathbf{z}, \cdot)$ and $g_j(\cdot)$ are convex functions on $\mathcal{S}$. Moreover, we assume that the following conditions hold:

- (i) For each $j \in I$, T_j is a compact set and upper Kuratowski semicontinuous on $\mathcal{S}$.
- (ii) For each $j \in I$ and $\mathbf{z} \in \mathbb{R}^n$, $F_j(\mathbf{z}, \cdot)$ is positively homogeneous and proper subodd on $\mathbb{R}^n$.
- (iii) For every $\mathbf{z} \in \mathcal{S}$, $(\mathbf{v}_j, \mathbf{z}) \mapsto F_j(\mathbf{v}_j, \mathbf{z})$ is upper semicontinuous on $\mathbb{R}^n \times \mathbb{R}^n$.
- (iv) There exists a non-empty compact subset (H) of $\mathcal{S}$ such that for every $\mathbf{y} \in \mathcal{S} \setminus H$ and for every $\mathbf{v}_j \in T_j(\mathbf{y})$ ($j \in I$), the following inclusion is satisfied for some $\tilde{\mathbf{z}} \in H$:

$$F_j(\mathbf{v}_j, \bar{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \bar{\mathbf{z}} - \mathbf{y} \rangle + \psi_j(\mathbf{y}, \bar{\mathbf{z}}) + g_j(\bar{\mathbf{z}}) - \psi_j(\mathbf{y}, \mathbf{y}) - g_j(\mathbf{y}) + P_j^\circ(\mathbf{y}; \bar{\mathbf{z}} - \mathbf{y}) < 0.$$

Then, GSMWVVI admits at least one solution in $\mathcal{S}$.

Proof. We define a set-valued map $G : \mathcal{S} \rightrightarrows \mathcal{S}$ as follows:

$$G(\mathbf{z}) := \{\mathbf{y} \in \mathcal{S} : \exists \mathbf{w}_j \in T_j(\mathbf{y}) (j \in I), \\ (F_1(\mathbf{w}_1, \mathbf{z} - \mathbf{y}) - \langle \Theta, \mathbf{z} - \mathbf{y} \rangle + \psi_1(\mathbf{y}, \mathbf{z}) + g_1(\mathbf{z}) - \psi_1(\mathbf{y}, \mathbf{y}) - g_1(\mathbf{y}) \\ + P_1^\circ(\mathbf{y}; \mathbf{z} - \mathbf{y}), \dots, F_p(\mathbf{w}_p, \mathbf{z} - \mathbf{y}) - \langle \Theta, \mathbf{z} - \mathbf{y} \rangle + \psi_p(\mathbf{y}, \mathbf{z}) \\ + g_p(\mathbf{z}) - \psi_p(\mathbf{y}, \mathbf{y}) - g_p(\mathbf{y}) + P_p^\circ(\mathbf{y}; \mathbf{z} - \mathbf{y})) \not\leq 0\}.$$

Owing to the fact that $\mathbf{z} \in G(\mathbf{z})$, it follows that $G(\mathbf{z}) \neq \emptyset$. To begin with, we claim that G is a KKM map. On the contrary, suppose that there exists a finite set $(\{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^m\} \subseteq \mathcal{S})$ such that

$$\text{co}\{\mathbf{y}^1, \dots, \mathbf{y}^m\} \not\subseteq \bigcup_{i=1}^m G(\mathbf{y}^i).$$

This implies that there exists $\hat{\mathbf{z}} \in \text{co}\{\mathbf{y}^1, \dots, \mathbf{y}^m\}$ such that $\hat{\mathbf{z}} \notin G(\mathbf{y}^i)$ for any $i = 1, 2, \dots, m$. Hence, there exist non-negative real numbers $(\alpha_i (i = 1, 2, \dots, m))$ with $\sum_{i=1}^m \alpha_i = 1$ such that

$$\hat{\mathbf{z}} = \sum_{i=1}^m \alpha_i \mathbf{y}^i.$$

For every $j \in I$ and $\mathbf{v}_j \in T_j(\hat{\mathbf{z}})$, it follows that

$$F_j(\mathbf{v}_j, \mathbf{y}^i - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y}^i - \hat{\mathbf{z}} \rangle + \psi_j(\hat{\mathbf{z}}, \mathbf{y}^i) + g_j(\mathbf{y}^i) - \psi_j(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_j(\hat{\mathbf{z}}) + P_j^\circ(\hat{\mathbf{z}}; \mathbf{y}^i - \hat{\mathbf{z}}) < 0. \quad (6)$$

For a fixed $j \in I$, multiplying by α_i in (6) and summing over i , we obtain the following due to the positive homogeneous property of $F_j(\mathbf{v}_j, \cdot)$ and $P_j^\circ(\hat{\mathbf{z}}; \cdot)$:

$$\sum_{i=1}^m F_j(\mathbf{v}_j, \alpha_i(\mathbf{y}^i - \hat{\mathbf{z}})) - \sum_{i=1}^m \alpha_i \langle \Theta, \mathbf{y}^i - \hat{\mathbf{z}} \rangle + \sum_{i=1}^m \alpha_i \psi_j(\hat{\mathbf{z}}, \mathbf{y}^i) + \sum_{i=1}^m \alpha_i g_j(\mathbf{y}^i) - \psi_j(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_j(\hat{\mathbf{z}}) + \sum_{i=1}^m P_j^\circ(\hat{\mathbf{z}}; \alpha_i(\mathbf{y}^i - \hat{\mathbf{z}})) < 0. \quad (7)$$

Since $\sum_{i=1}^m \alpha_i(\mathbf{y}^i - \hat{\mathbf{z}}) = \sum_{i=1}^m \alpha_i \mathbf{y}^i - \hat{\mathbf{z}} = 0$, $F_j(\mathbf{v}_j, \cdot)$, and $P_j^\circ(\hat{\mathbf{z}}; \cdot)$ are proper, subodd, and subadditive on $\mathbb{R}^n$, respectively,

$$\sum_{i=1}^m \alpha_i \psi_j(\hat{\mathbf{z}}, \mathbf{y}^i) + \sum_{i=1}^m \alpha_i g_j(\mathbf{y}^i) < \psi_j(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_j(\hat{\mathbf{z}}). \quad (8)$$

Moreover, from the given hypothesis, it follows that $\psi_j(\hat{\mathbf{z}}, \cdot)$ ($j \in I$) and $g_j(\cdot)$ ($j \in I$) are convex functions on $\mathcal{S}$. Hence,

$$g_j(\hat{\mathbf{z}}) = g_j\left(\sum_{i=1}^m \alpha_i \mathbf{y}^i\right) \leq \sum_{i=1}^m \alpha_i g_j(\mathbf{y}^i) \\ \psi_j(\hat{\mathbf{z}}, \hat{\mathbf{z}}) = \psi_j\left(\hat{\mathbf{z}}, \sum_{i=1}^m \alpha_i \mathbf{y}^i\right) \leq \sum_{i=1}^m \alpha_i \psi_j(\hat{\mathbf{z}}, \mathbf{y}^i). \quad (9)$$

In view of (9), we have

$$g_j(\hat{\mathbf{z}}) + \psi_j(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \leq \sum_{i=1}^m \alpha_i g_j(\mathbf{y}^i) + \sum_{i=1}^m \alpha_i \psi_j(\hat{\mathbf{z}}, \mathbf{y}^i), \quad (10)$$

which is a contradiction of (8). Therefore,

$$\text{co}\{\mathbf{y}^1, \dots, \mathbf{y}^m\} \subseteq \bigcup_{i=1}^m G(\mathbf{y}^i),$$

which claims that G is a KKM map.

In order to prove that $G(\mathbf{z})$ is a closed set for all $\mathbf{z} \in \mathcal{S}$, let $\{\mathbf{z}_s\}_{s=1}^\infty$ be any sequence in $G(\mathbf{z})$ such that $\mathbf{z}_s \rightarrow \mathbf{z}_0$ as $s \rightarrow \infty$. This implies that there exists $\mathbf{v}_j^s \in T_j(\mathbf{z}_s)$ ($j \in I$), satisfying

$$\begin{aligned} & (F_1(\mathbf{v}_1^s, \mathbf{z} - \mathbf{z}_s) - \langle \Theta, \mathbf{z} - \mathbf{z}_s \rangle + \psi_1(\mathbf{z}_s, \mathbf{z}) + g_1(\mathbf{z}) - \psi_1(\mathbf{z}_s, \mathbf{z}_s) - g_p(\mathbf{z}_s) \\ & + P_1^\circ(\mathbf{z}_s; \mathbf{z} - \mathbf{z}_s), \dots, F_p(\mathbf{v}_p^s, \mathbf{z} - \mathbf{z}_s) - \langle \Theta, \mathbf{z} - \mathbf{z}_s \rangle + \psi_p(\mathbf{z}_s, \mathbf{z}) + g_p(\mathbf{z}) - \psi_p(\mathbf{z}_s, \mathbf{z}_s) \\ & - g_p(\mathbf{z}_s) + P_p^\circ(\mathbf{z}_s; \mathbf{z} - \mathbf{z}_s)) \not\leq 0. \end{aligned}$$

It is evident from the given hypotheses that T_j ($j \in I$) is a compact set and upper Kuratowski semicontinuous on $\mathcal{S}$. This implies that there exists a subsequence $(\{\mathbf{v}_j^{s_k}\}_{k=1}^\infty)$ of $\{\mathbf{v}_j^s\}_{s=1}^\infty$, such that

$$\begin{aligned} \mathbf{v}_j^{s_k} & \in T_j(\mathbf{z}_{s_k}), \mathbf{v}_j^{s_k} \rightarrow \mathbf{v}_j^0 \text{ as } k \rightarrow \infty, j \in I, \\ \mathbf{v}_j^0 & \in T_j(\mathbf{z}_0), j \in I. \end{aligned}$$

Moreover, by employing the continuity of the inner product, Assumption (B3), the upper semicontinuity of $P_j^\circ(\cdot, \cdot)$, and $F_j(\cdot, \cdot)$ ($j \in I$), along with the continuity of $\psi_j(\cdot, \cdot)$ and lower semicontinuity of $g_j(\cdot)$, we infer that

$$\begin{aligned} & (F_1(\mathbf{v}_1^0, \mathbf{z} - \mathbf{z}_0) - \langle \Theta, \mathbf{z} - \mathbf{z}_0 \rangle + \psi_1(\mathbf{z}_0, \mathbf{z}) + g_1(\mathbf{z}) - \psi_1(\mathbf{z}_0, \mathbf{z}_0) - g_1(\mathbf{z}_0) \\ & + P_1^\circ(\mathbf{z}_0; \mathbf{z} - \mathbf{z}_0), \dots, F_p(\mathbf{v}_p^0, \mathbf{z} - \mathbf{z}_0) - \langle \Theta, \mathbf{z} - \mathbf{z}_0 \rangle + \psi_p(\mathbf{z}_0, \mathbf{z}) + g_p(\mathbf{z}) - \psi_p(\mathbf{z}_0, \mathbf{z}_0) \\ & - g_p(\mathbf{z}_0) + P_p^\circ(\mathbf{z}_0; \mathbf{z} - \mathbf{z}_0)) \not\leq 0. \end{aligned}$$

Thus, it follows that $\mathbf{z}_0 \in G(\mathbf{z})$; hence, $G(\mathbf{z})$ is a closed set for every $\mathbf{z} \in \mathcal{S}$. Now, we are left to show that $G(\mathbf{z}) \subseteq H$ for some $\mathbf{z} \in \mathcal{S}$. On the contrary, we suppose that for every $\mathbf{z} \in \mathcal{S}$, $G(\mathbf{z}) \not\subseteq H$, that is, for every $\mathbf{z} \in \mathcal{S}$, there exists $\mathbf{w} \in G(\mathbf{z}) \setminus H$. It follows that for every $\mathbf{z} \in \mathcal{S}$, there exists $\mathbf{w}_j \in T_j(\mathbf{w})$ ($j \in I$) such that

$$\begin{aligned} & (F_1(\mathbf{w}_1, \mathbf{z} - \mathbf{w}) - \langle \Theta, \mathbf{z} - \mathbf{w} \rangle + \psi_1(\mathbf{w}, \mathbf{z}) + g_1(\mathbf{z}) - \psi_1(\mathbf{w}, \mathbf{w}) - g_1(\mathbf{w}) \\ & + P_1^\circ(\mathbf{w}; \mathbf{z} - \mathbf{w}), \dots, F_p(\mathbf{w}_p, \mathbf{z} - \mathbf{w}) - \langle \Theta, \mathbf{z} - \mathbf{w} \rangle + \psi_p(\mathbf{w}, \mathbf{z}) \\ & + g_p(\mathbf{z}) - \psi_p(\mathbf{w}, \mathbf{w}) - g_p(\mathbf{w}) + P_p^\circ(\mathbf{w}; \mathbf{z} - \mathbf{w})) \not\leq 0. \end{aligned} \quad (11)$$

Since

$$G(\mathbf{z}) \subseteq \mathcal{S} \quad \text{and} \quad \mathbf{w} \in G(\mathbf{z}) \setminus H$$

, $\mathbf{w} \in \mathcal{S} \setminus H$. In view of the given hypothesis, there exists $\tilde{\mathbf{z}} \in H \subseteq \mathcal{S}$ such that the following inequality holds for every $j \in I$ and $\mathbf{w}_j \in T_j(\mathbf{w})$

$$F_j(\mathbf{w}_j, \tilde{\mathbf{z}} - \mathbf{w}) - \langle \Theta, \tilde{\mathbf{z}} - \mathbf{w} \rangle + \psi_j(\mathbf{w}, \tilde{\mathbf{z}}) + g_j(\tilde{\mathbf{z}}) - \psi_j(\mathbf{w}, \mathbf{w}) - g_j(\mathbf{w}) + P_j^\circ(\mathbf{w}; \tilde{\mathbf{z}} - \mathbf{w}) < 0,$$

which is a contradiction of (11). This implies that

$$G(\mathbf{z}) \subseteq H, \text{ for some } \mathbf{z} \in \mathcal{S}.$$

From Lemma 2, we infer that

$$\bigcap_{\mathbf{z} \in \mathcal{S}} G(\mathbf{z}) \neq \emptyset.$$

Hence, there exist $\hat{\mathbf{z}} \in \mathcal{S}$ and $\mathbf{v}_j^* \in T_j(\hat{\mathbf{z}})$ ($j \in I$) such that

$$\begin{aligned} & (F_1(\mathbf{v}_1^*, \mathbf{z} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{z} - \hat{\mathbf{z}} \rangle + \psi_1(\hat{\mathbf{z}}, \mathbf{z}) + g_1(\mathbf{z}) - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) + P_1^\circ(\hat{\mathbf{z}}; \mathbf{z} - \hat{\mathbf{z}}), \dots, \\ & F_p(\mathbf{v}_p^*, \mathbf{z} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{z} - \hat{\mathbf{z}} \rangle + \psi_p(\hat{\mathbf{z}}, \mathbf{z}) + g_p(\mathbf{z}) - \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_p(\hat{\mathbf{z}}) + P_p^\circ(\hat{\mathbf{z}}; \mathbf{z} - \hat{\mathbf{z}})) \not\leq 0, \end{aligned}$$

for every $\mathbf{z} \in \mathcal{S}$. This completes the proof. $\square$

Remark 5. If $\Theta = 0$, $I = \{1\}$, $T_1(\mathbf{z}) = \{\mathbf{z}\}$, and $\psi_1(\mathbf{z}, \mathbf{y}) = 0$ for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$ and P_1 is a constant function on $\mathbb{R}^n$, then $P_1^\circ(\mathbf{z}; \mathbf{y} - \mathbf{z}) = 0$ for every $\mathbf{z}, \mathbf{y} \in \mathbb{R}^n$, and Theorem 1 reduces to Theorem 2.2 as established by Mehta et al. [45].

In the following theorem, we establish sufficient conditions for the existence of solutions of GSMWVHIs under a compactness assumption on set $\mathcal{S}$.

Theorem 2. Let $\mathcal{S}$ be a compact subset of $\mathbb{R}^n$, such that for every $j \in I$ and $\mathbf{z} \in \mathcal{S}$, $\psi_j(\mathbf{z}, \cdot)$ and $g_j(\cdot)$ are convex functions on $\mathcal{S}$. Moreover, we assume that hypotheses (i)–(iii) in Theorem 1 are satisfied. Then, there exists $\hat{\mathbf{z}} \in \mathcal{S}$ such that $\hat{\mathbf{z}}$ is a solution of GSMWVHI.

Proof. We define a set-valued map $G : \mathcal{S} \rightrightarrows \mathcal{S}$ as follows:

$$\begin{aligned} G(\mathbf{z}) := \{ & \mathbf{y} \in \mathcal{S} : \exists \mathbf{w}_j \in T_j(\mathbf{y}) \ (j \in I), \\ & (F_1(\mathbf{w}_1, \mathbf{z} - \mathbf{y}) - \langle \Theta, \mathbf{z} - \mathbf{y} \rangle + \psi_1(\mathbf{y}, \mathbf{z}) + g_1(\mathbf{z}) - \psi_1(\mathbf{y}, \mathbf{y}) - g_1(\mathbf{y}) \\ & + P_1^\circ(\mathbf{y}; \mathbf{z} - \mathbf{y}), \dots, F_p(\mathbf{w}_p, \mathbf{z} - \mathbf{y}) - \langle \Theta, \mathbf{z} - \mathbf{y} \rangle + \psi_p(\mathbf{y}, \mathbf{z}) \\ & + g_p(\mathbf{z}) - \psi_p(\mathbf{y}, \mathbf{y}) - g_p(\mathbf{y}) + P_p^\circ(\mathbf{y}; \mathbf{z} - \mathbf{y})) \not\leq 0\}. \end{aligned}$$

Along the lines of the proof of Theorem 1, it follows that G is a KKM map and $G(\mathbf{z})$ is a closed set for every $\mathbf{z} \in \mathcal{S}$. Owing to the fact that for any $\mathbf{z} \in \mathcal{S}$, $G(\mathbf{z}) \subseteq \mathcal{S}$ and $\mathcal{S}$ is a bounded set, we infer that $G(\mathbf{z})$ is a compact set. Therefore, from Lemma 2, it follows that

$$\bigcap_{\mathbf{z} \in \mathcal{S}} G(\mathbf{z}) \neq \emptyset,$$

which, in turn, claims that GSMWVHI admits at least one solution in $\mathcal{S}$. $\square$

Remark 6. Theorem 2 generalizes Theorem 2.1, as deduced by Mehta et al. [45], from a mixed Stampacchia VI to a generalized Stampacchia mixed weak vector variational–hemivariational inequality, belonging to the broader classes of VVIs and variational–hemivariational inequalities.

The following example demonstrates the existence result for the solutions of GSMWVHIs established in Theorem 2.

Example 2. Let $\mathcal{S} := \{\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2 : 1 \leq z_1 \leq 3, 1 \leq z_2 \leq 2\}$, and for every $j \in I := \{1, 2\}$, let the bifunctions $(F_j : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R})$ be defined as follows:

$$F_j(\mathbf{v}, \mathbf{y}) := \left(10 - \frac{1}{\|\mathbf{v}\| + 1}\right) \langle \mathbf{v}, \mathbf{y} \rangle - y_2, \text{ for all } \mathbf{y}, \mathbf{v} \in \mathbb{R}^2.$$

We define $T_j : \mathcal{S} \rightrightarrows \mathbb{R}$ for every $j \in I$ as follows:

$$T_1(\mathbf{z}) := \{(z_1, z_2)\}, \quad T_2(\mathbf{z}) := \{(2z_1, 2z_2)\}, \text{ for all } \mathbf{z} \in \mathcal{S}.$$

It can be verified that $T_j(\cdot)$ ($j \in I$) are compact sets and upper Kuratowski semicontinuous on $\mathcal{S}$. Let $\psi_1, \psi_2 : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ be defined as follows:

$$\psi_1(\mathbf{z}, \mathbf{y}) := y_1 + 2y_2, \quad \psi_2(\mathbf{z}, \mathbf{y}) := 2y_2 + y_1 + e^2, \text{ for all } \mathbf{z}, \mathbf{y} \in \mathcal{S}.$$

Consider functions $g_1, g_2 : \mathcal{S} \rightarrow \mathbb{R}$, defined as follows:

$$g_1(\mathbf{z}) := z_1, \quad g_2(\mathbf{z}) := z_1 + 7e, \text{ for all } \mathbf{z} \in \mathcal{S}.$$

Now, we define functions $P_1, P_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows:

$$P_1(\mathbf{z}) := \begin{cases} -z_1, & \mathbf{z} \in \mathcal{S}, \\ -z_1 - e, & \text{otherwise,} \end{cases}; \quad P_2(\mathbf{z}) := -z_1 + e^2, \text{ for all } \mathbf{z} \in \mathbb{R}^2.$$

It can be verified that the Clarke directional derivatives of P_1 and P_2 at $\mathbf{z} \in \mathcal{S}$ in the direction of $\mathbf{y} - \mathbf{z}$ ($\mathbf{y} \in \mathcal{S}$) are given as follows:

$$P_1^\circ(\mathbf{z}; \mathbf{y} - \mathbf{z}) = -(y_1 - z_1), \quad P_2^\circ(\mathbf{z}; \mathbf{y} - \mathbf{z}) = -(y_1 - z_1).$$

Consider the following generalized Stampacchia and Minty mixed weak vector variational-hemivariational inequalities for $\Theta = (1, 1)$.

(GSMWVVHI1): Find $\hat{\mathbf{z}} \in \mathcal{S}$, such that there exists $\hat{\mathbf{v}}_j \in T_j(\hat{\mathbf{z}})$ ($j \in I$), satisfying

$$\begin{aligned} & (F_1(\hat{\mathbf{v}}_1, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \\ & F_2(\hat{\mathbf{v}}_2, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_2(\hat{\mathbf{z}}, \mathbf{y}) + g_2(\mathbf{y}) - \psi_2(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_2(\hat{\mathbf{z}}) + P_2^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\prec 0 \end{aligned}$$

for every $\mathbf{y} \in \mathcal{S}$.

Evidently, $g_j(\cdot)$ and $\psi_j(\mathbf{z}, \cdot)$ for every $j \in I$ are convex functions on $\mathcal{S}$ for every $\mathbf{z} \in \mathcal{S}$. Moreover, for every $j \in I$, F_j is positively homogeneous with respect to the second component and proper subodd. Therefore, all the hypotheses of Theorem 2 are satisfied.

Let us define a set (V) associated with P_j ($j \in I$):

$$V(P_j) := \{-\zeta_j : \mathbf{z} \in \mathcal{S}, \zeta_j \in \partial_c P_j(\mathbf{z})\}. \quad (12)$$

Under T_j -stably (g_j, ψ_j) -pseudomonotonicity assumptions on F_j ($j \in I$) and the hypotheses of Theorem 2, an existence result for solutions of GMMWVVHIs is derived in the following theorem.

Theorem 3. Let us assume that all the hypotheses in Theorem 2 are satisfied. Furthermore, we assume that for every $j \in I$, F_j is T_j -stably (g_j, ψ_j) -pseudomonotone with respect to set $V(P_j)$ and Θ . Then, there exists at least one solution to GMMWVVHI.

Proof. In view of Theorem 2, it follows that there exists at least one solution of GSMWVHI, e.g., $\hat{\mathbf{z}}$, that is, there exists $\hat{\mathbf{v}}_j \in T_j(\hat{\mathbf{z}})$ ($j \in I$) such that

$$\begin{aligned} & (F_1(\hat{\mathbf{v}}_1, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) \\ & + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\hat{\mathbf{v}}_p, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_p(\hat{\mathbf{z}}, \mathbf{y}) + g_p(\mathbf{y}) - \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \\ & - g_p(\hat{\mathbf{z}}) + P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0 \end{aligned}$$

for every $\mathbf{y} \in \mathcal{S}$. Equivalently, for every $\mathbf{y} \in \mathcal{S}$, there exists at least one component ($i \in I$) such that

$$\begin{aligned} & F_i(\hat{\mathbf{v}}_i, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_i(\hat{\mathbf{z}}, \mathbf{y}) + g_i(\mathbf{y}) - \psi_i(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_i(\hat{\mathbf{z}}) \\ & + P_i^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}) \geq 0. \end{aligned} \quad (13)$$

Since P_i ($i \in I$) is locally Lipschitz on $\mathbb{R}^n$ and the inner product $(\langle \cdot, \cdot \rangle)$ is continuous, it follows from Lemma 1 (iii)–(iv) that there exists $\hat{\zeta}_i \in \partial_c P_i(\hat{\mathbf{z}})$ such that

$$P_i^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}) = \langle \hat{\zeta}_i, \mathbf{y} - \hat{\mathbf{z}} \rangle. \quad (14)$$

This implies that (13) can be rewritten as follows:

$$\begin{aligned} & F_i(\mathbf{v}_i, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_i(\hat{\mathbf{z}}, \mathbf{y}) + g_i(\mathbf{y}) - \psi_i(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_i(\hat{\mathbf{z}}) \\ & - \langle -\hat{\zeta}_i, \mathbf{y} - \hat{\mathbf{z}} \rangle \geq 0. \end{aligned} \quad (15)$$

By employing the T_i -stably (g_i, ψ_i) -pseudomonotonicity of F_i with respect to set $V(P_i)$ as defined in (12) and Θ , we infer that for every $\mathbf{y} \in \mathcal{S}$ and for every $\mathbf{w}_i \in T_i(\mathbf{y})$, the following inequality holds:

$$\begin{aligned} & F_i(\mathbf{w}_i, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_i(\hat{\mathbf{z}}, \mathbf{y}) - g_i(\mathbf{y}) + \psi_i(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_i(\hat{\mathbf{z}}) \\ & - \langle \hat{\zeta}_i, \mathbf{y} - \hat{\mathbf{z}} \rangle \leq 0. \end{aligned} \quad (16)$$

Owing to the combination of (14) and (16), for every $\mathbf{y} \in \mathcal{S}$, there exists $i \in I$ such that for any $\mathbf{w}_i \in T_i(\mathbf{y})$, the following inequality is satisfied:

$$F_i(\mathbf{w}_i, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_i(\hat{\mathbf{z}}, \mathbf{y}) - g_i(\mathbf{y}) + \psi_i(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_i(\hat{\mathbf{z}}) - P_i^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}) \leq 0.$$

Hence, there exists $\hat{\mathbf{z}} \in \mathcal{S}$ such that for every $\mathbf{y} \in \mathcal{S}$ and $\mathbf{w}_j \in T_j(\mathbf{y})$ ($j \in I$), we have

$$\begin{aligned} & (F_1(\mathbf{w}_1, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_1(\hat{\mathbf{z}}, \mathbf{y}) - g_1(\mathbf{y}) + \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_1(\hat{\mathbf{z}}) \\ & - P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\mathbf{w}_p, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_p(\hat{\mathbf{z}}, \mathbf{y}) - g_p(\mathbf{y}) + \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \\ & + g_p(\hat{\mathbf{z}}) - P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0. \end{aligned}$$

Thus, the desired result follows. $\square$

The following corollary is an immediate consequence of Theorem 3.

Corollary 1. If for every $j \in I$, F_j is T_j -stably (g_j, ψ_j) -pseudomonotone with respect to set $V(P_j)$ and Θ , then every solution of GSMWVHI is a solution of GMMWVHI.

Remark 7. Corollary 1 generalizes Theorem 2.3 (i) in [45] from mixed VIs to broader classes of vector variational–hemivariational inequalities—namely, GSMWVHIs and GMMWVHIs.

In the following theorem, we establish an existence result for solutions of GSMWVHIs by relating them to GMMWVHIs under generalized hemicontinuity assumptions on bi-functions (F_j ($j \in I$)).

Theorem 4. For every $j \in I$ and $\mathbf{z} \in \mathcal{S}$, let $\psi_j(\mathbf{z}, \cdot)$ and $g_j(\cdot)$ be convex functions on $\mathcal{S}$. Moreover, we assume that the following conditions hold:

- (i) For each $j \in I$, T_j is a compact set and upper Kuratowski semicontinuous on $\mathcal{S}$.
- (ii) For each $j \in I$ and $\mathbf{z} \in \mathbb{R}^n$, $F_j(\mathbf{z}, \cdot)$ is positively homogeneous and proper subodd on $\mathbb{R}^n$.
- (iii) For each $j \in I$, F_j is generalized hemicontinuous on $T_j(\mathcal{S})$.

Then, $\hat{\mathbf{z}} \in \mathcal{S}$ is a solution of GSMWVHI, provided that $\hat{\mathbf{z}}$ is a solution of GMMWVHI.

Proof. In light of the given hypothesis, $\hat{\mathbf{z}}$ is a solution of GMMWVHI. This implies that for every $\mathbf{y} \in \mathcal{S}$ and $\mathbf{w}_j \in T_j(\mathbf{y})$ ($j \in I$), the following condition holds:

$$\begin{aligned} & (F_1(\mathbf{w}_1, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_1(\hat{\mathbf{z}}, \mathbf{y}) - g_1(\mathbf{y}) + \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_1(\hat{\mathbf{z}}) \\ & - P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\mathbf{w}_p, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_p(\hat{\mathbf{z}}, \mathbf{y}) - g_p(\mathbf{y}) + \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \\ & + g_p(\hat{\mathbf{z}}) - P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\geq 0. \end{aligned}$$

Let $\mathbf{y} \in \mathcal{S}$ be arbitrary and $t_m \subseteq (0, 1]$ such that $t_m \downarrow 0$ as $m \rightarrow \infty$. According to Assumption (B1), $\mathcal{S}$ is a convex set, which implies that $\gamma(t_m) := (1 - t_m)\hat{\mathbf{z}} + t_m\mathbf{y} \in \mathcal{S}$. Therefore, for every $\mathbf{w}_j^m \in T_j(\gamma(t_m))$ ($j \in I$), we have the following:

$$\begin{aligned} & (F_1(\mathbf{w}_1^m, \hat{\mathbf{z}} - \gamma(t_m)) - \langle \Theta, \hat{\mathbf{z}} - \gamma(t_m) \rangle - \psi_1(\hat{\mathbf{z}}, \gamma(t_m)) - g_1(\gamma(t_m)) + \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_1(\hat{\mathbf{z}}) \\ & - P_1^\circ(\hat{\mathbf{z}}; \gamma(t_m) - \hat{\mathbf{z}}), \dots, F_p(\mathbf{w}_p^m, \hat{\mathbf{z}} - \gamma(t_m)) - \langle \Theta, \hat{\mathbf{z}} - \gamma(t_m) \rangle - \psi_p(\hat{\mathbf{z}}, \gamma(t_m)) \\ & - g_p(\gamma(t_m)) + \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_p(\hat{\mathbf{z}}) - P_p^\circ(\hat{\mathbf{z}}; \gamma(t_m) - \hat{\mathbf{z}})) \not\geq 0. \end{aligned}$$

Since $\hat{\mathbf{z}} - \gamma(t_m) = t_m(\hat{\mathbf{z}} - \mathbf{y})$ and $\gamma(t_m) - \hat{\mathbf{z}} = t_m(\mathbf{y} - \hat{\mathbf{z}})$, $\psi_j(\hat{\mathbf{z}}, \cdot)$, $g_j(\cdot)$ are convex functions on $\mathcal{S}$, and $P_j^\circ(\hat{\mathbf{z}}; \cdot)$, $F_j(\mathbf{w}_j^m, \cdot)$ ($j \in I$) are positively homogeneous; therefore,

$$\begin{aligned} & (F_1(\mathbf{w}_1^m, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle + \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_1(\hat{\mathbf{z}}) - \psi_1(\hat{\mathbf{z}}, \mathbf{y}) - g_1(\mathbf{y}) \\ & - P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\mathbf{w}_p^m, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle + \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \\ & + g_p(\hat{\mathbf{z}}) - \psi_p(\hat{\mathbf{z}}, \mathbf{y}) - g_p(\mathbf{y}) - P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\geq 0. \end{aligned} \quad (17)$$

Moreover, owing to the proper subodd property of $F_j(\mathbf{w}_j^m, \cdot)$ ($j \in I$) and (17), we have

$$\begin{aligned} & (F_1(\mathbf{w}_1^m, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) \\ & + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\mathbf{w}_p^m, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle - \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \\ & - g_p(\hat{\mathbf{z}}) + \psi_p(\hat{\mathbf{z}}, \mathbf{y}) + g_p(\mathbf{y}) + P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0. \end{aligned} \quad (18)$$

It is given that T_j ($j \in I$) represents compact sets. Hence, there exists a subsequence $(\{\mathbf{w}_j^{m_k}\}_{k=1}^\infty)$ of $\{\mathbf{w}_j^m\}_{m=1}^\infty$ and $\{\gamma(t_{m_k})\}_{k=1}^\infty$ of $\{\gamma(t_m)\}_{m=1}^\infty$ such that $\mathbf{w}_j^{m_k} \in T_j(\gamma(t_{m_k}))$ and $\mathbf{w}_j^{m_k} \rightarrow \hat{\mathbf{w}}_j$ as $k \rightarrow \infty$. It follows from the upper Kuratowski semicontinuity property of T_j ($j \in I$) that $\hat{\mathbf{w}}_j \in T_j(\hat{\mathbf{z}})$. According to (18), the following inequality holds for every $\mathbf{y} \in \mathcal{S}$:

$$\begin{aligned} & (F_1(\mathbf{w}_1^{m_k}, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) \\ & + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\mathbf{w}_p^{m_k}, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle - \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \\ & - g_p(\hat{\mathbf{z}}) + \psi_p(\hat{\mathbf{z}}, \mathbf{y}) + g_p(\mathbf{y}) + P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0. \end{aligned} \quad (19)$$

Letting $k \rightarrow \infty$ in (19), it follows from the generalized hemicontinuous property of $F_j(\cdot, \mathbf{y} - \hat{\mathbf{z}})$ that

$$\begin{aligned} & (F_1(\hat{\mathbf{w}}_1, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) \\ & + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\hat{\mathbf{w}}_p, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle - \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \\ & - g_p(\hat{\mathbf{z}}) + \psi_p(\hat{\mathbf{z}}, \mathbf{y}) + g_p(\mathbf{y}) + P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0. \end{aligned}$$

Hence, $\hat{\mathbf{z}} \in \mathcal{S}$ is a solution of GSMWVVI. $\square$

Remark 8. If $\Theta = 0$, $I = \{1\}$, $T_1(\mathbf{z}) = \{\mathbf{z}\}$, and $\psi_1(\mathbf{z}, \mathbf{y}) = 0$ for every $\mathbf{z}, \mathbf{y} \in \mathcal{S}$ and P_1 is a constant function on $\mathbb{R}^n$, then $P_1^\circ(\mathbf{z}; \mathbf{y} - \mathbf{z}) = 0$ for every $\mathbf{z}, \mathbf{y} \in \mathbb{R}^n$, and Theorem 4 reduces to Theorem 2.3 (ii) derived by Mehta et al. [45].

The following corollary characterizes the relationship between the solutions of GSMWVVI and GMMWVVI by using T_j -stably (g_j, ψ_j) -pseudomonotonicity and generalized hemicontinuity assumptions on the involved bifunctions (F_j ($j \in I$)).

Corollary 2. Let every assumption in Theorems 3 and 4 be satisfied. Then, $\hat{\mathbf{z}} \in \mathcal{S}$ is a solution of GSMWVVI if and only if $\hat{\mathbf{z}} \in \mathcal{S}$ is a solution of GMMWVVI.

Remark 9. 1. Corollary 2 generalizes Proposition 4 and Theorem 3.1 deduced by Giannessi [8] and Lee [17], respectively, from VVI to generalized mixed-vector variational-hemivariational inequalities—namely, GSMWVVI and GMMWVVI.

2. It is worth noting that Proposition 4 in [8], Theorem 3.1 in [17], and Theorem 2.3 (i) in [45] are established by employing the monotonicity assumptions on the underlying mappings. However, the proof of Theorem 3 relies on T_j -stably (g_j, ψ_j) -pseudomonotonicity of F_j with respect to set $V(P_j)$ and Θ , which is weaker than the T_j -monotonicity assumptions on F_j . Consequently, the results derived in this paper are more widely applicable to VVI than the corresponding results available in the existing literature, even for particular reduced cases of GSMWVVI and GMMWVVI (see, for instance, [8,17]). We demonstrate this fact in the following example.

Example 3. Let $\mathcal{S} := \{\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2 : 1 \leq z_1 \leq 3, z_2 = 0\}$, and for every $j \in I := \{1, 2\}$, let the bifunctions expressed as $F_j : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as follows:

$$F_1(\mathbf{v}, \mathbf{y}) := (v_2 - 1)y_1, \quad F_2(\mathbf{v}, \mathbf{y}) := (v_1 + 1)y_1, \text{ for all } \mathbf{y}, \mathbf{v} \in \mathbb{R}^2.$$

We define $T_j : \mathcal{S} \rightrightarrows \mathbb{R}$ for every $j \in I$ as follows:

$$\begin{aligned} T_1(\mathbf{z}) &:= \{\mathbf{v} \in \mathbb{R}^2 : v_1 = tz_1, v_2 = -z_1 + 1, t \in [0, 1]\}, \text{ for all } \mathbf{z} \in \mathcal{S}, \\ T_2(\mathbf{z}) &:= \{\mathbf{v} \in \mathbb{R}^2 : v_1 = -z_1 - 1, v_2 = (1 - t)z_1, t \in [0, 1]\}, \text{ for all } \mathbf{z} \in \mathcal{S}. \end{aligned}$$

Let $\psi_1, \psi_2 : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ be defined as follows:

$$\psi_1(\mathbf{z}, \mathbf{y}) := z_1 + y_1^2, \quad \psi_2(\mathbf{z}, \mathbf{y}) := y_1^2 + z_1 + 1, \text{ for all } \mathbf{z}, \mathbf{y} \in \mathcal{S}.$$

Let $g_1, g_2 : \mathcal{S} \rightarrow \mathbb{R}$ be defined as follows:

$$g_1(\mathbf{z}) := z_1, \quad g_2(\mathbf{z}) := z_1 + 7, \quad \text{for all } \mathbf{z} \in \mathcal{S}.$$

We define functions $P_1, P_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows:

$$P_1(\mathbf{z}) := 2z_2^2, \quad P_2(\mathbf{z}) := -z_2^2, \quad \text{for all } \mathbf{z} \in \mathbb{R}^2.$$

It can be verified that the Clarke directional derivatives of P_1 and P_2 at $\mathbf{z} \in \mathbb{R}^2$ in the direction of $\mathbf{v} \in \mathbb{R}^2$ are given as follows:

$$P_1^\circ(\mathbf{z}; \mathbf{v}) = 4z_2v_2, \quad P_2^\circ(\mathbf{z}; \mathbf{v}) = -2z_2v_2.$$

Consider the following generalized Stampacchia and Minty mixed weak vector variational-hemivariational inequalities for $\Theta = (0, 1) \in \mathbb{R}^2$.

(GSMWVVHI2): Find $\hat{\mathbf{z}} \in \mathcal{S}$, such that there exists $\hat{\mathbf{v}}_j \in T_j(\hat{\mathbf{z}})$ ($j \in I$), satisfying

$$\begin{aligned} & (F_1(\hat{\mathbf{v}}_1, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \\ & F_2(\hat{\mathbf{v}}_2, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_2(\hat{\mathbf{z}}, \mathbf{y}) + g_2(\mathbf{y}) - \psi_2(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_2(\hat{\mathbf{z}}) + P_2^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0 \end{aligned}$$

for every $\mathbf{y} \in \mathcal{S}$.

(GMMWVVHI1): Find $\hat{\mathbf{z}} \in \mathcal{S}$, such that for every $\mathbf{y} \in \mathcal{S}$ and any $\mathbf{w}_j \in T_j(\mathbf{y})$ ($j \in I$), the following condition holds:

$$\begin{aligned} & (F_1(\mathbf{w}_1, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_1(\hat{\mathbf{z}}, \mathbf{y}) - g_1(\mathbf{y}) + \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_1(\hat{\mathbf{z}}) - P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \\ & F_2(\mathbf{w}_2, \hat{\mathbf{z}} - \mathbf{y}) - \langle \Theta, \hat{\mathbf{z}} - \mathbf{y} \rangle - \psi_2(\hat{\mathbf{z}}, \mathbf{y}) - g_2(\mathbf{y}) + \psi_2(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_2(\hat{\mathbf{z}}) - P_2^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0. \end{aligned}$$

Evidently, for every $j \in I$, $g_j(\cdot)$ and $\psi_j(\mathbf{z}, \cdot)$ are convex functions on $\mathcal{S}$ for every $\mathbf{z} \in \mathcal{S}$. Moreover, for every $j \in I$, F_j is positively homogeneous with respect to the second component and proper subadd. It can be verified that the bifunctions (F_j) ($j \in I$) are not strongly monotone on the set expressed as $K := \{(z, y) \in \mathcal{S} \times \mathcal{S} : z_1 \neq y_1\}$. However, one can verify that F_j ($j \in I$) are T_j -stably (g_j, ψ_j) -pseudomonotone with respect to the $V(P_j)$ and $\Theta = (0, 1)$. As a result, for such classes of problems, the strong monotonicity assumption on the corresponding bifunctions cannot be employed to establish the existence and uniqueness of solutions for these problems.

By imposing T_j -stably (g_j, ψ_j) -strong pseudomonotonicity assumptions on bifunctions (F_j) ($j \in I$), together with suitable conditions, a uniqueness result for the solution of GSMWVVHI is established in the following theorem.

Theorem 5. Let all the hypotheses of Theorem 2 be satisfied. Moreover, we assume that the following conditions hold:

- (i) There exists a positive constant (α_F) such that for every $j, i \in I$ and $\mathbf{z}, \mathbf{y} \in \mathcal{S}$, $\mathbf{v}_j \in T_j(\mathbf{z})$, $\mathbf{w}_i \in T_i(\mathbf{y})$, $\zeta_i \in V(P_i)$, $\zeta_j \in V(P_j)$, the following inequality holds:

$$\begin{aligned} & F_j(\mathbf{v}_j, \mathbf{y} - \mathbf{z}) - \langle \Theta, \mathbf{y} - \mathbf{z} \rangle + \psi_j(\mathbf{z}, \mathbf{y}) + g_j(\mathbf{y}) - \psi_j(\mathbf{z}, \mathbf{z}) - g_j(\mathbf{z}) - \langle \zeta_j, \mathbf{y} - \mathbf{z} \rangle \geq 0 \\ & \implies \\ & F_i(\mathbf{w}_i, \mathbf{z} - \mathbf{y}) - \langle \Theta, \mathbf{z} - \mathbf{y} \rangle - \psi_i(\mathbf{z}, \mathbf{y}) - g_i(\mathbf{y}) + \psi_i(\mathbf{z}, \mathbf{z}) + g_i(\mathbf{z}) + \langle \zeta_i, \mathbf{y} - \mathbf{z} \rangle \\ & \leq -\alpha_F \|\mathbf{y} - \mathbf{z}\|^2. \end{aligned}$$

(ii) For any $\mathbf{z}, \mathbf{x}, \mathbf{y}, \mathbf{w} \in S$, and $j \in I$, there exists $\alpha_\psi > 0$ such that

$$\psi_j(\mathbf{z}, \mathbf{w}) - \psi_j(\mathbf{z}, \mathbf{y}) + \psi_j(\mathbf{x}, \mathbf{y}) - \psi_j(\mathbf{x}, \mathbf{w}) \leq \alpha_\psi (\|\mathbf{z} - \mathbf{x}\|)(\|\mathbf{y} - \mathbf{w}\|).$$

(iii) For any $\mathbf{z}, \mathbf{y} \in S$ and $j \in I$, there exists a positive constant (α_P) such that

$$P_j^\circ(\mathbf{y}; \mathbf{y} - \mathbf{z}) + P_j^\circ(\mathbf{z}; \mathbf{z} - \mathbf{y}) \leq \alpha_P \|\mathbf{z} - \mathbf{y}\|^2.$$

In addition, we suppose that $\alpha_F > \alpha_\psi + \alpha_P$. Then, there exists a unique element ($\hat{\mathbf{z}} \in S$) such that $\hat{\mathbf{z}}$ is a solution of GSMWVVHI.

Proof. In view of Theorem 2, it follows that GSMWVVHI possesses at least one solution. Our aim is to show that GSMWVVHI admits a unique solution. On the contrary, suppose that $\hat{\mathbf{z}}, \tilde{\mathbf{z}} \in S$ are two distinct solutions of GSMWVVHI. This implies that there exist $\hat{\mathbf{v}}_j \in T_j(\hat{\mathbf{z}})$, $\tilde{\mathbf{v}}_j \in T_j(\tilde{\mathbf{z}})$ ($j \in I$) satisfying

$$\begin{aligned} & (F_1(\hat{\mathbf{v}}_1, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) \\ & + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\hat{\mathbf{v}}_p, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_p(\hat{\mathbf{z}}, \mathbf{y}) + g_p(\mathbf{y}) - \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \\ & - g_p(\hat{\mathbf{z}}) + P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0, \end{aligned}$$

$$\begin{aligned} & (F_1(\tilde{\mathbf{v}}_1, \mathbf{y} - \tilde{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \tilde{\mathbf{z}} \rangle + \psi_1(\tilde{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) - \psi_1(\tilde{\mathbf{z}}, \tilde{\mathbf{z}}) - g_1(\tilde{\mathbf{z}}) \\ & + P_1^\circ(\tilde{\mathbf{z}}; \mathbf{y} - \tilde{\mathbf{z}}), \dots, F_p(\tilde{\mathbf{v}}_p, \mathbf{y} - \tilde{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \tilde{\mathbf{z}} \rangle + \psi_p(\tilde{\mathbf{z}}, \mathbf{y}) + g_p(\mathbf{y}) - \psi_p(\tilde{\mathbf{z}}, \tilde{\mathbf{z}}) \\ & - g_p(\tilde{\mathbf{z}}) + P_p^\circ(\tilde{\mathbf{z}}; \mathbf{y} - \tilde{\mathbf{z}})) \not\leq 0 \end{aligned}$$

for every $\mathbf{y} \in S$. Equivalently, for every $\mathbf{y} \in S$, there exist $k, m \in I$ satisfying the following:

$$\begin{aligned} & F_k(\hat{\mathbf{v}}_k, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_k(\hat{\mathbf{z}}, \mathbf{y}) + g_k(\mathbf{y}) - \psi_k(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_k(\hat{\mathbf{z}}) \\ & + P_k^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}) \geq 0, \end{aligned} \quad (20)$$

$$\begin{aligned} & F_m(\tilde{\mathbf{v}}_m, \mathbf{y} - \tilde{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \tilde{\mathbf{z}} \rangle + \psi_m(\tilde{\mathbf{z}}, \mathbf{y}) + g_m(\mathbf{y}) - \psi_m(\tilde{\mathbf{z}}, \tilde{\mathbf{z}}) - g_m(\tilde{\mathbf{z}}) \\ & + P_m^\circ(\tilde{\mathbf{z}}; \mathbf{y} - \tilde{\mathbf{z}}) \geq 0. \end{aligned} \quad (21)$$

Since $\hat{\mathbf{z}}, \tilde{\mathbf{z}} \in S$, according to from (20) and (21), there exist $k_0, m_0 \in I$ such that the following inequalities hold:

$$\begin{aligned} & F_{k_0}(\hat{\mathbf{v}}_{k_0}, \tilde{\mathbf{z}} - \hat{\mathbf{z}}) - \langle \Theta, \tilde{\mathbf{z}} - \hat{\mathbf{z}} \rangle + \psi_{k_0}(\hat{\mathbf{z}}, \tilde{\mathbf{z}}) + g_{k_0}(\tilde{\mathbf{z}}) - \psi_{k_0}(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_{k_0}(\hat{\mathbf{z}}) \\ & + P_{k_0}^\circ(\hat{\mathbf{z}}; \tilde{\mathbf{z}} - \hat{\mathbf{z}}) \geq 0, \end{aligned} \quad (22)$$

$$\begin{aligned} & F_{m_0}(\tilde{\mathbf{v}}_{m_0}, \hat{\mathbf{z}} - \tilde{\mathbf{z}}) - \langle \Theta, \hat{\mathbf{z}} - \tilde{\mathbf{z}} \rangle + \psi_{m_0}(\tilde{\mathbf{z}}, \hat{\mathbf{z}}) + g_{m_0}(\hat{\mathbf{z}}) - \psi_{m_0}(\tilde{\mathbf{z}}, \tilde{\mathbf{z}}) - g_{m_0}(\tilde{\mathbf{z}}) \\ & + P_{m_0}^\circ(\tilde{\mathbf{z}}; \hat{\mathbf{z}} - \tilde{\mathbf{z}}) \geq 0. \end{aligned} \quad (23)$$

Since P_{k_0} is locally Lipschitz on $\mathbb{R}^n$ and the inner product $(\langle \cdot, \cdot \rangle)$ is continuous, it follows from Lemma 1 (iii)–(iv) that there exists $\hat{\zeta}_{k_0} \in \partial_c P_{k_0}(\hat{\mathbf{z}})$ such that

$$P_{k_0}^\circ(\hat{\mathbf{z}}; \tilde{\mathbf{z}} - \hat{\mathbf{z}}) := \langle \hat{\zeta}_{k_0}, \tilde{\mathbf{z}} - \hat{\mathbf{z}} \rangle. \quad (24)$$

Upon the combination of (22) and (24), we have

$$\begin{aligned} & F_{k_0}(\hat{\mathbf{v}}_{k_0}, \tilde{\mathbf{z}} - \hat{\mathbf{z}}) - \langle \Theta, \tilde{\mathbf{z}} - \hat{\mathbf{z}} \rangle + \psi_{k_0}(\hat{\mathbf{z}}, \tilde{\mathbf{z}}) + g_{k_0}(\tilde{\mathbf{z}}) - \psi_{k_0}(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_{k_0}(\hat{\mathbf{z}}) \\ & - \langle -\hat{\zeta}_{k_0}, \tilde{\mathbf{z}} - \hat{\mathbf{z}} \rangle \geq 0. \end{aligned} \quad (25)$$

In view of (25) and hypothesis (i), it can be inferred that

$$F_{m_0}(\hat{\mathbf{w}}_{m_0}, \hat{\mathbf{z}} - \tilde{\mathbf{z}}) - \langle \Theta, \hat{\mathbf{z}} - \tilde{\mathbf{z}} \rangle - \psi_{m_0}(\hat{\mathbf{z}}, \tilde{\mathbf{z}}) - g_{m_0}(\tilde{\mathbf{z}}) + \psi_{m_0}(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_{m_0}(\hat{\mathbf{z}}) - \langle \zeta_{m_0}, \tilde{\mathbf{z}} - \hat{\mathbf{z}} \rangle \leq -\alpha_F \|\hat{\mathbf{z}} - \tilde{\mathbf{z}}\|^2 \quad (26)$$

for every $\zeta_{m_0} \in \partial_c P_{m_0}(\mathbf{z})$. After invoking Lemma 1 (iii) in (26), the following inequality holds:

$$F_{m_0}(\hat{\mathbf{w}}_{m_0}, \hat{\mathbf{z}} - \tilde{\mathbf{z}}) - \langle \Theta, \hat{\mathbf{z}} - \tilde{\mathbf{z}} \rangle - \psi_{m_0}(\hat{\mathbf{z}}, \tilde{\mathbf{z}}) - g_{m_0}(\tilde{\mathbf{z}}) + \psi_{m_0}(\hat{\mathbf{z}}, \hat{\mathbf{z}}) + g_{m_0}(\hat{\mathbf{z}}) - P_{m_0}^\circ(\hat{\mathbf{z}}; \tilde{\mathbf{z}} - \hat{\mathbf{z}}) \leq -\alpha_F \|\hat{\mathbf{z}} - \tilde{\mathbf{z}}\|^2, \quad (27)$$

Using hypotheses (ii)–(iii), along with (23) and (27), we infer that

$$\alpha_F \|\tilde{\mathbf{z}} - \hat{\mathbf{z}}\|^2 \leq \alpha_\psi \|\tilde{\mathbf{z}} - \hat{\mathbf{z}}\|^2 + \alpha_P \|\tilde{\mathbf{z}} - \hat{\mathbf{z}}\|^2.$$

This implies that

$$(\alpha_F - \alpha_\psi - \alpha_P) \|\tilde{\mathbf{z}} - \hat{\mathbf{z}}\|^2 \leq 0. \quad (28)$$

From the given hypothesis, we have $(\alpha_F - \alpha_\psi - \alpha_P) > 0$. This yields

$$\|\tilde{\mathbf{z}} - \hat{\mathbf{z}}\|^2 = 0.$$

Hence, $\hat{\mathbf{z}} = \tilde{\mathbf{z}}$, which contradicts the assumption that $\hat{\mathbf{z}}$ and $\tilde{\mathbf{z}}$ are distinct solutions of GSMWVHI. This implies that the assertion holds. $\square$

In light of Theorems 4 and 5, the following corollary provides a uniqueness result for the solution of GMMWVHI.

Corollary 3. *If every hypothesis in Theorems 4 and 5 is satisfied, then GMMWVHI has a unique solution in $\mathcal{S}$.*

To highlight the significance of Theorem 5 and Corollary 3, we provide the following example.

Example 4. *For each $j \in I$, mappings T_j , F_j , ψ_j , P_j , and g_j are same as considered in Example 2. Moreover, it can be verified that hypotheses (i)–(iii) are satisfied in Theorem 5 with $\alpha_F = 9$, $\alpha_\psi = 1$, and $\alpha_P = \frac{1}{3}$. Furthermore,*

$$\alpha_F > \alpha_P + \alpha_\psi.$$

This implies that all the assumptions of Theorem 5 are satisfied. In addition to this, one can verify that F_j ($j \in I$) is generalized hemicontinuous on $T_j(\mathcal{S})$, which, in turn, implies that all the hypotheses in Corollary 3 are also satisfied.

It is worth mentioning that if $\alpha_F = \alpha_\psi + \alpha_P$, then according to (28), it follows that $0 \leq 0$.

This implies that the inequality in (28) is satisfied for any $\tilde{\mathbf{z}}$ and $\hat{\mathbf{z}}$, even when $\tilde{\mathbf{z}} \neq \hat{\mathbf{z}}$. Therefore, the uniqueness of the solution of GSMWVHI can no longer be guaranteed. To demonstrate this fact, we provide the following remark.

Remark 10. *Let us consider the set expressed as $\mathcal{S} := \{\mathbf{z} \in \mathbb{R} : -1 \leq \mathbf{z} \leq 1\}$. For every $j \in I := \{1, 2\}$, let the bifunctions $(F_j : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R})$ be defined as follows:*

$$F_1(\mathbf{v}, \mathbf{y}) := 3\mathbf{v}\mathbf{y}, \quad F_2(\mathbf{v}, \mathbf{y}) := 2\mathbf{v}\mathbf{y}, \text{ for all } \mathbf{y}, \mathbf{v} \in \mathbb{R}.$$

We define $T_j : \mathcal{S} \rightarrow \mathbb{R}$ for every $j \in I$ as follows:

$$T_1(\mathbf{z}) := \{2\mathbf{z}\}, \quad T_2(\mathbf{z}) := \{3\mathbf{z}\}, \text{ for all } \mathbf{z} \in \mathcal{S}.$$

For every $j \in I$, we consider the functions $\psi_j : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ and $g_j : \mathcal{S} \rightarrow \mathbb{R}$ to be defined as follows:

$$\begin{aligned} \psi_1(\mathbf{z}, \mathbf{y}) &:= -3\mathbf{z}\mathbf{y}, \quad \psi_2(\mathbf{z}, \mathbf{y}) := -3\mathbf{z}\mathbf{y} + e^2 + 1, \text{ for all } \mathbf{y}, \mathbf{z} \in \mathcal{S}, \\ g_1(\mathbf{z}) &:= \frac{1}{2}\mathbf{z}, \quad g_2(\mathbf{z}) := \frac{1}{2}\mathbf{z} + 2, \text{ for all } \mathbf{z} \in \mathcal{S}. \end{aligned}$$

Let functions $P_j : \mathbb{R} \rightarrow \mathbb{R}$ ($j \in I$) be defined as follows:

$$P_1(\mathbf{z}) := -\frac{3}{2}\mathbf{z}^2, \quad P_2(\mathbf{z}) := -\mathbf{z}^2 + e, \quad \mathbf{z} \in \mathbb{R}.$$

It can be verified that the Clarke directional derivatives of P_1 and P_2 at $\mathbf{z} \in \mathbb{R}$ in the direction of $\mathbf{v} \in \mathbb{R}$ are given as follows:

$$P_1^\circ(\mathbf{z}; \mathbf{v}) = -3\mathbf{z}\mathbf{v}, \quad P_2^\circ(\mathbf{z}; \mathbf{v}) = -2\mathbf{z}\mathbf{v}.$$

Consider the following generalized Stampacchia mixed weak vector variational–hemivariational inequalities for $\Theta = \frac{1}{2}$.

(GSMWVVI3): Find $\hat{\mathbf{z}} \in \mathcal{S}$, such that there exists $\hat{\mathbf{v}}_j \in T_j(\hat{\mathbf{z}})$ ($j \in I$), satisfying

$$\begin{aligned} (F_1(\hat{\mathbf{v}}_1, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \\ F_2(\hat{\mathbf{v}}_2, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_2(\hat{\mathbf{z}}, \mathbf{y}) + g_2(\mathbf{y}) - \psi_2(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_2(\hat{\mathbf{z}}) + P_2^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\prec 0 \end{aligned}$$

for every $\mathbf{y} \in \mathcal{S}$.

It can be verified that hypotheses (i)–(iii) in Theorem 5 are satisfied with $\alpha_F = 6$, $\alpha_\psi = 3$, $\alpha_P = 3$, which implies that

$$\alpha_F = \alpha_\psi + \alpha_P.$$

Furthermore, one can verify that every $\hat{\mathbf{z}} \in \mathcal{S}$ is a solution of GSMWVVI.

4. Stability Analysis for the Solution of GSMWVVI

In this section, we establish the stability of the solution of GSMWVVI.

Let $\epsilon > 0$ be a small perturbation parameter. To formulate the perturbed version of GSMWVVI, we make the following assumptions for every $\epsilon > 0$:

(B1 $_\epsilon$) Let $\mathcal{S}_\epsilon$ be a non-empty convex subset of $\mathbb{R}^n$ and $\Theta_\epsilon \in \mathbb{R}^n$.

(B2 $_\epsilon$) For each $j \in I$, let $T_j^\epsilon : \mathcal{S}_\epsilon \rightrightarrows \mathbb{R}^n$ be a set-valued map and $F_j^\epsilon : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a bifunction such that $F_j^\epsilon(\mathbf{v}, 0) = 0$ for all $\mathbf{v} \in \mathbb{R}^n$. Moreover, we assume that $P_j^\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ is a locally Lipschitz function for each $j \in I$.

(B3 $_\epsilon$) For each $j \in I$, let $\psi_j^\epsilon : \mathcal{S}_\epsilon \times \mathcal{S}_\epsilon \rightarrow \mathbb{R}$ and $g_j^\epsilon : \mathcal{S}_\epsilon \rightarrow \mathbb{R}$ be continuous and lower semicontinuous on $\mathcal{S}_\epsilon \times \mathcal{S}_\epsilon$ and $\mathcal{S}_\epsilon$, respectively.

The perturbed GSMWVVI (ϵ) is formulated as follows:

Find $\hat{\mathbf{z}}_\epsilon \in \mathcal{S}_\epsilon$ such that there exists $\mathbf{v}_j^\epsilon \in T_j^\epsilon(\hat{\mathbf{z}}_\epsilon)$ ($j \in I$) satisfying

$$\begin{aligned} (F_1^\epsilon(\hat{\mathbf{v}}_1^\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon) - \langle \Theta_\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon \rangle + \psi_1^\epsilon(\hat{\mathbf{z}}_\epsilon, \mathbf{y}_\epsilon) + g_1^\epsilon(\mathbf{y}_\epsilon) - \psi_1^\epsilon(\hat{\mathbf{z}}_\epsilon, \hat{\mathbf{z}}_\epsilon) - g_1^\epsilon(\hat{\mathbf{z}}_\epsilon) \\ + (P_1^\epsilon)^\circ(\hat{\mathbf{z}}_\epsilon; \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon), \dots, F_p^\epsilon(\hat{\mathbf{v}}_p^\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon) - \langle \Theta, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon \rangle + \psi_p^\epsilon(\hat{\mathbf{z}}_\epsilon, \mathbf{y}_\epsilon) \\ + g_p^\epsilon(\mathbf{y}_\epsilon) - \psi_p^\epsilon(\hat{\mathbf{z}}_\epsilon, \hat{\mathbf{z}}_\epsilon) - g_p^\epsilon(\hat{\mathbf{z}}_\epsilon) + (P_p^\epsilon)^\circ(\hat{\mathbf{z}}_\epsilon; \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon)) \not\prec 0 \end{aligned}$$

for every $\mathbf{y}_\epsilon \in \mathcal{S}_\epsilon$.

In the following theorems, we establish existence and uniqueness results for the solutions of GSMWVVI (ϵ) for every $\epsilon > 0$ by following the same lines of proof of Theorems 1 and 5, respectively.

Theorem 6. Let $\mathcal{S}_\epsilon$ be a closed subset of $\mathbb{R}^n$ such that for every $j \in I$ and $\mathbf{z}_\epsilon \in \mathcal{S}_\epsilon$, $\psi_j^\epsilon(\mathbf{z}_\epsilon, \cdot)$ and $g_j^\epsilon(\cdot)$ are convex functions on $\mathcal{S}_\epsilon$. Moreover, we assume that the following conditions hold:

- (i) For each $j \in I$, T_j^ϵ is a compact set and upper Kuratowski semicontinuous on $\mathcal{S}_\epsilon$.
- (ii) For each $j \in I$ and $\mathbf{z}_\epsilon \in \mathbb{R}^n$, $F_j^\epsilon(\mathbf{z}_\epsilon, \cdot)$ is positively homogeneous and proper subadd on $\mathbb{R}^n$.
- (iii) For every $\mathbf{z}_\epsilon \in \mathcal{S}_\epsilon$, $(\mathbf{v}_j^\epsilon, \mathbf{z}_\epsilon) \mapsto F_j^\epsilon(\mathbf{v}_j^\epsilon, \mathbf{z}_\epsilon)$ is upper semicontinuous on $\mathbb{R}^n \times \mathbb{R}^n$.
- (iv) There exists a non-empty compact subset (H_ϵ) of $\mathcal{S}_\epsilon$ such that for every $\mathbf{y}_\epsilon \in \mathcal{S}_\epsilon \setminus H_\epsilon$ and for every $\mathbf{v}_j^\epsilon \in T_j^\epsilon(\mathbf{y}_\epsilon)$ ($j \in I$), the following inclusion is satisfied for some $\tilde{\mathbf{z}}_\epsilon \in H_\epsilon$:

$$F_j^\epsilon(\mathbf{v}_j^\epsilon, \tilde{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon) - \langle \Theta_\epsilon, \tilde{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon \rangle + \psi_j^\epsilon(\mathbf{y}_\epsilon, \tilde{\mathbf{z}}_\epsilon) + g_j^\epsilon(\tilde{\mathbf{z}}_\epsilon) - \psi_j^\epsilon(\mathbf{y}_\epsilon, \mathbf{y}_\epsilon) - g_j^\epsilon(\mathbf{y}_\epsilon) + (P_j^\epsilon)^\circ(\mathbf{y}_\epsilon; \tilde{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon) < 0.$$

Then, GSMWVVI (ϵ) admits at least one solution in $\mathcal{S}_\epsilon$.

Theorem 7. Let all the hypotheses in Theorem 6 be satisfied. Moreover, we assume that the following conditions hold:

- (i) There exists a positive constant (α_F) such that for every $j, i \in I$, $\mathbf{z}, \mathbf{y} \in \mathcal{S}_\epsilon$, $\mathbf{v}_j^\epsilon \in T_j^\epsilon(\mathbf{z})$, $\mathbf{w}_i^\epsilon \in T_i^\epsilon(\mathbf{y})$, $\zeta_j^\epsilon \in V(P_j^\epsilon)$, and $\zeta_i^\epsilon \in V(P_i^\epsilon)$, the following implication holds:

$$\begin{aligned} F_j^\epsilon(\mathbf{v}_j^\epsilon, \mathbf{y} - \mathbf{z}) - \langle \Theta_\epsilon, \mathbf{y} - \mathbf{z} \rangle + \psi_j^\epsilon(\mathbf{z}, \mathbf{y}) + g_j^\epsilon(\mathbf{y}) - \psi_j^\epsilon(\mathbf{z}, \mathbf{z}) - g_j^\epsilon(\mathbf{z}) - \langle \zeta_j^\epsilon, \mathbf{y} - \mathbf{z} \rangle &\geq 0 \\ \implies \\ F_i^\epsilon(\mathbf{w}_i^\epsilon, \mathbf{z} - \mathbf{y}) - \langle \Theta_\epsilon, \mathbf{z} - \mathbf{y} \rangle - \psi_i^\epsilon(\mathbf{z}, \mathbf{y}) - g_i^\epsilon(\mathbf{y}) + \psi_i^\epsilon(\mathbf{z}, \mathbf{z}) + g_i^\epsilon(\mathbf{z}) + \langle \zeta_i^\epsilon, \mathbf{y} - \mathbf{z} \rangle &\leq -\alpha_F \|\mathbf{y} - \mathbf{z}\|^2. \end{aligned}$$

- (ii) For any $\mathbf{z}, \mathbf{x}, \mathbf{y}, \mathbf{w} \in \mathcal{S}_\epsilon$ and $j \in I$, there exists a positive constant (α_ψ) such that

$$\psi_j^\epsilon(\mathbf{z}, \mathbf{w}) - \psi_j^\epsilon(\mathbf{z}, \mathbf{y}) + \psi_j^\epsilon(\mathbf{x}, \mathbf{y}) - \psi_j^\epsilon(\mathbf{x}, \mathbf{w}) \leq \alpha_\psi \|\mathbf{z} - \mathbf{x}\| \|\mathbf{y} - \mathbf{w}\|.$$

- (iii) For any $\mathbf{z}, \mathbf{y} \in \mathcal{S}_\epsilon$ and $j \in I$, there exists a positive constant (α_P) such that

$$(P_j^\epsilon)^\circ(\mathbf{y}; \mathbf{y} - \mathbf{z}) + (P_j^\epsilon)^\circ(\mathbf{z}; \mathbf{z} - \mathbf{y}) \leq \alpha_P \|\mathbf{z} - \mathbf{y}\|^2.$$

In addition, suppose that $\alpha_F > \alpha_\psi + \alpha_P$. Then, there exists a unique element ($\hat{\mathbf{z}}_\epsilon \in \mathcal{S}_\epsilon$) such that $\hat{\mathbf{z}}_\epsilon$ is a solution of GSMWVVI(ϵ).

The following conditions will be employed in the subsequent stability analysis for the solution of GSMWVVI (see [56]):

- (C1) If $\mathbf{y}_\epsilon \in \mathcal{S}_\epsilon$ and $\mathbf{y}_\epsilon \rightarrow \mathbf{y}$ as $\epsilon \rightarrow 0$, then $\mathbf{y} \in \mathcal{S}$. Moreover, for any $\mathbf{z} \in \mathcal{S}$, there exists $\mathbf{z}_\epsilon \in \mathcal{S}_\epsilon$ such that $\mathbf{z}_\epsilon \rightarrow \mathbf{z}$ as $\epsilon \rightarrow 0$.
- (C2) There exists a function ($c_\psi : \mathbb{R} \rightarrow \mathbb{R}_+$) such that

$$c_\psi(\epsilon) \rightarrow 0, \text{ as } \epsilon \rightarrow 0.$$

Moreover, for every $\mathbf{y}, \mathbf{z} \in \mathcal{S}_\epsilon$ and $j \in I$, the following condition holds:

$$|\psi_j^\epsilon(\mathbf{z}, \mathbf{z}) - \psi_j^\epsilon(\mathbf{z}, \mathbf{y}) - \psi_j(\mathbf{z}, \mathbf{z}) + \psi_j(\mathbf{z}, \mathbf{y})| \leq c_\psi(\epsilon)(1 + \|\mathbf{z}\|)\|\mathbf{z} - \mathbf{y}\|.$$

(C3) There exists a function $(c_P : \mathbb{R} \rightarrow \mathbb{R}_+)$ such that

$$c_P(\epsilon) \rightarrow 0, \text{ as } \epsilon \rightarrow 0.$$

Moreover, for every $\mathbf{z}, \mathbf{y} \in \mathbb{R}^n$ and $j \in I$, we have

$$|(P_j^\epsilon)^\circ(\mathbf{z}; \mathbf{y} - \mathbf{z}) - P_j^\circ(\mathbf{z}; \mathbf{y} - \mathbf{z})| \leq c_P(\epsilon)(1 + \|\mathbf{z}\|)\|\mathbf{y} - \mathbf{z}\|.$$

(C4) Let $\Theta_\epsilon \rightarrow \Theta$ as $\epsilon \rightarrow 0$.

(C5) There exists a positive constant (β_P) such that for every $j \in I$ and $\mathbf{z} \in \mathbb{R}^n$,

$$\|\xi_j\| \leq \beta_P(1 + \|\mathbf{z}\|), \text{ for all } \xi_j \in \partial_c P_j(\mathbf{z}).$$

(C6) Assume that for every $j \in I$, $\mathbf{w} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^n$, the following inequality holds:

$$|F_j^\epsilon(\mathbf{w}, \mathbf{z})| \leq \|\mathbf{w}\| \|\mathbf{z}\|.$$

(C7) For every $j \in I$, let g_j^ϵ be Lipschitz continuous on $\mathcal{S}_\epsilon$ with rank L_j^g .

In the following theorem, we establish the stability of the solution of GSMWVVHI.

Theorem 8. *Let the hypotheses in Theorems 5, 6, and 7 be satisfied. Moreover, we assume that Conditions (C1)–(C7) are satisfied. Then, for solution $\hat{\mathbf{z}}$ of GSMWVVHI and solution $\hat{\mathbf{z}}_\epsilon$ of GSMWVVHI (ϵ), we have*

$$\hat{\mathbf{z}}_\epsilon \rightarrow \hat{\mathbf{z}}, \text{ as } \epsilon \rightarrow 0.$$

Proof. The proof is divided into the following steps:

Step 1. We show the boundedness of the set expressed as $\{\|\hat{\mathbf{z}}_\epsilon\| : \epsilon > 0\}$.

Since $\mathcal{S}$ is a non-empty subset of $\mathbb{R}^n$, there exists an element such that $\tilde{\mathbf{y}} \in \mathcal{S}$. In view of Condition (C1), there exists $\mathbf{y}_\epsilon \in \mathcal{S}_\epsilon$ such that

$$\mathbf{y}_\epsilon \rightarrow \tilde{\mathbf{y}}, \text{ as } \epsilon \rightarrow 0.$$

It follows that $\{\|\mathbf{y}_\epsilon\| : \epsilon > 0\}$ is a bounded set. Moreover, according to the given hypothesis, $\hat{\mathbf{z}}_\epsilon$ is a solution of GSMWVVHI (ϵ). This implies that there exists $\hat{\mathbf{v}}_j^\epsilon \in T_j^\epsilon$ ($j \in I$) such that

$$\begin{aligned} & (F_1^\epsilon(\hat{\mathbf{v}}_1^\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon) - \langle \Theta_\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon \rangle + \psi_1^\epsilon(\hat{\mathbf{z}}_\epsilon, \mathbf{y}_\epsilon) + g_1^\epsilon(\mathbf{y}_\epsilon) - \psi_1^\epsilon(\hat{\mathbf{z}}_\epsilon, \hat{\mathbf{z}}_\epsilon) - g_1^\epsilon(\hat{\mathbf{z}}_\epsilon) \\ & + (P_1^\epsilon)^\circ(\hat{\mathbf{z}}_\epsilon; \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon), \dots, F_p^\epsilon(\hat{\mathbf{v}}_p^\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon) - \langle \Theta, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon \rangle + \psi_p^\epsilon(\hat{\mathbf{z}}_\epsilon, \mathbf{y}_\epsilon) \\ & + g_p^\epsilon(\mathbf{y}_\epsilon) - \psi_p^\epsilon(\hat{\mathbf{z}}_\epsilon, \hat{\mathbf{z}}_\epsilon) - g_p^\epsilon(\hat{\mathbf{z}}_\epsilon) + (P_p^\epsilon)^\circ(\hat{\mathbf{z}}_\epsilon; \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon)) \not\leq 0. \end{aligned}$$

Hence, corresponding to every $\epsilon > 0$, there exists at least one $i_0 \in I$, for which the following inequality holds:

$$\begin{aligned} & F_{i_0}^\epsilon(\hat{\mathbf{v}}_{i_0}^\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon) - \langle \Theta_\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon \rangle + \psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \mathbf{y}_\epsilon) + g_{i_0}^\epsilon(\mathbf{y}_\epsilon) - \psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \hat{\mathbf{z}}_\epsilon) - g_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon) \\ & + (P_{i_0}^\epsilon)^\circ(\hat{\mathbf{z}}_\epsilon; \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon) \geq 0. \end{aligned}$$

According to hypothesis (i) of Theorem 7, for every $\mathbf{w}_{i_0}^\epsilon \in T_{i_0}(\mathbf{y}_\epsilon)$, we have

$$\begin{aligned} & F_{i_0}^\epsilon(\mathbf{w}_{i_0}^\epsilon, \hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon) + \langle \Theta_\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon \rangle - \psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \mathbf{y}_\epsilon) - g_{i_0}^\epsilon(\mathbf{y}_\epsilon) + \psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \hat{\mathbf{z}}_\epsilon) + g_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon) \\ & - (P_{i_0}^\epsilon)^\circ(\hat{\mathbf{z}}_\epsilon; \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon) \leq -\alpha_F \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2. \end{aligned} \quad (29)$$

After rearranging the terms in (29), it follows that

$$\alpha_F \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 \leq -F_{i_0}^\epsilon(\mathbf{w}_{i_0}^\epsilon, \hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon) - \langle \Theta_\epsilon, \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon \rangle + \psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \mathbf{y}_\epsilon) + g_{i_0}^\epsilon(\mathbf{y}_\epsilon) - \psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \hat{\mathbf{z}}_\epsilon) - g_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon) + (P_{i_0}^\epsilon)^\circ(\hat{\mathbf{z}}_\epsilon; \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon). \quad (30)$$

According to hypothesis (ii) in Theorem 7, we have

$$\psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \mathbf{y}_\epsilon) - \psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \hat{\mathbf{z}}_\epsilon) \leq \psi_{i_0}^\epsilon(\mathbf{y}_\epsilon, \mathbf{y}_\epsilon) - \psi_{i_0}^\epsilon(\mathbf{y}_\epsilon, \hat{\mathbf{z}}_\epsilon) + \alpha_\psi \|\mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon\|^2. \quad (31)$$

Moreover, from Condition (C2), we infer that

$$\psi_{i_0}^\epsilon(\mathbf{y}_\epsilon, \mathbf{y}_\epsilon) - \psi_{i_0}^\epsilon(\mathbf{y}_\epsilon, \hat{\mathbf{z}}_\epsilon) \leq \psi_i(\mathbf{y}_\epsilon, \mathbf{y}_\epsilon) - \psi_i(\mathbf{y}_\epsilon, \hat{\mathbf{z}}_\epsilon) + c_\psi(\epsilon)(1 + \|\mathbf{y}_\epsilon\|)\|\mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon\|. \quad (32)$$

By employing hypothesis (ii) in Theorem 5 and Condition (C1), we get the following inequality from (32):

$$\psi_{i_0}^\epsilon(\mathbf{y}_\epsilon, \mathbf{y}_\epsilon) - \psi_{i_0}^\epsilon(\mathbf{y}_\epsilon, \hat{\mathbf{z}}_\epsilon) \leq \psi_{i_0}(\tilde{\mathbf{y}}, \mathbf{y}_\epsilon) - \psi_{i_0}(\tilde{\mathbf{y}}, \hat{\mathbf{z}}_\epsilon) + \alpha_\psi \|\mathbf{y}_\epsilon - \tilde{\mathbf{y}}\| \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\| + c_\psi(\epsilon)(1 + \|\mathbf{y}_\epsilon\|)\|\mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon\|. \quad (33)$$

Since $\psi_{i_0}(\tilde{\mathbf{y}}, \cdot)$ is convex, according to [56], there exist two constants (c_1 and c_2 , depending on ψ_{i_0} and $\tilde{\mathbf{y}}$) such that

$$\psi_{i_0}(\tilde{\mathbf{y}}, \hat{\mathbf{z}}_\epsilon) \geq c_1 + c_2 \|\hat{\mathbf{z}}_\epsilon\|. \quad (34)$$

Upon the combination of (31)–(34), we obtain the following inequality:

$$\psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \mathbf{y}_\epsilon) - \psi_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon, \hat{\mathbf{z}}_\epsilon) \leq \alpha_\psi \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 + \alpha_\psi \|\mathbf{y}_\epsilon - \tilde{\mathbf{y}}\| \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\| + c_\psi(\epsilon)(1 + \|\mathbf{y}_\epsilon\|)\|\mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon\| + \psi_{i_0}(\tilde{\mathbf{y}}, \mathbf{y}_\epsilon) - (c_1 + c_2 \|\hat{\mathbf{z}}_\epsilon\|). \quad (35)$$

The Lipschitz continuity property of g_{i_0} with rank $L_{i_0}^g$ in Condition (C7) yields

$$g_{i_0}^\epsilon(\mathbf{y}_\epsilon) - g_{i_0}^\epsilon(\hat{\mathbf{z}}_\epsilon) \leq L_{i_0}^g \|\mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon\|. \quad (36)$$

According to hypothesis (iii) in Theorem 7, we have

$$(P_{i_0}^\epsilon)^\circ(\hat{\mathbf{z}}_\epsilon; \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon) \leq -(P_{i_0}^\epsilon)^\circ(\mathbf{y}_\epsilon; \hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon) + \alpha_P \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2. \quad (37)$$

In view of the continuity of the inner product and locally Lipschitz continuity of P_{i_0} , it follows from Lemma 1 (iii) that there exists $\xi_{i_0} \in \partial_c P_{i_0}(\hat{\mathbf{z}}_\epsilon)$ such that

$$(P_{i_0}^\epsilon)^\circ(\mathbf{y}_\epsilon; \hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon) = \langle \xi_{i_0}, \hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon \rangle. \quad (38)$$

From the Cauchy-Schwarz inequality in (38) and Condition (C5), we arrive at the following inequality:

$$|(P_{i_0}^\epsilon)^\circ(\mathbf{y}_\epsilon; \hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon)| \leq \beta_P(1 + \|\mathbf{y}_\epsilon\|)\|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|. \quad (39)$$

From (37) and (39), it follows that

$$(P_{i_0}^\epsilon)^\circ(\hat{\mathbf{z}}_\epsilon; \mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon) \leq \alpha_P \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 + \beta_P(1 + \|\mathbf{y}_\epsilon\|)\|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|. \quad (40)$$

Moreover, according to Condition (C6), (35), (36), and (40) and by applying the Cauchy–Schwarz inequality, we obtain the following:

$$\begin{aligned} \alpha_F \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 &\leq \|\mathbf{w}_{i_0}^\epsilon\| \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\| + \|\Theta_\epsilon\| \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\| + \alpha_\psi \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 \\ &\quad + \alpha_\psi \|\mathbf{y}_\epsilon - \tilde{\mathbf{y}}\| \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\| + c_\psi(\epsilon)(1 + \|\mathbf{y}_\epsilon\|) \|\mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon\| \\ &\quad + \psi_{i_0}(\tilde{\mathbf{y}}, \mathbf{y}_\epsilon) - (c_1 + c_2 \|\hat{\mathbf{z}}_\epsilon\|) + L_{i_0}^g \|\mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon\| + \alpha_P \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 \\ &\quad + \beta_P(1 + \|\mathbf{y}_\epsilon\|) \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|. \end{aligned} \quad (41)$$

For every $\epsilon > 0$, we define $A_\epsilon := \alpha_\psi \|\mathbf{y}_\epsilon - \tilde{\mathbf{y}}\| + c_\psi(\epsilon)(1 + \|\mathbf{y}_\epsilon\|) + L_{i_0}^g + \beta_P(1 + \|\mathbf{y}_\epsilon\|) + \|\mathbf{w}_{i_0}^\epsilon\| + \|\Theta_\epsilon\|$. Therefore, according to (41), one has the following:

$$(\alpha_F - \alpha_P - \alpha_\psi) \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 \leq A_\epsilon \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\| + \psi_{i_0}(\tilde{\mathbf{y}}, \mathbf{y}_\epsilon) - c_1 - c_2 \|\hat{\mathbf{z}}_\epsilon\|. \quad (42)$$

Applying the modified Cauchy–Schwarz inequality (see [56]), we have

$$A_\epsilon \|\mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon\| \leq \delta \|\mathbf{y}_\epsilon - \hat{\mathbf{z}}_\epsilon\|^2 + b_\delta A_\epsilon^2, \quad b_\delta = \frac{1}{4\delta}. \quad (43)$$

Moreover, we have

$$-c_2 \|\hat{\mathbf{z}}_\epsilon\| \leq \delta \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 + \tilde{b}_\delta, \quad (44)$$

for a constant $\tilde{b}_\delta$ depending on δ . Upon the combination of (42) and (44) and taking into account the fact that $\{A_\epsilon : \epsilon > 0\}$ is a bounded set, it follows that

$$(\alpha_F - \alpha_P - \alpha_\psi - 2\delta) \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 \leq A + |\psi_{i_0}(\tilde{\mathbf{y}}, \mathbf{y}_\epsilon)|, \quad (45)$$

where A is a positive constant such that $|b_\delta A_\epsilon^2 + \tilde{b}_\delta - c_1| \leq A$. Let $\delta := \frac{1}{4}(\alpha_F - \alpha_P - \alpha_\psi)$. Then, according to (45), we have

$$\frac{1}{2}(\alpha_F - \alpha_P - \alpha_\psi) \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 \leq A + |\psi_{i_0}(\tilde{\mathbf{y}}, \mathbf{y}_\epsilon)|.$$

Since $\psi_{i_0}(\tilde{\mathbf{y}}, \mathbf{y}_\epsilon) \rightarrow \psi_{i_0}(\tilde{\mathbf{y}}, \tilde{\mathbf{y}})$, as $\epsilon \rightarrow 0$, according to (45), we get

$$\frac{1}{2}(\alpha_F - \alpha_P - \alpha_\psi) \|\hat{\mathbf{z}}_\epsilon - \mathbf{y}_\epsilon\|^2 \leq A + |\psi_{i_0}(\tilde{\mathbf{y}}, \tilde{\mathbf{y}})|. \quad (46)$$

Due to the boundedness of the set expressed as $\{\|\mathbf{y}_\epsilon\| : \epsilon > 0\}$ and $\alpha_F > \alpha_P + \alpha_\psi$, we conclude from (46) that $\{\|\hat{\mathbf{z}}_\epsilon\| : \epsilon > 0\}$ is a bounded set. Therefore, there exist $\hat{\mathbf{w}} \in \mathbb{R}^n$ and $\{\epsilon'\} \subseteq \{\epsilon\}$ such that $\hat{\mathbf{z}}_{\epsilon'} \rightarrow \hat{\mathbf{w}}$ as $\epsilon' \rightarrow 0$. According to Condition (C1), we have $\hat{\mathbf{w}} \in \mathcal{S}$.

Step 2. In this step, we show that $\hat{\mathbf{w}} \in \mathcal{S}$ is a solution of GSMWVVI, in view of the fact that $\hat{\mathbf{z}}_{\epsilon'} \in \mathcal{S}_{\epsilon'}$ and $\hat{\mathbf{z}}_{\epsilon'}$ is a solution of GSMWVVI (ϵ'). It follows that there exists $\hat{\mathbf{v}}_j^{\epsilon'} \in T_j^{\epsilon'}(\hat{\mathbf{z}}_{\epsilon'})$ ($j \in I$) such that

$$\begin{aligned} &\left(F_1^{\epsilon'}(\hat{\mathbf{v}}_1^{\epsilon'}, \mathbf{y}_{\epsilon'} - \hat{\mathbf{z}}_{\epsilon'}) - \langle \Theta_{\epsilon'}, \mathbf{y}_{\epsilon'} - \hat{\mathbf{z}}_{\epsilon'} \rangle + \psi_1^{\epsilon'}(\hat{\mathbf{z}}_{\epsilon'}, \mathbf{y}_{\epsilon'}) + g_1^{\epsilon'}(\mathbf{y}_{\epsilon'}) - \psi_1^{\epsilon'}(\hat{\mathbf{z}}_{\epsilon'}, \hat{\mathbf{z}}_{\epsilon'}) \right. \\ &\quad \left. - g_1^{\epsilon'}(\hat{\mathbf{z}}_{\epsilon'}) + (P_1^{\epsilon'})^\circ(\hat{\mathbf{z}}_{\epsilon'}; \mathbf{y}_{\epsilon'} - \hat{\mathbf{z}}_{\epsilon'}), \dots, \right. \\ &F_p^{\epsilon'}(\hat{\mathbf{v}}_p^{\epsilon'}, \mathbf{y}_{\epsilon'} - \hat{\mathbf{z}}_{\epsilon'}) - \langle \Theta, \mathbf{y}_{\epsilon'} - \hat{\mathbf{z}}_{\epsilon'} \rangle + \psi_p^{\epsilon'}(\hat{\mathbf{z}}_{\epsilon'}, \mathbf{y}_{\epsilon'}) + g_p^{\epsilon'}(\mathbf{y}_{\epsilon'}) - \psi_p^{\epsilon'}(\hat{\mathbf{z}}_{\epsilon'}, \hat{\mathbf{z}}_{\epsilon'}) \\ &\quad \left. - g_p^{\epsilon'}(\hat{\mathbf{z}}_{\epsilon'}) + (P_p^{\epsilon'})^\circ(\hat{\mathbf{z}}_{\epsilon'}; \mathbf{y}_{\epsilon'} - \hat{\mathbf{z}}_{\epsilon'}) \right) \not\leq 0. \end{aligned} \quad (47)$$

Since $T_j(\cdot)$ ($j \in I$) are compact sets, for each $j \in I$, $\{\|\mathbf{v}_j^\epsilon\| : \epsilon > 0\}$ is a bounded set. This implies that there exist $\hat{\mathbf{v}}_j \in \mathbb{R}^n$ ($j \in I$) and $\{\epsilon''\} \subseteq \{\epsilon'\}$ such that

$$\hat{\mathbf{v}}_j^{\epsilon''} \in T_j(\hat{\mathbf{z}}_{\epsilon''}), \text{ and } \hat{\mathbf{v}}_j^{\epsilon''} \rightarrow \hat{\mathbf{v}}_j, \text{ as } \epsilon'' \rightarrow 0.$$

From the upper Kuratowski semicontinuity property of T_j ($j \in I$) and the fact that $\hat{\mathbf{z}}_{\epsilon''} \rightarrow \hat{\mathbf{z}}$ as $\epsilon'' \rightarrow 0$, we infer that $\hat{\mathbf{v}}_j \in T_j(\hat{\mathbf{z}})$ ($j \in I$). According to (47), we have the following:

$$\begin{aligned} & \left(F_1^{\epsilon'}(\hat{\mathbf{v}}_1^{\epsilon''}, \mathbf{y}_{\epsilon''} - \hat{\mathbf{z}}_{\epsilon''}) - \langle \Theta_{\epsilon''}, \mathbf{y}_{\epsilon''} - \hat{\mathbf{z}}_{\epsilon''} \rangle + \psi_1^{\epsilon''}(\hat{\mathbf{z}}_{\epsilon''}, \mathbf{y}_{\epsilon''}) + g_1^{\epsilon''}(\mathbf{y}_{\epsilon''}) \right. \\ & \quad \left. - \psi_1^{\epsilon''}(\hat{\mathbf{z}}_{\epsilon''}, \hat{\mathbf{z}}_{\epsilon''}) - g_1^{\epsilon'}(\hat{\mathbf{z}}_{\epsilon''}) + (P_1^{\epsilon''})^\circ(\hat{\mathbf{z}}_{\epsilon''}; \mathbf{y}_{\epsilon''} - \hat{\mathbf{z}}_{\epsilon''}), \dots, \right. \\ & F_p^{\epsilon''}(\hat{\mathbf{v}}_p^{\epsilon''}, \mathbf{y}_{\epsilon''} - \hat{\mathbf{z}}_{\epsilon''}) - \langle \Theta, \mathbf{y}_{\epsilon''} - \hat{\mathbf{z}}_{\epsilon''} \rangle + \psi_p^{\epsilon''}(\hat{\mathbf{z}}_{\epsilon''}, \mathbf{y}_{\epsilon''}) + g_p^{\epsilon''}(\mathbf{y}_{\epsilon''}) - \psi_p^{\epsilon''}(\hat{\mathbf{z}}_{\epsilon''}, \hat{\mathbf{z}}_{\epsilon''}) \\ & \quad \left. - g_p^{\epsilon'}(\hat{\mathbf{z}}_{\epsilon''}) + (P_p^{\epsilon''})^\circ(\hat{\mathbf{z}}_{\epsilon''}; \mathbf{y}_{\epsilon''} - \hat{\mathbf{z}}_{\epsilon''}) \right) \not\leq 0. \end{aligned} \quad (48)$$

Letting $\epsilon'' \rightarrow 0$ on both sides of (48) and using the upper semicontinuity property of $F_j(\cdot, \cdot)$, $P_j^\circ(\cdot, \cdot)$ ($j \in I$) and the continuity property of $\psi_j(\cdot, \cdot)$, $g_j(\cdot)$ ($j \in I$), we get

$$\begin{aligned} & (F_1(\hat{\mathbf{v}}_1, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) \\ & \quad + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \dots, F_p(\hat{\mathbf{v}}_p, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_p(\hat{\mathbf{z}}, \mathbf{y}) \\ & \quad + g_p(\mathbf{y}) - \psi_p(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_p(\hat{\mathbf{z}}) + P_p^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}})) \not\leq 0. \end{aligned}$$

This implies that $\hat{\mathbf{w}}$ is a solution of GSMWVHI. In view of the given hypothesis, Theorem 5 is satisfied, which, in turn, implies that $\hat{\mathbf{w}} = \hat{\mathbf{z}}$. Moreover, since the limit of $\hat{\mathbf{w}} = \hat{\mathbf{z}}$ does not depend on the subsequence of the solutions of perturbed problems, we have the convergence of the entire family, that is,

$$\hat{\mathbf{z}}_\epsilon \rightarrow \hat{\mathbf{z}}, \text{ as } \epsilon \rightarrow 0.$$

This completes the proof. $\square$

Remark 11. Let $\epsilon > 0$ be a small perturbation parameter. In view of the proof of Theorem 8, it follows that the boundedness of the set expressed as $\{\|\hat{\mathbf{z}}_\epsilon\| : \epsilon > 0\}$ plays an important role in establishing the stability of the solution of GSMWVHI. Now, if $\alpha_F = \alpha_P + \alpha_\psi$, then in light of (46), the following two cases may arise:

Case 1. If the set expressed as $\{\|\hat{\mathbf{z}}_\epsilon\| : \epsilon > 0\}$ is bounded, then using arguments similar to those employed in the proof of Theorem 8, one can conclude the stability of the solution of GSMWVHI.

Case 2. If the set expressed as $\{\|\hat{\mathbf{z}}_\epsilon\| : \epsilon > 0\}$ is unbounded, then the solution of GSMWVHI may or may not be stable. Hence, no conclusion can be drawn.

5. Application to the Sustainable Electricity Generation Planning Problem

Let us consider a regional electricity-generation system consisting of n ($\in \mathbb{N}$) renewable-energy generators interconnected through a transmission network, thereby representing a sustainable electricity-generation system based on renewable energy sources. For each $i \in J := \{1, 2, \dots, n\}$, let z_i denote the electrical power generated by generator i . Moreover, let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{R}_+^n$ represent a generation schedule. The system operator aims to determine a sustainable generation policy that balances economic efficiency and network reliability while satisfying the electricity demand.

In modern electric power systems, aggregate electricity demand exhibits a persistent upward trend driven by population growth, industrial development, the electrification of transportation networks, the expansion of data centers, and the widespread adoption of digital technologies. Let $D_t > 0$ denote the forecasted electricity demand during planning period $t \in \mathbb{N}$.

Unlike conventional short-term economic dispatch models, long-term sustainable electricity planning must account for future capacity expansion through the deployment of renewable-energy facilities, energy-storage systems, and additional generation units. Consequently, imposing a fixed upper bound on the total electricity generation capacity is generally inappropriate in a long-term planning framework. Rather, the primary requirement is that the aggregate electricity generation be adequate to meet the forecasted electricity demand. For every $t \in \mathbb{N}$, let $\mathcal{S}_t$ denote the set of all feasible generation plans, that is,

$$\mathcal{S}_t = \left\{ \mathbf{z} \in \mathbb{R}_+^n : \sum_{j=1}^n z_j \geq D_t \right\}.$$

Evidently, $\mathcal{S}_t$ is a non-empty, closed, convex, and noncompact subset of $\mathbb{R}^n$ for each $t \in \mathbb{N}$.

Let $t \in \mathbb{N}$ be arbitrary but fixed.

For each $j \in \{1, 2\}$, let $T_j : \mathcal{S}_t \rightrightarrows \mathbb{R}^n$, denote the congestion-price mapping, defined as follows (see [57]):

$$T_j(\mathbf{z}) := \left\{ \frac{\mathbf{z}}{2} \right\}.$$

It is easy to verify that for every $j \in \{1, 2\}$, T_j is a compact set and upper Kuratowski semicontinuous on $\mathcal{S}$.

Associated with the economic efficiency of the aforementioned system, the quantity expressed as $\langle \mathbf{v}_1, \mathbf{z} - \mathbf{y} \rangle$, $\mathbf{v}_1 \in T_1(\mathbf{y})$ measures the variation in congestion-related expenditures when the generation schedule changes from $\mathbf{y}$ to $\mathbf{z}$. Nevertheless, modifying a generation schedule often requires reserve deployment, generator ramping, and market redispatch actions. In order to account for the additional expenditures associated with these operational adjustments, we define a bifunction ($F_1 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$) as follows (see [58]):

$$F_1(\mathbf{v}_1, \mathbf{z} - \mathbf{y}) := \langle \mathbf{v}_1, \mathbf{z} - \mathbf{y} \rangle + \alpha_1 \|\mathbf{z} - \mathbf{y}\|,$$

where $\alpha_1 > 0$ is a redispatch-cost coefficient.

The operating expenditure associated with economic efficiency is represented as a function ($g_1 : \mathcal{S}_t \rightarrow \mathbb{R}$) defined as follows (see [58]):

$$g_1(\mathbf{z}) := \sum_{i \in J} (a_i z_i + b_i z_i^2),$$

where $a_i, b_i > 0$.

To quantify the network reliability of the system, we define the bifunction expressed as $F_2 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ as follows:

$$F_2(\mathbf{v}_2, \mathbf{z} - \mathbf{y}) := \langle \mathbf{v}_2, \mathbf{z} - \mathbf{y} \rangle + \alpha_2 \|\mathbf{z} - \mathbf{y}\|_\infty,$$

where $\alpha_2 > 0$ is a coefficient that measures the impact of abrupt local production changes on network reliability.

It is worth noting that the norm term of $\|\mathbf{z} - \mathbf{y}\|_\infty := \max\{|z_i - y_i| : i \in J\}$ appears in the definition of F_2 due to the fact that network reliability is often governed by the largest local generation adjustment rather than by the total aggregate adjustment (see [59]).

The reliability-support expenditure is represented as the $g_2 : \mathcal{S}_t \rightarrow \mathbb{R}$ function, which is defined as follows (see [59]):

$$g_2(\mathbf{z}) := \rho_2 \sum_{i \in J} z_i^2,$$

where $\rho_2 > 0$ denotes the reliability-support cost coefficient.

Additional network-management expenditures generated by congestion mitigation, reserve coordination, and transmission reconfiguration are represented by the $\psi_j : \mathcal{S} \rightarrow \mathbb{R}$ ($j = 1, 2$) functions, which are defined as follows (see [58]):

$$\psi_j(\mathbf{z}, \mathbf{y}) := \sigma_j \|\mathbf{y} - \mathbf{z}\|, \quad j = 1, 2,$$

where $\sigma_1 > 0$ denotes the congestion-management cost coefficient associated with transmission redispatch and congestion-relief actions, whereas $\sigma_2 > 0$ denotes the schedule-adjustment cost coefficient associated with reserve rescheduling, dispatch coordination, and transmission-security actions required during the transition between generation schedules.

Moreover, generator start-up, shut-down, commitment-adjustment, and ramping operations generate nonsmooth expenditures that cannot be adequately represented by smooth cost functions. Let $\bar{z}_i$ denote the reference generation level of generator i determined according to a previously approved renewable-energy expansion strategy or a long-term regulatory planning benchmark. Deviations of the current planning decision (z_i) from the reference level ($\bar{z}_i$) may result in additional adjustment costs associated with infrastructure planning, transmission expansion, energy-storage deployment, and reserve allocation. These costs are generally nonsmooth, as substantial expenditures may arise when the generation plan crosses prescribed planning thresholds. To capture these effects, we introduce the following switching-cost potential:

$$P_j(\mathbf{z}) := \eta_j \sum_{i \in J} |z_i - \bar{z}_i|, \quad j = 1, 2,$$

where $\eta_j > 0$ denotes the adjustment-cost coefficient associated with the j th component.

To characterize the generalized first-order variation of switching expenditures associated with a transition from the current generation plan ($\mathbf{z}$) to another feasible generation plan ($\mathbf{y}$), we incorporate the Clarke generalized directional derivative of P_j at $\mathbf{z}$ in the direction of $\mathbf{y} - \mathbf{z}$ for each $j \in \{1, 2\}$, that is, $P_j^\circ(\mathbf{z}; \mathbf{y} - \mathbf{z})$.

Environmental regulations, carbon taxes, renewable-energy incentives, and governmental sustainability policies are represented by a policy vector: $\Theta \in \mathbb{R}^n$. The term $\langle \Theta, \mathbf{y} - \mathbf{z} \rangle$ quantifies the variation in policy-related expenditures or incentives resulting from a change in the generation plan.

From the above discussions, it follows that the sustainable electricity generation planning problem (in short, SEGPP) can be formulated as the following generalized mixed weak vector variational–hemivariational inequality: find $\hat{\mathbf{z}} \in S$ and $\hat{\mathbf{v}}_j \in T_j(\hat{\mathbf{z}})$, $j = 1, 2$ such that

$$\begin{aligned} & \left(F_1(\hat{\mathbf{v}}_1, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_1(\hat{\mathbf{z}}, \mathbf{y}) + g_1(\mathbf{y}) - \psi_1(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_1(\hat{\mathbf{z}}) + P_1^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}), \right. \\ & \left. F_2(\hat{\mathbf{v}}_2, \mathbf{y} - \hat{\mathbf{z}}) - \langle \Theta, \mathbf{y} - \hat{\mathbf{z}} \rangle + \psi_2(\hat{\mathbf{z}}, \mathbf{y}) + g_2(\mathbf{y}) - \psi_2(\hat{\mathbf{z}}, \hat{\mathbf{z}}) - g_2(\hat{\mathbf{z}}) + P_2^\circ(\hat{\mathbf{z}}; \mathbf{y} - \hat{\mathbf{z}}) \right) \not\prec \mathbf{0}, \end{aligned}$$

for all $\mathbf{y} \in \mathcal{S}_t$.

It is clear from the above that hypothesis (i) of Theorem 1 is satisfied. Moreover, it can be verified that hypotheses (ii), (iii), and (iv) of Theorem 1 are also satisfied. Therefore, all the assumptions of Theorem 1, including the relaxed compactness hypothesis, are satisfied. However, bifunctions F_1 and F_2 are not monotone on the sets expressed as $K_1 := \{(\mathbf{z}, \mathbf{y}) \in$

$\mathcal{S}_t \times \mathcal{S}_t : 0 < \|\mathbf{z} - \mathbf{y}\| < 2\alpha_1\}$ and $K_2 := \{(\mathbf{z}, \mathbf{y}) \in \mathcal{S}_t \times \mathcal{S}_t : \|\mathbf{z} - \mathbf{y}\|^2 < 2\alpha_2\|\mathbf{z} - \mathbf{y}\|_\infty\}$, respectively. On the other hand, in view of Remark 4, it follows that several VVIs studied in the literature (see [8,17]) cannot, in general, be modeled as the SEGPP, whereas the SEGPP could be effectively solved by employing the results established in this paper.

6. Conclusions and Future Research Directions

In this paper, we have investigated generalized Stampacchia and generalized Minty mixed weak vector variational–hemivariational inequalities (GSMWVVHIs and GMMWVVHIs) in terms of bifunctions. We established existence results for the solutions of GSMWVVHI by employing the KKM-Fan lemma without using any monotonicity hypotheses on bifunctions and relaxed compactness assumptions. Moreover, we deduced an existence theorem for solutions of GMMWVVHI under T_j -stably (g_j, ψ_j) -pseudomonotonicity assumptions on F_j ($j \in I$). Subsequently, we characterized the relationship between the solutions of GSMWVVHI and GMMWVVHI by using T_j -stably (g_j, ψ_j) -pseudomonotonicity and generalized hemicontinuity assumptions on F_j ($j \in I$). In addition to this, uniqueness results for the solutions of GSMWVVHI and GMMWVVHI were established under T_j -stably (g_j, ψ_j) -strong pseudomonotonicity of bifunctions and some suitable assumptions. The stability of the solution of GSMWVVHI was established. Finally, it was demonstrated that the sustainable electricity generation planning problem can be naturally formulated as GSMWVVHI.

The results established in this paper generalize various well-known results in the existing literature. Specifically, the results established in this paper generalize the corresponding results deduced by Lee [17] from VVIs to broader classes of vector variational–hemivariational inequalities—namely, GSMWVVHI and GMMWVVHI. Moreover, the results with respect to the existence of solutions of GSMWVVHI and GMMWVVHI generalize the corresponding results established by Mehta et al. [45] from mixed VIs to generalized mixed-vector variational–hemivariational inequalities. In view of the fact that bifunctions need not be linear and the T_j -stably (g_j, ψ_j) -pseudomonotonicity of F_j ($j \in I$) is weaker than the T_j -monotonicity hypotheses on F_j ($j \in I$), the results established in this paper can be applied to address a wider range of VVIs than those available in the existing literature (see, for instance, [17,21] and the references cited therein).

The results presented in this paper present some potential future research avenues. For instance, in view of the work of Ansari and Lee [41], we can establish relationships between the solutions of the considered vector variational–hemivariational inequalities and nonsmooth multiobjective optimization problems.

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