

Article

Some Rigidity Results Related to Conformal Vector Fields

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Abstract

This article explores properties of conformal vector fields on a Riemannian manifold, focusing on conditions that lead to the manifold being isometric to the Euclidean space. Given a conformal vector ζ with conformal factor σ on a Riemannian manifold (N, g) , there is naturally associated a skew-symmetric tensor χ to ζ called the *essential tensor* of ζ . It is shown that the essential tensor χ plays a vital role in our study. We intend to analyze when a conformal vector field becomes a Killing vector field. In a first result of this article, we obtain a necessary and sufficient geometric condition on a complete and connected Riemannian manifold (N, g) admitting a conformal vector field ζ so that ζ is a Killing vector field. In the rest of the article, we obtain characterizations of a Euclidean space using conformal vector fields. In the first such result, it is shown that an n -dimensional complete and connected Riemannian manifold (N, g) , $n > 2$ admits a conformal vector field ζ with conformal factor $\sigma \neq 0$ and essential tensor χ such that the affinity tensor of $\nabla\sigma$ is zero, the function $\zeta(\sigma)$ is a constant, and ζ annihilates χ if and only if (N, g) is isometric to the Euclidean space E^n . Similarly, in a second characterization of the Euclidean space E^n using a conformal vector field ζ , we use the following conditions: ζ annihilates the Ricci operator S , $\nabla\sigma$ annihilates χ , and the vector field $\chi(\zeta)$ is incompressible. Finally, we consider a conformal vector field ζ with conformal factor $\sigma \neq 0$ and essential tensor χ on a complete and connected Riemannian manifold (N, g) such that the Hessian operator H_σ is invariant under the local flow of ζ so that the function $\zeta(\sigma) - \sigma^2$ is a subharmonic function and ζ annihilates χ , and show that (N, g) is isometric to the Euclidean space E^n . The converse holds as well.

Keywords: conformal vector field; conformal factor; essential tensor; ghairaffinity function; isometric to Euclidean space

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1. Introduction

A conformal vector field ζ on a Riemannian manifold (N, g) is one with a local flow that consists of local conformal diffeomorphisms of (N, g) , or equivalently

$$\frac{1}{2}L_\zeta g = \sigma g,$$

where L_ζ is the Lie derivative with respect to ζ and σ is a smooth function on N , called the conformal factor. If the conformal factor $\sigma = 0$, then ζ is a Killing vector field. It is known that by a conformal change of the metric, any conformal vector field on a Riemannian manifold (N, g) that has no zeros can be made into a Killing vector field. Such a conformal

vector field ζ having no zeros on (N, g) is called an inessential conformal vector field. A conformal vector field that is not inessential is called an essential conformal vector field. Since inessential conformal vector fields on a Riemannian manifold (N, g) can be made into Killing vector fields through conformal change of metric, our focus is on the study of essential conformal vector fields. If the Riemannian manifold (N, g) is compact, the famous conjecture of Lichnerowicz (cf. [1]) about essential conformal vector fields states that a compact Riemannian manifold (N, g) $\dim N \geq 3$ admitting an essential conformal vector field is conformally equivalent to a sphere. This conjecture, which is now a proved result in a more general form (cf. [2]), states that an n -dimensional complete Riemannian manifold (N, g) that admits a complete and essential conformal vector field is conformally equivalent to either the sphere S^n or to the Euclidean space E^n . It is worth noting that conformal vector fields not only reflect on the geometry of the manifold on which they are defined, but also play a significant role in physics and relativity (cf. [3–9]).

It is well known that conformal vector fields influence the geometry of the Riemannian manifold on which they exist. Among conformal vector fields, closed conformal vector fields on a Riemannian manifold have been extensively studied (cf. [2,5–7,9–24]). Observe that the defining equation implies that the conformal factor σ of the conformal vector ζ on an n -dimensional Riemannian manifold (N, g) is given by $\sigma = n^{-1} \operatorname{div} \zeta$; in the case that (N, g) is Einstein and ζ is a closed conformal vector field, it turns out that $\nabla \sigma$ is again a conformal vector field and gives a differential equation, which is studied in detail in [15].

If a conformal vector field ζ with conformal factor σ on a Riemannian manifold (N, g) is not closed, then using the 1-form β dual to ζ , one comes across a skew-symmetric tensor χ on (N, g) defined by

$$d\beta(F_1, F_2) = 2g(\chi F_1, F_2)$$

for smooth vector fields F_1, F_2 on N . This skew-symmetric tensor χ is called the *essential tensor* of the conformal vector field ζ , and plays a vital role in the study of non-closed conformal vector fields (cf. [13,25,26]). In the next section, we discuss examples of non-closed conformal vector fields.

Note that spaces such as the sphere $S^n(c)$, the Euclidean space E^n , the hyperbolic space $H^n(-c)$, warped products $I \times_f S^n$, $S^1 \times_f S^n$, and pseudo-umbilical submanifolds of complex space forms $M^{2n}(c)$ are equipped with conformal vector fields (cf. [11,13,14,25]). Taking a clue from the role of position vector fields of submanifolds of the Euclidean space, there has been interest in studying submanifolds of the Riemannian manifold (N, g) that admits a conformal vector field (changing the role of position vector field of Euclidean space to the conformal vector field on (N, g)); cf. [11] and references therein.

An inessential conformal vector field ζ on a Riemannian manifold (N, g) is made Killing on $(N, \bar{g})$, where $\bar{g}$ is conformal to g . Instead of imposing the condition $\sigma = 0$ resulting in making a conformal vector field ζ on a Riemannian manifold (N, g) Killing, one would be interested in exploring geometric conditions on (N, g) that will force $\sigma = 0$. Such a question will require a specific geometric condition satisfied by a Killing vector field ζ on a Riemannian manifold (N, g) . In this article, we show that a Killing vector field ζ on a Riemannian manifold (N, g) satisfies

$$\operatorname{Ric}(\zeta, \zeta) = \operatorname{div}(\nabla_\zeta \zeta) + \|\chi\|^2, \quad (1)$$

where Ric is the Ricci tensor and χ is the essential tensor of the Killing vector field ζ . A natural question that arises is whether the property (1) is the characteristic property of a Killing vector field, or equivalently, whether condition (1) is satisfied by a conformal vector field ζ with essential tensor χ on a complete and connected Riemannian manifold (N, g) ,

and if this necessarily implies that ζ is a Killing vector field. We show that the answer to this question is in the affirmative (see Theorem 2).

We mentioned earlier that there is a focus on studying submanifolds of a Riemannian manifold (N, g) that admits a conformal vector field (as a replacement of the role of position vector field for submanifolds of a Euclidean space). This initiates the question of finding conditions on an n -dimensional Riemannian manifold (N, g) that admits a conformal vector field ζ with conformal factor σ and essential tensor χ so that (N, g) is isometric to the Euclidean space E^n . In the rest of this article, we focus on this issue and find that the essential tensor χ plays an important role in this pursuit. We recall that for a smooth function f on a Riemannian manifold (N, g) , the affinity tensor of ∇f is the tensor (cf. [8])

$$(L_{\nabla f} \nabla),$$

and we call the function f a *gairaffinity function* if the affinity tensor is zero. It is known that the Euclidean space E^n admits a conformal vector field ζ where the conformal factor σ is a *ghairaffinity function* (see next section). This motivates the question of finding conditions on a complete and connected Riemannian manifold (N, g) that admits a conformal vector field ζ with a conformal factor σ that is a *ghairaffinity function* and essential tensor χ so that (N, g) is isometric to the Euclidean space E^n . We answer this question and show that an n -dimensional complete and connected Riemannian manifold (N, g) , $n > 2$ admits a conformal vector field ζ with conformal factor $\sigma \neq 0$ a *ghairaffinity function* such that the function $\zeta(\sigma)$ is a constant and ζ annihilating the essential tensor χ is necessarily isometric to the Euclidean space E^n . The converse also holds (see Theorem 3).

Note that on the Euclidean space E^n , the vector field

$$\zeta = \lambda \sum_i u^i \frac{\partial}{\partial u^i},$$

where λ is a nonzero constant and $u^1, \dots, u^n$ are Euclidean coordinates, is a conformal vector field on E^n with conformal factor $\sigma = \lambda \neq 0$ and essential tensor $\chi = 0$. We see that this example satisfies the following set of conditions:

- (a) (i) $S(\zeta) = 0$ (as $S = 0$ for E^n), (ii) $\chi(\nabla \sigma) = 0$, and (iii) $\text{div}(\chi(\zeta)) = 0$
- (b) (i) $\text{div}(\chi(\zeta)) = 0$, (ii) $\zeta(\sigma) = 0$, and (iii) $\text{Ric}(\zeta, \zeta) \leq 0$.

This naturally initiates the question of whether a complete and connected Riemannian manifold (N, g) admitting a conformal vector field ζ with conformal factor $\sigma \neq 0$ and essential tensor χ (not necessarily zero), satisfying any one of the above set of conditions, is necessarily isometric to the Euclidean space E^n . We answer this question in the affirmative.

In the next result, we show that an n -dimensional complete and connected Riemannian manifold (N, g) , $n > 2$ admits a conformal vector field ζ with conformal factor $\sigma \neq 0$ and essential tensor χ satisfying the following: (i) ζ annihilates the Ricci operator S , (ii) the gradient $\nabla \sigma$ annihilates the essential operator χ , and (iii) the vector $\chi(\zeta)$ is incompressible and isometric to the Euclidean space E^n , with the converse also holding (see Theorem 4). In yet another characterization of the Euclidean space, we use a conformal vector field ζ on an n -dimensional complete and connected Riemannian manifold (N, g) , $n > 2$ with conformal factor σ and the essential tensor χ , satisfying: (i) the vector $\chi(\zeta)$ is incompressible, (ii) the conformal factor σ is a constant along the integral curves of ζ , and (iii) the Ricci curvature $\text{Ric}(\zeta, \zeta) \leq 0$ (see Theorem 5).

Note that Tashiro [27] considered the closed conformal vector field ζ on an n -dimensional complete Riemannian manifold (N, g) , $n \geq 2$ and obtained conditions under which the manifold is isometric to the Euclidean space. Also, in [14] the authors considered

a φ -analytic conformal vector field ζ , that is, a conformal vector field ζ on an n -dimensional complete and connected Riemannian manifold (N, g) satisfying

$$L_{\zeta}\chi = 0,$$

where χ is the essential tensor of the conformal vector field and, under additional assumptions that $\chi(\zeta) = 0$ and $\text{Ric}(\zeta, \zeta) = 0$, showed that (N, g) is isometric to the Euclidean space E^n and that the converse also holds. In addition to this result in [14], the authors also showed that if ζ is a nontrivial φ -analytic conformal vector field and satisfies $\Delta\zeta = 0$, where Δ is the Laplace operator acting on vector fields, then (N, g) is isometric to the Euclidean space E^n and the converse holds as well.

Finally, we consider an n -dimensional complete and connected Riemannian manifold (N, g) admitting a conformal vector field ζ with conformal factor $\sigma \neq 0$ and essential tensor χ . We focus on the Hessian operator H_{σ} of the conformal factor σ . We are interested in the operator H_{σ} being invariant under the local flow of the conformal vector field ζ , that is, equivalent to

$$L_{\zeta}H_{\sigma} = 0,$$

and in this case we say that H_{σ} is invariant under ζ .

Tashiro in [27] considered an n -dimensional complete Riemannian manifold (N, g) , $n \geq 2$ admitting a closed conformal vector field ζ with conformal factor σ satisfying

$$H_{\sigma} = bI,$$

where b is a nonzero constant. There, it was shown that (N, g) is isometric to the Euclidean space E^n (see Theorem 2). The above condition on the Hessian operator satisfies

$$L_{\zeta}H_{\sigma} = 0,$$

which raises the natural question of what additional conditions on a conformal vector field ζ with conformal factor σ and essential tensor χ (not necessarily zero) on an n -dimensional complete and connected Riemannian manifold (N, g) will imply that (N, g) is isometric to the Euclidean space E^n . We answer this question in the last result of this article, where we show that the n -dimensional complete and connected Riemannian manifold (N, g) admitting a conformal vector field ζ with conformal factor $\sigma \neq 0$ and essential tensor χ with H_{σ} is invariant under ζ ; the function $(\zeta(\sigma) - \sigma^2)$ is a subharmonic function, and $\chi(\zeta) = 0$ is necessarily isometric to the Euclidean space E^n , with the converse also holding.

2. Preliminaries

On a Riemannian manifold (N, g) with Riemannian metric g , we denote by ∇ the Riemannian connection on (N, g) . A vector field ζ on (N, g) is called a conformal vector field if

$$\frac{1}{2}L_{\zeta}g = \sigma g, \quad (2)$$

where L_{ζ} stands for Lie derivative with respect to ζ and σ is a smooth function on N called the conformal factor of the conformal vector field ζ . It follows that if $\sigma = 0$, then ζ is a Killing vector field. We denote by β the 1-form dual to the conformal vector field ζ , that is, $\beta(F) = g(\zeta, F)$ for $F \in \mathfrak{X}(N)$, where $\mathfrak{X}(N)$ is the set of smooth vector fields on N . It is natural to assign to a conformal vector field a skew-symmetric $(1, 1)$ -tensor field χ on N defined by

$$d\beta(F_1, F_2) = 2g(\chi F_1, F_2), \quad F_1, F_2 \in \mathfrak{X}(N). \quad (3)$$

This tensor χ assigned to the conformal vector field ζ is called the *essential tensor* of ζ , and has vital role to play in the geometry of the conformal vector field ζ . Using the formulas

$$L_{\zeta}g(F_1, F_2) = g(\nabla_{F_1}\zeta, F_2) + g(\nabla_{F_2}\zeta, F_1), \quad d\beta(F_1, F_2) = g(\nabla_{F_1}\zeta, F_2) - g(\nabla_{F_2}\zeta, F_1)$$

and adding them while using Equations (2) and (3), one immediately concludes

$$\nabla_F\zeta = \sigma F + \chi F, \quad F \in \mathfrak{X}(N). \quad (4)$$

The curvature tensor of the Riemannian manifold (N, g)

$$R(F_1, F_2)F_3 = \nabla_{F_1}\nabla_{F_2}F_3 - \nabla_{F_2}\nabla_{F_1}F_3 - \nabla_{[F_1, F_2]}F_3, \quad (5)$$

for $F_i \in \mathfrak{X}(N)$, $i = 1, 2, 3$, and on taking a local frame $\{e_1, \dots, e_n\}$ on (N, g) , the Ricci tensor Ric of (N, g) is defined by

$$Ric(F_1, F_2) = \sum_i g(R(e_i, F_1)F_2, e_i), \quad F_1, F_2 \in \mathfrak{X}(N). \quad (6)$$

Through the symmetries of the Riemannian curvature tensor, it follows that Ric is a symmetric tensor. The Ricci operator S of (N, g) is defined by

$$Ric(F_1, F_2) = g(SF_1, F_2), \quad F_1, F_2 \in \mathfrak{X}(N), \quad (7)$$

which is also a symmetric operator. The scalar curvature τ of (N, g) is given by

$$\tau = \sum_i Ric(e_i, e_i). \quad (8)$$

On using the formula

$$(\nabla_{F_1}S)(F_2) = \nabla_{F_1}SF_2 - S(\nabla_{F_1}F_2),$$

we have the following expression for the gradient $\nabla\tau$ of the scalar curvature τ (cf. [9]):

$$\frac{1}{2}\nabla\tau = \sum_i (\nabla_{e_i}S)(e_i). \quad (9)$$

Given a conformal vector field ζ on an n -dimensional Riemannian manifold (N, g) with essential tensor χ , using Equation (4), we have

$$\operatorname{div}\zeta = n\sigma, \quad (10)$$

where we have used the skew-symmetry of the essential operator χ . We find the following expression for the covariant derivative of the essential tensor χ , though it is not new; yet, as it is so important to our work, we list it below as a Lemma.

Lemma 1. *Let ζ be a conformal vector field on a Riemannian manifold (N, g) with conformal factor σ and essential tensor χ . Then,*

$$(\nabla_{F_1}\chi)(F_2) = F_2(\sigma)F_1 - g(F_1, F_2)\nabla\sigma + R(F_1, \zeta)F_2, \quad F_1, F_2 \in \mathfrak{X}(N).$$

Proof. On differentiating Equation (4), we have

$$\nabla_{F_1}\nabla_{F_2}\zeta = F_1(\sigma)F_2 + (\nabla_{F_1}\chi)(F_2) + \nabla_{\nabla_{F_1}F_2}\zeta,$$

which in view of Equation (5) reveals

$$R(F_1.F_2)\zeta = F_1(\sigma)F_2 - F_2(\sigma)F_1 + (\nabla_{F_1}\chi)(F_2) - (\nabla_{F_2}\chi)(F_1). \quad (11)$$

Note that the 2-form $d\beta$ in Equation (3) is closed; therefore, we have

$$g((\nabla_{F_1}\chi)(F_2), F_3) + g((\nabla_{F_2}\chi)(F_3), F_1) + g((\nabla_{F_3}\chi)(F_1), F_2) = 0,$$

which, on using skew-symmetry of the essential tensor in second term and then using Equation (11), yields

$$g(R(F_1.F_2)\zeta - F_1(\sigma)F_2 + F_2(\sigma)F_1, F_3) + g((\nabla_{F_3}\chi)(F_1), F_2) = 0.$$

Transvecting the slot F_2 in the above equation, we get the result in the Lemma. $\square$

Recall that the Riemannian curvature tensor of a Riemannian manifold (N, g) is defined by

$$R(E_1, E_2; E_3, E_4) = h(R(E_1, E_2)E_3, E_4), \quad E_i \in \mathfrak{X}(N), i = 1, \dots, 4.$$

Lemma 2. Let ζ be a conformal vector field on a Riemannian manifold (N, g) with conformal factor σ and essential tensor χ . Then,

$$-R(\zeta, F_1; F_2, \zeta) + \frac{1}{2}(L_{\nabla_{\zeta}\zeta}g)(F_1, F_2) + g(\chi F_1, \chi F_2) = \zeta(\sigma)g(F_1, F_2) + \sigma^2 g(F_1, F_2)$$

for smooth vector fields F_1, F_2 on N .

Proof. Following the techniques described in [28], on using

$$(L_{\zeta}g)(F_1, F_2) = g(\nabla_{F_1}\zeta, F_2) + g(\nabla_{F_2}\zeta, F_1)$$

we have that

$$\begin{aligned} (L_{\zeta}L_{\zeta}g)(F_1, F_2) &= \zeta(g(\nabla_{F_1}\zeta, F_2) + g(\nabla_{F_2}\zeta, F_1)) - (L_{\zeta}g)([\zeta, F_1], F_2) \\ &\quad - (L_{\zeta}g)(F_1, [\zeta, F_2]). \end{aligned} \quad (12)$$

Also, we use

$$(L_{\nabla_{\zeta}\zeta}g)(F_1, F_2) = g(\nabla_{F_1}\nabla_{\zeta}\zeta, F_2) + g(\nabla_{F_2}\nabla_{\zeta}\zeta, F_1)$$

in Equation (12), and upon simplification arrive at

$$(L_{\zeta}L_{\zeta}g)(F_1, F_2) = 2R(\zeta, F_1; \zeta, F_2) + (L_{\nabla_{\zeta}\zeta}g)(F_1, F_2) + 2g(\nabla_{F_1}\zeta, \nabla_{F_2}\zeta).$$

Using Equation (4) and the skew-symmetry of the tensor χ in the above equation, we obtain

$$\begin{aligned} (L_{\zeta}L_{\zeta}g)(F_1, F_2) &= 2R(\zeta, F_1; \zeta, F_2) + (L_{\nabla_{\zeta}\zeta}g)(F_1, F_2) + 2\sigma^2 g(F_1, F_2) \\ &\quad + 2g(\chi F_1, \chi F_2). \end{aligned} \quad (13)$$

Now, taking the Lie derivative in Equation (2), we have

$$L_{\zeta}L_{\zeta}g = 2\zeta(\sigma)g + 4\sigma^2 g,$$

and inserting this into Equation (13) gives the required expression. $\square$

Let ζ be a conformal vector field on a Riemannian manifold (N, g) with essential tensor χ . Then, the squared length of χ is given by

$$\|\chi\|^2 = \sum_i g(\chi e_i, \chi e_i) \quad (14)$$

for a local frame $\{e_1, \dots, e_n\}$ on (N, g) .

Note that if the conformal factor $\sigma = 0$, then ζ is called a Killing vector field. Thus, for a Killing vector field ζ on a Riemannian manifold (N, g) , Equation (4) takes the form

$$\nabla_F \zeta = \chi F, \quad F \in \mathfrak{X}(N). \quad (15)$$

Now, we derive the following important geometric characteristic property of a Killing vector field ζ on a Riemannian manifold (N, g) , which will be useful in deriving geometric conditions under which a conformal vector field is Killing.

Theorem 1. *Let ζ be a Killing vector field on a Riemannian manifold (N, g) with essential tensor χ . Then, the following holds:*

$$\text{Ric}(\zeta, \zeta) = \text{div}(\nabla_\zeta \zeta) + \|\chi\|^2.$$

Proof. As ζ is a Killing vector field, Lemma 1 gives

$$(\nabla_{F_1} \chi)(F_2) = R(F_1, \zeta)F_2, \quad F_1, F_2 \in \mathfrak{X}(N). \quad (16)$$

Taking a local frame $\{e_1, \dots, e_n\}$ on (N, g) and using the expression

$$S(\zeta) = \sum_i R(\zeta, e_i)e_i$$

in Equation (16), we get

$$S(\zeta) = -\sum_i (\nabla_{e_i} \chi)(e_i),$$

that is,

$$\text{Ric}(\zeta, \zeta) = -\sum_i g(\zeta, (\nabla_{e_i} \chi)(e_i)). \quad (17)$$

Now, using Equation (15) and the skew-symmetry of the essential tensor χ , we have

$$\begin{aligned} \text{div}(\nabla_\zeta \zeta) &= \text{div}(\chi \zeta) = \sum_i g(\nabla_{e_i} \chi \zeta, e_i) = \sum_i g((\nabla_{e_i} \chi)(\zeta) + \chi(\nabla_{e_i} \zeta), e_i) \\ &= \sum_i g((\nabla_{e_i} \chi)(\zeta), e_i) - \sum_i g(\chi e_i, \chi e_i). \end{aligned}$$

Thus, using Equations (14) and (17) in the above equation gives the desired result. $\square$

Recall that the affinity tensor of a vector field F on a Riemannian manifold (N, g) is given by (cf. [9])

$$(L_F \nabla)(F_1, F_2) = L_F \nabla_{F_1} F_2 - \nabla_{L_F F_1} F_2 - \nabla_{F_1} L_F F_2, \quad F_1, F_2 \in \mathfrak{X}(N),$$

and we call a vector field F on (N, g) a *ghairaffinity vector* if

$$(L_F \nabla) = 0.$$

Also, we call a function ρ on a Riemannian manifold (N, g) a *ghairaffinity function* if

$$(L_{\nabla\rho}\nabla) = 0,$$

which is equivalent to

$$R(F_1, \nabla\rho)F_2 = \nabla_{F_1}\nabla_{F_2}\nabla\rho - \nabla_{\nabla_{F_1}F_2}\nabla\rho, \quad F_1, F_2 \in \mathfrak{X}(N). \quad (18)$$

Examples of *ghairaffinity functions* are in abundance on Euclidean spaces. For example, on the Euclidean space E^n with Euclidean coordinates $u^1, \dots, u^n$, Euclidean metric g , and Euclidean connection ∇ , taking the smooth function $\rho = \frac{1}{2}(u^i)^2$ for a Euclidean coordinate u^i gives

$$\nabla\rho = u^i \frac{\partial}{\partial u^i},$$

and we compute

$$\nabla_{F_2}\nabla\rho = F_2(u^i) \frac{\partial}{\partial u^i} \quad \text{and} \quad \nabla_{F_1}\nabla_{F_2}\nabla\rho = F_1F_2(u^i) \frac{\partial}{\partial u^i}.$$

Consequently, we arrive at

$$\nabla_{F_1}\nabla_{F_2}\nabla\rho - \nabla_{\nabla_{F_1}F_2}\nabla\rho = \text{Hess}(u^i)(F_1, F_2) \frac{\partial}{\partial u^i}, \quad (19)$$

where $\text{Hess}(f)$ of a smooth function f is defined by

$$\text{Hess}(f)(F_1, F_2) = g(\nabla_{F_1}\nabla f, F_2)$$

for smooth vector fields F_1, F_2 . Since for the Euclidean coordinate u^i we see that $\text{Hess}(u^i)(F_1, F_2) = 0$ and the curvature tensor of the Euclidean space E^n is zero, we conclude through Equations (18) and (19) that the function ρ is a *ghairaffinity function* on E^n .

For a smooth function ρ on a Riemannian manifold (N, g) , the symmetric operator H_ρ defined by

$$H_\rho(F) = \nabla_F\nabla\rho, \quad F \in \mathfrak{X}(N) \quad (20)$$

is called the Hessian operator, and is related to $\text{Hess}(\rho)$ by

$$\text{Hess}(\rho)(F_1, F_2) = g(H_\rho(F_1), F_2), \quad F_1, F_2 \in \mathfrak{X}(N). \quad (21)$$

Now, we wish to explore some examples of non-closed conformal vector fields, which will help in supporting the results in this article.

(1) Consider a $(2n - 1)$ -dimensional Riemannian manifold (N, g) that has Sasakian structure (ϕ, ξ, η) , where ϕ is a $(1, 1)$ -tensor field, ξ is a unit vector field, η is a 1-form dual to ξ satisfying (cf. [10])

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1,$$

$$g(\phi F_1, \phi F_2) = g(F_1, F_2) - \eta(F_1)\eta(F_2), \quad F_1, F_2 \in \mathfrak{X}(N),$$

and

$$(\nabla_{F_1}\phi)(F_2) = g(F_1, F_2)\xi - \eta(F_2)F_1, \quad F_1, F_2 \in \mathfrak{X}(N).$$

Choose a closed conformal vector field $\mathbf{u}$ with conformal factor σ on (N, g) , that is, satisfying

$$\nabla_F\mathbf{u} = \sigma F, \quad F \in \mathfrak{X}(N);$$

is this possible? The answer is yes: the unit sphere S^{2n-1} has many closed conformal vector fields along with a Sasakian structure. Now, defining a vector field ζ on the Sasakian manifold (N, g) by

$$\zeta = \mathbf{u} + \tilde{\zeta},$$

it follows that

$$\nabla_F \zeta = \sigma F - \phi(F), \quad F \in \mathfrak{X}(N),$$

that is, ζ is a conformal vector field with conformal factor σ and essential tensor $\chi = -\phi$. As $\chi(\zeta) = -\phi(\mathbf{u})$, it is straightforward to compute

$$\operatorname{div} \chi(\zeta) = (2n - 2)g(\mathbf{u}, \tilde{\zeta}).$$

Thus, if $\mathbf{u}$ is orthogonal to $\tilde{\zeta}$, we get

$$\operatorname{div} \chi(\zeta) = 0.$$

(2) Consider the even-dimensional Euclidean space (E^{2n}, J, g) , where J is the complex structure and g is the Euclidean metric, which is Hermitian on E^{2n} . Consider the position vector field

$$\tilde{\zeta} = \sum_i u^i \frac{\partial}{\partial u^i},$$

where $u^1, \dots, u^{2n}$ are the Euclidean coordinates on E^{2n} . Now, defining a vector field ζ on E^{2n} by

$$\zeta = \tilde{\zeta} + J\tilde{\zeta},$$

it follows that

$$\nabla_F \zeta = F + JF, \quad F \in \mathfrak{X}(E^{2n}).$$

This proves that ζ is a conformal vector field on (E^{2n}, J, g) with conformal factor $\sigma = 1$ and essential tensor $\chi = J$. We have $S(\zeta) = 0$, $\chi(\nabla\sigma) = 0$, and σ is a ghairaffinity function; in addition, we have

$$\operatorname{div} \chi(\zeta) = -2n.$$

3. Conformal Vector Fields on Complete Manifolds

We wish to explore geometric conditions on a complete Riemannian manifold (N, g) so that a conformal vector field ζ with conformal factor σ and essential tensor χ on (N, g) becomes a Killing vector field. In particular, we shall establish that the geometric property described in Theorem 1 is an essential characteristic of a manifold admitting a Killing vector field, as seen in the following.

Theorem 2. *Let ζ be a conformal vector field on a complete connected Riemannian manifold (N, g) with essential tensor χ . If the condition*

$$\operatorname{Ric}(\zeta, \zeta) = \operatorname{div}(\nabla_{\zeta} \zeta) + \|\chi\|^2$$

holds, then ζ is a Killing vector field. The converse is also true.

Proof. Let ζ be a conformal vector field on a complete connected Riemannian manifold (N, g) with conformal factor σ and essential tensor χ with $\dim N = n$. Suppose the condition

$$\operatorname{Ric}(\zeta, \zeta) = \operatorname{div}(\nabla_{\zeta} \zeta) + \|\chi\|^2 \quad (22)$$

holds. Then, by Lemma 2 we have

$$-R(\zeta, F_1; F_2, \zeta) + \frac{1}{2} \left(L_{\nabla_{\zeta} \zeta} g \right) (F_1, F_2) + g(\chi F_1, \chi F_2) = \zeta(\sigma) g(F_1, F_2) + \sigma^2 g(F_1, F_2) \quad (23)$$

for smooth vector fields F_1, F_2 on N . Taking the trace in the above equation and using Equation (6), we get

$$-Ric(\zeta, \zeta) + div(\nabla_{\zeta} \zeta) + \|\chi\|^2 = n(\zeta(\sigma) + \sigma^2).$$

Thus, in view of Equation (22), we get

$$\zeta(\sigma) + \sigma^2 = 0. \quad (24)$$

Let $\alpha(s)$ be an integral curve of ζ . Then, on taking $\zeta(s) = \zeta(\alpha(s))$ and $\sigma(s) = (\sigma \circ \alpha)(s)$, Equation (24) assumes the form

$$\frac{d\sigma}{ds} = -\sigma^2(s). \quad (25)$$

Assume that $\sigma(0) = \sigma_0$; then, the unique solution of the initial value problem

$$\frac{d\sigma}{ds} = -\sigma^2(s), \quad \sigma(0) = \sigma_0$$

is

$$\sigma(s) = \frac{\sigma_0}{s\sigma_0 + 1}. \quad (26)$$

Now, suppose that the constant $\sigma_0 \neq 0$. Then, we compute

$$\frac{d}{ds} \|\zeta(s)\| = \frac{1}{\|\zeta(s)\|} g(\nabla_{\zeta(s)} \zeta(s), \zeta(s)),$$

which in view of Equation (4) and the skew-symmetry of the essential tensor χ yields

$$\frac{d}{ds} \|\zeta(s)\| = \sigma(s) \|\zeta(s)\|.$$

Using Equation (26) in the above equation, we arrive at

$$\frac{d}{ds} \|\zeta(s)\| = \frac{\sigma_0}{s\sigma_0 + 1} \|\zeta(s)\|,$$

that is,

$$\frac{d}{ds} (\ln \|\zeta(s)\|) = \frac{d}{ds} (\ln(s\sigma_0 + 1)).$$

Integrating above equation yields

$$\|\zeta(s)\| = (s\sigma_0 + 1) \|\zeta(0)\|. \quad (27)$$

Note that the solution $\sigma(s)$ in Equation (26) blows away at

$$\bar{s} = -\frac{1}{\sigma_0},$$

and we let $L_0^{\bar{s}}$ be the length of the curve $\alpha(s)$ from 0 to $\bar{s}$. Then, using Equation (27), we have

$$L_0^{\bar{s}} = \int_0^{\bar{s}} \|\zeta(s)\| ds = \frac{1}{2\sigma_0} \|\zeta(0)\| (0 - 1) = \frac{1}{2} \|\zeta(0)\| \bar{s} < \infty.$$

Hence, as (N, g) is complete, there exists a point $x \in N$ such that $\alpha(\bar{s}) = x$. Note that as $s \rightarrow \bar{s}$, $\alpha(s) \rightarrow x$. Now, using Equation (26), we can see that

$$\sigma(x) = \lim_{s \rightarrow \bar{s}} \sigma(\alpha(s)) = \lim_{s \rightarrow \bar{s}} \frac{\sigma_0}{s\sigma_0 + 1} = \infty,$$

which contradicts the fact that σ is continuous (being a smooth function). Thus, the assumption $\sigma_0 \neq 0$ leads to a contradiction. Hence, $\sigma_0 = 0$, which in view of Equation (26) implies that $\sigma(s) = 0$ along each integral curve of ζ . As N is connected, it leads to $\sigma = 0$ on N , that is, ζ is a Killing vector field. The converse follows by Theorem 1. $\square$

4. Conformal Vector Fields and Euclidean Spaces

The Euclidean space E^n is known to admit many conformal vector fields. For instance, for a constant λ , the vector field given by

$$\zeta = \lambda \sum_i u^i \frac{\partial}{\partial u^i}$$

is a conformal vector field with conformal factor $\sigma = \lambda$ and essential tensor $\chi = 0$. Similarly, on the even-dimensional Euclidean space E^{2n} with complex structure J , for a constant μ the vector field

$$\mathbf{u} = \mu \sum_i \left(u^i \frac{\partial}{\partial u^i} + u^i J \frac{\partial}{\partial u^i} \right)$$

is a conformal vector field with conformal factor $\sigma = \mu$ and essential tensor $\chi = \mu J$. In both examples, it is easy to see that the conformal factor σ is a *ghairaffinity function*. This naturally raises a question: Under what conditions is a complete and connected Riemannian manifold (N, g) admitting a conformal vector field ζ and with conformal factor σ being a *ghairaffinity function* isometric to a Euclidean space? We answer this question in the following.

Theorem 3. *An n -dimensional complete connected Riemannian manifold (N, g) , $n > 2$ admits a conformal vector field ζ with conformal factor $\sigma \neq 0$ a *ghairaffinity function* such that the function $\zeta(\sigma)$ is a constant and ζ annihilates the essential tensor χ if and only if (N, g) is isometric to the Euclidean space E^n .*

Proof. Since the conformal factor σ is a *ghairaffinity function*, by Equation (18) we have

$$R(F_1, \nabla \sigma) F_2 = \nabla_{F_1} \nabla_{F_2} \nabla \sigma - \nabla_{\nabla_{F_1} F_2} \nabla \sigma, \quad F_1, F_2 \in \mathfrak{X}(N).$$

Using Equation (20) in the above equation gives

$$R(F_1, \nabla \sigma) F_2 = (\nabla_{F_1} H_\sigma)(F_2). \quad (28)$$

Since H_σ is symmetric, taking the trace in above equation leads to

$$\text{Ric}(\nabla \sigma, F_2) = \sum_i g(F_2, (\nabla_{e_i} H_\sigma)(e_i)). \quad (29)$$

Now, using a local frame $\{e_1, \dots, e_n\}$ of normal coordinates and noting

$$\Delta\sigma = \sum_i g(H_\sigma(e_i), e_i),$$

we have

$$F(\Delta\sigma) = \sum_i Fg(H_\sigma(e_i), e_i) = \sum_i g((\nabla_F H_\sigma)(e_i), e_i). \quad (30)$$

Now, using the identity

$$R(F_1, F_2)\nabla\sigma = (\nabla_{F_1} H_\sigma)(F_2) - (\nabla_{F_2} H_\sigma)(F_1)$$

in Equation (30), we conclude

$$\begin{aligned} F(\Delta\sigma) &= \sum_i g(R(F, e_i)\nabla\sigma + (\nabla_{e_i} H_\sigma)(F), e_i) \\ &= -Ric(F, \nabla\sigma) + \sum_i g(F, (\nabla_{e_i} H_\sigma)(e_i)), \end{aligned}$$

where we have used the symmetry of H_σ . Thus, we have

$$\sum_i g(F, (\nabla_{e_i} H_\sigma)(e_i)) = F(\Delta\sigma) + Ric(F, \nabla\sigma), \quad (31)$$

which upon comparison with Equation (29) yields

$$F(\Delta\sigma) = 0, \quad F \in \mathfrak{X}(N). \quad (32)$$

Now, we can proceed to see the impact of the condition that the function $\zeta(\sigma)$ be a constant, which in particular implies $Fg(\zeta, \nabla\sigma) = 0, F \in \mathfrak{X}(N)$. Using Equation (4) in the last equation, we get

$$g(\sigma F + \chi F, \nabla\sigma) + g(\zeta, H_\sigma(F)) = 0, \quad F \in \mathfrak{X}(N),$$

that is,

$$\sigma\nabla\sigma + H_\sigma(\zeta) = \chi(\nabla\sigma). \quad (33)$$

Note that

$$\operatorname{div}(\sigma\nabla\sigma) = \|\nabla\sigma\|^2 + \sigma\Delta\sigma \quad (34)$$

and

$$\operatorname{div}H_\sigma(\zeta) = \sum_i g(\nabla_{e_i} H_\sigma(\zeta), e_i) = \sum_i g((\nabla_{e_i} H_\sigma)(\zeta) + H_\sigma(\nabla_{e_i} \zeta), e_i).$$

Using Equation (4) and the symmetry of the operator H_σ , we get

$$\operatorname{div}H_\sigma(\zeta) = \sigma\Delta\sigma + \sum_i g(\zeta, (\nabla_{e_i} H_\sigma)(e_i)),$$

where we have used $\operatorname{Tr}(H_\sigma \circ \chi) = 0$. Using Equations (31) and (32) in the above equation yields

$$\operatorname{div}H_\sigma(\zeta) = \sigma\Delta\sigma + Ric(\zeta, \nabla\sigma). \quad (35)$$

Next, we compute

$$\operatorname{div}\chi(\nabla\sigma) = \sum_i g(\nabla_{e_i} \chi(\nabla\sigma), e_i) = \sum_i g((\nabla_{e_i} \chi)(\nabla\sigma) + \chi(H_\sigma e_i), e_i)$$

and, using the skew symmetry of the essential tensor, we get

$$\operatorname{div} \chi(\nabla \sigma) = - \sum_i g(\nabla \sigma, (\nabla_{e_i} \chi)(e_i)). \quad (36)$$

Using Lemma 1, we have

$$\sum_i (\nabla_{e_i} \chi)(e_i) = -(n-1) \nabla \sigma - S \zeta$$

which can be inserted into Equation (36) to yield

$$\operatorname{div} \chi(\nabla \sigma) = (n-1) \|\nabla \sigma\|^2 + \operatorname{Ric}(\nabla \sigma, \zeta). \quad (37)$$

Taking divergence in Equation (33) and using Equations (34), (35), and (37), we conclude

$$\|\nabla \sigma\|^2 + \sigma \Delta \sigma + \sigma \Delta \sigma + \operatorname{Ric}(\zeta, \nabla \sigma) = (n-1) \|\nabla \sigma\|^2 + \operatorname{Ric}(\nabla \sigma, \zeta),$$

that is,

$$(n-2) \|\nabla \sigma\|^2 = 2\sigma \Delta \sigma. \quad (38)$$

Also, taking the inner product in Equation (33) by $\nabla \sigma$ and noting that the essential tensor χ is skew-symmetric, we arrive at

$$\sigma \|\nabla \sigma\|^2 + g(H_\sigma(\zeta), \nabla \sigma) = 0,$$

which is equivalent to

$$\sigma \|\nabla \sigma\|^2 + \frac{1}{2} \zeta \left(\|\nabla \sigma\|^2 \right) = 0. \quad (39)$$

Note that $\zeta(\sigma) = c$ is a constant (statement) and, by Equation (32), $\Delta \sigma = \bar{c}$ is another constant. As $n > 2$, Equation (38) implies

$$\|\nabla \sigma\|^2 = \frac{2}{n-2} \bar{c} \sigma. \quad (40)$$

Also, the operation $\frac{1}{2} \zeta$ on Equation (40) provides

$$\frac{1}{2} \zeta \left(\|\nabla \sigma\|^2 \right) = \frac{1}{n-2} \bar{c} c$$

and inserting the above equation in Equation (39) gives

$$\sigma \|\nabla \sigma\|^2 = -\frac{1}{n-2} \bar{c} c.$$

Multiplying Equation (40) by σ and using it in the above equation, we conclude that

$$\frac{2}{n-2} \bar{c} \sigma^2 = -\frac{1}{n-2} \bar{c} c,$$

that is,

$$\bar{c} (2\sigma^2 + c) = 0.$$

We now have two choices: (i) if $\bar{c} = 0$, then Equation (30) conveys that σ is a constant; (ii) if $\bar{c} \neq 0$, then the above equation implies that $2\sigma^2 + c = 0$. The last choice gives $4\sigma F(\sigma) = 0$, $F \in \mathfrak{X}(N)$; since $\sigma \neq 0$ and N is connected, we must have $F(\sigma) = 0$, $F \in \mathfrak{X}(N)$, that is, in the second choice σ is also a constant.

Finally, we define a smooth function f on N by

$$f = \frac{1}{2} \|\zeta\|^2,$$

which by virtue of Equation (4) and $\chi\zeta = 0$ (statement) has the gradient

$$\nabla f = \sigma\zeta, \quad (41)$$

where σ is a constant (indeed, a nonzero constant by statement). Observe that if f were to be a constant, we get $\zeta = 0$, which by Equation (4) puts us in the following situation:

$$\sigma F = -\chi F$$

with trace $n\sigma = 0$, contrary to the assumption $\sigma \neq 0$. Hence, f is a non-constant function, which by Equation (41) satisfies

$$H_f(F) = \sigma^2 F + \sigma\chi F.$$

Taking the inner product by F in the above equation gives

$$\text{Hess}(f)(F, F) = \sigma^2 g(F, F), \quad F \in \mathfrak{X}(N).$$

Polarization in the above equation yields

$$\text{Hess}(f) = \sigma^2 g$$

for a non-constant function f and nonzero constant σ^2 . Hence, (N, g) is isometric to the Euclidean space E^n (cf. [29]).

The converse is trivial, as the Euclidean space E^n admits a conformal vector field given by

$$\zeta = \lambda \sum_i u^i \frac{\partial}{\partial u^i}$$

for a constant λ with conformal factor $\sigma = \lambda$, which satisfies Equation (18) and $\zeta(\sigma) = 0$, which is a constant and the essential tensor $\chi = 0$, guaranteeing that ζ annihilates χ . $\square$

It is interesting to note that if ζ is a conformal vector field with conformal factor σ and essential tensor χ on an n -dimensional Riemannian manifold (N, g) such that σ is a ghairaffinity function, then one easily obtains that $\Delta\sigma$ is a constant. In particular, if (N, g) is compact, it will imply $\Delta\sigma = 0$, and consequently that σ is a constant. However, on non-compact (N, g) , there are examples of non-constant function σ with $\Delta\sigma$ being a constant, for instance the function

$$f = \frac{1}{2} \|\xi\|^2,$$

where ξ is a position vector on the Euclidean space E^n . We see that $\Delta f = n$ is a constant and f a non-constant function. The combination where (i) σ is a ghairaffinity function and (ii) $\zeta(\sigma)$ is a constant implies that σ is a constant, as seen in the proof of Theorem 3. Thus, the role of the combination where (i) σ is a ghairaffinity function and (ii) (N, g) is compact is achieved in the non-compact case by the combination where (i) σ is a ghairaffinity function and (ii) $\zeta(\sigma)$ is a constant.

On a Euclidean space E^n , there are conformal vector fields and the Ricci operator of E^n is trivial; thus, it is natural to seek conditions on a complete and connected Riemannian manifold (N, g) admitting a conformal vector field ζ that satisfies $S(\zeta) = 0$ (a less restrictive

condition than $S = 0$ on Euclidean space E^n) so that (N, g) is isometric to E^n . Recall that a vector field F on a Riemannian manifold (N, g) is said to incompressible if

$$\operatorname{div} F = 0.$$

Our next result in this direction is provided below.

Theorem 4. *Let (N, g) be an n -dimensional, $n > 1$, complete and connected Riemannian manifold that admits a conformal vector field ζ with conformal factor $\sigma \neq 0$ and essential tensor χ such that (i) ζ annihilates the Ricci operator S , (ii) $\nabla\sigma$ annihilates the essential tensor χ , and (iii) the vector field $\chi(\zeta)$ is incompressible. Then, (N, g) is isometric to the Euclidean space E^n . The converse also holds.*

Proof. Since ζ annihilates the Ricci operator S , we have

$$\operatorname{Ric}(\zeta, F) = 0, \quad F \in \mathfrak{X}(N). \quad (42)$$

Now, $\nabla\sigma$ annihilates the essential tensor χ , that is, $\chi(\nabla\sigma) = 0$, which on differentiation gives

$$(\nabla_E \chi)(\nabla\sigma) + \chi(H_\sigma E) = 0.$$

Utilizing Lemma 1 in the above equation yields

$$\|\nabla\sigma\|^2 E - E(\sigma)\nabla\sigma + R(E, \zeta)\nabla\sigma + \chi(H_\sigma E) = 0, \quad F \in \mathfrak{X}(N),$$

which on taking the trace yields

$$(n-1)\|\nabla\sigma\|^2 + \operatorname{Ric}(\zeta, \nabla\sigma) = 0.$$

Inserting Equation (42) in the above equation and noting $n > 1$, we conclude that σ is a constant. Now, we can use Equation (4) in computing

$$\begin{aligned} \operatorname{div} \chi(\zeta) &= \sum_i g(\nabla_{e_i} \chi(\zeta), e_i) = \sum_i g((\nabla_{e_i} \chi)(\zeta) + \chi(\sigma e_i + \chi(e_i)), e_i) \\ &= -\|\chi\|^2 - \sum_i g(\zeta, (\nabla_{e_i} \chi)(e_i)), \end{aligned} \quad (43)$$

where we have used the skew-symmetry of the essential tensor χ . Using Lemma 1, we have the following on account of σ being a constant:

$$\sum_i (\nabla_{e_i} \chi)(e_i) = \sum_i R(e, \zeta) e_i$$

and taking the inner product with ζ in the above equation yields

$$\sum_i g(\zeta, (\nabla_{e_i} \chi)(e_i)) = -\operatorname{Ric}(\zeta, \zeta).$$

Using Equation (42), we conclude

$$\sum_i g(\zeta, (\nabla_{e_i} \chi)(e_i)) = 0.$$

Inserting this equation in Equation (43) and noting that the vector field $\chi(\zeta)$ is incompressible, we immediately confirm $\chi = 0$. Finally, on taking the function f defined by

$$f = \frac{1}{2} \|\zeta\|^2$$

together with $\sigma \neq 0$ a constant and $\chi = 0$, as in the proof of Theorem 3, we confirm that (N, g) is isometric to the Euclidean space E^n . The converse is trivial. $\square$

We observe that one of the strategies for showing an n -dimensional complete and connected Riemannian manifold (N, g) admitting a conformal vector field ζ with conformal factor σ and essential tensor χ isometric to the Euclidean space E^n requires (i) σ a nonzero constant and (ii) $\chi = 0$. Thus, in Theorem 4 we see that the first two conditions and $n > 1$ imply that σ is a constant and that the remaining requirement is fulfilled by the restriction

$$\operatorname{div} \chi(\zeta) = 0.$$

It is possible to achieve the remaining requirement by some other suitable restriction; therefore, it cannot be said that the condition that the vector field $\chi(\zeta)$ be incompressible is an optimal condition.

Finally, in the following result we observe that the incompressibility of the vector field $\chi(\zeta)$ appears to be an interesting tool in obtaining a characterization of the Euclidean space.

Theorem 5. *Let (N, g) be an n -dimensional complete and connected Riemannian manifold that admits a conformal vector field ζ with conformal factor $\sigma \neq 0$ and essential tensor χ such that (i) the vector field $\chi(\zeta)$ is incompressible, (ii) the conformal factor σ is constant along the integral curves of ζ , and (iii) the Ricci curvature $\operatorname{Ric}(\zeta, \zeta)$ is nonpositive. Then (N, g) is isometric to the Euclidean space E^n . The converse also holds.*

Proof. Using Equations (4) and (10), we have

$$\nabla_{\zeta} \zeta = \sigma \zeta + \chi(\zeta). \quad (44)$$

Since, $\chi(\zeta)$ is incompressible, using Equation (10) in (44) yields

$$\operatorname{div}(\nabla_{\zeta} \zeta) = \zeta(\sigma) + n\sigma^2. \quad (45)$$

Now, Lemma 2 gives

$$-R(\zeta, F_1; F_2, \zeta) + \frac{1}{2} (L_{\nabla_{\zeta} \zeta} g)(F_1, F_2) + g(\chi F_1, \chi F_2) = \zeta(\sigma)g(F_1, F_2) + \sigma^2 g(F_1, F_2)$$

for smooth vector fields F_1, F_2 on N . Taking the trace in the above equation, we get

$$-\operatorname{Ric}(\zeta, \zeta) + \operatorname{div}(\nabla_{\zeta} \zeta) + \|\chi\|^2 = n\zeta(\sigma) + n\sigma^2,$$

and inserting Equation (45) into the above gives

$$-\operatorname{Ric}(\zeta, \zeta) + \|\chi\|^2 = (n-1)\zeta(\sigma).$$

As σ is a constant along the integral curves of ζ , the above equation takes the form

$$-\operatorname{Ric}(\zeta, \zeta) + \|\chi\|^2 = 0. \quad (46)$$

Since, $Ric(\zeta, \zeta) \leq 0$, we get $\chi = 0$. Thus, by Lemma 1, we have

$$R(F_1, \zeta)F_2 = -F_2(\sigma)F_1 + g(F_1, F_2)\nabla\sigma,$$

and we can use $F_2 = \zeta$ and $\zeta(\sigma) = 0$ in the above equation and conclude

$$R(F_1, \zeta)\zeta = g(F_1, \zeta)\nabla\sigma.$$

Note that $R(F_1, \zeta)\zeta$ is symmetric; in view of this fact, the above equation implies

$$g(F_1, \zeta)g(\nabla\sigma, F_2) = g(F_2, \zeta)g(\nabla\sigma, F_1), \quad F_1, F_2 \in \mathfrak{X}(N).$$

Consequently, we have

$$g(F_1, \zeta)\nabla\sigma = g(\nabla\sigma, F_1)\zeta,$$

and taking $F_1 = \zeta$ in above equation yields

$$\|\zeta\|^2\nabla\sigma = 0. \quad (47)$$

If $\zeta = 0$, then the first equation in (44) will imply $\sigma = 0$, a contradiction to the hypothesis that $\sigma \neq 0$. Hence, on connected N , Equation (47) confirms that σ is a nonzero constant, and we have already established that the essential tensor $\chi = 0$. Thus, as in the proof of Theorem 3, (N, g) is isometric to the Euclidean space E^n . The converse is trivial. $\square$

5. An Invariance Under a Conformal Vector Field

Given a conformal vector field ζ with conformal factor σ and essential tensor χ on a Riemannian manifold (N, g) , there is a natural operator, namely, the Hessian operator H_σ associated with the conformal vector field. Note that if the operator H_σ is invariant under the local flow of ζ , we say that H_σ is invariant under the conformal vector field ζ , which is equivalent to

$$L_\zeta H_\sigma = 0. \quad (48)$$

Note that the Euclidean space E^n has a conformal vector field ζ with conformal factor $\sigma = \lambda$ a constant and $H_\sigma = 0$, which satisfies Equation (48). This naturally raises a question: Under what conditions is a complete and connected Riemannian manifold (N, g) possessing a conformal vector field with a conformal factor satisfying condition (48) isometric to a Euclidean space? Recall that a smooth function f on a Riemannian manifold (N, g) is said to be a subharmonic function if it satisfies

$$\Delta f \geq 0. \quad (49)$$

In this section, we use the notion of a subharmonic function to answer the above question through the following result.

Theorem 6. *Let (N, g) be an n -dimensional complete and connected Riemannian manifold that admits a conformal vector field ζ with conformal factor $\sigma \neq 0$ and essential tensor χ such that (i) H_σ is invariant under the conformal vector field ζ , (ii) the function $\zeta(\sigma) - \sigma^2$ is subharmonic, and (iii) ζ annihilates the essential tensor χ . Then, (N, g) is isometric to the Euclidean space E^n . The converse also holds.*

Proof. As H_σ is invariant under the conformal vector field ζ , through Equation (48) we get

$$[\zeta, H_\sigma F] = H_\sigma[\zeta, F], \quad F \in \mathfrak{X}(N),$$

and using Equation (4) in the above equation, we arrive at

$$(\nabla_{\zeta} H_{\sigma})(F) = \chi H_{\sigma} F - H_{\sigma} \chi F, \quad F \in \mathfrak{X}(N). \quad (50)$$

Since, the trace $Tr H_{\sigma} = \Delta \sigma$, on taking the trace in Equation (50) and noting that $Tr \chi H_{\sigma} = Tr H_{\sigma} \chi = 0$, we conclude

$$\zeta(\nabla \sigma) = 0. \quad (51)$$

Now, consider the function $\varphi = \zeta(\sigma)$; on using Equation (4), we can compute the gradient $\nabla \varphi$ to be

$$\nabla \varphi = \sigma \nabla \sigma - \chi(\nabla \sigma) + H_{\sigma} \zeta. \quad (52)$$

As we are interested computing $\Delta \varphi$, we shall compute the divergence of each term in Equation (52). We have

$$\operatorname{div}(\sigma \nabla \sigma) = \|\nabla \sigma\|^2 + \sigma \Delta \sigma \quad (53)$$

and

$$\begin{aligned} \operatorname{div} \chi(\nabla \sigma) &= \sum_i g(\nabla_{e_i} \chi(\nabla \sigma), e_i) = \sum_i g((\nabla_{e_i} \chi)(\nabla \sigma) + \chi H_{\sigma} e_i, e_i) \\ &= \sum_i g((\nabla_{e_i} \chi)(\nabla \sigma), e_i) = - \sum_i g(\nabla \sigma, (\nabla_{e_i} \chi)(e_i)), \end{aligned}$$

where we have used the skew-symmetry of χ and $Tr \chi H_{\sigma} = 0$. Now, using Lemma 1 in the above equation, we reach

$$\begin{aligned} \operatorname{div} \chi(\nabla \sigma) &= g(\nabla \sigma, (n-1) \nabla \sigma) - \sum_i g(\nabla \sigma, R(e_i, \zeta) e_i) \\ &= (n-1) \|\nabla \sigma\|^2 + Ric(\nabla \sigma, \zeta). \end{aligned} \quad (54)$$

Also, using Equation (4), we have

$$\begin{aligned} \operatorname{div} H_{\sigma} \zeta &= \sum_i g(\nabla_{e_i} H_{\sigma} \zeta, e_i) = \sum_i g((\nabla_{e_i} H_{\sigma})(\zeta) + H_{\sigma}(\sigma e_i + \chi e_i), e_i) \\ &= \sigma \Delta \sigma + \sum_i g(\zeta, (\nabla_{e_i} H_{\sigma})(e_i)), \end{aligned}$$

where we have used $Tr H_{\sigma} = \Delta \sigma$, $Tr H_{\sigma} \chi = 0$, and the symmetry of H_{σ} . Next, we take $F = \zeta$ in Equation (31) and use it in above equation to confirm

$$\operatorname{div} H_{\sigma} \zeta = \sigma \Delta \sigma + \zeta(\Delta \sigma) + Ric(\zeta, \nabla \sigma),$$

which in view of Equation (51) changes to

$$\operatorname{div} H_{\sigma} \zeta = \sigma \Delta \sigma + Ric(\zeta, \nabla \sigma). \quad (55)$$

Thus, on combining Equations (53)–(55) with Equation (52), we have the conclusion

$$\Delta \varphi = 2\sigma \nabla \sigma - (n-2) \|\nabla \sigma\|^2,$$

which on using

$$\Delta \sigma^2 = 2\sigma \Delta \sigma + 2 \|\nabla \sigma\|^2$$

gives

$$\Delta \varphi = \Delta \sigma^2 - n \|\nabla \sigma\|^2.$$

Thus, we conclude

$$\Delta(\zeta(\sigma) - \sigma^2) = -n\|\nabla\sigma\|^2.$$

Now, invoking the fact that the function $\zeta(\sigma) - \sigma^2$ is subharmonic, we get that σ is a constant. Though we could follow the line taken in the proof of Theorem 3 that using $\chi(\zeta) = 0$ implies $\chi = 0$, we instead exhibit another proof of this proposition in the following arguments.

Using Equation (4) and $\chi(\zeta) = 0$, we have

$$\nabla_{\zeta}\zeta = \sigma\zeta$$

with σ being a constant. Differentiating the above equation, we get

$$\nabla_F\nabla_{\zeta}\zeta = \sigma^2F + \sigma\chi F, \quad F \in \mathfrak{X}(N),$$

which implies

$$R(F, \zeta)\zeta + \nabla_{\zeta}\nabla_F\zeta + \nabla_{[F, \zeta]}\zeta = \sigma^2F + \sigma\chi F,$$

and on using Equation (4) with constant σ , after simple steps we get

$$R(F, \zeta)\zeta + (\nabla_{\zeta}\chi)(F) + \sigma\chi F + \chi^2F = 0, \quad F \in \mathfrak{X}(N). \quad (56)$$

Note that reflecting Lemma 1 onto $(\nabla_{\zeta}\chi)(F)$, with σ being a constant, yields

$$(\nabla_{\zeta}\chi)(F) = 0.$$

Hence, Equation (56) becomes

$$R(F, \zeta)\zeta + \chi^2F = -\sigma\chi F, \quad F \in \mathfrak{X}(N).$$

Since, $R(F, \zeta)\zeta$ is a symmetric operator and χ^2F is as well, it is apparent that $R(F, \zeta)\zeta + \chi^2F$ is symmetric, while the operator $\sigma\chi F$ is skew-symmetric; this gives

$$\sigma\chi F = 0, \quad F \in \mathfrak{X}(N).$$

Next, invoking the hypothesis that $\sigma \neq 0$, the above equation implies $\chi = 0$. Hence, as in the proof of Theorem 3, we get that (N, g) is isometric to the Euclidean space E^n . The converse is trivial. $\square$

6. Concluding Remarks

In this article, the result in Theorem 2 characterizes a Killing vector field using a conformal vector field ζ with essential tensor χ on a complete and connected Riemannian manifold (N, g) . This may be a humble beginning of the fact that the property

$$Ric(\zeta, \zeta) = \operatorname{div}(\nabla_{\zeta}\zeta) + \|\chi\|^2 \quad (57)$$

is an exclusive property of a Killing vector field. Note that for any nonzero vector field ζ on a Riemannian manifold (N, g) , the essential tensor χ of ζ is related to the 1-form β dual to ζ by

$$d\beta(F_1, F_2) = 2g(\chi F_1, F_2), \quad (58)$$

meaning that on using Equation (14) we have

$$\begin{aligned}\|d\beta\|^2 &= \sum_{i,j} d\beta(e_i, e_j)^2 = 4 \sum_{i,j} g(\chi e_i, e_j)^2 = 4 \sum_i \sum_j g(\chi e_i, g(\chi e_i, e_j) e_j) \\ &= 4 \sum_i g(\chi e_i, \chi e_i) = 4\|\chi\|^2.\end{aligned}$$

Thus, the condition in (57) for any arbitrary vector field ζ on (N, g) takes the form

$$\text{Ric}(\zeta, \zeta) = \text{div}(\nabla_\zeta \zeta) + \frac{1}{4}\|d\beta\|^2.$$

Note that a closed Killing vector field is parallel; therefore, for a nontrivial Killing vector field, we require that it should not be closed. We propose the following interesting problem.

Problem 1. A nonzero non-closed vector field ζ on an n -dimensional complete and connected Riemannian manifold (N, g) with dual 1-form β satisfying

$$\text{Ric}(\zeta, \zeta) = \text{div}(\nabla_\zeta \zeta) + \frac{1}{4}\|d\beta\|^2$$

is a Killing vector field.

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