

Article

Optimal Control of an Innovation Diffusion Model in Smartphone Markets

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Abstract

Competitive smartphone markets are shaped by social influence, ecosystem loyalty, technological obsolescence, and marketing interventions. This study develops a nonlinear innovation diffusion model to describe the interactions among potential consumers, Android users, iPhone users, and dissatisfied consumers in a competitive smartphone market. The model incorporates social influence, ecosystem switching, market saturation, and consumer dissatisfaction effects and is calibrated using United States smartphone market share data from 2016 to 2026 obtained from StatCounter and Statista. Time-dependent advertising controls are then introduced for both smartphone ecosystems, and an optimal control problem is formulated and solved using Pontryagin's Maximum Principle and a forward–backward sweep algorithm. The calibration results indicate a stronger effective diffusion capability for the iPhone ecosystem. Numerical simulations show that early promotional interventions improve long-term market penetration and that the effectiveness of advertising strategies depends strongly on ecosystem valuation, market saturation, and diffusion parameters. Under the estimated parameter set, Android-oriented interventions generate the largest gains in the objective functional, despite the stronger baseline position of the iPhone ecosystem. These results demonstrate how nonlinear diffusion modelling and optimal control theory can provide quantitative support for strategic marketing decisions in competitive technology markets.

Keywords: innovation diffusion; optimal control; smartphone ecosystems; nonlinear diffusion model; technology adoption

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1. Introduction

The smartphone industry constitutes one of the most dynamic and competitive technology markets of the last two decades. Rapid technological innovation, short replacement cycles, strong ecosystem loyalty, and significant social interaction effects continuously shape smartphone adoption patterns worldwide. Although smartphone platforms are now comprised of mature technologies rather than newly emerging innovations, the competition between Android-based devices and Apple's iPhone ecosystem remains a representative example of technology adoption dynamics driven simultaneously by consumer behaviour, network effects, platform competition, and ecosystem lock-in mechanisms.

Understanding the mechanisms governing technology adoption and market penetration has therefore become an important research topic in applied mathematics, economics, marketing science, and management. Mathematical models of innovation diffusion provide a quantitative framework for analysing how technologies spread through populations over time and how social influence mechanisms affect long-term market behaviour. Since the pioneering contributions of Bass [1] and Rogers [2], diffusion theory has played a central role in understanding adoption dynamics, word-of-mouth effects, and consumer imitation processes.

Over the years, diffusion models have been extensively applied to the analysis of product adoption, technological substitution, word-of-mouth effects, and market competition. Comprehensive surveys and reviews of diffusion modelling approaches may be found in [3,4]. In addition to classical diffusion theory, modern smartphone markets exhibit several characteristics associated with platform competition and network economics. Consumers derive value not only from device characteristics but also from software ecosystems, application availability, interoperability, and social prestige. Consequently, network externalities and ecosystem lock-in effects play a central role in shaping adoption trajectories and long-term market shares [5–7]. These mechanisms motivate the use of nonlinear diffusion models capable of capturing interactions between competing technological ecosystems.

However, classical diffusion formulations frequently assume homogeneous adoption mechanisms and static market structures, limiting their ability to capture the complex interactions observed in modern digital ecosystems.

While the classical Bass model successfully describes the diffusion of a single innovation through innovation and imitation effects, it does not explicitly account for competition between technological ecosystems, consumer dissatisfaction, or marketing interventions. The proposed framework extends this classical paradigm by incorporating competing platforms, ecosystem switching, and optimal control strategies, thereby providing a more suitable representation of mature smartphone markets.

In highly competitive smartphone markets, consumers may migrate between competing ecosystems according to perceived utility, technological innovation, ecosystem integration, social prestige, and purchasing power. Empirical studies consistently report stronger customer retention rates within the Apple ecosystem and a persistent net migration tendency from Android to iOS in the United States market [8,9]. Motivated by this evidence, the present study incorporates a simplified unidirectional migration mechanism that captures the dominant long-term switching trend while maintaining analytical tractability.

Recent studies have therefore incorporated nonlinear interaction mechanisms and competitive effects into innovation diffusion models [10–13]. Such approaches allow a more realistic representation of the complex behavioural dynamics characterizing modern technology markets. At the same time, firms operating in smartphone ecosystems continuously adjust advertising intensity and promotional strategies in order to influence consumer adoption trajectories and sustain competitive advantage.

These considerations naturally motivate the incorporation of optimal control techniques into innovation diffusion models. Optimal control theory provides a rigorous mathematical framework for determining time-dependent intervention strategies capable of improving diffusion performance while balancing implementation costs. Applications of optimal control methods have become increasingly relevant in economics, epidemiology, social dynamics, and technological diffusion processes [14–17]. In particular, advertising and marketing dynamics have long been studied through dynamic models, beginning with classical contributions such as the Vidale–Wolfe advertising model [18].

Although previous studies have successfully explored innovation diffusion, technology adoption, and optimal control strategies, most existing approaches treat these

phenomena separately or focus on only a subset of the mechanisms driving competition in smartphone markets. In particular, the combined influence of consumer dissatisfaction, asymmetric switching behaviour between competing ecosystems, network effects, and dynamic advertising interventions remains insufficiently addressed within a unified mathematical framework. Capturing these interactions is essential for understanding how firms can influence market evolution and sustain competitive advantages in mature technology markets. This observation highlights the need for more comprehensive modelling approaches capable of jointly describing adoption dynamics and strategic marketing decisions.

To address these limitations, we develop a nonlinear compartmental model that explicitly incorporates potential consumers, Android users, iPhone users, and dissatisfied consumers, while accounting for ecosystem switching and advertising-based control strategies. To the best of our knowledge, this is one of the first studies to integrate these elements within a single optimal control framework for smartphone market competition. The proposed framework describes the interactions among potential consumers, Android users, iPhone users, and dissatisfied consumers through a system of nonlinear differential equations incorporating social influence effects, migration between ecosystems, consumer dissatisfaction, and market exit dynamics. The model parameters are calibrated using smartphone market share data from the United States between 2016 and 2026.

The proposed model should be interpreted as a phenomenological representation of aggregate market dynamics rather than a detailed behavioural model of individual consumers. Its purpose is to capture the dominant mechanisms governing ecosystem competition, social influence, switching behaviour, and consumer dissatisfaction while preserving mathematical tractability.

An optimal control problem is subsequently formulated by introducing time-dependent advertising and promotional controls associated with both smartphone ecosystems. Using Pontryagin's Maximum Principle, the corresponding optimality system is derived and numerically solved through a forward–backward sweep algorithm. Several control scenarios are investigated in order to analyse how ecosystem valuation and advertising costs influence long-term adoption dynamics. The numerical results indicate that early-stage promotional interventions substantially improve market penetration, while the effectiveness of advertising depends strongly on the calibrated diffusion parameters, customer retention mechanisms, and market saturation levels.

The remainder of the paper is organized as follows. Section 2 introduces the mathematical formulation of the smartphone diffusion model. Section 3 presents the data sources and parameter estimation procedure. Section 4 formulates the optimal control problem and derives the associated optimality conditions. Section 5 presents the numerical simulations and discusses the optimal control scenarios. Finally, Section 6 summarizes the main conclusions and outlines possible directions for future research.

2. Mathematical Formulation of the Innovation Diffusion Model

Mathematical approaches originally developed in epidemiology have been increasingly adapted to describe diffusion and adoption processes in social, technological, and economic systems. The analogy between epidemic spreading and innovation adoption arises from the fact that both phenomena are driven by interaction mechanisms between individuals within a population. In particular, social influence, imitation effects, and word-of-mouth interactions play a role in technology adoption dynamics analogous to contact transmission mechanisms in epidemiological systems [1–3]. Consequently, compartmental dynamical systems originally proposed in epidemiology provide a useful mathematical framework for modelling innovation diffusion in competitive technological markets.

Recent studies have successfully employed nonlinear compartmental formulations to investigate software adoption, competing technologies, behavioural diffusion, and social influence dynamics [10–12]. Within this perspective, the present work adopts and extends a nonlinear diffusion structure inspired by epidemic-type interaction models in order to describe the competition between two major smartphone ecosystems.

In this study, we consider a competitive smartphone market composed of two major technological ecosystems: a mass-market platform represented by Android-based devices and a premium platform represented by Apple's iOS ecosystem. The objective is to describe the temporal evolution of smartphone adoption under the combined influence of social interaction, ecosystem competition, consumer dissatisfaction, and migration effects.

Let the total market population at time t be denoted by

$$P(t) = N(t) + A_1(t) + A_2(t) + R(t),$$

where

- $N(t)$ represents consumers who are not currently active participants in the smartphone replacement or adoption process. This compartment includes potential first-time adopters, delayed upgraders, and consumers temporarily inactive in the competitive smartphone market.
- $A_1(t)$ denotes the population of Android users.
- $A_2(t)$ denotes the population of iOS users.
- $R(t)$ represents dissatisfied or temporarily disengaged consumers.

The compartment $R(t)$ does not necessarily represent permanent abandonment of smartphone usage. Instead, it models consumers temporarily disengaged from active participation in the competitive diffusion process due to technological dissatisfaction, delayed replacement decisions, switching hesitation, or temporary withdrawal from active purchasing behaviour. Similar intermediate behavioural compartments have been considered in several diffusion and behavioural dynamical systems models [11,12].

The model assumes that smartphone adoption is primarily driven by social interaction and imitation effects. Potential consumers adopt smartphone platforms through interactions with existing users. The parameters α_1 and α_2 denote the effective diffusion rates associated with Android and iOS users, respectively.

The nonlinear interaction terms $\alpha_1 N A_1$ and $\alpha_2 N A_2$ follow the classical mass-action principle frequently employed in epidemic-type diffusion models. In the present context, these terms quantify the intensity of social influence generated by interactions between active users and potential adopters. Although simplified, this formulation provides a tractable approximation of word-of-mouth effects and imitation dynamics commonly observed in technology adoption processes [1,2,19]. Because all state variables are subsequently normalized with respect to a fixed reference population, the interaction terms may be interpreted as effective diffusion rates describing aggregate social influence mechanisms.

Moreover, consumers may migrate from the Android ecosystem toward the premium iOS ecosystem due to perceived quality improvement, ecosystem integration, social prestige, or changes in purchasing power. This migration mechanism is represented by switching parameter β .

This simplification is adopted to avoid introducing additional parameters that cannot be reliably identified from the available aggregate market-share data.

Users of both ecosystems may also become dissatisfied due to technological obsolescence, software limitations, battery degradation, pricing perception, or unmet consumer expectations. The corresponding dissatisfaction rates are denoted by γ_1 and γ_2 .

Finally, dissatisfied consumers may progressively return to the pool of potential adopters at rate δ , while all compartments are subject to a natural market exit rate μ , representing permanent market withdrawal, replacement cycles, or demographic turnover.

Under these assumptions, the innovation diffusion dynamics are described by the following nonlinear differential system:

$$\begin{cases} \frac{dN}{dt} = \Lambda - \alpha_1 N A_1 - \alpha_2 N A_2 + \delta R - \mu N \\ \frac{dA_1}{dt} = \alpha_1 N A_1 - \beta A_1 - \gamma_1 A_1 - \mu A_1 \\ \frac{dA_2}{dt} = \alpha_2 N A_2 + \beta A_1 - \gamma_2 A_2 - \mu A_2 \\ \frac{dR}{dt} = \gamma_1 A_1 + \gamma_2 A_2 - \delta R - \mu R \end{cases} \quad (1)$$

Here, Λ denotes the inflow rate of new potential consumers entering the market. Summing all equations yields

$$\frac{dP}{dt} = \Lambda - \mu P,$$

which implies that the total market population remains bounded and converges asymptotically to Λ/μ . Consequently, the feasible region

$$\Omega = \left\{ (N, A_1, A_2, R) \in \mathbb{R}_+^4 : P(t) \leq \frac{\Lambda}{\mu} \right\}$$

is positively invariant. Furthermore, for nonnegative initial conditions, standard differential equation arguments guarantee the positivity and boundedness of all state variables for all $t \geq 0$ [17].

The effective diffusion indicators associated with the Android and iPhone ecosystems are given by

$$R_{01} = \frac{\alpha_1 \Lambda}{\mu(\beta + \gamma_1 + \mu)}, \quad R_{02} = \frac{\alpha_2 \Lambda}{\mu(\gamma_2 + \mu)}.$$

These effective diffusion indicators are analogous to basic reproduction numbers commonly used in epidemiological modelling. In the context of innovation diffusion, they measure the effective capacity of each ecosystem to generate new adopters in a predominantly non-adoptive market environment. More precisely, they quantify the average diffusion potential generated by a representative adopter through social interaction mechanisms.

The overall diffusion threshold is defined as

$$R_0 = \max\{R_{01}, R_{02}\}.$$

From a qualitative perspective, values of R_0 below unity are associated with weak diffusion capacity and may hinder long-term ecosystem persistence. Conversely, values of R_0 above unity favour sustained diffusion dynamics and long-term market presence [11].

Although the proposed framework remains intentionally simplified, it captures several important qualitative mechanisms frequently observed in competitive technology markets, including social influence, migration between ecosystems, dissatisfaction effects, and market saturation. The resulting compartmental system provides a mathematically tractable basis for the optimal control analysis developed in the following sections.

3. Data and Parameter Estimation

3.1. Data Sources

The empirical analysis considered in this work is based on smartphone market data from the United States between January 2016 and January 2026. Monthly market-share information associated with the Android and iOS ecosystems was obtained from StatCounter Global Stats [20], while estimates for the total smartphone user base were collected from Statista reports [21].

The United States smartphone market constitutes an appropriate case study for innovation diffusion analysis due to its high technological penetration, strong consumer purchasing power, rapid replacement cycles, and persistent competition between the Android and iOS ecosystems.

Let $s_1(t)$ and $s_2(t)$ denote the monthly market shares associated with the Android and iOS ecosystems, respectively, where

$$0 \leq s_i(t) \leq 1, \quad i = 1, 2.$$

The corresponding active-user populations were approximated through

$$A_i(t) = s_i(t)T(t), \quad i = 1, 2,$$

where $T(t)$ denotes the estimated total smartphone user base at time t .

The quantities $A_1(t)$ and $A_2(t)$ therefore represent approximations of the active populations associated with the Android and iOS ecosystems, respectively. Throughout the simulations, the variables were normalized with respect to the reference population

$$S = 300 \text{ million},$$

which approximately corresponds to the size of the active smartphone market considered during the analysed period.

The compartment $N(t)$ represents potential consumers temporarily outside the active smartphone ecosystems considered in the model. Since direct observations for this population are not publicly available, $N(t)$ was approximated using aggregate demographic and market-participation information and should therefore be regarded as a proxy rather than an independently observed quantity. Similarly, the compartment $R(t)$ represents dissatisfied or temporarily disengaged consumers. Since no direct observations are available for this population, $R(t)$ is treated as a latent variable generated endogenously by the dynamical system. Consequently, both $N(t)$ and $R(t)$ should be interpreted as phenomenological variables introduced to capture market inactivity and temporary disengagement effects within the proposed diffusion framework.

The resulting dataset should be interpreted as a phenomenological representation of aggregate smartphone diffusion dynamics rather than a precise demographic description of the market. In particular, StatCounter data measure web-usage shares rather than exact numbers of active smartphone users. Consequently, the variables $A_1(t)$ and $A_2(t)$ should be regarded as proxies for the relative diffusion behaviour of the competing ecosystems.

Since only Android and iOS populations are directly observable, some parameters associated with latent compartments may not be uniquely identifiable. Consequently, the estimated values should be interpreted as effective parameters reproducing the observed diffusion trajectories rather than direct behavioural measurements. Despite these limitations, the available data remain suitable for analysing aggregate diffusion trends, competitive interaction effects, and the qualitative impact of marketing interventions on smartphone ecosystem dynamics. Therefore, the calibration should be interpreted as a

practical fitting procedure rather than as a full structural identifiability analysis. The available data support the estimation of an effective parameter set capable of reproducing the observed Android and iOS trajectories, but they do not allow us to claim uniqueness of all individual parameter values.

3.2. Parameter Estimation

The parameter estimation procedure aims to identify parameter values capable of reproducing the observed competitive dynamics between the Android and iOS ecosystems over the considered time horizon.

Let

$$\theta = (\Lambda, \alpha_1, \alpha_2, \beta, \gamma_1, \gamma_2, \delta, \mu)^\top$$

denote the vector of model parameters to be estimated.

The calibration process was performed by minimizing the discrepancy between the model trajectories and the observed monthly populations associated with the Android and iOS ecosystems. More precisely, the estimation procedure is based on the weighted least-squares functional

$$J_{\text{est}} = 2.5 \sum_k [\hat{A} * 1(t_k) - A * 1, k]^2 + 2.5 \sum_k [\hat{A} * 2(t_k) - A * 2, k]^2 + 0.3 \sum_k [\hat{N}(t_k) - N_k]^2,$$

where $\hat{A} * 1(t_k)$, $\hat{A} * 2(t_k)$, and $\hat{N}(t_k)$ denote the model solutions evaluated at the observation times t_k , while $A * 1, k$, $A * 2, k$, and N_k represent the corresponding calibration data.

The weighting factors were selected to prioritize the fit of the observable smartphone ecosystems while assigning lower importance to the indirectly estimated non-user compartment. In particular, larger weights were assigned to the Android and iOS populations, which are supported by market-share observations, whereas a smaller weight was used for $N(t)$, reflecting its proxy nature. The numerical values of the weighting coefficients (2.5, 2.5, and 0.3) were selected empirically to reflect the different levels of confidence associated with the available data rather than as statistically optimal weights. Their purpose is to ensure that the calibration primarily reproduces the observed Android and iOS market-share trajectories, while preventing the approximately estimated non-user compartment from exerting a disproportionate influence on the objective function. Although alternative weighting schemes could be adopted, the present choice provides a reasonable balance between data fidelity and model robustness for the purposes of the subsequent optimal control analysis.

Since no direct observations are available for the compartment $R(t)$, this variable was not included as a calibration target.

The parameter estimation procedure was implemented using a bound-constrained weighted least-squares optimization approach combined with numerical solutions of the differential system obtained through a Runge–Kutta–Fehlberg integration scheme. The optimization was performed in logarithmic parameter space, where lower and upper bounds were imposed on all optimization variables. The resulting optimization problem was solved using the L-BFGS-B algorithm with a multi-start strategy based on four different initial parameter sets. This approach improves numerical conditioning, accounts for parameters spanning several orders of magnitude, and reduces sensitivity to the initial parameter values, thereby increasing the robustness of the calibration procedure.

3.3. Calibration Results and Model Fitting

The parameter values obtained through the calibration procedure are summarized in Table 1. These parameters characterize the effective diffusion intensity, migration behaviour

between ecosystems, dissatisfaction effects, and market re-entry dynamics associated with the considered smartphone ecosystems.

Table 1. Estimated parameter values obtained from the weighted least-squares calibration.

Parameter	Estimated Value
Λ	0.08516
α_1	4.71360
α_2	61.9650
β	0.00001
γ_1	0.00100
γ_2	0.87349
δ	8.00000
μ	0.06036

The estimation results suggest that the iOS ecosystem exhibits a stronger effective diffusion capability than the Android ecosystem, as reflected by the larger estimated value of the parameter α_2 . This behaviour is consistent with the strong ecosystem retention and premium-user loyalty commonly observed in the United States smartphone market.

The estimated migration parameter β is small but positive, suggesting that migration effects play a secondary role compared with the diffusion mechanisms governed by α_1 and α_2 .

The estimated parameters γ_1 and γ_2 should be interpreted as effective phenomenological quantities rather than direct measures of consumer dissatisfaction. The larger value of γ_2 reflects the aggregate transition intensity required by the calibrated model to reproduce the observed diffusion dynamics of the iOS ecosystem.

Large parameter values arise naturally from the normalization procedure and the adopted calibration framework, including the bound-constrained optimization. Consequently, the estimated parameter values should be interpreted within the context of the normalized model rather than as directly measurable physical or behavioural rates.

Similarly, the relatively small value of the market exit parameter μ reflects the long-term persistence of users within the global smartphone ecosystem rather than the physical replacement cycle of individual devices.

Figure 1 compares the calibrated model trajectories with the observed annual average populations associated with the Android and iOS ecosystems.

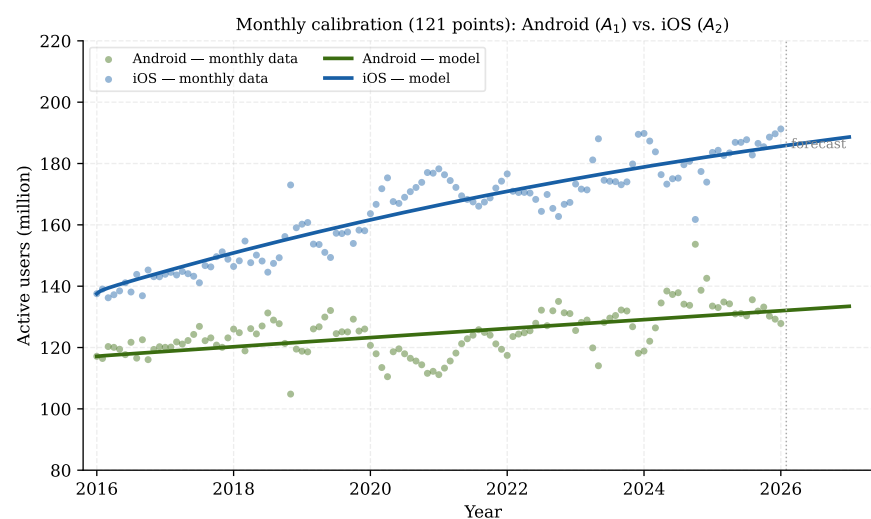


Figure 1. Model fit to the observed monthly Android and iOS populations.

The calibrated model successfully reproduces the main competitive diffusion dynamics observed during the analysed period. In particular, the model captures the sustained growth of the iOS ecosystem and the comparatively slower evolution of the Android ecosystem within the United States market.

To provide a more comprehensive assessment of the calibration quality, four complementary goodness-of-fit metrics were computed using the annual mean market data. Although the model parameters were estimated from the complete monthly dataset, the goodness-of-fit indicators were evaluated using annual averages because the proposed model is intended to reproduce the long-term evolution of smartphone ecosystem adoption rather than short-term market fluctuations. Monthly observations may exhibit temporary variations associated with new smartphone releases, pricing promotions, marketing campaigns, and seasonal purchasing behaviour, whereas annual averages provide a more stable assessment of the model's ability to capture the underlying diffusion trend.

Let y_i denote the observed value, $\hat{y}_i$ denote the corresponding model prediction, $\bar{y}$ denote the sample mean of the observed values, and n denote the number of observations. The adopted metrics are defined as

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2}, \quad (2)$$

$$\text{MAE} = \frac{1}{n} \sum_{i=1}^n |y_i - \hat{y}_i|, \quad (3)$$

$$R^2 = 1 - \frac{\sum_{i=1}^n (y_i - \hat{y}_i)^2}{\sum_{i=1}^n (y_i - \bar{y})^2}, \quad (4)$$

$$\text{MAPE} = \frac{100}{n} \sum_{i=1}^n \left| \frac{y_i - \hat{y}_i}{y_i} \right|. \quad (5)$$

The four metrics provide complementary information on the quality of the calibration. RMSE and MAE quantify the prediction error in absolute terms, whereas R^2 and MAPE evaluate the explained variability and the relative prediction error, respectively. The corresponding values are reported in Table 2. Overall, the obtained indicators confirm a satisfactory calibration of both ecosystems. As expected from Figure 1, the iOS ecosystem exhibits a higher goodness-of-fit than Android, particularly in terms of the coefficient of determination and percentage error, reflecting its smoother long-term market evolution.

Table 2. Goodness-of-fit metrics based on annual mean market data. RMSE and MAE are expressed in millions of users.

Ecosystem	RMSE (M)	MAE (M)	R^2	MAPE (%)
Android (A_1)	3.90	3.18	0.52	2.59
iOS (A_2)	3.23	2.71	0.95	1.65

Figure 2 illustrates the temporal evolution of all state variables generated by the calibrated model.

The numerical simulations suggest that the iOS ecosystem progressively consolidates its market position throughout the analysed period, while the Android ecosystem exhibits slower growth and stronger sensitivity to migration effects. The monthly observations exhibit noticeable short-term fluctuations, reflecting seasonal effects, market-specific events, and variations in consumer behaviour. In contrast, the model captures the long-term diffusion trends rather than these high-frequency variations.

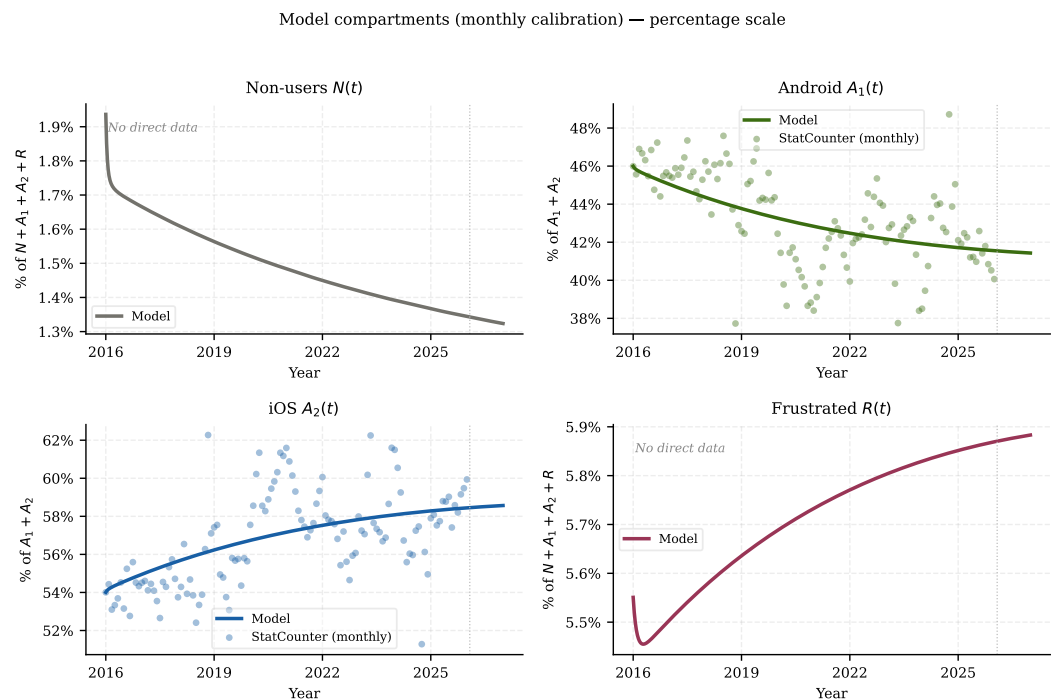


Figure 2. Temporal evolution of the calibrated model compartments.

Although the proposed framework remains intentionally simplified, the calibration results demonstrate that the model is capable of reproducing the principal aggregate diffusion patterns observed in the competitive smartphone ecosystem market. The discrepancy between the model and the observations remains relatively limited, suggesting that the estimated parameters successfully capture the principal diffusion mechanisms governing smartphone adoption. Moreover, the forecast generated beyond the calibration interval indicates the continuation of the observed competitive structure, with the premium ecosystem preserving its market advantage under the current diffusion dynamics.

4. Optimal Control Problem

In highly competitive smartphone markets, firms continuously adjust marketing intensity, advertising investment, and promotional strategies in order to maximize product adoption and long-term profitability. Since consumer behaviour evolves dynamically over time, optimal managerial decisions must account for the temporal evolution of market diffusion and competitive interaction effects.

In order to represent managerial actions capable of influencing consumer adoption, we introduce two time-dependent control variables. The control $u_1(t)$ represents the intensity of promotional and advertising efforts directed towards the Android ecosystem, including marketing campaigns, customer acquisition programs, and brand visibility initiatives. Similarly, $u_2(t)$ represents the intensity of promotional and strategic actions favouring the iOS ecosystem, such as advertising campaigns, ecosystem promotion, customer retention programs, and pricing incentives. Both controls are normalised and interpreted as dimensionless intervention levels bounded between 0 and 1, where 0 corresponds to the absence of additional intervention and 1 represents the maximum feasible managerial effort.

From a modelling perspective, the controls act as effective intervention variables that aggregate the combined effects of advertising, promotional campaigns, pricing policies, customer engagement initiatives, and other managerial actions capable of influencing adoption behaviour. Therefore, each control should be interpreted as a synthetic measure of market intervention rather than a single operational decision variable.

The admissible control set is defined by

$$\mathcal{U} = \{(u_1, u_2) \in L^\infty([0, T], \mathbb{R}^2) : 0 \leq u_i(t) \leq u_{\max}\}, \quad u_{\max} = 1.0, \quad T = 10 \text{ years}. \quad (6)$$

Here, T denotes the final time horizon of the optimization process.

Under the action of the control variables, the smartphone diffusion dynamics become

$$\begin{cases} \frac{dN}{dt} = \Lambda - (1 + u_1)\alpha_1 N A_1 - (1 + u_2)\alpha_2 N A_2 + \delta R - \mu N \\ \frac{dA_1}{dt} = (1 + u_1)\alpha_1 N A_1 - \beta A_1 - \gamma_1 A_1 - \mu A_1 \\ \frac{dA_2}{dt} = (1 + u_2)\alpha_2 N A_2 + \beta A_1 - \gamma_2 A_2 - \mu A_2 \\ \frac{dR}{dt} = \gamma_1 A_1 + \gamma_2 A_2 - \delta R - \mu R \end{cases}. \quad (7)$$

The controls u_1 and u_2 amplify the effective social influence and adoption word-of-mouth rates associated with each ecosystem through advertising, digital campaigns, and promotional exposure.

The main objective is to maximize weighted smartphone adoption while minimizing advertising costs:

$$J(u_1, u_2) = \int_0^T \left[C_1 A_1(t) + C_2 A_2(t) - \frac{1}{2} (B_1 u_1^2 + B_2 u_2^2) \right] dt. \quad (8)$$

The positive coefficients C_1 and C_2 represent the relative strategic value associated with Android and iOS users, respectively, while $B_1 > 0$ and $B_2 > 0$ quantify the marginal costs of advertising and promotional interventions. Different choices of these parameters allow the investigation of alternative competitive scenarios.

The quadratic structure of the control costs guarantees convexity of the optimization problem and prevents unrealistically large control intensities.

The optimal control problem consists of determining control functions

$$(u_1^*, u_2^*) \in \mathcal{U}$$

such that

$$J(u_1^*, u_2^*) = \max_{(u_1, u_2) \in \mathcal{U}} J(u_1, u_2). \quad (9)$$

According to Pontryagin's Maximum Principle, the Hamiltonian associated with the optimal control problem is given by

$$\begin{aligned} H = & C_1 A_1 + C_2 A_2 - \frac{1}{2} (B_1 u_1^2 + B_2 u_2^2) \\ & + \lambda_1 [\Lambda - (1 + u_1)\alpha_1 N A_1 - (1 + u_2)\alpha_2 N A_2 + \delta R - \mu N] \\ & + \lambda_2 [(1 + u_1)\alpha_1 N A_1 - \beta A_1 - \gamma_1 A_1 - \mu A_1] \\ & + \lambda_3 [(1 + u_2)\alpha_2 N A_2 + \beta A_1 - \gamma_2 A_2 - \mu A_2] \\ & + \lambda_4 [\gamma_1 A_1 + \gamma_2 A_2 - \delta R - \mu R], \end{aligned} \quad (10)$$

where

$$\lambda_i(t), \quad i = 1, 2, 3, 4,$$

are the adjoint variables associated with the state variables.

Since the state system is bilinear in the controls, the admissible control set is convex and closed, the state variables remain bounded in the positively invariant region Ω , and the

integrand is concave with respect to the controls, standard existence results for optimal control problems apply (see Lenhart and Workman [17]).

Theorem 1 (Pontryagin's Maximum Principle [22]). *Let (u_1^*, u_2^*) be optimal with state (N^*, A_1^*, A_2^*, R^*) . There exist adjoint functions $\lambda_i : [0, T] \rightarrow \mathbb{R}$ satisfying*

$$\begin{cases} \dot{\lambda}_1 = (1 + u_1)\alpha_1 A_1(\lambda_1 - \lambda_2) + (1 + u_2)\alpha_2 A_2(\lambda_1 - \lambda_3) + \mu\lambda_1 \\ \dot{\lambda}_2 = -C_1 + (1 + u_1)\alpha_1 N(\lambda_1 - \lambda_2) + \beta(\lambda_2 - \lambda_3) + \gamma_1(\lambda_2 - \lambda_4) + \mu\lambda_2 \\ \dot{\lambda}_3 = -C_2 + (1 + u_2)\alpha_2 N(\lambda_1 - \lambda_3) + \gamma_2(\lambda_3 - \lambda_4) + \mu\lambda_3 \\ \dot{\lambda}_4 = \delta(\lambda_4 - \lambda_1) + \mu\lambda_4 \end{cases}, \quad (11)$$

with transversality conditions $\lambda_i(T) = 0, i = 1, \dots, 4$, and optimal controls

$$\begin{cases} u_1^* = \min\left\{\max\left\{\frac{\alpha_1 N A_1(\lambda_2 - \lambda_1)}{B_1}, 0\right\}, u_{\max}\right\}, \\ u_2^* = \min\left\{\max\left\{\frac{\alpha_2 N A_2(\lambda_3 - \lambda_1)}{B_2}, 0\right\}, u_{\max}\right\}. \end{cases} \quad (12)$$

Proof. Setting $\partial H / \partial u_i = 0$:

$$\frac{\partial H}{\partial u_1} = \alpha_1 N A_1(\lambda_2 - \lambda_1) - B_1 u_1 = 0 \Rightarrow u_1 = \frac{\alpha_1 N A_1(\lambda_2 - \lambda_1)}{B_1},$$

and analogously for u_2 . Projection onto $[0, u_{\max}]$ yields (12). Since $\partial^2 H / \partial u_i^2 = -B_i < 0$, the Hamiltonian is strictly concave with respect to each control variable, confirming the existence of a unique maximizer. $\square$

5. Numerical Results and Discussion

The optimality system is solved iteratively using the Forward–Backward Sweep Method [17]: forward integration of (7) with fourth-order Runge–Kutta ($\Delta t = T/1000$) and backward integration of (11) with $\lambda_i(T) = 0$. Convergence ($\text{tol} = 10^{-7}$) is achieved in fewer than 30 iterations for all scenarios. The parameters adopted for the simulations are reported in Table 1, with their values taken from Section 3.

For the calibrated parameter set, the estimated thresholds satisfy

$$R_{02} > R_{01} > 1,$$

indicating that both smartphone ecosystems persist in the market, while the premium iPhone ecosystem exhibits stronger long-term diffusion capability. The diffusion indicators obtained in this study are relatively high because the underlying smartphone market experienced a very large expansion during the analysed period. In particular, the active smartphone user population in the United States increased by approximately 100 million users over the last decade, reflecting rapid technological adoption, strong network effects, and continuous market growth. Consequently, the diffusion process operates in a large and highly connected population, which naturally amplifies the estimated diffusion thresholds. Therefore, the quantities R_{01} and R_{02} should not be interpreted as epidemiological reproduction numbers in the classical sense, but rather as effective market diffusion indicators measuring the potential of each ecosystem to generate new adopters in a rapidly expanding technological market.

Five scenarios are considered, all satisfying $C_1 + C_2 = 1$ and $B_1 + B_2 = 1$:

- Case 1 ($C_1 = C_2 = B_1 = B_2 = 0.5$): symmetric benchmark—equal value and equal cost for both platforms.

- Case 2 ($C_1 = 0.1, C_2 = 0.9, B_1 = B_2 = 0.5$): iOS-oriented strategy—premium market valued much more highly, symmetric costs.
- Case 3 ($C_1 = 0.9, C_2 = 0.1, B_1 = B_2 = 0.5$): Android-oriented strategy—volume market valued more highly.
- Case 4 ($C_1 = C_2 = 0.5, B_1 = 0.1, B_2 = 0.9$): symmetric value—Android advertising much cheaper.
- Case 5 ($C_1 = C_2 = 0.5, B_1 = 0.9, B_2 = 0.1$): symmetric value—iOS advertising much cheaper.

Table 3 summarizes the outcomes of the five optimal-control scenarios. For each case, the table reports the value weights (C_1, C_2), the advertising cost weights (B_1, B_2), the optimal objective value J_{ctrl} , the uncontrolled baseline J_{base} , and the relative gain

$$\frac{J_{ctrl} - J_{base}}{J_{base}} \times 100\%.$$

The gain measures the additional net benefit generated by the optimal advertising strategy relative to the uncontrolled case. Comparing the different scenarios highlights the influence of platform valuation and advertising costs on the optimal allocation of marketing effort.

Table 3. Summary of the five optimal control scenarios. $u_{\max} = 1.0, T = 10$ years, $S = 300$ M.

C_1	C_2	B_1	B_2	J_{base}	J_{ctrl}	Gain	$\max u_1^*$	$\max u_2^*$
0.5	0.5	0.5	0.5	1448.3	1448.6	0.02%	0.033	0.013
0.1	0.9	0.5	0.5	1609.5	1642.7	2.06%	0.000	0.362
0.9	0.1	0.5	0.5	1287.1	1350.7	4.93%	0.580	0.001
0.5	0.5	0.1	0.9	1448.3	1449.5	0.08%	0.168	0.007
0.5	0.5	0.9	0.1	1448.3	1448.5	0.02%	0.018	0.062

Figure 3 shows the time evolution of the two adopter populations, $A_1(t)$ (Android) and $A_2(t)$ (iOS), under the optimal advertising policy for each of the five scenarios, together with the uncontrolled baseline. These trajectories are the state response that results from applying the Pontryagin-optimal controls $u_1^*(t)$ and $u_2^*(t)$ to the diffusion dynamics, and they show how the advertising effort reshapes the adoption paths relative to the do-nothing case. In every controlled scenario the targeted population lies at or above its uncontrolled trajectory, the gap widening over the horizon as the cumulative effects of sustained advertising accumulate. The size and timing of this gap depend on which platform the value weights (C_1, C_2) favour: scenarios that reward the lower-priced segment steer more effort towards A_1 , while those that reward the premium segment lift A_2 . The state trajectories thus make explicit the mechanism behind the performance gains reported in Table 3: the additional adoption visible here is precisely what feeds the weighted-adoption term of the objective functional and drives the improvement achieved under optimal control. The resulting cumulative benefit is discussed after the control profiles are introduced.

Figure 4 shows the optimal advertising intensities $u_1^*(t)$ (Android) and $u_2^*(t)$ (iOS) as functions of time for the five scenarios. These are the controls obtained from the Pontryagin optimality conditions, and their shape reveals when and how strongly the planner should advertise each platform over the horizon. In every scenario the controls are highest at the start and decline towards the terminal time, eventually reaching zero at $t = T$: this is the expected consequence of the transversality conditions, since near the end of the horizon there is too little time left for additional advertising to pay off before the optimization horizon ends. The relative magnitude of u_1^* and u_2^* is governed by the interplay between the value weights (C_1, C_2) and the cost weights (B_1, B_2): a platform is advertised more

aggressively when its adopters are highly valued (large C_i) or when advertising it is cheap (small B_i). The quadratic cost structure ensures the controls respond smoothly rather than bang–bang, with each u_i^* scaling inversely with its cost weight B_i . These profiles are the inputs that generate the state trajectories of Figure 3 and, ultimately, the cumulative benefit shown in Figure 5.

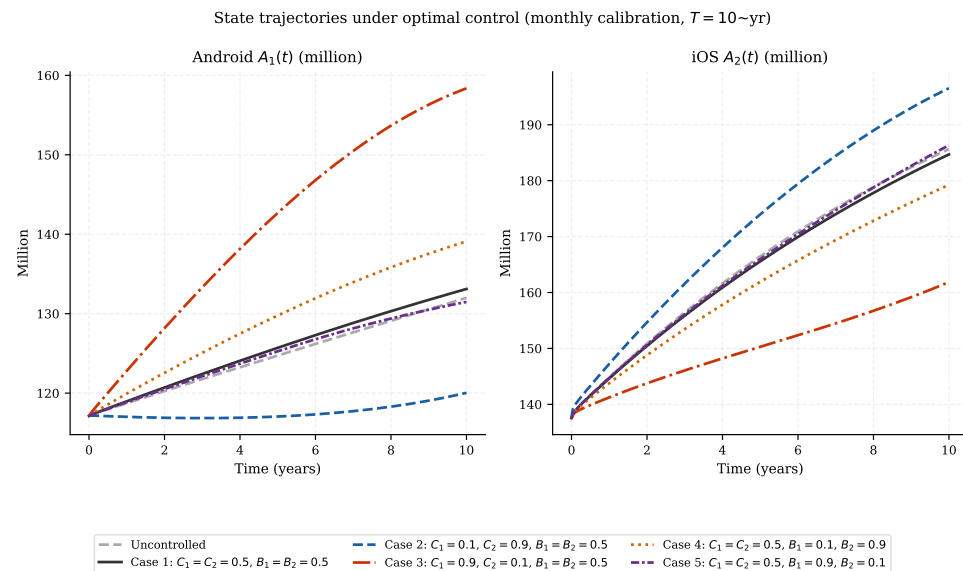


Figure 3. State trajectories $A_1(t)$ (left) and $A_2(t)$ (right) for all five scenarios vs. the uncontrolled baseline (grey dashed). Case 2 produces the largest iOS gain; Case 3 produces the largest Android gain. In Case 4 (B_1 small) the cheap Android campaign drives Android sales up while barely affecting iOS sales.

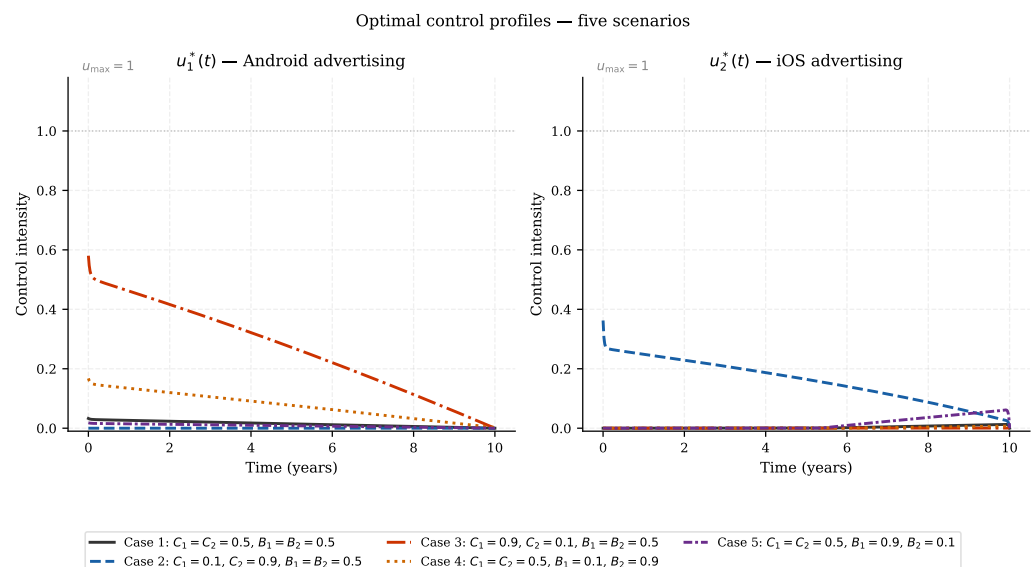


Figure 4. Optimal control profiles $u_1^*(t)$ (left) and $u_2^*(t)$ (right) for all five scenarios. Controls are front-loaded and decay towards zero as $t \rightarrow T$, consistent with the transversality conditions $\lambda_i(T) = 0$.

While the scalar value $J = J(T)$ provides a compact measure of performance, Figure 5 shows the cumulative objective functional over time for all scenarios. The controlled trajectories remain above the uncontrolled baseline throughout the simulation horizon, indicating that the additional marketing expenditure is more than compensated by the increase in weighted adoption generated by the optimal strategy. The separation between

the curves becomes progressively more pronounced, reflecting the cumulative effect of sustained advertising interventions.

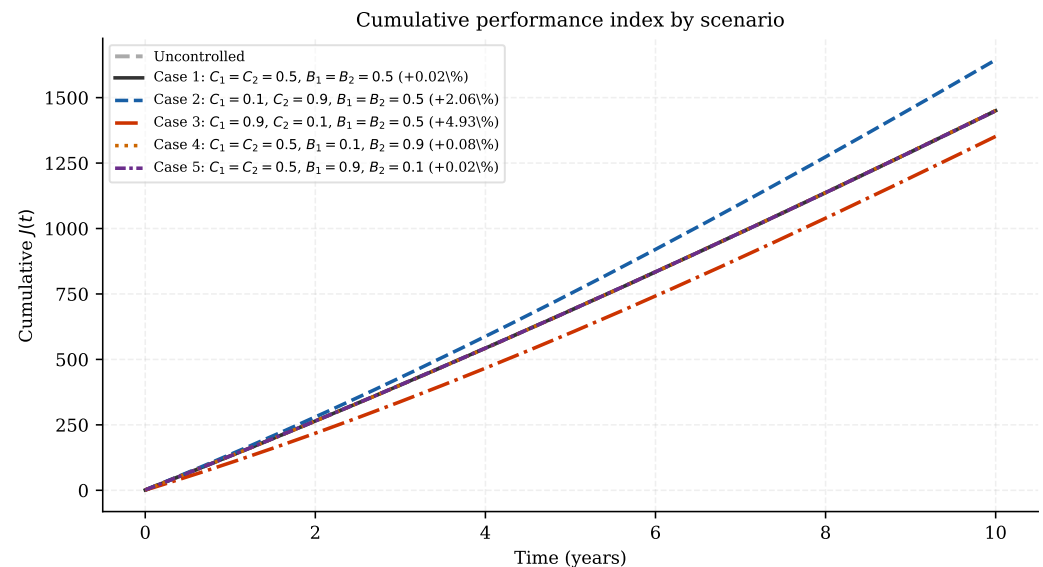


Figure 5. Cumulative performance index $J(t)$ for the five control scenarios versus the uncontrolled baseline, where $J(t)$ is the net economic benefit (weighted adoption minus advertising cost) accumulated up to time t . The vertical gap to the baseline is the net gain of the intervention.

Figure 6 summarises the relative gain in the performance index, $(J_{\text{ctrl}} - J_{\text{base}})/J_{\text{base}}$ expressed as a percentage, achieved by the optimal advertising policy in each of the five scenarios. Whereas the cumulative-index curves show how the payoff accumulates over time, this bar chart condenses each strategy into a single figure of merit, making the comparison across scenarios immediate. The height of each bar is the net economic benefit that the optimal policy delivers over the do-nothing baseline, so taller bars identify the configurations of value weights (C_1, C_2) and cost weights (B_1, B_2) for which active advertising is most worthwhile. The comparison provides qualitative insight into the relative influence of value weights and advertising costs on the resulting optimal strategy.

Table 3 and Figures 3–6 reveal that the effectiveness of the optimal intervention strategy depends strongly on the relative economic value assigned to each ecosystem. Although the calibrated market is already close to saturation, the different weighting structures still generate distinct adoption trajectories and control profiles.

The most noticeable result is obtained in Case 3, where a higher value is attributed to Android users ($C_1 = 0.9, C_2 = 0.1$). Under these conditions, the optimal strategy yields the largest increase in the objective functional, corresponding to a gain of approximately 4.93% relative to the uncontrolled scenario. The associated control profile allocates most of the intervention effort to $u_1(t)$, particularly during the initial stages of the planning horizon. This behaviour suggests that, within the calibrated market conditions, Android users represent the segment with the greatest remaining growth potential. Since the iOS ecosystem already occupies a dominant and mature position, additional investment directed towards Android adoption generates a larger marginal return in terms of total market expansion.

A different pattern emerges in Case 2, where the economic value of iOS users is prioritised. Although the corresponding control strategy increases iPhone adoption, the overall improvement remains more modest, reaching approximately 2.06%. This result is consistent with the parameter estimates obtained during calibration, particularly the high value associated with the iOS adoption coefficient. In practical terms, the diffusion process already

favours the premium ecosystem even in the absence of strong intervention, limiting the additional gains that can be achieved through further promotional effort.

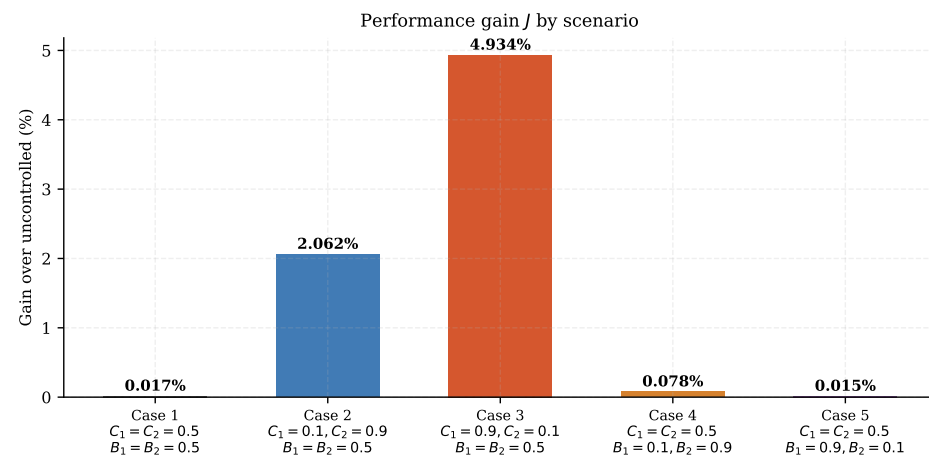


Figure 6. Percentage gain in J over the uncontrolled baseline for each scenario. Case 3 (Android-oriented value weights, $C_1 = 0.9$) yields by far the largest gain (4.93%), followed by Case 2 (2.06%). This reflects the fact that the Android ecosystem retains greater growth potential within the remaining non-user segment, whereas the iOS ecosystem is already close to saturation under the calibrated parameter set.

The scenarios involving asymmetric control costs (Cases 4 and 5) produce only marginal improvements relative to the baseline solution. Even when one type of intervention becomes substantially cheaper, the resulting gains remain small. This behaviour reflects the mature state of the market rather than the inefficiency of the controls themselves. Because the pool of non-adopters is already limited, reductions in intervention costs cannot generate substantial increases in adoption once the market approaches saturation.

Another feature common to all scenarios is the temporal distribution of the optimal controls. The control functions attain their highest values near the beginning of the planning horizon and progressively decrease over time. Economically, this indicates that promotional investments are most effective when implemented early, while a delayed intervention strategy yields diminishing returns as the number of potential adopters decreases. Such behaviour is consistent with the diffusion dynamics of mature technology markets, where opportunities for expansion become increasingly scarce as market penetration grows.

Taken together, these results suggest that the primary determinant of successful intervention is not the absolute intensity of advertising expenditure, but rather the identification of the market segment where additional adoption can still be generated. Contrary to the intuitive expectation that the premium ecosystem should always receive the greatest promotional emphasis, the calibrated model indicates that under current market conditions, the Android ecosystem offers greater scope for marginal expansion. This conclusion arises directly from the estimated diffusion dynamics and the near-saturated state of the United States smartphone market.

6. Conclusions

In this paper, we developed and analysed a nonlinear innovation diffusion model describing competition between smartphone ecosystems under the influence of social interactions, consumer dissatisfaction, and marketing interventions. The proposed framework provides a unified mathematical approach for analysing the interaction between innovation diffusion, ecosystem competition, and optimal advertising strategies in mature smartphone markets.

The mathematical analysis established the positivity and boundedness of solutions and identified the equilibrium conditions governing the coexistence and persistence of competing ecosystems. In particular, the diffusion thresholds (R_{01}) and (R_{02}) were shown to play a central role in determining the long-term market dynamics. Model calibration using United States smartphone market data from 2016 to 2026 produced trajectories that reproduce the observed evolution of Android and iPhone adoption with satisfactory accuracy, as confirmed by the reported goodness-of-fit metrics.

Beyond the quality of the numerical fit, the results provide insight into the mechanisms driving market evolution. The calibrated model suggests that long-term smartphone diffusion is influenced primarily by social interaction and user retention processes, while direct migration between competing ecosystems plays a comparatively limited role. This finding is consistent with the small estimated value of the switching parameter and with the sensitivity analysis, which indicates that adoption and retention mechanisms exert a stronger influence on market outcomes than ecosystem switching effects.

The optimal control analysis further reveals that the effectiveness of marketing interventions depends not only on their intensity but also on the growth potential of the targeted ecosystem. Under the calibrated market conditions, the greatest improvement was obtained when a higher value was assigned to Android-oriented interventions, despite the stronger baseline position of the iOS ecosystem. This result reflects the mature and highly saturated nature of the United States smartphone market, where additional gains are more likely to arise from expanding adoption opportunities in segments that still retain growth potential than from reinforcing already dominant positions. More generally, the results suggest that, in mature technology markets, the marginal effectiveness of advertising depends critically on market structure and remaining adoption capacity.

Several limitations should nevertheless be acknowledged. The analysis relies on aggregate market data and includes latent compartments that are not directly observable, implying that some calibrated parameters should be interpreted as effective diffusion coefficients rather than direct behavioural measurements. Moreover, the model was calibrated using data from a single national market and therefore its conclusions should not be automatically generalised to other geographical contexts or technology sectors. The simplifying assumption of predominantly unidirectional migration between ecosystems also represents an abstraction of more complex consumer behaviour. This assumption may underestimate occasional reverse switching between competing ecosystems, particularly over short time horizons. Nevertheless, because the model is intended to capture the dominant long-term diffusion trends rather than individual switching events, this simplification is not expected to alter the main qualitative conclusions regarding market evolution and optimal marketing interventions.

Despite these limitations, the proposed framework demonstrates how nonlinear diffusion modelling and optimal control theory can be integrated to analyse strategic decision-making in technology-intensive industries. By combining ecosystem competition, consumer dissatisfaction, social influence mechanisms and optimal marketing interventions within a single calibrated framework, the model provides a useful analytical tool for studying innovation diffusion and market dynamics.

Future research may extend the present work by incorporating stochastic effects, heterogeneous consumer segments, multiple competing brands, fractional-order dynamics, or game-theoretical interactions between firms. Particular attention will also be devoted to practical parameter identifiability and uncertainty quantification, especially as additional market data become available. Such developments may provide a more rigorous assessment of parameter uncertainty while further improving the predictive capability and reliability of the proposed modelling framework.

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