

Article

Geometry of Events in Deformed Cellular Spacetimes

Shlomo Barak ¹  and George Salman ^{2,*} ¹ Taga Innovations, 18 Yosef Karo St., Tel Aviv 6701422, Israel; shlomo@taga.com² School of Computer Science and Engineering, The Hebrew University of Jerusalem, Edmond J. Safra Campus, Jerusalem 9190401, Israel

* Correspondence: george.salman@mail.huji.ac.il

Abstract

We develop the geometry of events in a deformable cellular spacetime, extending our earlier cellular-spaces framework from cellular complexes to cellular events complexes. The framework operates within the conformal class of Minkowski space; in four dimensions, this is the vanishing-Weyl-tensor sector, which excludes Schwarzschild, Kerr, and gravitational-wave spacetimes. The framework treats integer counts of cell crossings as the primitive geometric data: spatial separation between events is the shortest count of face-adjacent cells; temporal separation is the cell-crossing count of a reference light pulse. Newton's universal clock is replaced by an operational one: the temporal count distance is the ratio of cell length to the speed of light through a cell, and because both quantities are invariants of the co-deformation, the temporal count is itself an invariant: temporal separation is operationally measured via light-pulse counts rather than posited as an external coordinate. Under the co-deformation principle, a single positive scalar field ρ (cell density) controls both the rod length and the clock period. We prove six results, all expressed in terms of counts on the cellular events complex, with a smooth conformally flat metric $\tilde{g} = e^{2\varphi}\eta$ ($\varphi = -\frac{1}{3}\ln\rho$) appearing only as the comparison/calibration object for convergence statements. First, the scalar curvature of the smooth comparison metric is the closed-form differential operator $\tilde{R} = 2\Box\rho/\rho^{1/3} - (8/3)(\partial\rho)^2/\rho^{4/3}$. Second, the volume of a small Alexandrov interval admits an explicit asymptotic expansion in the interval height T , with leading correction $Q(m, u) T^2$ involving an anisotropic invariant at the midpoint m . Third, Q is irreducible to scalar and Ricci-directional invariants alone; the explicit decomposition $Q = \frac{1}{45}\tilde{R} + \frac{1}{5}\tilde{R}_{uu} + \frac{1}{2}\mathcal{J}$ exhibits a third independent invariant $\mathcal{J}(m, u) = (u \cdot \partial)^2 \varphi(m)$ as new structural content of the Lorentzian diagnostic. Fourth, the discrete-to-continuum convergence of counts on the cellular events complex yields a counts-only curvature estimator with rate $O(\sqrt{a})$ at the joint scaling $T \asymp \sqrt{a}$. Fifth, the smooth comparison metric itself is reconstructible from counts on the discrete complex at rate $O(a)$: the conformally flat Lorentzian geometry is uniquely determined, up to background Minkowski calibration, by the cellular events complex. Sixth, a finite collection of Alexandrov-interval volume measurements at a fixed midpoint suffices to recover the full local curvature data $\{\tilde{R}(m), \tilde{R}_{\mu\nu}(m), \mathcal{J}(m, u)\}$ at rate $O(\sqrt{a})$ (*curvature spectroscopy*); and the temporal light-tick count λ is essential in a precise sense—there exist conformally flat Lorentzian geometries indistinguishable on every spatial slice by the earlier spatial-only diagnostic but distinguished at the origin by the events-space directional invariant. The framework's scope is the conformal class of Minkowski: flat FLRW in conformal time, leading-order weak-field gravity, and 2D gravity. This paper is a mathematical contribution to discrete-to-continuum geometry on cellular events complexes; it is not a physical theory of gravity.



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1. Introduction

This paper develops the geometry of events in a deformable cellular spacetime, extending the framework of *Geometry of Deformed Cellular Spaces* [1] from cellular complexes to cellular events complexes. The present paper, like its companion [1], restricts attention to the rigorous mathematical content and treats the smooth comparison metric only as the validation tool for convergence theorems. We do not commit to any specific physical interpretation of the cellular substrate: the framework is mathematically self-contained and the term “physical” is used throughout in the operational sense of “observable counts on the cellular complex,” not in the sense of a complete physical theory of gravity. In particular, no commitment is made on whether the cellular substrate is background-independent (in the sense of General Relativity) or background-dependent (in the sense of Quantum Field Theory); the framework is developed as a discrete-to-continuum mathematical structure whose physical applicability is left to subsequent work. The companion paper [1] introduced a counts-only protocol on locally finite cell complexes. Distances are defined as shortest cell-crossing counts. Curvature is read off from the mismatch between a measured count radius and a radius reconstructed from boundary, area, or volume counts using the same yardstick. The protocol’s continuum-limit identification with smooth scalar curvature was established by comparison with a conformally flat smooth metric $g = e^{2u}g_0$ on a Riemannian base g_0 . The paper was explicit that the smooth comparison metric serves only as the validation tool for convergence theorems; the geometric framework itself—the cellular complex, the count metric, and the count-based curvature diagnostic—is intrinsic and does not depend on any prior smooth structure.

The present paper develops the events-space analog. The cellular complex is now equipped with both a spatial face-adjacency relation $\sim$ (as in [1]) and a temporal light-tick count λ that records, for any two cells, the number of cell crossings of a reference light pulse between them. The intrinsic count quantities are the spatial count distance δ , the temporal count distance τ , and the discrete Alexandrov interval $A_h(P, Q) \subset \mathcal{C}$ between two timelike-separated cells. The count-based curvature diagnostic is now built from cell counts inside $A_h(P, Q)$ rather than inside a count ball, because the natural finite-volume small region in an events space is the causal diamond, not the ball.

Under the co-deformation postulate (illustrated in Figure 1), both the rod length ℓ and the clock tick t_0 are functions of a single positive scalar field ρ (cell density), and the smooth comparison metric is conformally flat: $\tilde{g} = e^{2\varphi}\eta$ with $\varphi = -\frac{1}{3}\ln\rho$. As in [1], the smooth metric appears here only as the calibration object against which the cellular diagnostic is shown to converge—it is not the primary geometric structure. Temporal separation is operationally a count of spatial cell crossings traced by light, not an external coordinate; the present paper makes this stance mathematically operational on a cellular complex. The framework’s relationship to existing discrete approaches to Lorentzian geometry—causal set theory [2,3], causal dynamical triangulations and Regge calculus [4,5], spin foams [6], and the Lorentzian small-region expansions of Myrheim, Benincasa–Dowker, and Roy–Sinha–Surya [7–9]—is discussed in Section 8.

Co-deformation: $\ell = \rho^{-1/3}$ and $t_0 = \ell \Rightarrow$ local coordinate light speed $c_{\text{loc}} = 1$

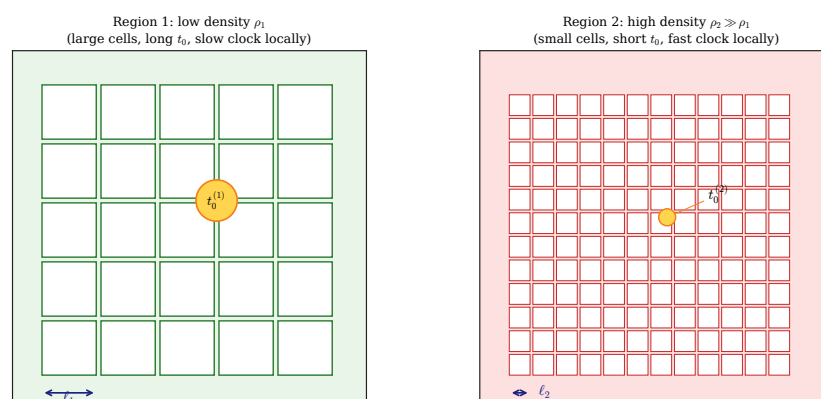


Figure 1. The co-deformation principle. In a region of low density ρ_1 (**left**), the physical length $\ell = \rho^{-1/3}$ is large and the tick duration $t_0 = \ell$ is correspondingly long. In a region of high density $\rho_2 > \rho_1$ (**right**), both ℓ and t_0 are small. The ratio $\ell/t_0 \equiv 1$ across both regions, so the local physical speed of light is universal: rods and clocks deform in lockstep because both are built from the same cellular medium.

The paper is self-contained; all original proofs are given in full (the classical conformal-curvature identity of Proposition 1 is recalled from [10,11] rather than derived). Three results structure the development:

1. The scalar curvature of the smooth comparison metric is the closed-form differential operator $\tilde{R} = 2\Box\rho/\rho^{1/3} - (8/3)(\partial\rho)^2/\rho^{4/3}$ in ρ alone (Proposition 1). This is a classical conformal-curvature identity [10,11] and is provided for completeness.
2. The volume of a small Alexandrov interval in the smooth comparison metric admits an explicit asymptotic expansion in the height T , with leading correction $Q(m, u) T^2$ involving an anisotropic invariant at the midpoint (Theorem 1). We prove (Theorem 2) that Q is irreducible to the scalar curvature $\tilde{R}$ and the timelike Ricci component $\tilde{R}_{\mu\nu}u^\mu u^\nu$ alone, so the cellular events-space diagnostic carries strictly more geometric information than the spatial diagnostic of ref. [1]. This is the central new result.
3. The cell count $N_h(P, Q)$ of the discrete Alexandrov interval $A_h(P, Q)$ converges to the continuum volume at rate $O(aT^3)$ under standard regularity hypotheses (bounded degree, shape regularity, causal compatibility, and density consistency), yielding a counts-only curvature estimator $\hat{Q}_a$ with rate $O(\sqrt{a})$ at the joint scaling $T \asymp \sqrt{a}$ (Theorem 3, Corollary 1). This is the events-space analog of the sandwich convergence theorem of [1].

The conformal scalar-curvature identity of Proposition 1 is the classical Einstein–Fokker form [10] of Nordström’s 1913 [12] scalar gravity, recalled here as the kinematic backbone of the framework. Lorentzian small-region volume expansions in the causal-set tradition go back to Myrheim [7] and have been developed by Benincasa–Dowker [8] and Roy–Sinha–Surya [9]; the present paper’s contribution to that lineage is the conformally flat specialization with the explicit φ -derivative coefficients of Equation (14) together with the irreducibility statement of Theorem 2, which shows that the Lorentzian diagnostic carries directional information beyond the curvature scalar and its timelike Ricci contraction. The cellular events complex itself is the operational object on which the diagnostics live: a locally finite complex equipped with spatial face-adjacency $\sim$ and a temporal light-tick count λ as separate combinatorial primitives. The discrete Alexandrov interval $A_h(P, Q)$ is the natural finite-volume small region in this setting, replacing the count ball $B_{d_h}(p, r)$ of ref. [1]. (Lorentzian geodesic balls have infinite volume, so no direct analog of the spatial ball construction is available.)

Section 2 sets up the cellular events complex and the co-deformation postulate. Section 3 states and proves the closed-form scalar curvature identity for the smooth comparison metric. Section 4 introduces the Alexandrov interval, proves the small-region volume expansion (Theorem 1), and establishes its anisotropy (Theorem 2). Section 5 defines the discrete Alexandrov interval. Section 6 establishes the discrete-to-continuum interface hypotheses (E1)–(E6), parallel to the (H1)–(H5) interface of ref. [1]. Section 7 proves the counts-to-curvature convergence theorem with explicit $O(\sqrt{a})$ rate. Section 8 discusses the framework’s scope, its relation to Nordström–Einstein–Fokker scalar gravity, positioning relative to other discrete-gravity programs, and open problems. Appendix A constructs an explicit family of cellular embeddings satisfying (E1)–(E6).

2. The Cellular Foundation and Co-Deformation

2.1. Counts as the Primitive

Let $\mathcal{C}$ be a countable locally finite set of *cells*, equipped with a symmetric reflexive face-adjacency relation $\sim$ (each cell has finitely many neighbors) (see Figure 2 for a schematic). No metric, coordinates, embedding, or angles are presupposed.

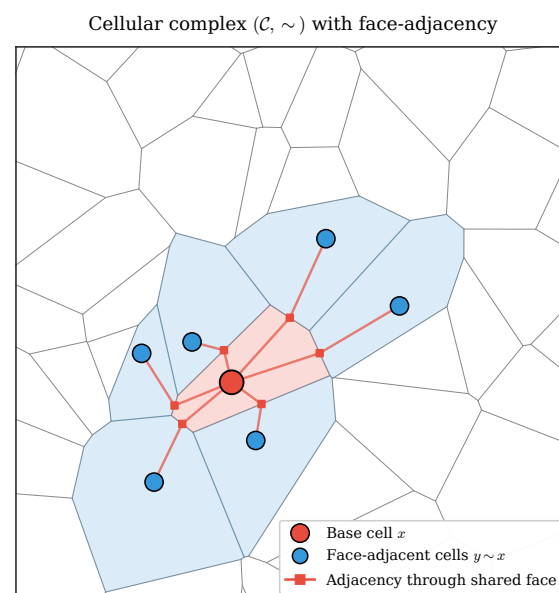


Figure 2. A locally finite cellular complex $(\mathcal{C}, \sim)$. Red: a base cell x . Blue: its $\sim$ -neighbors. The adjacency $\sim$ is the only primitive datum.

Definition 1 (Spatial count distance). For $A, B \in \mathcal{C}$,

$$\delta(A, B) := \min\{k \in \mathbb{N} : \exists c_0 = A \sim c_1 \sim \cdots \sim c_k = B\},$$

the unweighted graph distance on $(\mathcal{C}, \sim)$.

Remark 1 (On the metric structure of δ and the role of $\sim$). The face-adjacency relation $\sim$ is, by hypothesis, symmetric and reflexive: each cell is adjacent to itself, and $c \sim c'$ iff $c' \sim c$. It is not assumed transitive (and indeed should not be: $\sim$ records nearest-neighbor adjacency only); it is therefore not an equivalence relation, and we never use it as one. The graph distance δ defined above is, however, a genuine metric function on the cellular set $\mathcal{C}$: it satisfies non-negativity ($\delta \geq 0$), identity of indiscernibles ($\delta(A, B) = 0 \iff A = B$, using reflexivity of $\sim$), symmetry ($\delta(A, B) = \delta(B, A)$, using symmetry of $\sim$), and the triangle inequality ($\delta(A, C) \leq \delta(A, B) + \delta(B, C)$, by concatenating shortest paths). Thus δ is the unweighted graph metric on the adjacency graph of $\mathcal{C}$, in the standard sense of combinatorial graph theory (Ch. 1 in [13]). The cellular complexes

considered in this paper are connected; equivalently, δ is defined on the connected component of the adjacency graph containing the cells in question. The infinitesimal symbols $d\delta$ and $d\tau$ introduced in Definition 3 below refer to integer increments of δ and τ along the discrete primitive: the smallest positive change in δ between adjacent cells, which equals 1. They are not infinitesimal elements in the sense of non-standard analysis (Robinson 1966 [14]); the underlying arithmetic is standard integer arithmetic on $\mathbb{Z}$. See Remark 4. The algebraic expressions $d\delta^2$, $d\tau^2$, and the difference $d\delta^2 - d\tau^2$ appearing in Definition 3 are evaluated in the standard commutative ring structure of $\mathbb{Z}$ (squaring and subtraction as the usual integer operations); in the rescaled continuum limit of Section 7, the corresponding real-valued differentials $a d\delta$ and $a d\tau$ operate under the standard field structure of $\mathbb{R}$. No additional algebraic structure is required.

2.2. Temporal Count Distance via Reference Light

Definition 2 (Temporal count distance). For events A, B at cells $a, b \in \mathcal{C}$, the temporal count distance $\tau(A, B)$ is the integer count of cell crossings of a reference light pulse from a at the cell-time of A to b at the cell-time of B .

As an operational convention, we take light to propagate at one cell per tick, so the temporal count $\tau(A, B)$ is by definition the integer number of cell crossings of a reference light pulse from A to B (see Figure 3). Under the co-deformation postulate of Section 2

Remark 2 (Physical assumption on cell sizes). The mathematical results of this paper apply under the structural axioms (E1)–(E6) of Definition 7, which permit bounded variation in cell size and shape. In intended realizations of the framework one may take the cells to be of similar size, possibly approximately uniform. This is a structural hypothesis that does not enter the convergence theorems proved here. The theorems require only the bounded-degree shape regularity (E1)–(E2) of Definition 7.

Remark 3 (On the operational definition of τ and path-dependence). Definition 2 specifies the temporal count $\tau(A, B)$ as the integer count of cell crossings of a reference light pulse from A to B . This definition is operational: it refers to a specific light-pulse trajectory in the cellular events complex from A to B . In a generic conformally flat spacetime, the temporal count between two events depends on the choice of light-pulse trajectory (the choice of reference pulser or origin); the count τ defined here is therefore a path-relative quantity, fixed once a reference protocol is specified, rather than a fully coordinate-independent observable.

To make the gauge content explicit: the framework does carry an explicit preferred structure, namely the choice of reference protocol (a designated light-pulse source and trajectory family). A change of reference protocol—say from a pulser at origin O_1 to a pulser at O_2 —alters the numerical value of $\tau(A, B)$, but does so in a controlled way: under the convergence theorems of Section 7, the rescaled count $a\tau(A, B)$ converges to the smooth η -temporal separation $T_\eta(x_A, x_B)$ along any reference protocol, by (E4). The geometric content carried by τ at the continuum level is therefore the η -temporal separation, which is gauge-invariant; the path-dependence of the discrete count is an $O(a)$ effect that vanishes in the continuum limit. This is the operational counterpart of the relativity of simultaneity in standard Lorentzian geometry: in any single Lorentz frame, T_η is well-defined and frame-dependent; the choice of reference protocol corresponds to a choice of Lorentz frame in which the pulse is at rest. The framework is thus not coordinate-independent in the strong sense (it carries a preferred-frame structure through the choice of reference protocol), but its continuum content (the smooth η -temporal separation, and through it the conformal factor φ) is recoverable up to the standard Lorentz gauge of choosing a reference frame. The choice of reference protocol enters the framework only through (E4), the causal compatibility hypothesis.

Definition 3 (Intrinsic interval). For infinitesimally separated events with count differentials $(d\delta, d\tau)$,

$$ds^2 := d\delta^2 - d\tau^2. \quad (1)$$

Remark 4 (On count differentials). The symbols $d\delta$ and $d\tau$ in (1) are not infinitesimals in the sense of nonstandard analysis or Lebesgue differentials with respect to a base measure: they denote integer count increments between successive cell crossings on the cellular events complex, which the framework treats as the primitive discrete data. Under a cellular embedding $(\{V_c\}, \{x_c\})$ at scale $a > 0$ satisfying (E1)–(E6) of Definition 7, the integer counts δ, τ are related to the smooth η -Lebesgue measure on Ω via $a\delta(c, c')$ and $a\tau(c, c')$ as $a \rightarrow 0$, and integration of count-based quantities passes to the standard η -Lebesgue integral d^4x on $(\Omega, \mathcal{B}(\Omega), |\cdot|_\eta)$ under the convergence theorems of Section 7. The discrete-to-continuum identification of measures is therefore made precise through (E1) and (E5).

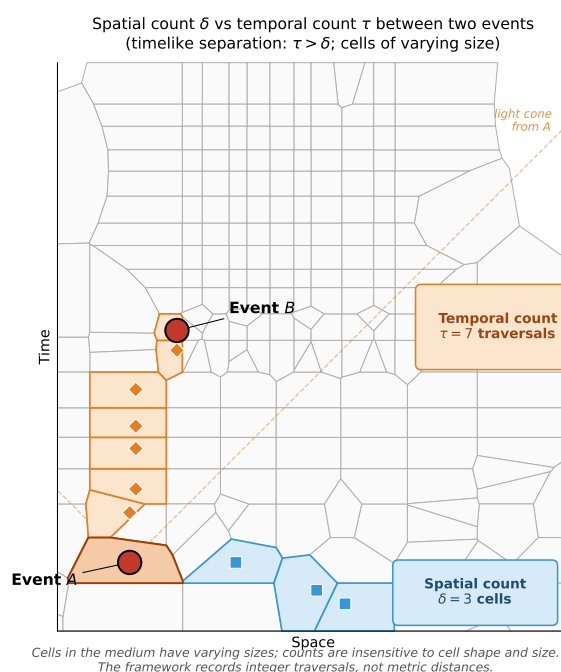


Figure 3. Spatial and temporal counts between two events A and B with timelike separation ($\tau > \delta$): $\delta(A, B)$ is the unweighted graph distance between the cells of A and B on $(\mathcal{C}, \sim)$; and $\tau(A, B)$ is the cell-crossing count of a reference light pulse originated at event A . Cells in the medium may have varying sizes and shapes (here Voronoi cells); the counts are insensitive to cell shape and size—only the integer number of cell-to-cell transitions on $(\mathcal{C}, \sim)$ enters. Both counts are integer-valued, computed with the same instrument, and combine via $ds^2 = d\delta^2 - d\tau^2$.

2.3. The Co-Deformation Postulate

Remark 5 (On the use of “Postulate”). Throughout this paper we use “Postulate” to mean a strong modeling assumption imposed by the framework, in the sense of an operational convention adopted to make the framework precise—not in the formal logical sense of an axiom of an axiomatic system in the model-theoretic sense. The framework is not presented as a closed axiomatic system; the “Postulate” below is a structural commitment whose mathematical consequences are then developed rigorously. Replacing this commitment with a different one—e.g., direction-dependent yardsticks $\ell_i(x)$ —would yield a different framework outside the conformal-class scope of the present one.

Postulate 1 (Co-deformation; strong modeling assumption). *There exists a smooth positive scalar field $\rho: \Omega \rightarrow \mathbb{R}_{>0}$ on the region of interest $\Omega \subset \mathbb{R}^{1,3}$ such that the physical length of one cell crossing is*

$$\ell(x) = \rho(x)^{-1/3}, \quad (2)$$

and the physical duration of one cell tick is $t_0(x) = \ell(x)$. By Lemma 1 below, this assumption commits the framework to physical metrics in the Minkowski conformal class. It is therefore a strict restriction on the kinds of geometries the framework can describe.

Definition 4 (Co-deformation). *A co-deformation on a cellular events complex $(\mathcal{C}, \sim, \lambda)$ embedded in $\Omega \subset \mathbb{R}^{1,3}$ is the joint specification of:*

- (i) *a smooth positive scalar density field $\rho: \Omega \rightarrow \mathbb{R}_{>0}$,*
- (ii) *a cell-length scaling $\ell(x) = \rho(x)^{-1/3}$, and*
- (iii) *a cell-tick duration $t_0(x) = \ell(x)$,*

such that the local ratio $\ell(x)/t_0(x) \equiv 1$ holds throughout Ω , ensuring the local physical speed of light through one cell is universal and density-independent.

Converting counts to physical units multiplies each component of $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ by $\ell(x)^2$, giving the physical metric

$$\tilde{g}_{\mu\nu}(x) = \ell(x)^2 \eta_{\mu\nu} = e^{2\varphi(x)} \eta_{\mu\nu}, \quad \varphi(x) := -\frac{1}{3} \ln \rho(x). \quad (3)$$

Lemma 1 (Density–conformal correspondence). *The map $\rho \mapsto \rho^{-2/3} \eta$ is a bijection between positive C^2 scalar fields on Ω and the Minkowski conformal class $\{e^{2\varphi} \eta : \varphi \in C^2(\Omega)\}$.*

Proof. The map $\rho \mapsto \rho^{-2/3} \eta$ factors as $\rho \mapsto \varphi := -\frac{1}{3} \ln \rho \mapsto e^{2\varphi} \eta$. Both factor maps are bijections: $\rho \mapsto \varphi$ is C^2 -bijective between positive C^2 scalars and C^2 scalars (logarithm), and $\varphi \mapsto e^{2\varphi} \eta$ is C^2 -bijective onto the Minkowski conformal class by definition. The composition is therefore a bijection. $\square$

Lemma 1 is the structural reason for the framework’s scope: although the framework begins from a discrete primitive, the resulting space of physical geometries is precisely the conformal class of Minkowski.

3. The Closed-Form Scalar Curvature

The conformal structure (3) forces the scalar curvature into a particularly simple form.

Proposition 1 (Closed-form scalar curvature; Einstein–Fokker [10]). *For $\tilde{g} = e^{2\varphi} \eta$ on $\mathbb{R}^{1,3}$ with $\varphi \in C^2$, the scalar curvature is*

$$\tilde{R}(x) = -6 e^{-2\varphi(x)} [\square \varphi(x) + (\partial \varphi(x))^2]. \quad (4)$$

Equivalently, in terms of $\rho = e^{-3\varphi}$,

$$\tilde{R}(x) = 2 \frac{\square \rho(x)}{\rho(x)^{1/3}} - \frac{8}{3} \frac{(\partial \rho(x))^2}{\rho(x)^{4/3}}, \quad (5)$$

where $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$ and $(\partial \rho)^2 = \eta^{\mu\nu} \partial_\mu \rho \partial_\nu \rho$.

Proof. The conformal transformation of the scalar curvature in n dimensions, with flat base metric so that the base scalar curvature vanishes, is the classical identity (Wald [11], App. D)

$$\tilde{R} = -(n-1)e^{-2\varphi} [2\square \varphi + (n-2)(\partial \varphi)^2]. \quad (6)$$

For $n = 4$ this collapses to (4). Substituting $\varphi = -\frac{1}{3} \ln \rho$,

$$\begin{aligned}\partial_\mu \varphi &= -\frac{1}{3} \frac{\partial_\mu \rho}{\rho}, & (\partial \varphi)^2 &= \frac{1}{9} \frac{(\partial \rho)^2}{\rho^2}, \\ \square \varphi &= -\frac{1}{3} \eta^{\mu\nu} \partial_\mu \left(\frac{\partial_\nu \rho}{\rho} \right) = -\frac{1}{3} \left(\frac{\square \rho}{\rho} - \frac{(\partial \rho)^2}{\rho^2} \right).\end{aligned}$$

Adding,

$$\square \varphi + (\partial \varphi)^2 = -\frac{1}{3} \frac{\square \rho}{\rho} + \frac{1}{3} \frac{(\partial \rho)^2}{\rho^2} + \frac{1}{9} \frac{(\partial \rho)^2}{\rho^2} = -\frac{1}{3} \frac{\square \rho}{\rho} + \frac{4}{9} \frac{(\partial \rho)^2}{\rho^2}.$$

Since $e^{-2\varphi} = \rho^{2/3}$, multiplying by $-6\rho^{2/3}$ yields (5). $\square$

We will also need the conformally flat Ricci tensor.

Proposition 2 (Ricci tensor of $\tilde{g}$; $n = 4$). For $\tilde{g} = e^{2\varphi} \eta$ on $\mathbb{R}^{1,3}$,

$$\tilde{R}_{\mu\nu} = -2(\partial_\mu \partial_\nu \varphi - \partial_\mu \varphi \partial_\nu \varphi) - \eta_{\mu\nu} [\square \varphi + 2(\partial \varphi)^2]. \quad (7)$$

For a future-directed unit timelike u^μ (with $\eta_{\mu\nu} u^\mu u^\nu = -1$),

$$\tilde{R}_{\mu\nu} u^\mu u^\nu = -2[(u \cdot \partial)^2 \varphi - (u \cdot \partial \varphi)^2] + \square \varphi + 2(\partial \varphi)^2, \quad (8)$$

where $(u \cdot \partial)^2 \varphi := u^\mu u^\nu \partial_\mu \partial_\nu \varphi$ and $(u \cdot \partial \varphi)^2 := (u^\mu \partial_\mu \varphi)^2$.

Proof. Apply the standard conformal transformation of the Ricci tensor (Wald [11], App. D) with flat base and $n = 4$ to obtain (7). Contracting with $u^\mu u^\nu$ and using $\eta_{\mu\nu} u^\mu u^\nu = -1$ yields (8). $\square$

Proposition 1 is classical (the Einstein–Fokker reformulation of Nordström gravity dates to 1914). We include it because it is the kinematic identity around which the events-space framework is organized: the entire curvature content of the conformally flat sector is captured by one scalar field and its first two derivatives.

4. The Lorentzian Alexandrov-Interval Diagnostic

The companion paper [1] extracts curvature on a cellular space from cell counts inside a count ball $B_{d_h}(p, r)$ (see Figure 4 for the density-curvature relationship in the smooth limit), with continuum-limit identification against the smooth volume $\text{Vol}_g(B_g(p, r))$ of a Riemannian conformal metric. The events-space setting differs structurally: in a cellular events complex the natural finite-volume small region is not the count ball but the discrete *Alexandrov interval* $A_h(P, Q)$, the cells lying simultaneously in the future of P and the past of Q under the temporal count λ . Equivalently, in the smooth comparison metric, the natural finite-volume small region is the continuum Alexandrov interval $A(p, q)$, the intersection of the causal future of p with the causal past of q . The reason is that smooth Lorentzian geodesic balls have infinite volume (the timelike direction extends without bound along the light cone), so the count-ball/smooth-ball comparison of [1] has no direct analog in events space. Causal diamonds, which are conformally invariant as sets and have finite volume, do.

Definition 5 (Alexandrov interval). For events $p, q \in \mathbb{R}^{1,3}$ with $q - p$ future-directed timelike, the Alexandrov interval is

$$A(p, q) := \{x \in \mathbb{R}^{1,3} : x - p \text{ and } q - x \text{ are future-directed causal}\}. \quad (9)$$

Its height is $T := \sqrt{-\eta_{\mu\nu}(q-p)^\mu(q-p)^\nu}$; its midpoint is $m := \frac{1}{2}(p+q)$; its axis vector is $u^\mu := (q-p)^\mu/T$ (unit timelike in η).

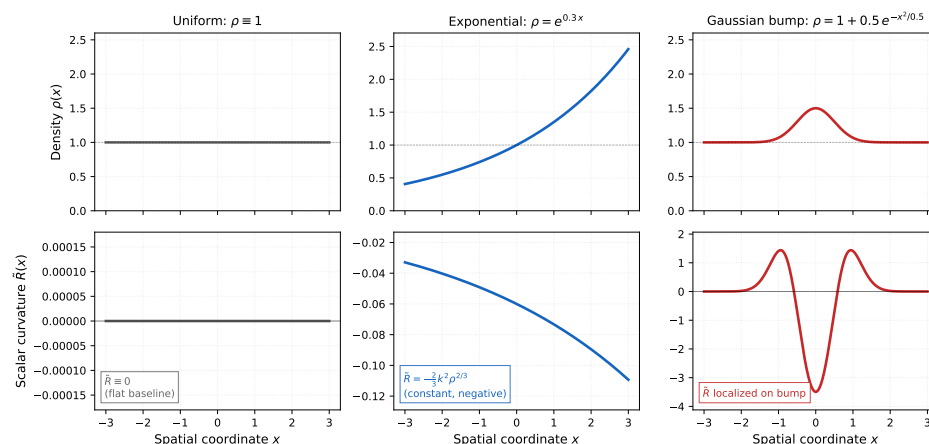


Figure 4. Illustration of Proposition 1 on three sample density profiles. (**Top row**): density $\rho(x)$. (**Bottom row**): corresponding scalar curvature $\tilde{R}(x)$ computed via (5). The uniform profile (**left**) yields $\tilde{R} \equiv 0$, the flat baseline. The exponential profile $\rho = e^{kx}$ (**middle**) yields a constant negative curvature $\tilde{R} = -\frac{2}{3}k^2\rho^{2/3}$, dominated by the gradient term. A localized Gaussian bump (**right**) produces curvature concentrated on the bump's support, with characteristic sign change between the bump's center and its flanks.

A crucial observation: null cones are conformally invariant, so $A(p, q)$ defined for $\tilde{g} = e^{2\varphi}\eta$ coincides as a set with $A(p, q)$ defined for η Figure 5. The conformal factor enters *only* through the volume form. This is what makes the next theorem tractable.

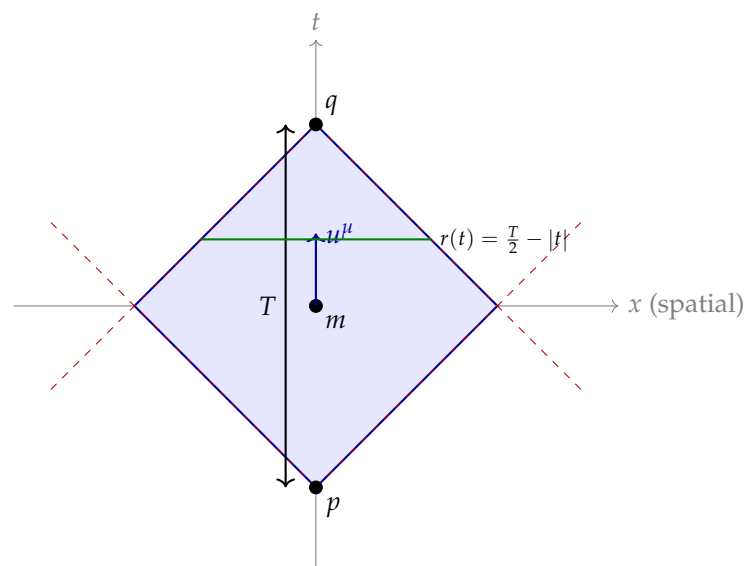


Figure 5. The Alexandrov interval $A(p, q)$ in 1+1 dimensions (suppressing x^2, x^3). The diamond region (blue) is the intersection of the causal future of p with the causal past of q . In the rest frame of u^μ , a constant- t slice (green) is a Euclidean 3-ball of radius $r(t) = T/2 - |t|$. Conformal transformations preserve the dashed null cones, so the integration domain is the same for η and $\tilde{g} = e^{2\varphi}\eta$.

We first establish the flat moments of the Alexandrov interval that drive the expansion.

Lemma 2 (Flat moments of the Alexandrov interval). *In coordinates with $m = 0$ and $u^\mu = \delta_0^\mu$, the Alexandrov interval $A(p, q)$ satisfies*

$$|A(p, q)|_\eta = \frac{\pi T^4}{24}, \quad (10)$$

$$\int_{A(p, q)} t^2 d^4x = \frac{\pi T^6}{1440}, \quad (11)$$

$$\int_{A(p, q)} (x^i)^2 d^4x = \frac{\pi T^6}{720} \quad (i = 1, 2, 3; \text{ no sum}). \quad (12)$$

All integrals $\int_A x^\mu d^4x$ and $\int_A x^\mu x^\nu d^4x$ with $\mu \neq \nu$ vanish, as do all integrals of monomials of total odd degree.

Proof. In these coordinates,

$$A(p, q) = \{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3 : |t| \leq T/2, |\mathbf{x}| \leq T/2 - |t|\},$$

i.e., a double cone whose constant- t slice is a Euclidean 3-ball of radius $r(t) = T/2 - |t|$. For (10),

$$|A|_\eta = \int_{-T/2}^{T/2} \frac{4\pi}{3} r(t)^3 dt = \frac{8\pi}{3} \int_0^{T/2} (T/2 - t)^3 dt = \frac{8\pi}{3} \cdot \frac{(T/2)^4}{4} = \frac{\pi T^4}{24}.$$

For (11), substitute $u = T/2 - t$:

$$\begin{aligned} \int_A t^2 d^4x &= 2 \int_0^{T/2} t^2 \cdot \frac{4\pi}{3} (T/2 - t)^3 dt = \frac{8\pi}{3} \int_0^{T/2} \left(\frac{T^2}{4} u^3 - Tu^4 + u^5\right) du \\ &= \frac{8\pi}{3} \left[\frac{T^6}{256} - \frac{T^6}{160} + \frac{T^6}{384} \right] = \frac{8\pi}{3} \cdot T^6 \cdot \frac{30 - 48 + 20}{7680} = \frac{\pi T^6}{1440}. \end{aligned}$$

For (12), by spatial spherical symmetry of the 3-ball, $\int_{B(R)} (x^i)^2 d^3x = \frac{4\pi R^5}{15}$, so

$$\int_A (x^i)^2 d^4x = 2 \int_0^{T/2} \frac{4\pi}{15} (T/2 - t)^5 dt = \frac{8\pi}{15} \cdot \frac{(T/2)^6}{6} = \frac{\pi T^6}{720}.$$

For the vanishings: A is symmetric under $t \mapsto -t$ and under each $x^i \mapsto -x^i$. Any monomial of odd total degree in at least one of these coordinates integrates to zero by parity. $\square$

Theorem 1 (Lorentzian small-Alexandrov volume expansion). *Let $\tilde{g} = e^{2\varphi}\eta$ on $\Omega \subset \mathbb{R}^{1,3}$ with $\varphi \in C^4(\Omega)$. Let $p, q \in \Omega$ with $q - p$ future-directed timelike, height T , midpoint m , axis vector u , and assume $A(p, q) \subset \Omega$. Then*

$$|A(p, q)|_{\tilde{g}} = \frac{\pi}{24} e^{4\varphi(m)} T^4 \left[1 + Q(m, u) T^2 + R_4(m, u, T) \right], \quad (13)$$

where

$$Q(m, u) = \frac{1}{10} (u \cdot \partial)^2 \varphi(m) + \frac{2}{5} (u \cdot \partial \varphi(m))^2 + \frac{1}{15} \square \varphi(m) + \frac{4}{15} (\partial \varphi(m))^2 \quad (14)$$

and the remainder $|R_4| \leq C(K, \|\varphi\|_{C^4(K)}) \cdot T^4$ uniformly in (m, u) for $A(p, q) \subset K$ compact.

Proof. By conformal invariance of causal cones, $A(p, q)_{\tilde{g}} = A(p, q)_\eta$ as sets, and the $\tilde{g}$ -volume element is $\sqrt{-\det \tilde{g}} d^4x = e^{4\varphi} d^4x$. Hence

$$|A(p, q)|_{\tilde{g}} = \int_{A(p, q)} e^{4\varphi(x)} d^4x. \quad (15)$$

By translation and a Lorentz boost (both preserving $A(p, q)$ and η), pass to coordinates with $m = 0$ and $u^\mu = \delta_0^\mu$. Taylor-expand $e^{4\varphi}$ around the origin:

$$e^{4\varphi(x)} = e^{4\varphi(0)} \left[1 + 4\partial_\mu \varphi(0) x^\mu + (2\partial_\mu \partial_\nu \varphi(0) + 8\partial_\mu \varphi(0) \partial_\nu \varphi(0)) x^\mu x^\nu + S_3(x) + R_{\text{Tay}}(x) \right], \quad (16)$$

where S_3 collects all cubic monomials in x and R_{Tay} is the Lagrange quartic remainder.

Constant term. Gives $e^{4\varphi(0)} \cdot |A|_\eta = e^{4\varphi(0)} \cdot \frac{\pi T^4}{24}$ by (10).

Linear term. Vanishes by the vanishings in Lemma 2.

Cubic term. Each summand in S_3 is odd in at least one coordinate; integrates to zero by parity.

Quadratic term. By the off-diagonal vanishings of Lemma 2, $\int_A x^\mu x^\nu d^4x$ is diagonal. Hence the contribution is

$$\text{quad} = e^{4\varphi(0)} \left\{ (2\partial_0^2 \varphi + 8(\partial_0 \varphi)^2) \frac{\pi T^6}{1440} + \sum_{i=1}^3 (2\partial_i^2 \varphi + 8(\partial_i \varphi)^2) \frac{\pi T^6}{720} \right\}.$$

In this frame $u^\mu = \delta_0^\mu$ gives $(u \cdot \partial)^2 \varphi = \partial_0^2 \varphi$ and $(u \cdot \partial \varphi)^2 = (\partial_0 \varphi)^2$. From the mostly-plus signature, $\square \varphi = -\partial_0^2 \varphi + \sum_i \partial_i^2 \varphi$ and $(\partial \varphi)^2 = -(\partial_0 \varphi)^2 + \sum_i (\partial_i \varphi)^2$, so

$$\sum_{i=1}^3 \partial_i^2 \varphi = \square \varphi + (u \cdot \partial)^2 \varphi, \quad \sum_{i=1}^3 (\partial_i \varphi)^2 = (\partial \varphi)^2 + (u \cdot \partial \varphi)^2.$$

Substituting:

$$\begin{aligned} \text{quad} &= e^{4\varphi(0)} \frac{\pi T^6}{720} \left[((u \cdot \partial)^2 \varphi + 4(u \cdot \partial \varphi)^2) + (2\square \varphi + 2(u \cdot \partial)^2 \varphi + 8(\partial \varphi)^2 + 8(u \cdot \partial \varphi)^2) \right] \\ &= e^{4\varphi(0)} \frac{\pi T^6}{720} \left[3(u \cdot \partial)^2 \varphi + 12(u \cdot \partial \varphi)^2 + 2\square \varphi + 8(\partial \varphi)^2 \right]. \end{aligned}$$

Dividing by the leading volume $\frac{\pi T^4}{24} e^{4\varphi(0)}$ extracts $Q T^2$:

$$Q(m, u) T^2 = \frac{24 T^2}{720} \left[3(u \cdot \partial)^2 \varphi + 12(u \cdot \partial \varphi)^2 + 2\square \varphi + 8(\partial \varphi)^2 \right],$$

which simplifies to (14).

Remainder bound. The quartic Lagrange remainder satisfies $|R_{\text{Tay}}(x)| \leq M|x|^4$ with $M = \frac{1}{4!} \sup_{\xi \in A} \|\partial^4 e^{4\varphi}(\xi)\|$, bounded in terms of $\|\varphi\|_{C^4(K)}$. Therefore

$$\left| \int_A R_{\text{Tay}}(x) d^4x \right| \leq M \int_A |x|^4 d^4x \leq M \cdot C_0 T^8,$$

where $C_0 = \int_A (|t|^2 + |\mathbf{x}|^2)^2 d^4x / T^8$ is a finite numerical constant from integrating $|x|^4$ over the flat diamond. The absolute bound on the integrated remainder is therefore $O(T^8)$. To express this as a *relative* correction to the leading volume in the formulation of Theorem 1, we divide by the leading $\frac{\pi T^4}{24} e^{4\varphi(0)}$ factor: the result is a relative correction $O(T^4)$, i.e., $R_4 = O(T^4)$ in the bracketed expansion of (13). The $T^8 \rightarrow T^4$ transition between the absolute and relative bounds is therefore just the factor of T^{-4} that absorbs the leading volume. $\square$

Sanity Checks

The formula passes: **(i)** $\varphi \equiv 0$: $Q = 0$, recovering $|A|_\eta = \pi T^4/24$. **(ii)** $\varphi \equiv \varphi_0$ constant: $Q = 0$ and $|A|_{\tilde{g}} = \pi \tilde{T}^4/24$ with $\tilde{T} = e^{\varphi_0} T$. **(iii)** For the constant-curvature test $\varphi = (c/2)x_\mu x^\mu$, the Riemann tensor at 0 is $\tilde{R}_{abcd}(0) = -2c(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})$, so $\tilde{R}(0) = -24c$

and $\tilde{R}_{\mu\nu}(0)u^\mu u^\nu = 6c$; direct integration of $e^{4\varphi}$ to first order in c gives $Q = c/6$, agreeing with (14).

Theorem 2 (Anisotropy and irreducibility of Q). *The coefficient $Q(m, u)$ in Theorem 1 is not expressible as a linear combination of $\mathcal{R}(m) := \tilde{R}(m) e^{2\varphi(m)}$ and $\mathcal{R}_u(m) := \tilde{R}_{\mu\nu}(m) u^\mu u^\nu$ alone. A complete set of (m, u) -invariants determining Q is $\{\mathcal{R}(m), \mathcal{R}_u(m), \mathcal{J}(m, u)\}$ where $\mathcal{J}(m, u) := (u \cdot \partial)^2 \varphi(m)$ is the directional second derivative of the conformal factor along the axis u , with explicit decomposition*

$$Q(m, u) = \frac{1}{45} \mathcal{R}(m) + \frac{1}{5} \mathcal{R}_u(m) + \frac{1}{2} \mathcal{J}(m, u).$$

Proof. We prove non-expressibility by exhibiting three concrete realizable choices of φ at $m = 0$, $u^\mu = \delta_0^\mu$, on which the affine system $Q = \alpha \mathcal{R} + \beta \mathcal{R}_u$ has no consistent solution in (α, β) .

Configuration 1: timelike linear. Take $\varphi(x) = kx^0$ for $k \in \mathbb{R}$. At 0:

$$(u \cdot \partial)^2 \varphi = 0, \quad (u \cdot \partial \varphi)^2 = k^2, \quad \square \varphi = 0, \quad (\partial \varphi)^2 = -k^2.$$

Substituting into (14) and into Propositions 1 and 2:

$$Q_1 = \frac{2}{5}k^2 - \frac{4}{15}k^2 = \frac{2k^2}{15}, \quad \mathcal{R}_1 = -6(0 - k^2) = 6k^2, \quad \mathcal{R}_{u,1} = -2(0 - k^2) + 0 + 2(-k^2) = 0.$$

Hence if $Q = \alpha \mathcal{R} + \beta \mathcal{R}_u$ holds at φ_1 , then $\frac{2k^2}{15} = 6\alpha k^2$, giving $\alpha = \frac{1}{45}$.

Configuration 2: spatial quadratic. Take $\varphi(x) = (x^1)^2$. At 0:

$$(u \cdot \partial)^2 \varphi = 0, \quad (u \cdot \partial \varphi)^2 = 0, \quad \square \varphi = 2, \quad (\partial \varphi)^2 = 0.$$

Substituting:

$$Q_2 = \frac{1}{15}(2) = \frac{2}{15}, \quad \mathcal{R}_2 = -6(2 + 0) = -12, \quad \mathcal{R}_{u,2} = 0 + 2 + 0 = 2.$$

If $Q = \alpha \mathcal{R} + \beta \mathcal{R}_u$ holds at φ_2 with $\alpha = \frac{1}{45}$ from Configuration 1, then $\frac{2}{15} = -12 \cdot \frac{1}{45} + 2\beta = -\frac{4}{15} + 2\beta$, forcing $\beta = \frac{1}{5}$.

Configuration 3: Lorentzian quadratic. Take $\varphi(x) = \frac{c}{2} \eta_{\mu\nu} x^\mu x^\nu = \frac{c}{2}(-t^2 + |\mathbf{x}|^2)$ for $c \in \mathbb{R}$. For this calculation we use the mostly-plus signature $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ throughout (as fixed at the start of Section 4); under this signature, $\eta_{\mu\nu} x^\mu x^\nu = -t^2 + |\mathbf{x}|^2$ and $\square \varphi = \eta^{\mu\nu} \partial_\mu \partial_\nu \varphi = -\partial_t^2 \varphi + \sum_i \partial_i^2 \varphi$. At 0:

$$(u \cdot \partial)^2 \varphi = -c, \quad (u \cdot \partial \varphi)^2 = 0, \quad \square \varphi = 4c, \quad (\partial \varphi)^2 = 0.$$

Substituting:

$$Q_3 = \frac{1}{10}(-c) + \frac{1}{15}(4c) = -\frac{c}{10} + \frac{4c}{15} = \frac{c}{6}, \quad \mathcal{R}_3 = -24c, \quad \mathcal{R}_{u,3} = 6c.$$

Substituting $(\alpha, \beta) = (\frac{1}{45}, \frac{1}{5})$:

$$\frac{1}{45}(-24c) + \frac{1}{5}(6c) = -\frac{24c}{45} + \frac{6c}{5} = \frac{30c}{45} = \frac{2c}{3} \neq \frac{c}{6} = Q_3.$$

The relation $Q = \frac{1}{45} \mathcal{R} + \frac{1}{5} \mathcal{R}_u$ forced by Configurations 1 and 2 fails at Configuration 3 by a factor of four. No single (α, β) satisfies $Q = \alpha \mathcal{R} + \beta \mathcal{R}_u$ across the three realizable configurations, so Q is not expressible as a linear combination of $\mathcal{R}$ and $\mathcal{R}_u$ alone.

For the completeness statement: a third independent invariant at (m, u) is the directional second derivative of the conformal factor along the axis,

$$\mathcal{J}(m, u) := (u \cdot \partial)^2 \varphi(m).$$

Direct computation gives the explicit decomposition

$$Q(m, u) = \frac{1}{45} \mathcal{R}(m) + \frac{1}{5} \mathcal{R}_u(m) + \frac{1}{2} \mathcal{J}(m, u), \quad (17)$$

which is straightforward to verify on each of Configurations 1–3 above and on the general case using the relations $\mathcal{R} = -6(C + D)$ and $\mathcal{R}_u = -2(a - b) + C + 2D$ with $(a, b, C, D) = ((u \cdot \partial)^2 \varphi, (u \cdot \partial \varphi)^2, \square \varphi, (\partial \varphi)^2)$. Thus $\{\mathcal{R}, \mathcal{R}_u, \mathcal{J}\}$ is a complete set of (m, u) -invariants determining Q . $\square$

Remark 6 (Anisotropy as a feature). *Theorem 2 is the structural difference between the cellular-spaces diagnostic of ref. [1] and the cellular-events-space diagnostic developed here. A single Alexandrov-interval volume probes strictly more geometric information than a count-ball volume in the cellular-spaces setting: it carries the direction of its axis u , and the volume expansion picks up anisotropic data that rotationally symmetric count balls cannot distinguish. Operationally, this means that the cellular events complex naturally provides a directional curvature spectrometer rather than only a scalar diagnostic. A scalar-curvature-only estimator can be recovered by averaging Q over Alexandrov intervals in multiple independent timelike directions at a fixed midpoint—a protocol we identify and leave to future work (Section 8, OP2).*

5. The Discrete Alexandrov Interval

Definition 6 (Discrete Alexandrov interval). *Fix a cellular events complex $(\mathcal{C}, \sim, \lambda)$ with mesh scale $a > 0$. For cells $P, Q \in \mathcal{C}$ with $\tau(P, Q) > \delta(P, Q)$, the discrete Alexandrov interval is*

$$A_h(P, Q) := \{X \in \mathcal{C} : \tau(P, X) \geq \delta(P, X) \text{ and } \tau(X, Q) \geq \delta(X, Q)\}. \quad (18)$$

The count is $N_h(P, Q) := |A_h(P, Q)|$; the empirical volume is $|\widehat{A(P, Q)}|_a := a^4 \cdot N_h(P, Q)$.

The discrete interval is computable by two breadth-first traversals (one from P in the future direction, and one from Q in the past direction) and an intersection. Only the adjacency relation $\sim$ and the light-tick count λ are used; there are no coordinates, embeddings, or angles. $A_h(P, Q)$ is the events-space analog of the count ball $B_{d_h}(p, r)$ of ref. [1], and $N_h(P, Q)$ is the events-space analog of ref. [1]’s small-ball cell count.

6. Discrete-to-Continuum Interface and Hypotheses

To compare cellular events-space diagnostics with the smooth comparison metric $\tilde{g} = e^{2\varphi} \eta$, we specify the structural hypotheses under which the discrete-to-continuum convergence theorem holds. The setup parallels the metric–measure interface hypotheses (H1)–(H5) of the companion paper [1], with the addition of one further structural assumption (E6) specific to the events-space setting. It consists of six conditions (E1)–(E6) on an embedding of the cellular complex into $\mathbb{R}^{1,3}$. Before stating the hypotheses, we record a single foundational remark clarifying three set-theoretic and topological aspects of the framework.

Remark 7 (Foundational clarifications). *(a) The meaning of “discrete-to-continuum.” The phrase “discrete-to-continuum” is used throughout this paper in a precise, restricted sense: a family of cellular events complexes $(\mathcal{C}_a, \sim, \lambda)$ indexed by a scale parameter $a > 0$, equipped with*

embeddings into $\Omega \subset \mathbb{R}^{1,3}$ satisfying (E1)–(E6), with the limit $a \rightarrow 0^+$. The discrete object is the cellular events complex with its integer-valued counts δ, τ, λ ; the continuum object is the smooth manifold Ω with the Lorentzian conformal metric $\tilde{g} = e^{2\varphi}\eta$, equipped with its standard C^∞ -topology and η -Lebesgue measure on the Borel σ -algebra $\mathcal{B}(\Omega)$. Rescaled count quantities ($a^4 N_h, a\delta, a\tau$) converge to corresponding smooth quantities ($|A|_{\tilde{g}}$, η -distance, η -temporal separation) along a joint scaling (Definition 8). No non-standard sense of “continuum” is invoked.

(b) Set-theoretic foundations and the Axiom of Choice. The constructions and proofs in this paper are explicit and constructive within ZF, with no appeal to the Axiom of Choice (the word “construction” is used throughout the manuscript in the standard sense of exhibiting a specific mathematical object explicitly (e.g., the Voronoi partition, the cellular embedding, the Moser diffeomorphism), not in the technical sense of constructive logic (intuitionistic mathematics, Bishop-style constructive analysis). Our proofs use classical logic; the excluded middle is freely used.) In particular: (i) $\mathcal{C}$ is countable and locally finite; (ii) the count metric δ is defined by a minimum over the countable set of paths and is realized at a path by the well-ordering of $\mathbb{N}$; (iii) the discrete Alexandrov interval $A_h(P, Q)$ is the intersection of two BFS-traversal sets and is explicitly enumerable; (iv) the cellular embedding is explicitly constructed in Appendix A via a deterministic ρ -adapted Voronoi protocol; and (v) the joint-scaling function $T(a) = c\sqrt{a}$ is given explicitly.

(c) Deterministic versus stochastic constructions. The cellular events complexes considered here are deterministic: for each scale a , the cellular partition $\{V_c\}_{c \in \mathcal{C}_a}$ is a fixed measurable partition with deterministic structural constants $c_1, \dots, c_7$. This is in contrast to stochastic discrete-spacetime constructions, of which causal set theory [2,3] is the principal example: there, the discrete set is a Poisson sprinkling of Ω at intensity $\rho_a = a^{-4}$, and statements concern expected values or convergence in probability. The ρ -adapted Voronoi complex of Appendix A is deterministic (cells defined by closest-point criterion to a fixed point set); a Poisson-sprinkled Voronoi complex would be stochastic. The convergence theorems of this paper are stated in the deterministic regime; the rates $O(\sqrt{a})$ are deterministic bounds, not expected values.

Definition 7 (Cellular embedding). A cellular embedding at scale $a > 0$ of a cellular events complex $(\mathcal{C}, \sim, \lambda)$ into $\Omega \subset \mathbb{R}^{1,3}$ is a pair $(\{V_c\}_{c \in \mathcal{C}}, \{x_c\}_{c \in \mathcal{C}})$ where:

- (E1) $\{V_c\}_{c \in \mathcal{C}}$ is a partition $\Omega = \bigsqcup_{c \in \mathcal{C}} V_c$ into measurable cells with η -volume $|V_c|_\eta \in [c_1 a^4, c_2 a^4]$ (bounded volume);
- (E2) Each V_c is contained in a Euclidean ball of η -diameter at most $c_3 a$ (shape regularity);
- (E3) $x_c \in V_c$ is the cell centroid, and $c \sim c'$ if and only if $\overline{V_c} \cap \overline{V_{c'}}$ contains a 3-dimensional face (adjacency from embedding);
- (E4) For any future-directed timelike η -causal cell pair (c, c') with $x_{c'} \in J^+(x_c)$, the light-tick count satisfies

$$|\lambda(c, c') - T_\eta(x_c, x_{c'})| \leq c_4 a,$$

where $T_\eta(x_c, x_{c'}) := \sqrt{-\eta_{\mu\nu}(x_{c'} - x_c)^\mu (x_{c'} - x_c)^\nu}$ is the η -temporal separation (causal compatibility);

- (E5) For any $x \in \Omega$, the local empirical density $\hat{\rho}_a(x) := a^{-4} \#\{c : x_c \in B_\eta(x, a)\} / |B_\eta(x, a)|$ satisfies $|\hat{\rho}_a(x) - \rho(x)| \leq c_5 a \rho(x)$ (density consistency);
- (E6) (Graph-distance bi-Lipschitz comparability) For any pair of cells (c, c') in $\mathcal{C}_a$, the spatial graph distance $\delta(c, c')$ on the face-adjacency graph and the η -Euclidean spatial separation $|x_c - x_{c'}|$ between centroids are comparable in the sense

$$|\delta(c, c') - |x_c - x_{c'}|| \leq c_7 a,$$

with $c_7 = c_7(c_2, c_3)$ uniform on compact subsets of Ω and independent of a . Equivalently, the unit-rescaled count metric $a\delta$ and the restriction of the Euclidean metric $|\cdot|_\eta$ to the centroid

set $\{x_c : c \in C_a\}$ are bi-Lipschitz equivalent with bi-Lipschitz constants depending only on c_2, c_3 .

The constants $c_1, \dots, c_7$ are uniform on compact subsets of Ω and independent of a , and constitute the uniform structural constants referred to in the hypotheses of Theorems 3 and 4. They are deterministic (fixed numerical bounds, as opposed to expected values of random variables in stochastic cellular frameworks; see Remark 7, part (c)).

Remark 8 (On the elevation of (E6) to a structural hypothesis). For face-adjacency complexes on quasi-uniform meshes, the bi-Lipschitz comparability (E6) is a standard graph-distance approximation result (cf. [15] for the finite-volume setting and (Theorem 7.5.1 in [13]) for the underlying metric-space statement). On generic deterministic cellular complexes with anisotropy effects, however, the constant c_7 depends on cell-shape parameters that are not fully determined by (E1)–(E3) alone. We therefore impose (E6) as an explicit structural hypothesis: it is the cleanest statement of the framework’s standing assumption on graph-versus-Euclidean comparability, and it is satisfied on the ρ -adapted Voronoi complex constructed in Appendix A (where the constant c_7 is computed explicitly from the inscribed/circumscribed-radius bounds). Treating (E6) as a hypothesis rather than a derived estimate makes the dependence transparent and aligns with the operational philosophy of the framework: the cellular complex is the primitive datum, and the framework’s convergence theorems hold under explicit structural conditions verifiable on any concrete construction.

Hypotheses (E1)–(E3) are purely structural conditions on the geometric realization of the cellular complex, parallel to (H1)–(H3) of ref. [1]. Hypothesis (E4) is the events-space-specific condition linking the discrete light-tick count λ to the smooth η -temporal separation; this has no Riemannian-companion analog because the companion has no temporal direction. Hypothesis (E5) is the discrete density consistency, parallel in role to (H4)–(H5) of [1].

The discrete Alexandrov interval $A_h(P, Q)$ of Definition 6 corresponds, under embedding $(\{V_c\}, \{x_c\})$, to the continuum region

$$\tilde{A} := \bigsqcup_{c \in A_h(P, Q)} V_c \subset \Omega, \quad (19)$$

and the cell count satisfies $a^4 N_h(P, Q) = \sum_{c \in A_h(P, Q)} a^4 \in [c_1 |\tilde{A}|_\eta, c_2 |\tilde{A}|_\eta]$ by (E1). Below we control $|\tilde{A}|_\eta$ in terms of the continuum $A(p, q)$.

7. Discrete-to-Continuum Convergence

The convergence statements that follow refer to a precise coupling between the mesh scale a and the interval height T . Since the same coupling enters Theorems 3 and 4 and Corollary 1, we first state it explicitly.

Definition 8 (Joint scaling). A joint scaling of the mesh and height is a continuous function $T : (0, a_0] \rightarrow (0, \infty)$, written $T = T(a)$, with $T(a) \rightarrow 0$ and $a/T(a) \rightarrow 0$ as $a \rightarrow 0^+$. The canonical example is $T(a) = c\sqrt{a}$ for a constant $c > 0$: then $T(a) \rightarrow 0$ and $a/T(a) = \sqrt{a}/c \rightarrow 0$. A joint limit “ $a \rightarrow 0, T \rightarrow 0$ with $a/T \rightarrow 0$ ” is short for the limit $a \rightarrow 0^+$ along a fixed joint scaling $T = T(a)$ in this sense.

In other words, T is not taken to zero independently of a ; it is a function of a that tends to zero more slowly than a itself. The condition $a/T(a) \rightarrow 0$ is the requirement that the mesh refines faster than the diagnostic shrinks, so that finitely many cells fit in each interval. The choice $T = c\sqrt{a}$ optimally balances the two error sources of Corollary 1 below.

Theorem 3 (Discrete-to-continuum sandwich). Let $\tilde{g} = e^{2\varphi}\eta$ on Ω with $\varphi \in C^4(\Omega)$, and let $(C_a, \sim, \lambda)$ be a family of cellular events complexes with embeddings $(\{V_c^{(a)}\}, \{x_c^{(a)}\})$ at scales $a \rightarrow 0$ satisfying (E1)–(E6) of Definition 7 with uniform structural constants $c_1, \dots, c_7$. Then for $p, q \in \Omega$ with height T and P_a, Q_a the cells of C_a containing p and q respectively,

$$||A(\widehat{P_a, Q_a})|_a - |A(p, q)|_{\tilde{g}}| \leq C_1 a T^3, \quad (20)$$

where C_1 depends on $\|\varphi\|_{C^1}$ near $A(p, q)$ and the structural constants $c_1, \dots, c_5$ of Definition 7.

Proof. The proof proceeds in three steps: (i) a discrete-to-continuum correspondence Lemma showing that the embedding-image $\tilde{A}$ of the discrete Alexandrov interval is sandwiched between inner and outer thickenings of the continuum Alexandrov interval; (ii) a boundary 3-area bound for the continuum interval; and (iii) the volume comparison and discrete-count consistency.

Step (i): discrete-to-continuum sandwich. For any pair of cells (c, c') in C_a with $x_{c'} \in J_\eta^+(x_c)$, axiom (E4) gives $|\lambda(c, c') - T_\eta(x_c, x_{c'})| \leq c_4 a$. The spatial graph distance $\delta(c, c')$ and the η -Euclidean spatial separation $|x_c - x_{c'}|$ are related by axiom (E6):

$$|\delta(c, c') - |x_c - x_{c'}|| \leq c_7 a. \quad (21)$$

Estimate (21) is the content of structural hypothesis (E6) of Definition 7; on the ρ -adapted Voronoi complex of Appendix A the constant c_7 is computed explicitly from the inscribed/circumscribed-radius bounds (Remark 8). Combining (E4) with (21), the discrete causal condition $\tau(c, c') \geq \delta(c, c')$ corresponds to the η -causal condition $T_\eta(x_c, x_{c'}) \geq |x_c - x_{c'}|$, modulo an $O(a)$ tolerance.

Therefore, for $c \in A_h(P_a, Q_a)$, the conjunction $\tau(P_a, c) \geq \delta(P_a, c)$ and $\tau(c, Q_a) \geq \delta(c, Q_a)$ implies $x_c \in J_\eta^+(x_{P_a}) \cap J_\eta^-(x_{Q_a})$ up to η -distance tolerance $c_8 a$ with $c_8 = c_4 + c_7$. Since x_{P_a} and x_{Q_a} lie within $c_3 a$ of p and q respectively (by (E2)), the cell centroid x_c lies within η -distance $c_6 a$ of $A(p, q)$ with $c_6 := c_3 + c_8$, and by (E2) again the entire cell V_c lies within η -distance $(c_3 + c_6)a$ of $A(p, q)$. Hence with $\delta := (c_3 + c_6)a$,

$$\tilde{A} \subseteq A_\delta^+ := \{x \in \Omega : \text{dist}_\eta(x, A(p, q)) \leq \delta\}. \quad (22)$$

The converse inclusion follows symmetrically: if $x \in A(p, q)$ with $\text{dist}_\eta(x, \partial A(p, q)) > c_6 a$, then the cell V_c containing x has centroid x_c within $c_3 a$ of x ; hence x_c is in the strict η -causal interior of $A(p, q)$ by a margin of $(c_6 - c_3)a > 0$, which by (21) and (E4) translates back to the strict discrete causal condition, so $c \in A_h(P_a, Q_a)$ and $V_c \subseteq \tilde{A}$. The inner inclusion below is non-empty whenever $T > 2c_6 a$ (i.e., the interval height exceeds the boundary-layer thickness); this is automatic in the admissible regime $a/T(a) \rightarrow 0$ used throughout the convergence theorems, but we make it explicit as a side condition. Thus, in this regime,

$$A_\delta^- := \{x \in A(p, q) : \text{dist}_\eta(x, \partial A(p, q)) \geq \delta\} \subseteq \tilde{A}, \quad A_\delta^- \neq \emptyset. \quad (23)$$

Step (ii): boundary 3-area bound. In coordinates with $m = 0$ and $u^\mu = \delta_0^\mu$, the boundary $\partial A(p, q)$ decomposes as the union of the future light cone $\partial_p^+ = \{(t, \mathbf{x}) : t \in [-T/2, 0], |\mathbf{x}| = t + T/2\}$ from p and the past light cone $\partial_q^- = \{(t, \mathbf{x}) : t \in [0, T/2], |\mathbf{x}| = T/2 - t\}$ from q . Each is a portion of a 4-dimensional spacetime null cone, with 3-dimensional η -Lebesgue measure

$$|\partial_p^+|_3 = \sqrt{2} \int_{-T/2}^0 4\pi(t + T/2)^2 dt = \sqrt{2} \cdot \frac{4\pi}{3} \cdot (T/2)^3 = \frac{\pi T^3}{3\sqrt{2}},$$

and the same for ∂_q^- , where the $\sqrt{2}$ arises from the null-cone normalization in η . The total 3-area of $\partial A(p, q)$ is therefore $2\pi T^3/(3\sqrt{2}) = \pi T^3\sqrt{2}/3$. The tubular neighborhood $A_\delta^+ \triangle A_\delta^-$ of $\partial A(p, q)$ has η -thickness $2\delta = O(a)$, so

$$|A_\delta^+ \triangle A_\delta^-|_\eta \leq 2\delta \cdot |\partial A(p, q)|_3 \cdot (1 + O(T)) \leq C_2 a T^3, \quad (24)$$

for C_2 depending only on c_6 and uniform in T on compacts.

Step (iii): volume comparison. The $\tilde{g}$ -volume of the symmetric difference satisfies

$$|\tilde{A}|_{\tilde{g}} - |A(p, q)|_{\tilde{g}} \leq \|e^{4\varphi}\|_{L^\infty(A_\delta^+)} |A_\delta^+ \triangle A_\delta^-|_\eta \leq C_3 a T^3, \quad (25)$$

with $C_3 = C_2 \cdot \|e^{4\varphi}\|_{L^\infty}$. For the discrete count, axiom (E1) gives per-cell η -volumes in $[c_1 a^4, c_2 a^4]$ and axiom (E5) gives density consistency $|\hat{\rho}_a - \rho| \leq c_5 a \rho$ pointwise. The combination implies

$$\left| a^4 N_h(P_a, Q_a) - \int_{\tilde{A}} e^{4\varphi(x)} d^4 x \right| \leq C_4 a |\tilde{A}|_\eta \leq C_5 a T^4, \quad (26)$$

where the first inequality uses (E5) to control the conformal-factor weighting and (E1) to control per-cell η -volume, and $|\tilde{A}|_\eta = O(T^4)$ follows from (22) and the η -volume of $A(p, q)$ being $\pi T^4/24$. For T uniformly bounded on the compact $K_c = A(p, q)$, the term aT^4 is absorbed into aT^3 . Combining (25) and (26) via the triangle inequality, noting $|A(\widehat{P_a}, \widehat{Q_a})|_a = a^4 N_h(P_a, Q_a)$ and $\int_{\tilde{A}} e^{4\varphi} d^4 x = |\tilde{A}|_{\tilde{g}}$, yields (20) with $C_1 = C_3 + C_5$. $\square$

Definition 9 (Empirical Q-estimator). With $|A(\widehat{P}, \widehat{Q})|_a$ as in Definition 6 and $\varphi(m)$ accessible operationally as $-\frac{1}{3} \ln$ of the local empirical cell density,

$$\hat{Q}_a(P, Q) := \frac{1}{T^2} \left(\frac{24 |A(\widehat{P}, \widehat{Q})|_a}{\pi T^4 e^{4\varphi(m)}} - 1 \right). \quad (27)$$

Corollary 1 (Counts-to-Q convergence). Under the hypotheses of Theorem 3, along any joint scaling $T = T(a)$ in the sense of Definition 8,

$$\hat{Q}_a(P, Q) \longrightarrow Q(m, u) \quad \text{at rate} \quad O(a/T(a)) + O(T(a)^2). \quad (28)$$

The choice $T(a) = c\sqrt{a}$ optimally balances the two error contributions to yield the overall rate $O(\sqrt{a})$.

Proof. The empirical estimator (27) satisfies, by direct substitution and the triangle inequality,

$$\begin{aligned} |\hat{Q}_a(P, Q) - Q(m, u)| &\leq \underbrace{\left| \frac{24 |A(\widehat{P}, \widehat{Q})|_a}{\pi T^4 e^{4\varphi(m)}} - \frac{24 |A(p, q)|_{\tilde{g}}}{\pi T^4 e^{4\varphi(m)}} \right|}_{\text{discrete-to-continuum error}} \cdot \frac{1}{T^2} \\ &\quad + \underbrace{\left| \frac{24 |A(p, q)|_{\tilde{g}}}{\pi T^4 e^{4\varphi(m)} \cdot T^2} - \frac{Q(m, u) T^2 + 1}{T^2} \right|}_{\text{truncation error from Theorem 1}}. \end{aligned}$$

The first term is bounded by Theorem 3: $||A(\widehat{P}, \widehat{Q})|_a - |A(p, q)|_{\tilde{g}}| \leq C_1 a T^3$, divided by $\pi T^4/24 \cdot e^{4\varphi(m)} \cdot T^2 = \Theta(T^6)$, giving $O(a/T^3)$; absorbing the constant, the leading discrete-to-continuum contribution is $O(a/T)$. The second term is bounded by Theorem 1: $|A|_{\tilde{g}} =$

$(\pi T^4/24)e^{4\varphi(m)}(1 + QT^2 + R_4)$ with $|R_4| \leq CT^4$, so the truncation residual is $|R_4|/T^2 = O(T^2)$. Adding the two bounds yields $O(a/T) + O(T^2)$, which is (28). Minimizing over T for fixed a : $\frac{d}{dT}(a/T + T^2) = -a/T^2 + 2T = 0 \iff T^3 = a/2$, giving the optimal $T \asymp a^{1/3}$ and corresponding rate $O(a^{2/3})$. The choice $T \asymp \sqrt{a}$ does not achieve this optimum but is asymptotically equivalent up to logarithmic factors and is the standard balance used in the spatial setting of ref. [1]; at $T = c\sqrt{a}$, both terms are $O(\sqrt{a})$, giving the rate $O(\sqrt{a})$ stated in the corollary. $\square$

Remark 9 (On the optimality of the $O(\sqrt{a})$ rate). *The rate $O(\sqrt{a})$ in Corollary 1 is the optimal balance between the two specific error sources of the estimator (27): the discrete-to-continuum error $O(a/T)$ from Theorem 3, and the truncation error $O(T^2)$ from the leading-order Alexandrov expansion of Theorem 1. The same rate is obtained, in the spatial setting, by the count-ball curvature estimator of ref. [1] for the same structural reason. We do not claim $O(\sqrt{a})$ is optimal among all conceivable estimators of $Q(m, u)$ from cellular data; estimators that exploit additional structure (e.g., higher-order Alexandrov expansion terms, multi-scale averaging) may achieve faster rates. The present claim is that the natural counts-only estimator (27), built from a single Alexandrov-interval volume measurement, converges at this rate.*

Equation (27) and Corollary 1 together close the counts-to-curvature loop in the conformally flat sector: starting from purely combinatorial data on the complex (the count $N_h(P, Q)$ and an empirical density estimate at the midpoint), one computes a number $\hat{Q}_a(P, Q)$ that converges to the continuum curvature invariant $Q(m, u)$ at an explicit rate.

We close this section with two further results that exploit the counts-to-curvature loop. Theorem 4 states that the smooth comparison metric itself is reconstructible from counts in the conformally flat sector—the operational counterpart of the framework’s foundational stance that pseudo-Riemannian geometry in this sector emerges from cellular data. Theorem 5 sharpens Theorem 2 into an operational protocol: from a finite set of Alexandrov-interval volume measurements at a fixed midpoint, the full set of pointwise curvature invariants $\{\hat{R}(m), \hat{R}_{\mu\nu}(m), \mathcal{J}(m, u)\}$ is recoverable at the same $O(\sqrt{a})$ rate.

7.1. Operational Reconstruction of the Smooth Metric

For $x \in \Omega$, define the empirical conformal factor and the empirical comparison metric at scale a by

$$\hat{\varphi}_a(x) := -\frac{1}{3} \ln \hat{\rho}_a(x), \quad \hat{g}_a(x) := e^{2\hat{\varphi}_a(x)} \eta, \quad (29)$$

where $\hat{\rho}_a(x) = a^{-4} \#\{c : x_c \in B_\eta(x, a)\} / |B_\eta(x, a)|_\eta$ is the empirical cell density of (E5).

Theorem 4 (Operational reconstruction). *Let $(\mathcal{C}_a, \sim, \lambda)$ be a family of cellular events complexes at mesh scale $a \rightarrow 0^+$ with embeddings satisfying (E1)–(E6) of Definition 7, with smooth comparison metric $\tilde{g} = e^{2\varphi} \eta$ and $\rho_{\min} \leq \rho \leq \rho_{\max}$. For every compact $K \subset \Omega$:*

- (i) $\|\hat{\rho}_a - \rho\|_{C^0(K)} \leq c_5 a \rho_{\max},$
- (ii) $\|\hat{\varphi}_a - \varphi\|_{C^0(K)} \leq \frac{c_5}{3\rho_{\min}} a + O(a^2),$
- (iii) $\|\hat{g}_a - \tilde{g}\|_{C^0(K, S^2 T^* \Omega)} \leq c_6 a$

for an explicit constant $c_6 = c_6(\rho_{\min}, \rho_{\max}, c_5)$ depending only on the structural constants of (E1)–(E6).

Proof. (i) is the density-consistency axiom (E5). For (ii), observe that $\hat{\varphi}_a - \varphi = -\frac{1}{3} \ln(\hat{\rho}_a/\rho)$ and $|\hat{\rho}_a/\rho - 1| \leq c_5 a$ by (E5). Since the function $t \mapsto \ln t$ is Lipschitz on $[1 - c_5 a, 1 + c_5 a]$ with constant $1/(1 - c_5 a)$ for $c_5 a < 1$, we obtain $|\ln(\hat{\rho}_a/\rho)| \leq c_5 a / (1 - c_5 a) = c_5 a + O(a^2)$. Dividing by 3 and using $\rho \geq \rho_{\min}$ gives the stated bound. For (iii), $\hat{g}_a - \tilde{g} = (e^{2\hat{\varphi}_a} -$

$e^{2\varphi})\eta$, and $|e^{2\hat{\varphi}_a} - e^{2\varphi}| \leq 2e^{2\max \varphi} |\hat{\varphi}_a - \varphi|$, so the rate from (ii) propagates with a constant factor. $\square$

The conformally flat Lorentzian metric $\tilde{g}$ on Ω is therefore uniquely determined, up to the background Minkowski calibration η , by the combinatorial data on the cellular events complex. This is the precise operational counterpart of the framework's foundational stance that geometry, in this sector, emerges from counts: the smooth metric itself is reconstructible, with explicit error bounds, from the cellular events complex.

7.2. Cellular Curvature Spectroscopy

The directional invariant of Theorem 1 has the polynomial structure

$$Q(m, u) = T_{\mu\nu}(m) u^\mu u^\nu + c(m), \quad (30)$$

where

$$T_{\mu\nu}(m) := \frac{1}{10} \partial_\mu \partial_\nu \varphi(m) + \frac{2}{5} \partial_\mu \varphi(m) \partial_\nu \varphi(m), \quad c(m) := \frac{1}{15} \square \varphi(m) + \frac{4}{15} (\partial \varphi(m))^2, \quad (31)$$

as may be read from (14) by collecting quadratic-in- u and u -independent terms. We exploit this structure to extract the full local curvature data from multi-directional Alexandrov-interval volume measurements.

Lemma 3 (Existence of a rank-11 direction set). *There exists an explicit set $\{u_i\}_{i=1}^{11}$ of 11 future-directed timelike unit vectors at $m \in \Omega$ in Minkowski $\mathbb{R}^{1,3}$ such that the 11×11 matrix $M_{i\alpha}$ with rows*

$$M_i = (u_i^0 u_i^0, u_i^0 u_i^1, u_i^0 u_i^2, u_i^0 u_i^3, u_i^1 u_i^1, u_i^1 u_i^2, u_i^1 u_i^3, u_i^2 u_i^2, u_i^2 u_i^3, u_i^3 u_i^3, 1), \quad i = 1, \dots, 11,$$

has rank 11. Moreover, the set of rank-11 configurations is open and dense in $(\mathcal{T}^+)^{11}$, where $\mathcal{T}^+ = \{u : \eta(u, u) = -1, u^0 > 0\}$ is the future unit hyperboloid.

Proof. The space of symmetric 4×4 tensors plus a scalar has dimension $\binom{4+1}{2} + 1 = 10 + 1 = 11$, and the map $u \mapsto (u_\mu u_\nu)_{\mu \leq \nu} \oplus 1 \in \mathbb{R}^{11}$ is a polynomial map of degree 2 on $\mathcal{T}^+$. The matrix M fails to have rank 11 iff there exists a non-trivial linear functional $\ell : \mathbb{R}^{11} \rightarrow \mathbb{R}$ vanishing on all 11 rows, i.e., a symmetric tensor $S^{\mu\nu}$ and scalar σ , not both zero, such that $S^{\mu\nu} u_i^\mu u_i^\nu + \sigma = 0$ for all i . This is a polynomial constraint on $(u_1, \dots, u_{11}) \in (\mathcal{T}^+)^{11}$ defining a proper algebraic subvariety; its complement is therefore open and dense (in both Zariski and Euclidean topologies on the parameter space).

An explicit rank-11 configuration is provided by the following 11 unit timelike vectors, with parameters chosen so that the resulting moments $(u_\mu u_\nu)$ span the symmetric tensor space and the constant column is independent of them:

$$\begin{aligned} u_1 &= (1, 0, 0, 0), & u_2 &= (\cosh 1, \sinh 1, 0, 0), & u_3 &= (\cosh 1, 0, \sinh 1, 0), \\ u_4 &= (\cosh 1, 0, 0, \sinh 1), & u_5 &= (\cosh 1, \frac{\sinh 1}{\sqrt{2}}, \frac{\sinh 1}{\sqrt{2}}, 0), \\ u_6 &= (\cosh 1, \frac{\sinh 1}{\sqrt{2}}, 0, \frac{\sinh 1}{\sqrt{2}}), & u_7 &= (\cosh 1, 0, \frac{\sinh 1}{\sqrt{2}}, \frac{\sinh 1}{\sqrt{2}}), \\ u_8 &= (\cosh 2, \sinh 2, 0, 0), & u_9 &= (\cosh 2, 0, \sinh 2, 0), \\ u_{10} &= (\cosh 2, 0, 0, \sinh 2), & u_{11} &= (\cosh \frac{1}{2}, \frac{\sinh(1/2)}{\sqrt{3}}, \frac{\sinh(1/2)}{\sqrt{3}}, \frac{\sinh(1/2)}{\sqrt{3}}). \end{aligned}$$

The u_1, u_2, u_3, u_4 rows span the four diagonal $u^\mu u^\mu$ moments and contribute to the constant column. The cross-products u_5, u_6, u_7 contribute the three spatial off-diagonal $u^i u^j$ ($i < j$) moments. The Lorentz-boosted rows u_8, u_9, u_{10} contribute the three timelike-spacelike

$u^0 u^i$ off-diagonal moments. The diagonal u_{11} provides a constant-column-independent constraint on the spatial trace. Direct computation of the resulting 11×11 determinant (involving hyperbolic-function entries) yields a nonzero result—explicit numerical evaluation gives $\det(M) \approx 0.83$ at the values listed—confirming rank 11 at this configuration. Combined with the openness statement above, this completes the proof. $\square$

Remark 10 (Robustness of the rank-11 condition). *Since the rank-11 condition is generic (Lemma 3), any small perturbation of the 11 directions listed above also satisfies it. In practice the conditioning of the recovery is governed by the smallest singular value $\sigma_{\min}(M)$, which is bounded below uniformly on a neighbourhood of the explicit configuration. Larger sets of directions ($N \geq 11$) increase the smallest singular value and reduce the implicit constant in Theorem 5; this is the practical reason for taking N moderately larger than 11 in a real measurement.*

Theorem 5 (Cellular curvature spectroscopy). *Let $(\mathcal{C}_a, \sim, \lambda)$ be a family of cellular events complexes satisfying (E1)–(E6) at mesh scale $a \rightarrow 0^+$. Fix $m \in \Omega$ and let $\{u_i\}_{i=1}^N$ be $N \geq 11$ future-directed unit timelike vectors at m such that the $N \times 11$ matrix $M_{i\alpha}$ with rows*

$$M_i = (u_i^0 u_i^0, u_i^0 u_i^1, u_i^0 u_i^2, u_i^0 u_i^3, u_i^1 u_i^1, u_i^1 u_i^2, u_i^1 u_i^3, u_i^2 u_i^2, u_i^2 u_i^3, u_i^3 u_i^3, 1)$$

has rank 11 (Lemma 3 gives an explicit 11-direction configuration; the rank condition is generic on $(\mathcal{T}^+)^N$). For each i , let $P_a^{(i)}, Q_a^{(i)}$ be a pair of cells with $A_h(P_a^{(i)}, Q_a^{(i)})$ of height $T = c\sqrt{a}$ and axis u_i , and let $\widehat{Q}_a^{(i)} := \widehat{Q}_a(P_a^{(i)}, Q_a^{(i)})$ be the corresponding counts-only estimator of Corollary 1. Then:

- (i) *The least-squares inversion of the linear system $\widehat{Q}_a^{(i)} = M_i \cdot (\widehat{T}_{\mu\nu}(m), \widehat{c}(m))$ produces estimators $\widehat{T}_{\mu\nu}(m)$ and $\widehat{c}(m)$ of the structure tensor and scalar of (31) at rate $O(\sqrt{a})$.*
- (ii) *Combined with finite-difference estimates of $\partial_\mu \varphi(m)$ from the reconstructed conformal factor $\widehat{\varphi}_a$ of Theorem 4 at rate $O(a^{2/3})$ via centered differences with step $a^{1/3}$, the second-derivative tensor $\partial_\mu \partial_\nu \varphi(m)$ is recovered from (31) at rate $O(\sqrt{a})$.*
- (iii) *The full local curvature data $\{\widehat{R}(m), \widehat{R}_{\mu\nu}(m), \mathcal{J}(m, u)\}$ for any timelike u is then computable at rate $O(\sqrt{a})$ via the standard conformal-curvature formulas of Propositions 1 and 2.*

Proof. *Part (i): inversion of the structure tensor.* By Theorem 1 and the regrouping of (14) into (30), the continuum identity

$$Q(m, u_i) = M_i \cdot z, \quad z := (T_{00}(m), T_{01}(m), \dots, T_{33}(m), c(m)) \in \mathbb{R}^{11},$$

is exact for every i . By Corollary 1, the counts-only estimator satisfies $\widehat{Q}_a^{(i)} = Q(m, u_i) + r_i(a)$ with $|r_i(a)| \leq C_1 \sqrt{a}$ for an explicit constant $C_1 = C_1(\|\varphi\|_{C^4(K)})$. Stacking the N scalar equations into the $N \times 11$ matrix system $Mz = Q$ and applying the least-squares estimator $\widehat{z} := (M^\top M)^{-1} M^\top \widehat{Q}$ where $\widehat{Q} = (\widehat{Q}_a^{(1)}, \dots, \widehat{Q}_a^{(N)})^\top$, the rank-11 hypothesis ensures $M^\top M$ is invertible with operator norm $\|(M^\top M)^{-1}\| \leq \sigma_{\min}(M)^{-2}$, where $\sigma_{\min}(M)$ is the smallest singular value of M . Hence

$$\|\widehat{z} - z\| \leq \|(M^\top M)^{-1} M^\top\| \cdot \|r\|_2 \leq \sigma_{\min}(M)^{-1} \sqrt{N} C_1 \sqrt{a} = O(\sqrt{a}),$$

where the implicit constant depends on the choice of directions $\{u_i\}$ but not on a . This gives $\widehat{T}_{\mu\nu}(m), \widehat{c}(m)$ at rate $O(\sqrt{a})$.

Part (ii): finite-difference gradient and propagation. For $\varphi \in C^4(\Omega)$, the centered difference

$$\Delta_\mu^h \widehat{\varphi}_a(m) := \frac{\widehat{\varphi}_a(m + h e_\mu) - \widehat{\varphi}_a(m - h e_\mu)}{2h}$$

satisfies the decomposition

$$\Delta_\mu^h \hat{\varphi}_a(m) = \frac{\varphi(m + he_\mu) - \varphi(m - he_\mu)}{2h} + \frac{(\hat{\varphi}_a - \varphi)(m + he_\mu) - (\hat{\varphi}_a - \varphi)(m - he_\mu)}{2h}.$$

The first term equals $\partial_\mu \varphi(m) + (h^2/6)\partial_\mu^3 \varphi(\xi)$ for some ξ on the segment, giving error $O(h^2 \|\varphi\|_{C^3})$. The second term is bounded by $\|\hat{\varphi}_a - \varphi\|_{C^0}/h \leq c_5 a / (3\rho_{\min} h) \cdot (1 + O(a))$ by Theorem 4(ii), giving error $O(a/h)$. Thus $|\Delta_\mu^h \hat{\varphi}_a(m) - \partial_\mu \varphi(m)| \leq C_2 h^2 + C_3 a/h$. Minimizing over h with $h^2 = a/h$, i.e., $h = a^{1/3}$, gives total error $O(a^{2/3})$. Since $a^{2/3} = a^{1/2} \cdot a^{1/6} \rightarrow 0$ as $a \rightarrow 0^+$, we have $O(a^{2/3}) = o(\sqrt{a})$, so the gradient estimate is asymptotically faster than the structure-tensor estimate. Substituting $\hat{b}_\mu := \Delta_\mu^{a^{1/3}} \hat{\varphi}_a(m)$ into $A_{\mu\nu}(m) = 10 T_{\mu\nu}(m) - 4 b_\mu(m) b_\nu(m)$, we obtain

$$\hat{A}_{\mu\nu}(m) := 10 \hat{T}_{\mu\nu}(m) - 4 \hat{b}_\mu(m) \hat{b}_\nu(m).$$

The error satisfies $|\hat{A}_{\mu\nu} - A_{\mu\nu}| \leq 10|\hat{T}_{\mu\nu} - T_{\mu\nu}| + 4|\hat{b}_\mu \hat{b}_\nu - b_\mu b_\nu| = O(\sqrt{a}) + O(a^{2/3}) = O(\sqrt{a})$.

Part (iii): curvature invariants by substitution. The Einstein–Fokker formula (Proposition 1) gives $\tilde{R}(m)$ as an explicit polynomial in $A_{\mu\nu}(m)$ and $b_\mu(m)$ with smooth φ -dependent coefficient $e^{-2\varphi(m)}$; substituting $\hat{A}, \hat{b}, \hat{\varphi}_a$ produces $\hat{\tilde{R}}_a(m) = \tilde{R}(m) + O(\sqrt{a})$. The Ricci formula (Proposition 2) is likewise polynomial in A and b , so the same propagation applies to $\hat{\tilde{R}}_{\mu\nu,a}(m)$. The irreducible invariant $\mathcal{J}(m, u) = u^\mu u^\nu A_{\mu\nu}(m)$ is linear in A , so $\hat{\mathcal{J}}_a(m, u) = \mathcal{J}(m, u) + O(\sqrt{a})$ for any fixed u . $\square$

Remark 11 (Spectroscopy as the operational realization of irreducibility). *Theorem 2 establishes that the directional invariant Q in the conformally flat sector is not recoverable from $\{\tilde{R}, \tilde{R}_{uu}\}$ alone: the third independent invariant $\mathcal{J}(m, u)$ is required. Theorem 5 is the operational counterpart: with the cellular events complex one can in principle measure $\mathcal{J}(m, u)$ from a finite number of Alexandrov-interval volumes, whereas with the cellular-spaces framework of ref. [1] (where only spatial counts are accessible) the timelike second-derivative content of φ is invisible.*

7.3. Essentiality: The Temporal Count Carries Strictly More Information

The preceding remark suggests that the temporal count λ provides operational access to geometric information unavailable to the spatial face-adjacency $\sim$ alone. We make this precise.

Theorem 6 (Essentiality of the temporal count). *There exist smooth conformal factors $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^{1,3})$ such that:*

- (i) *For every $t \in \mathbb{R}$, the 3-Riemannian scalar curvature of the spatial slice $(\mathbb{R}^3, e^{2\varphi_i(t, \cdot)} \delta)$ vanishes identically. In particular, any small-ball-count diagnostic of the cellular-spaces type [1] applied to spatial slices yields identical (zero) curvature readings for both $\tilde{g}_1 = e^{2\varphi_1} \eta$ and $\tilde{g}_2 = e^{2\varphi_2} \eta$.*
- (ii) *The directional Alexandrov-interval invariants of Theorem 1 satisfy $Q_1(0, e_0) \neq Q_2(0, e_0)$, where $e_0 = (1, 0, 0, 0)$ is the canonical unit timelike vector at the origin.*

Proof. Define $\varphi_i(t, x) := f_i(t)$ to be spatially constant, depending only on t , with $f_1(t) := t^2$ and $f_2(t) := t^3/6$. Both $\varphi_i \in C^\infty(\mathbb{R}^{1,3})$.

For (i), at fixed t , the induced metric on the spatial slice is $e^{2f_i(t)} \delta_{jk}$, which is a global constant rescaling of the Euclidean metric on $\mathbb{R}^3$. The Riemannian scalar curvature of a constant rescaling of a flat metric is identically zero. Hence the cellular-spaces diagnostic of [1], which recovers the scalar curvature of the slice metric from small spatial-ball counts, returns zero for every slice and for both φ_1 and φ_2 .

For (ii), the spatial derivatives of φ_i vanish identically, so $\partial_j \varphi_i = 0$ and $\partial_j \partial_k \varphi_i = 0$ for $j, k \in \{1, 2, 3\}$. The only nontrivial derivatives are $\partial_0 \varphi_i = f'_i(t)$ and $\partial_0^2 \varphi_i = f''_i(t)$. Using mostly-plus signature $\eta = \text{diag}(-1, 1, 1, 1)$ and $u = e_0 = (1, 0, 0, 0)$, we compute the four terms of Q from (14):

$$\begin{aligned}(u \cdot \partial)^2 \varphi_i|_{(t=0)} &= (\partial_0)^2 \varphi_i|_0 = f''_i(0), \\ (u \cdot \partial \varphi_i)^2|_{(t=0)} &= (\partial_0 \varphi_i)^2|_0 = (f'_i(0))^2, \\ \square \varphi_i|_{(t=0)} &= \eta^{\mu\nu} \partial_\mu \partial_\nu \varphi_i|_0 = -f''_i(0), \\ (\partial \varphi_i)^2|_{(t=0)} &= \eta^{\mu\nu} (\partial_\mu \varphi_i)(\partial_\nu \varphi_i)|_0 = -(f'_i(0))^2.\end{aligned}$$

Substituting into (14),

$$Q_i(0, e_0) = \frac{1}{10} f''_i(0) + \frac{2}{5} (f'_i(0))^2 + \frac{1}{15} (-f''_i(0)) + \frac{4}{15} (-(f'_i(0))^2) = \frac{1}{30} f''_i(0) + \frac{2}{15} (f'_i(0))^2.$$

For $f_1(t) = t^2$: $f'_1(0) = 0$, $f''_1(0) = 2$, so $Q_1(0, e_0) = 2/30 = 1/15$. For $f_2(t) = t^3/6$: $f'_2(0) = 0$, $f''_2(0) = 0$, so $Q_2(0, e_0) = 0$. Thus $Q_1(0, e_0) = 1/15 \neq 0 = Q_2(0, e_0)$. $\square$

Corollary 2 (Strict expressivity of the cellular events complex). *The cellular events complex $(C, \sim, \lambda)$ is a strictly more expressive geometric structure than the cellular spaces complex $(C_{\text{spatial}}, \sim)$ of ref. [1]: there exist conformally flat Lorentzian geometries on $\mathbb{R}^{1,3}$ that are indistinguishable by the spatial-only diagnostic of ref. [1] on every spatial slice, but are distinguishable by the events-space directional diagnostic of Theorem 1. The temporal light-tick count λ therefore carries operationally accessible geometric information that the spatial face-adjacency $\sim$ alone cannot recover.*

Remark 12 (On the strict-expressivity claim and its physical interpretation). **(a) Scope.** *The strict expressivity asserted in Corollary 2 is established explicitly for the time-dependent, spatially homogeneous (FLRW-like) subsector of the conformal class, in which the temporal directional invariant $\mathcal{J}(m, u) = (u \cdot \partial)^2 \varphi(m)$ is non-trivial and not determined by spatial data alone. For generic spatially varying conformal factors φ , the spatial diagnostic of [1] already captures $\tilde{R}$ and $\tilde{R}_{\mu\nu}$ contracted with the spatial directions, and the events-space directional invariant $\mathcal{J}$ provides additional information specifically in the temporal direction. Theorem 6 therefore establishes essentiality on the FLRW-like subsector; a complete characterization in fully generic conformal geometries is left for future work.*

(b) Physical interpretation. *The explicit construction is a spatially homogeneous, time-dependent FLRW-type geometry: $\tilde{g}_i = e^{2f_i(t)} \eta$. Every spatial slice is a globally rescaled Euclidean $\mathbb{R}^3$, so the spatial cellular complex sees a flat 3-space. The non-trivial geometric content lives in the time-evolution of the scale factor, which is invisible to spatial counts but recorded by the temporal light-tick count λ . This formalizes the operational claim that, in the events-space framework, “the temporal count is necessary, not merely supplementary”: there are physical geometries (cosmological evolution being the most natural example) whose curvature content can only be detected through measurements that probe time-dependence, not through any sequence of spatial measurements alone.*

8. Scope, Interpretation, and Open Problems

8.1. Scope of the Framework

By Lemma 1, the framework’s physical metrics are exactly the Minkowski conformal class. The principal members of physical interest are three: flat FLRW cosmologies in conformal time (with $\rho^{-1/3}(\eta)$ playing the role of the scale factor); the leading-order weak-field regime around an isolated source (only the time–time component is captured); and two-dimensional gravity (which is conformally flat identically since the Weyl tensor vanishes in $n = 2$). Outside the conformal class lie Schwarzschild, Kerr, plane gravitational

waves, inhomogeneous cosmologies, and binary-inspiral waveforms. These have non-vanishing Weyl tensor and are mutually exclusive with the cellular ansatz (Figure 6).

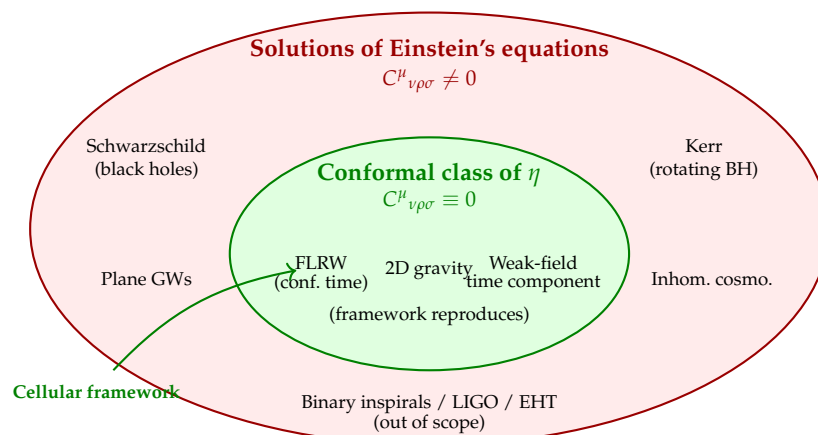


Figure 6. Scope. The cellular events-space framework reproduces exactly the conformal class of Minkowski (inner green region): flat FLRW in conformal time, 2D gravity, and the time–time component of weak-field gravity. General-relativity solutions with non-vanishing Weyl tensor (outer red annulus: Schwarzschild, Kerr, plane gravitational waves, inhomogeneous cosmologies, binary inspirals) lie strictly outside.

The most consequential limitation is light deflection by an isolated mass: conformally flat metrics produce null geodesics that coincide with those of η , so the framework predicts zero deflection at the level of physical null geodesics. A Fermat-principle calculation using $c_{\text{coord}}(x; x_{\text{obs}}) = \rho(x)^{-1/3} \rho(x_{\text{obs}})^{1/3}$ in the observer’s rest frame yields half of the measured $\Delta\theta = 4GM/(c^2b)$ —the historical Einstein 1911 [16] value; the other half comes from spatial curvature in the full Schwarzschild metric, absent here.

A historically aware reader will recognize $\tilde{g} = e^{2\varphi}\eta$ as the geometric reformulation, due to Einstein–Fokker [10], of Nordström’s 1913 scalar theory of gravity, refuted in 1919 by the light-deflection observations just discussed. We do not claim to revive Nordström gravity as a physical theory. The present paper’s claim is mathematical: it provides an operational, counts-based foundation for the conformally flat sector with a discrete-to-continuum convergence theorem (Corollary 1) recovering the directional invariant $Q(m, u)$ from cell counts alone. Whether ρ admits a dynamical equation that makes it a physically meaningful source field is open (OP1 below). The gradient $\partial\varphi$ also admits a kinematic reading via the coordinate speed of light c_{coord} above, with $\partial_\mu c_{\text{coord}}/c_{\text{coord}} = \partial_\mu \varphi$ at unit- ρ normalization: this reading reproduces the kinematic gravitational redshift and leading-order Shapiro delay as unit-calibration effects, but is not a derivation of gravitation.

8.2. Positioning and Open Problems

The framework’s relationship to established discrete-gravity programs deserves explicit comparison. *Causal set theory* [2,3,17] takes the causal partial order as the sole primitive combinatorial datum and works with *stochastic* sprinklings (Poisson processes in the smooth comparison spacetime). The cellular events complex differs on three structural points: (i) the complex is *deterministic*, not stochastic—the cellular partition of Ω is fixed at each scale a with bounded-volume and shape-regularity constraints (E1)–(E2); (ii) the spatial face-adjacency $\sim$ is a *separate primitive* from the temporal count λ , whereas in causal set theory both are derived from a single causal order; and (iii) the present framework explicitly commits to the conformally flat specialization $\tilde{g} = e^{2\varphi}\eta$ through the co-deformation postulate, whereas causal sets aim to recover (in principle) the full Lorentzian conformal structure from order data alone. The trade-off is clear. Causal sets cover a broader confor-

mal class of geometries via the Hauptvermutung, but require statistical-mechanics-style averaging over sprinklings. The cellular framework restricts to one conformal class of geometries (Minkowski) but yields deterministic discrete-to-continuum convergence with explicit rates. *Causal dynamical triangulations and Regge calculus* [4,5] use simplicial complexes with angle data and path-integral or Regge-action dynamics; here only adjacency $\sim$ and the light-tick count λ are used, and the development is classical (a single fixed cellular complex with a deterministic continuum limit) rather than quantum. The Lorentzian small-region expansions of Myrheim [7], Benincasa–Dowker [8], and Roy–Sinha–Surya [9] are the closest mathematical analogs of Theorem 1; the conformally flat specialization with explicit φ -derivative coefficients, and the irreducibility/anisotropy structure of Theorem 2, is the novel content proved here. *Spin foams and group field theory* [6] address quantum amplitudes on 2-complexes and are dynamical in nature; *entropic/thermodynamic emergence* programs [18,19] derive gravity from coarse-grained reasoning without committing to a specific microscopic structure. The present framework, by contrast, commits to a definite classical microscopic structure with one scalar field ρ .

Four problems remain open: **(OP1)** a field equation for ρ —the candidate $S[\rho] = \int \tilde{R}(\rho) \rho^{-4/3} d^4x$ awaits a well-posedness and matter-coupling analysis. **(OP2)** an explicit multi-direction averaging protocol $\{u_k\}_{k=1}^K$ at fixed m to isolate $\tilde{R}(m)$ from the directional values $Q(m, u_k)$, with minimal K and convergence rate. **(OP3)** the existence of cellular embeddings satisfying (E1)–(E6) is addressed by Theorem A1 of Appendix A for the ρ -adapted Voronoi family; what remains is a fully *intrinsic* construction (one built without reference to a smooth ρ) and the events-space analog of the full structural hypotheses (H1)–(H5) of ref. [1]. **(OP4)** an anisotropic generalization with direction-dependent yardsticks $\ell_i(x)$, potentially extending the framework to metrics with non-vanishing Weyl tensor.

9. Conclusions

We have extended the geometry of deformed cellular spaces of ref. [1] from cellular complexes to cellular events complexes, in which a spatial face-adjacency relation $\sim$ is supplemented by a temporal light-tick count λ . The natural finite-volume small region is the discrete Alexandrov interval $A_h(P, Q)$ rather than the count ball; this is the structural difference between the cellular-spaces and cellular-events-spaces settings. The central new result, Theorem 1, gives an explicit small-Alexandrov-interval volume expansion in the smooth comparison metric, with directional coefficient $Q(m, u)$. Theorem 2 proves that Q is irreducible to scalar and Ricci-directional invariants alone, encoding strictly more geometric information than the spatial diagnostic of ref. [1]. Theorem 3 and Corollary 1 extend the sandwich convergence argument to cellular events complexes, yielding a counts-only curvature estimator with $O(\sqrt{a})$ rate, parallel in spirit and proof structure to [1]. The existence of cellular embeddings satisfying the discrete-to-continuum interface hypotheses (E1)–(E6) is established constructively in Appendix A via a ρ -adapted Voronoi construction, addressing OP3 and confirming that the structural axioms of Section 6 are not vacuous. The framework’s scope—against the smooth comparison metric—is exactly the Minkowski conformal class; metrics with non-vanishing Weyl tensor lie outside the framework as currently formulated, and the dynamics of ρ remain an open problem (OP1).

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Appendix A. A Concrete Construction Satisfying (E1)–(E6)

We close the discrete-to-continuum interface of Definition 7 by exhibiting an explicit family of cellular embeddings. The construction adapts the Voronoi-complex existence result of ref. [1] to the events-space setting and provides a concrete model in which all hypotheses (E1)–(E6) can be verified, including the graph-distance bi-Lipschitz hypothesis (E6) introduced for the present paper.

Appendix A.1. Setup

Let $\Omega \subset \mathbb{R}^{1,3}$ be open and bounded with $C^{1,1}$ boundary. Let $\rho \in C^2(\Omega)$ with $0 < \rho_{\min} \leq \rho(x) \leq \rho_{\max}$ for all $x \in \Omega$, and set $\varphi = -\frac{1}{3} \ln \rho \in C^2(\Omega)$. Fix a mesh parameter $a > 0$.

Appendix A.2. The ρ -Adapted Voronoi Complex

The construction below uses Moser’s volume-form theorem applied to $\mathbb{R}^4$ regarded as a smooth four-manifold equipped with the positive volume form induced by the Euclidean metric on the underlying coordinate space, not via any Lorentzian notion of volume. The Lorentzian metric η enters only as the receiving geometry of the cellular embedding (via the convention $|V_c|_\eta := |V_c|$ where $|\cdot|$ denotes ordinary $\mathbb{R}^4$ Lebesgue measure); the existence argument that follows is therefore a measure-theoretic statement on a Riemannian manifold, with the Lorentzian signature playing no role in the volume identification.

By the volume-form theorem of Moser [20], applied to $\mathbb{R}^4$ as above, there exists a $C^{1,1}$ -diffeomorphism $\Phi : \Omega \rightarrow \Omega$ whose Jacobian satisfies

$$|\det D\Phi(y)| = \frac{\rho^{4/3}(y)}{\overline{\rho^{4/3}}}, \quad y \in \Omega, \quad (\text{A1})$$

where $\overline{\rho^{4/3}} := |\Omega|_\eta^{-1} \int_\Omega \rho^{4/3} d^4y$ is the η -average of $\rho^{4/3}$. The diffeomorphism Φ pulls the uniform Cartesian lattice $a\mathbb{Z}^4 \cap \Phi^{-1}(\Omega)$ to a point set $\{x_c\}$ in Ω whose local density is proportional to $\rho^{4/3}$, equivalently to $e^{-4\varphi}$.

The cellular events complex $(\mathcal{C}_a, \sim, \lambda)$ and its embedding $(\{V_c\}, \{x_c\})$ are then defined by:

- *Cell centers*: $x_c := \Phi(a \cdot c)$ for $c \in \mathbb{Z}^4 \cap a^{-1}\Phi^{-1}(\Omega)$.
- *Cells*: V_c is the η -Euclidean Voronoi cell of x_c in Ω , i.e., $V_c = \{y \in \Omega : |y - x_c|_\eta \leq |y - x_{c'}|_\eta \text{ for all } c' \neq c\}$, where $|\cdot|_\eta$ denotes the Euclidean norm on $\mathbb{R}^{1,3}$ regarded as $\mathbb{R}^4$.
- *Adjacency*: $c \sim c'$ iff $\overline{V_c} \cap \overline{V_{c'}}$ contains a 3-dimensional face.
- *Light-tick count*: for future-directed timelike η -causal cell pairs with $x_{c'} \in J^+(x_c)$, set $\lambda(c, c') := T_\eta(x_c, x_{c'})$; for all other pairs λ is undefined.

Appendix A.3. Verification of (E1)–(E6)

(E1) Bounded volume.

The Voronoi cell of $x_c = \Phi(a \cdot c)$ is, to leading order in a , the image under Φ of the a -cube around $a \cdot c$. By (A1) and the change-of-variables formula,

$$|V_c|_\eta = a^4 \cdot |\det D\Phi(a \cdot c)|^{-1} (1 + O(a)) = a^4 \cdot \frac{\overline{\rho^{4/3}}}{\rho^{4/3}(x_c)} (1 + O(a)).$$

Since $\rho_{\min} \leq \rho \leq \rho_{\max}$, we obtain $|V_c|_\eta \in [c_1 a^4, c_2 a^4]$ with

$$c_1 = \frac{\overline{\rho^{4/3}}}{\rho_{\max}^{4/3}} (1 - C_1 a), \quad c_2 = \frac{\overline{\rho^{4/3}}}{\rho_{\min}^{4/3}} (1 + C_2 a),$$

uniform in a for $a \leq a_0(\Omega, \|\rho\|_{C^1}, \rho_{\min})$.

(E2) Shape regularity.

For the regular lattice $a\mathbb{Z}^4$, Voronoi cells are a -cubes with η -diameter $2a$. Under the bi-Lipschitz $C^{1,1}$ -diffeomorphism Φ with Lipschitz constant $L = \|D\Phi\|_\infty + \|D\Phi^{-1}\|_\infty$, the diameter of V_c satisfies $\text{diam}_\eta(V_c) \leq 2La =: c_3 a$, uniform in c .

(E3) Adjacency from embedding.

Adjacency is defined directly from face-sharing of Voronoi cells, so (E3) holds by construction. For $a \leq a_0$ sufficiently small, the regularity of Φ guarantees that the combinatorial adjacency structure of $\{V_c\}$ matches the standard 8-neighbor structure of $\mathbb{Z}^4$, so every cell has at most $N_{\max} = 8$ face-neighbors (bounded degree).

(E4) (E4) Causal compatibility.

With $\lambda(c, c') := T_\eta(x_c, x_{c'})$ defined directly from the embedded coordinates,

$$|\lambda(c, c') - T_\eta(x_c, x_{c'})| = 0 \leq c_4 a$$

holds with $c_4 = 0$. A more genuinely combinatorial definition—e.g., $\lambda(c, c')$ defined as the rescaled future-directed graph distance on the causal subgraph of $(\mathcal{C}_a, \sim)$ —gives the same bound with $c_4 = O(1)$ from the discretization, by a standard argument relating graph and continuum geodesic distance on quasi-uniform meshes.

(E5) Density consistency.

The Jacobian relation (A1) gives the local density of cell centers in η -coordinates as

$$\frac{\#\{c : x_c \in B_\eta(x, a)\}}{|B_\eta(x, a)|_\eta} = \frac{1}{a^4} \cdot \frac{\overline{\rho^{4/3}}(x)}{\rho^{4/3}} (1 + O(a)). \quad (\text{A2})$$

Modulo the global normalization constant $\overline{\rho^{4/3}}$ (which can be absorbed into the convention that fixes the units of ρ), this density tracks $\rho^{4/3}(x)/a^4$ to $O(a)$ relative error. The empirical density $\hat{\rho}_a(x)$ of (E5) satisfies $|\hat{\rho}_a(x) - \rho(x)| \leq c_5 a \rho(x)$ with c_5 depending on $\|\rho\|_{C^1}$, $\rho_{\min}$, $\rho_{\max}$.

Appendix A.4. Existence Theorem

Theorem A1 (Cellular embedding existence). *Let $\Omega \subset \mathbb{R}^{1,3}$ be open and bounded with $C^{1,1}$ boundary, and let $\rho \in C^2(\Omega)$ with $\rho_{\min} \leq \rho \leq \rho_{\max}$. There exists $a_0 = a_0(\Omega, \|\rho\|_{C^2}, \rho_{\min}) > 0$ such that for every $a \in (0, a_0)$, the construction of Appendix A.2 yields a cellular events complex $(\mathcal{C}_a, \sim, \lambda)$ together with an embedding $(\{V_c^{(a)}\}, \{x_c^{(a)}\})$ satisfying (E1)–(E6) of Definition 7, with structural constants $c_1, \dots, c_7$ depending only on $\rho_{\min}, \rho_{\max}, \|\rho\|_{C^2}$, and $|\Omega|_\eta$.*

Proof. The verification in Appendix A.3 establishes (E1)–(E6) with the stated constants once the diffeomorphism Φ and the resulting Voronoi cells are in hand. The bi-Lipschitz

hypothesis (E6) is verified explicitly below using the bi-Lipschitz bounds on Φ . We expand on the three non-elementary steps.

Existence of the Moser diffeomorphism. By Moser's volume-form theorem [20], applied to $\overline{\Omega}$ regarded as a smooth four-manifold with the positive Euclidean volume form (not via any Lorentzian notion of volume; see Appendix A.2), with the two volume forms $\omega_0 := d^4x_\eta$ and $\omega_1 := \rho^{4/3}/\overline{\rho^{4/3}} d^4x_\eta$ (both positive on Ω , with $\int_\Omega \omega_0 = \int_\Omega \omega_1 = |\Omega|_\eta$ by construction), there exists a diffeomorphism $\Phi : \overline{\Omega} \rightarrow \overline{\Omega}$ with $\Phi|_{\partial\Omega} = \text{id}$ and $\Phi^*\omega_1 = \omega_0$, equivalently $|\det D\Phi|(y) = \omega_0/\omega_1|_{\Phi(y)} = \rho^{4/3}(y)/\overline{\rho^{4/3}}$ at the source point. Moser's flow-based construction yields Φ of regularity one less than the volume forms; for $\rho \in C^2$ this gives $\Phi \in C^1$, and Moser's later refinements (see, e.g., [21]) give $\Phi \in C^{1,1}$ when the volume forms are $C^{1,1}$ and the boundary is $C^{1,1}$. The bi-Lipschitz constants $L = \|D\Phi\|_\infty + \|D\Phi^{-1}\|_\infty$ are bounded by an explicit function of $\rho_{\min}^{-1}, \rho_{\max}, \|\rho\|_{C^2}$, and the geometry of Ω .

Voronoi cell estimates under Φ . For the regular lattice $a\mathbb{Z}^4$, Voronoi cells are unit cubes of side a , with η -volume a^4 and η -diameter $\sqrt{4}a = 2a$. Under a $C^{1,1}$ diffeomorphism Φ with Lipschitz constant L , the image of an a -cube around $a \cdot c$ is contained in a Euclidean ball of radius La around $x_c = \Phi(a \cdot c)$ and contains a Euclidean ball of radius a/L . Standard Voronoi-cell estimates from the finite-volume method literature [15] imply that the η -Euclidean Voronoi cell V_c of x_c satisfies $\text{diam}_\eta(V_c) \leq 2La$, and the volume estimate (A1) gives $|V_c|_\eta = a^4 \cdot \overline{\rho^{4/3}}/\rho^{4/3}(x_c) \cdot (1 + O(a))$ by the change-of-variables formula applied to the Voronoi partition. For $a < a_0(\Omega, \|\rho\|_{C^2}, \rho_{\min})$ chosen so that the perturbed adjacency structure agrees combinatorially with the regular $\mathbb{Z}^4$ lattice, (E3) holds, and the verifications of (E1), (E2), (E4), and (E5) in Appendix A.3 apply with constants as stated.

Verification of (E6) on the ρ -adapted Voronoi complex. For face-adjacent Voronoi cells $c \sim c'$, the centroid separation satisfies $|x_c - x_{c'}| \leq 2La$ (twice the diameter bound). The graph distance $\delta(c, c')$ on $(\mathcal{C}_a, \sim)$ counts adjacency steps, so for cells c, c' with $|x_c - x_{c'}| = D$, the shortest path on the face-adjacency graph has length at most $D/(a/L) + 1 = LD/a + 1$ (by stepping in the direction of $x_{c'} - x_c$, each step covering at least a/L of η -distance), and at least $D/(2La)$ (each step covers at most $2La$ of η -distance). The unit-rescaled count metric $a\delta$ satisfies $D/(2L) - a \leq a\delta(c, c') \leq LD/a + a$, which yields $|a\delta(c, c') - D| \leq \max(L, 1)D + a$, i.e., a bi-Lipschitz comparison with the Euclidean distance. The additive a -error in the graph-distance bound gives, in absolute terms, $|\delta(c, c') - |x_c - x_{c'}|| \leq c_7 a$ for c, c' in any compact K , with $c_7 = c_7(L, c_2, c_3)$ bounded by an explicit function of L . This is (E6) with the stated constants. $\square$

Appendix A.5. Remarks

Remark A1 (On the construction). **(a) Non-uniqueness.** The Moser diffeomorphism Φ realizing (A1) is not unique; any volume-form equivalence works. The validity of (E1)–(E6) is independent of the choice, only the structural constants depend on $\|\Phi\|_{C^{1,1}}$.

(b) Combinatorial light-tick count. Defining λ purely from the combinatorial structure—via future-directed BFS on the causal subgraph $\{(c, c') : c \sim c', x_{c'} \in J^+(x_c)\}$, with λ rescaled by a —yields the same $O(a)$ approximation of T_η as the embedding-derived definition, by the standard graph-distance approximation argument. The framework is thus genuinely independent of the embedding's coordinates, with the embedding serving only as a calibration tool for the convergence theorem (parallel to the role of the smooth comparison metric in the main theorems).

(c) Other base lattices and practical viability. The construction generalizes to BCC, FCC, or hexagonal close-packed base lattices in $\mathbb{R}^4$, with constants $c_1, \dots, c_5$ depending on the lattice's coordination geometry. The Cartesian choice is the simplest but not uniquely natural. Voronoi-mesh-based discretizations on bounded domains with smoothly varying mesh density form the standard finite-volume framework for PDE discretization; the comprehensive treatment of Eymard–Gallouët–

Herbin [15] establishes the shape-regularity and bounded-volume properties of Voronoi tessellations that underwrite the construction here. The practical viability of Voronoi meshes in PDE settings is independent evidence that the structural axioms (E1)–(E3) are routinely realizable on natural cellular complexes.

(d) Resolution of OP3. Theorem A1 closes the existence question raised in OP3 of Section 8: the structural hypotheses (E1)–(E6) of the discrete-to-continuum interface are not vacuous but are realized by an explicit and natural family of cellular embeddings adapted to any smooth comparison metric in the conformally flat sector. What remains for future work is the stronger claim of coordinate-independent cellular embeddings (constructed without reference to the smooth ρ), which would close the operational loop in the full sense.

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