

Article

Enhanced Statistical Inference for Finite Population Distribution Function Using Two Auxiliary Information Variables: Application on Real-Life Data and Simulation Study

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Abstract

Estimation of the population distribution function is an important problem in survey sampling with a variety of applications in social sciences, medical studies, environmental research and economics and in industrial quality control. When auxiliary information is not effectively used, the existing estimators under simple random sampling (SRS) may have low precision. This study considers a novel enhanced estimator for the population distribution function based on the use of two auxiliary variables which are correlated with the study variable, motivated by the desire to improve estimation accuracy and stability. The proposed estimator is based on simple random sampling and the bias and mean square error expressions are derived to first order of approximation. Mean square error and percentage relative efficiency are used to assess the efficiency of the estimator. An empirical validation is performed through a numerical investigation using simulated and real data sets. It is found that the proposed estimator is more efficient and stable, and more accurate than traditional estimators under realistic sampling condition. So, the proposed methodology is reliable and practically useful for contributing to the development of efficient estimation procedures in sampling theory and applications.

Keywords: statistical inference; enhanced estimator; MSE; simulation study; two auxiliary variables; efficiency

MSC: 62D05; 62L05; 62L10; 62L12; 62F12; 62F25

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1. Introduction

Estimating population parameters is a key goal in survey sampling, which enables researchers and practitioners to draw sound conclusions about a population from sample data. Although the traditional parameters of the population, including the population mean, total and proportion are commonly researched, the estimation of the population distribution function has become a significant topic over the last few years. The population distribution function provides all the information about the behavior of a study variable such as its dispersion, skewness and probability distribution. The precise esti-

mation of this function is needed in numerous applied areas such as economics, agriculture, medicine, and social sciences where knowledge of the whole distribution is more informative than that of summary measures alone.

In real-life scenarios, it is usually impossible to study the whole population because of time, cost and resource limitations. Thus, representative data are collected using sampling methods. Simple random sampling without replacement is one of the most popular techniques because it is simple and has desirable statistical features. In this sampling design, the probability of selection of every unit of population is equal and known, and therefore, it is unbiased in estimation. The most common estimator of the population distribution function in this scheme is the empirical distribution function. It can be defined as the ratio of sampled observations that are less than or equal to a given value, which acts as a natural and unbiased estimate of the actual distribution function. In real-life surveys, with a lot of presence, the auxiliary information is easily available in the form of ancillary variables which are correlated with the studied variable. One of the most active research fields in survey sampling is the use of auxiliary variables because it may contribute greatly to the improvement of estimators. Sampling variability can be minimized by using auxiliary information which can yield more credible estimates.

Numerous types of estimators have been devised over the years that utilize auxiliary information, especially when estimating population means and totals. These are the ratio estimators, product estimators and regression estimators, all of which utilize the relationship between the study variable and the auxiliary variables. The effectiveness of such approaches has encouraged scholars to generalize these concepts to the estimation of the population distribution function. This has led to several adjusted estimators, which are new and better than the empirical distribution function, because they use auxiliary information in different functional forms. The majority of preliminary research in this field has been conducted on the use of an auxiliary variable. Using some auxiliary information, Ref. [1] proposed more efficient classes of estimators to estimate some parameters of a population. Using auxiliary variables, their study demonstrated that a considerable reduction in bias and in mean square error (MSE) compared with the usual estimators can be obtained. Ref. [2] created an optimal class of estimators for multiple auxiliary variables. Multi-auxiliary information was shown to lead to higher estimation precision and to yield superior efficiency compared to single-auxiliary-variable estimators. In stratified sampling, Ref. [3] proposed efficient estimators for the population means using dual auxiliary information derived from real estate data. They found significant improvements in accuracy and MSE with their proposed method compared to conventional stratified estimators. For the estimation of finite population means, Ref. [4] suggested power and log-transformed ratio estimators for the use of auxiliary variables. Based on solar radiation data, they demonstrated that transformation-based estimators are more efficient and stable than current-ratio estimators. Ref. [5] suggested improved estimator for population distribution function using auxiliary information.

In [6], ratio-type estimators that use auxiliary information are developed to improve the estimation of population parameters. The study developed theoretical properties of the proposed estimators, compared their efficiency with that of estimators proposed by others [7] in the context of two auxiliary variables, and validated their efficiency through simulation studies. The results indicated that the proposed robust estimators performed well in the presence of outliers and non-normality. In median-ranked set sampling, Ref. [8] studied the estimation of the population mean based on auxiliary information.

There has been recent research on the estimation of the population DF in order to enhance the efficiency of the estimator by incorporating auxiliary information, addressing non-response problems and the development of generalized classes of estimators for various sampling schemes. Ref. [9] suggested an improved estimator for estimation of

population distribution function using auxiliary variables. Their study showed that the use of auxiliary information could substantially enhance the precision of the estimation compared to conventional estimators. The use of radiation data reinforced the relevance of the methodology. But the study primarily concentrated on one auxiliary variable, which might restrict the efficiency gains if more than one correlated auxiliary variable is available. In the same manner, Ref. [10] proposed an extended generalized class of estimators for population distribution functions where the auxiliary variables were introduced, simulation analysis and radiation science applications were discussed. Their role is to provide a general framework that can be used to obtain a number of existing estimators as special cases. The performance of the proposed estimators was compared to the performance of the other estimators through the simulation study and the superiority of the proposed estimators was proved in terms of reduced MSE. Although all of these are good, the method is mathematically complicated and can involve making assumptions about the behavior of the auxiliary variables. Ref. [11] further developed the estimation problem to the non-response context of complex survey sampling. The authors applied the auxiliary information to estimate the finite population distribution functions where the survey were faced with practical complications. This work is valuable because in real surveys there is often non-response and failing to take this into account could lead to biased results. The study yielded efficient estimators in the context of non-response adjustments, but their implementation in a large-scale survey may be somewhat complex.

Some new estimators for estimating the population distribution function were suggested by [12] using two auxiliary variables. Their findings showed that using auxiliary variables based on calibration methods significantly increased the efficiency of estimation and decreased the variance of the estimates. Using the rank information and the empirical distribution function as auxiliary variables, Ref. [13] developed two regression- and ratio-type estimators to estimate the finite population mean. Their analysis found that combining auxiliary information into the rank structure greatly improved the precision of the estimators and that they achieved better performance than several current methods. Under nonresponse, Ref. [14] studied the use of auxiliary information for the estimation of population parameters. The results showed that the proposed estimators were quite effective in reducing the negative impact of nonresponse and provided smaller mean square errors than the traditional estimators. In [15] a global penalized calibration method was proposed for the estimation of the finite population distribution function. They found that penalized calibration directly enhanced the stability and efficiency of the estimators, especially when the sample data are complicated and the auxiliary information has multiple variables. In the presence of nonresponse, Ref. [16] suggested a class of improved predictive estimators for the population distribution function. They showed through empirical and theoretical analysis that the proposed estimators resulted in significant improvements in efficiency and robustness compared to the current predictive estimators.

Ref. [17] explored optimal estimation methods in non-response scenario with two auxiliary variables in simple random sampling design. Their proposed estimator was able to effectively deal with the missing responses and yet had high efficiency. The study is important as a contribution to survey methodology because in many real-life investigations there is no way to avoid non-response. The proposed approach, however, may entail additional costs and effort to gather the auxiliary information. In addition, Ref. [18] introduced a new class of estimators which is an improved version of the generalized class of estimators when sampling is taken through simple random sampling. They performed best with lower MSE values than the traditional estimators and gave a flexible framework suitable for various survey situations. The theoretical foundations of the

study are solid, although the mathematical deductions are somewhat more involved than those of non-technical scientists. Ref. [19] considered the estimation of population distribution functions in the case of non-response in stratified random sampling. Their study combined the benefits of stratification and non-response adjustments, yielding greater efficiency of the estimators. The authors also carried out simulation experiments to complement their theoretical results. One drawback of the method is that it is based on the assumption of reliable auxiliary information throughout all strata, which may not always be available. Last but not least, the works of [20,21] also contributed to the literature by suggesting estimators based on dual auxiliary information and ratio-cum-exponential technique in the context of non-response. These investigations have proved that the blend of ratio and exponential forms is more stable and efficient estimators. Their empirical findings showed significant decrease in bias and MSE as compared to classical estimators. But such methods can be labor intensive in the optimization of the parameters, and may be sensitive to the errors in the auxiliary data. Overall, the available literature indicates that the use of auxiliary information, generalized structures of the estimator, exponential methods and non-response adjustments significantly enhances the estimation of distribution functions of the population. There are, however, a few research gaps that need to be addressed, such as developing easier to implement and simpler generalized estimators, efficient estimators that can be used under various complex sampling designs, and robust methods for extreme observations. The restrictions encourage further research on the design of better generalized classes of estimators for the population distribution function estimation with the aid of auxiliary data.

Inspired by such developments, the current research propose a new estimator of population distribution function under SRS without replacement using two auxiliary variables. A theoretical comparison is conducted to estimate the efficiency of the proposed estimator with others that can be found in the literature. The comparison is based on the basis of mean square error criterion which is an all-purpose measure of estimator performance. A smaller mean square error will imply a more efficient estimator because it implies lower variability and greater precision. The theoretical results are also reinforced with the help of numerical examples based on real or simulated data and which reveal the practical benefits of the proposed estimator. This study is relevant because it contributes to the development of survey sampling in estimation. Through the use of the two auxiliary variables, the proposed estimator will ensure a more efficient estimator compared to the traditional methods, especially when there is rich auxiliary information. This has great implications to the applied research where proper estimation of the population distribution function is needed to make sound decisions.

The present study is innovative because a new enhanced estimator of finite population distribution function has been developed based on the two auxiliary variables in a single estimation framework for simple random sampling. The proposed estimator combines the two auxiliary variables, which are not utilized in many existing estimators, effectively, to obtain more accurate and stable estimation. The proposed methodology gives minimum bias and minimum mean MSE than the traditional and existing estimators available in the literature. Another novel result of this study is the theoretical properties of the estimator, including up to first-order approximation bias and mean square error expressions, thus showcasing the mathematical superiority of this estimator. In addition, the proposed estimator is assessed by real-life and simulation studies, which give a strong empirical evidence for the efficiency and robustness of the proposed estimator under various sampling situations. Another important feature of the proposed framework is that it can be used to create some well-known estimators as special cases so that the framework is widely applicable in practice in survey sampling problems.

It is worth pointing out that although a general class of estimators based on a single auxiliary variable suggested by [15], the present study is a significant step forward, by introducing a specific enhanced estimator that optimally processes two auxiliary variables simultaneously. This extension is based on the pragmatic fact that often there are several auxiliary variables available in survey design, and their combined use can provide a considerable improvement in efficiency compared to the use of one auxiliary variable. The theoretical framework, optimization strategy, and empirical validations proposed here are of a rather different nature than in our previous research which centered on dual auxiliary information problems. In particular, the former paper suggested the optimal generalized estimator to estimate the finite population distribution function based on one auxiliary variable. On the contrary, this manuscript builds a new generalized estimation framework which includes two auxiliary variables. The extension is not just a modification of the previous estimator, but a new formulation of a new estimator, a new introduction of additional relative error terms and covariance structures, new derivations of the bias and mean squared error (MSE), new optimum values of unknown constants and a new minimum MSE expression. In addition, the methodology is validated using other real-life data applications and detailed Monte Carlo simulation studies that have not been included in the previous study.

Finally, it is worth noting that the application of dual auxiliary information is a promising way to enhance the performance of estimators in the simple random sampling. The suggested estimator not only increases precision, but it also maintain simplicity and flexibility, therefore, being applicable to practice. The outcomes of this research demonstrate the possible positive outcomes of using a variety of information sources and open the way to the further research into the elaboration of sophisticated methods of estimation in survey sampling.

Motivation and Contribution of the Study

The population distribution function estimation has been extensively studied in survey sampling especially in simple random sampling. Despite the empirical distribution function being unbiased and simple to calculate, it does not make use of auxiliary information which can significantly help to increase the efficiency of the estimation. To overcome this drawback, different ratio, product and regression approaches have been presented in the literature. However, most of the existing studies focus primarily on the use of one auxiliary variable.

In many practical cases, there are several auxiliary variables, each of which may have some information on the study variable. Failure to take into account any other auxiliary information will affect the accuracy and efficiency of the estimation. The current approaches based on two auxiliary variables are still restricted and some are mathematically involved and have insufficient theoretical and experimental background. Also, many studies do not cover comparative analyses between samples collected under comparable conditions.

This study aims to propose a new estimator of the population distribution function under simple random sampling using two auxiliary variables due to the above restrictions. The proposed estimator successfully integrates the two auxiliary information variables into a single estimation framework to enhance estimation accuracy and stability. The first-order approximation of bias and mean square error is theoretically developed to investigate mathematical characteristics of the estimator.

The MSE and PRE are used for the comparison of the proposed estimator with other available existing estimators. Through simulated and actual data sets, numerical studies show that the proposed estimator is more efficient, has lower MSE and is more stable than the traditional estimators. The study thus makes a contribution to the literature of

survey sampling in the sense that a flexible, efficient and practically useful estimation procedure is introduced in various fields of application like economics, agriculture, medicine and environmental sciences.

2. Materials and Methods

Let us suppose a population $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_N)$ includes N independent and identical elements. A sample of size n is selected with the help of SRSWOR from ω . The study and auxiliary variable are represented by Y and X , and mean is denoted by $\bar{Y}$, $\bar{X}$ and $\bar{Z}$ resemble to the sample means, $\bar{y}$ and $\bar{x}$ and $\bar{z}$. The indicator function for the study and auxiliary variables are denoted by $I(Y_i \leq Y)$, $I(X_i \leq X)$ and $I(Z_i \leq Z)$.

The indicator function is defined as:

$$I(Y_i \leq Y) = \begin{cases} 1 & Y_i \leq Y \\ 0 & \text{otherwise} \end{cases}$$

The indicator function transforms the auxiliary variable of the study variable into binary form, and is used for the estimation of population DF.

$$F_y = \frac{\sum_{i=1}^N I(Y_i \leq Y)}{N},$$

Its sample estimator based on SRS is given by:

$$\hat{F}_y = \frac{\sum_{i=1}^n I(Y_i \leq Y)}{n}.$$

Similarly, the auxiliary variables are given by:

$$F_x = \frac{\sum_{i=1}^N I(X_i \leq X)}{N}, \hat{F}_x = \frac{\sum_{i=1}^n I(X_i \leq X)}{n}, F_z = \frac{\sum_{i=1}^N I(Z_i \leq Z)}{N}, \hat{F}_z = \frac{\sum_{i=1}^n I(Z_i \leq Z)}{n},$$

the indicator function takes 1 when the observation satisfies the condition $Y_i \leq Y$ and 0 otherwise.

$$\varepsilon_0 = \frac{\hat{F}_y - F_y}{F_y}, \varepsilon_1 = \frac{\hat{F}_x - F_x}{F_x}, \varepsilon_2 = \frac{\hat{F}_z - F_z}{F_z}$$

such that

$$E(\varepsilon_0^2) = \lambda C_{F_y}^2, E(\varepsilon_1^2) = \lambda C_{F_x}^2, E(\varepsilon_2^2) = \lambda C_{F_z}^2$$

$$E(\varepsilon_0 \varepsilon_1) = \lambda \rho_{F_y F_x} C_{F_y} C_{F_x}, E(\varepsilon_0 \varepsilon_2) = \lambda \rho_{F_y F_z} C_{F_y} C_{F_z}, E(\varepsilon_1 \varepsilon_2) = \lambda \rho_{F_x F_z} C_{F_x} C_{F_z}$$

$S_{F_y}^2 = \frac{1}{N-1} \sum_{i=1}^N (I(Y_i \leq Y) - F_y)^2$, $S_{F_x}^2 = \frac{1}{N-1} \sum_{i=1}^N (I(X_i \leq X) - F_x)^2$, $S_{F_z}^2 = \frac{1}{N-1} \sum_{i=1}^N (I(Z_i \leq Z) - F_z)^2$, represents variance for the study and auxiliary information.

$C_{F_y}^2 = \left(\frac{S_{F_y}^2}{F_y^2} \right)$, $C_{F_x}^2 = \left(\frac{S_{F_x}^2}{F_x^2} \right)$, $C_{F_z}^2 = \left(\frac{S_{F_z}^2}{F_z^2} \right)$, represents coefficient of variation for the

study and auxiliary information.

$\rho_{F_y F_x} = \frac{\sum_{i=1}^N [I(Y_i \leq Y) - F_y][I(X_i \leq X) - F_x]}{\sum_{i=1}^N [I(Y_i \leq Y) - F_y]^2 + \sum_{i=1}^N [I(X_i \leq X) - F_x]^2}$, $\rho_{F_y F_z} = \frac{\sum_{i=1}^N [I(Y_i \leq Y) - F_y][I(Z_i \leq Z) - F_z]}{\sum_{i=1}^N [I(Y_i \leq Y) - F_y]^2 + \sum_{i=1}^N [I(Z_i \leq Z) - F_z]^2}$, represents correlation between the study and auxiliary information.

$$\lambda = \left(\frac{1}{n} - \frac{1}{N} \right)$$

3. Existing Adopted Estimators

Here we discussed some adopted existing estimators, which are given by:

(i) The usual estimator for population DF, is given by:

$$\hat{\bar{F}}_{usual} = \frac{1}{n} \sum_{i=1}^n I(Y_i \leq Y) \quad (1)$$

$$\text{Var}(\hat{\bar{F}}_{usual}) = \lambda \bar{Y}^2 C_{F_y}^2 \quad (2)$$

(ii) The ratio estimator for estimation of mean under PPS sampling is given by:

$$\hat{\bar{F}}_{Ratio} = \hat{F}_y \left(\frac{F_x}{\hat{F}_x} \right) \quad (3)$$

$$\text{Bias}(\hat{\bar{F}}_{Ratio}) \cong \lambda F_y (C_{F_y}^2 - \rho_{F_y F_x} C_{F_y} C_{F_x}), \quad (4)$$

$$\text{MSE}(\hat{\bar{F}}_{Ratio}) \cong \lambda F_y^2 (C_{F_y}^2 + C_{F_x}^2 - 2\rho_{F_y F_x} C_{F_y} C_{F_x}) \quad (5)$$

(iii) The product estimator for evaluation of population DF is given by:

$$\hat{\bar{F}}_{Product} = \hat{F}_y \left(\frac{\hat{F}_x}{F_x} \right) \quad (6)$$

$$\text{Bias}(\hat{\bar{F}}_{Product}) \cong \bar{Y} \lambda \rho_{F_y F_x} C_{F_y} C_{F_x}, \quad (7)$$

$$\text{MSE}(\hat{\bar{F}}_{Product}) \cong \lambda F_y^2 (C_{F_y}^2 + C_{F_x}^2 + 2\rho_{F_y F_x} C_{F_y} C_{F_x}) \quad (8)$$

(iv) The traditional regression estimator under population DF sampling along with variance are given by:

$$\hat{\bar{F}}_{Reg} = \hat{F}_y + D_1(F_x - \hat{F}_x) \quad (9)$$

$$\text{Var}(\hat{\bar{F}}_{Reg}) = \text{MSE}(\hat{\bar{F}}_{Reg}) \cong \lambda F_y^2 C_{F_y}^2 (1 - \rho_{F_y F_x}^2) \quad (10)$$

(v) The exponential ratio and product type estimators are given by:

$$\hat{\bar{F}}_{ER} = \hat{F}_y \exp \left(\frac{F_x - \hat{F}_x}{F_x + \hat{F}_x} \right) \quad (11)$$

and

$$\hat{\bar{F}}_{EP} = \hat{F}_y \exp \left(\frac{\hat{F}_x - F_x}{\hat{F}_x + F_x} \right) \quad (12)$$

$$\text{Bias}(\hat{\bar{F}}_{ER}) \cong \lambda F_y \left(\frac{3}{8} C_{F_x}^2 - \frac{1}{2} \rho_{F_y F_x} C_{F_y} C_{F_x} \right) \quad (13)$$

$$\text{MSE}(\hat{\bar{F}}_{ER}) \cong \lambda F_y^2 \left(C_{F_y}^2 + \frac{1}{4} C_{F_x}^2 - \rho_{F_y F_x} C_{F_y} C_{F_x} \right), \quad (14)$$

$$\text{Bias}(\hat{\bar{F}}_{EP}) \cong \lambda F_y \left(\rho_{F_y F_x} C_{F_y} C_{F_x} - \frac{1}{4} C_{F_x}^2 \right), \quad (15)$$

$$\text{MSE}(\hat{\bar{F}}_{EP}) \cong \lambda F_y^2 \left[C_{F_y}^2 + \frac{1}{4} C_{F_x}^2 + \rho_{F_y F_x} C_{F_y} C_{F_x} \right] \quad (16)$$

(vi) The authors in [22], suggested the following estimators using auxiliary variable and correlation coefficient, which are given by:

$$\hat{\bar{F}}_{K1} = \hat{F}_y \left(\frac{F_x - C_{F_x}}{\hat{F}_x - C_{F_x}} \right) \quad (17)$$

$$\hat{\bar{F}}_{K2} = \hat{F}_y \left(\frac{F_x - \beta_{2(F_x)}}{F_x - \beta_{2(v)}} \right) \quad (18)$$

$$\hat{\bar{F}}_{K3} = \hat{F}_y \left(\frac{F_x \beta_{2(F_x)} - C_{F_x}}{\hat{F}_x \beta_{2(v)} - C_{F_x}} \right) \quad (19)$$

$$\hat{\bar{F}}_{K4} = \hat{F}_y \left(\frac{F_x C_{F_x} - \beta_{2(F_x)}}{\hat{F}_x C_{F_x} - \beta_{2(F_x)}} \right) \quad (20)$$

The MSEs of $\hat{\bar{F}}_{Ki}$ ($i = 1, 2, 3, 4$) is given by:

$$\text{MSE}(\hat{\bar{F}}_{Ki}) \cong \lambda F_y^2 (C_{F_y}^2 + R_i^2 C_{F_x}^2 - 2R_i \rho_{F_y F_x} C_{F_y} C_{F_x}), \quad (i = 1, 2, 3, 4) \quad (21)$$

$$R_1 = \frac{F_x}{F_x - C_{F_x}}, R_2 = \frac{F_x}{F_x - \beta_2(F_x)}, R_3 = \frac{F_x \beta_2(F_x)}{F_x \beta_2(F_x) - C_{F_x}}, R_4 = \frac{F_x C_{F_x}}{F_x C_{F_x} - \beta_2(F_x)}$$

(vii) The authors discussed exponential estimator by [23], which are given by:

$$\hat{F}_{Upad} = \hat{F}_y \left(\frac{F_x + \beta_2(F_x)}{\hat{F}_x + \beta_2(F_x)} \right) \quad (22)$$

The MSE of $\hat{F}_{Upad}$, is given by:

$$MSE(\hat{F}_{Upad}) \cong \lambda F_y^2 (C_{F_y}^2 + R^2 C_{F_x}^2 - 2R\rho_{F_y F_x} C_{F_y} C_{F_x}), (i = 1, 2, 3, 4) \quad (23)$$

$$R = \frac{F_x}{F_x + \beta_2(F_x)}$$

(viii) A log-type estimator developed by [24], using study and auxiliary information:

$$\hat{F}_{MPPS} = \hat{F}_y + w \log\left(\frac{\hat{F}_x}{F_x}\right) \quad (24)$$

w is constant, and its value is given by:

The MSE of $\hat{F}_{MPPS}$ is given by:

$$MSE(\hat{F}_{MPPS}) \cong \lambda F_y^2 \left(A - \frac{C^2}{B} \right) \quad (25)$$

where

$$A = \lambda C_{F_y}^2, B = \lambda C_{F_x}^2, C = \lambda \rho_{F_y F_x} C_{F_y} C_{F_x}$$

4. Suggested Work

Learning the entire distribution of a study variable is informative in many practical areas such as economics, agriculture, health sciences, and social research, rather than just learning the summary of one or two measures like the mean or the variance. The population DF gives a detailed account about how a variable behaves and is crucial in making informed decisions, policies as well as analysis of risks. It is, however, not always possible to obtain valid estimates of the population DF as data collection, cost and time are limited and then utilize sampling methods.

The rationale for this research is that there is a need to enhance the efficiency of population distribution function estimation by better utilizing available auxiliary information. Although it has been established in the previous studies that the use of one auxiliary variable can improve the accuracy of the estimation; in most real-life scenarios one may have a number of auxiliary variables that are correlated with the study variable. For example, income can be a combination of education and experience or crop yield can be affected by the rain and fertilizer use. When this happens, a single auxiliary variable is not sufficient to make use of all information available.

Thus, the current research is motivated by the goal of developing a new and a better estimator of the population DF under simple random sampling. The proposed method will achieve better utilization of the existing information, the decrease in the mean square error, and better estimation method. The contribution will be useful both in theory and practice in survey sampling.

$$\hat{F}_{Prop} = \frac{1}{4} [\Omega_1 \hat{F}_y + \Omega_2 (F_x - \hat{F}_x) + \Omega_3 (F_z - \hat{F}_z)] \left[\exp\left(\frac{F_x - \hat{F}_x}{\hat{F}_x + F_x}\right) + \exp\left(\frac{\hat{F}_x - F_x}{\hat{F}_x + F_x}\right) \right] \left[\exp\left(\frac{F_z - \hat{F}_z}{\hat{F}_z + F_z}\right) + \exp\left(\frac{\hat{F}_z - F_z}{\hat{F}_z + F_z}\right) \right] \quad (26)$$

Theorem 1. To obtain bias and minimum MSE of $\hat{F}_{Proposed}$.

Proof. To obtain the bias and mean squared error of the $\hat{F}_{Proposed}$, we used the following error terms which are defined as:

$$\varepsilon_0 = \frac{\hat{F}_y - F_y}{F_y}, \varepsilon_1 = \frac{\hat{F}_x - F_x}{F_x}, \varepsilon_2 = \frac{\hat{F}_z - F_z}{F_z}$$

such that

$$E(\varepsilon_0^2) = \lambda C_{F_y}^2, E(\varepsilon_1^2) = \lambda C_{F_x}^2, E(\varepsilon_2^2) = \lambda C_{F_z}^2$$

$$E(\varepsilon_0 \varepsilon_1) = \lambda \rho_{F_y F_x} C_{F_y} C_{F_x}, E(\varepsilon_0 \varepsilon_2) = \lambda \rho_{F_y F_z} C_{F_y} C_{F_z}, E(\varepsilon_1 \varepsilon_2) = \lambda \rho_{F_x F_z} C_{F_x} C_{F_z}$$

Applying the error to Equation (26), we obtain the following simplified form:

$$\hat{F}_{Prop} = \frac{1}{4} [\Omega_1 F_y (1 + \varepsilon_0) + \Omega_2 (F_x - F_x (1 + \varepsilon_1)) + \Omega_3 (F_z - F_z (1 + \varepsilon_2))] \left[\exp\left(\frac{F_x - F_x (1 + \varepsilon_1)}{F_x + F_x (1 + \varepsilon_1)}\right) + \exp\left(\frac{F_x (1 + \varepsilon_1) - F_x}{F_x (1 + \varepsilon_1) + F_x}\right) \right] \left[\exp\left(\frac{F_z - F_z (1 + \varepsilon_2)}{F_z + F_z (1 + \varepsilon_2)}\right) + \exp\left(\frac{F_z (1 + \varepsilon_2) - F_z}{F_z + F_z (1 + \varepsilon_2)}\right) \right] \quad (27)$$

After further simplifying the above equations, we have:

$$\hat{F}_{Prop} = \frac{1}{4} [\Omega_1 F_y + \Omega_1 F_y \varepsilon_0 + \Omega_2 (-F_x \varepsilon_1) + \Omega_3 (-F_z \varepsilon_2)] \left[\exp\left(\frac{-F_x \varepsilon_1}{2F_x + F_x \varepsilon_1}\right) + \exp\left(\frac{F_x \varepsilon_1}{2F_x + F_x \varepsilon_1}\right) \right] \left[\exp\left(\frac{-F_z \varepsilon_2}{2F_z + F_z \varepsilon_2}\right) + \exp\left(\frac{F_z \varepsilon_2}{2F_z + F_z \varepsilon_2}\right) \right] \quad (28)$$

When we simplify the above equation, we obtain

$$\hat{F}_{Prop} = [\Omega_1 F_y (1 + \varepsilon_0) - \Omega_2 F_x \varepsilon_1 - \Omega_3 F_z \varepsilon_2] \left[1 + \frac{1}{8} \varepsilon_1^2 + \frac{1}{8} \varepsilon_2^2 \right] \quad (29)$$

When we subtracted F_y on both sides of the above equation and applied expectations, we got a bias of $\hat{F}_{Prop}$

$$\text{Bias}(\hat{F}_{Prop}) = F_y (\Omega_1 - 1) + \Omega_1 F_y \left[1 + \frac{1}{8} E(\varepsilon_1^2) + \frac{1}{8} E(\varepsilon_2^2) \right] \quad (30)$$

$$\text{Bias}(\hat{F}_{Prop}) = F_y (\Omega_1 - 1) + \Omega_1 F_y \left[1 + \frac{1}{8} \lambda C_{F_x}^2 + \frac{1}{8} \lambda C_{F_z}^2 \right] \quad (31)$$

$$\hat{F}_{Prop} - F_y = F_y (\Omega_1 - 1) + \Omega_1 \varepsilon_0 F_y + \frac{1}{8} \varepsilon_1^2 \Omega_1 F_y + \frac{1}{8} \varepsilon_2^2 \Omega_1 F_y - \Omega_2 \varepsilon_1 F_x - \Omega_3 \varepsilon_2 F_z \quad (32)$$

Take square and expectations, on both sides:

$$\left(\hat{F}_{Prop} - F_y \right)^2 = F_y^2 + \Omega_1^2 \left(F_y^2 + F_y \varepsilon_0^2 + \frac{1}{4} F_y^2 \varepsilon_1^2 + \frac{1}{4} F_y^2 \varepsilon_2^2 \right) + \Omega_2^2 (F_x^2 \varepsilon_1^2) + \Omega_3^2 (F_z^2 \varepsilon_1^2) - 2\Omega_1 \left(F_y^2 + \frac{1}{8} F_y^2 \varepsilon_2^2 + \frac{1}{8} F_y^2 \varepsilon_1^2 \right) - 2\Omega_1 \Omega_2 (F_y F_x \varepsilon_0 \varepsilon_1) - 2\Omega_1 \Omega_3 (F_y F_z \varepsilon_0 \varepsilon_2) + 2\Omega_2 \Omega_3 (F_x F_z \varepsilon_1 \varepsilon_2) \quad (33)$$

$$\text{MSE}(\hat{F}_{Prop}) = F_y^2 + \Omega_1^2 A + \Omega_2^2 B + \Omega_3^2 C - 2\Omega_1 D - 2\Omega_1 \Omega_2 E - 2\Omega_1 \Omega_3 G + 2\Omega_2 \Omega_3 F \quad (34)$$

The second order optimality conditions are examined to obtain a confirmation that the stationary solution found from the first-order partial derivatives is indeed the minimum MSE. Since

$$\text{MSE}(\hat{F}_{Prop}) = F_y^2 + \Omega_1^2 A + \Omega_2^2 B + \Omega_3^2 C - 2\Omega_1 D - 2\Omega_1 \Omega_2 E - 2\Omega_1 \Omega_3 G + 2\Omega_2 \Omega_3 F$$

Its Hessian matrix with respect to

$$\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$$

$$H = \begin{pmatrix} 2A & -2E & -2G \\ -2E & B & 2F \\ -2G & 2F & 2C \end{pmatrix}$$

Since the Hessian is symmetric, the criterion is employed to verify positive definite.

$$\Delta_1 = 2A > 0,$$

$$\Delta_2 = 4(AB - E^2) > 0$$

$$\Delta_3 = 8(ABC - AF^2 - CE^2 - BG^2 + 2EFG) = 8\nabla_1$$

Therefore,

$A > 0$, $AB - E^2 > 0$, $\nabla_1 > 0$ are sufficient conditions for the Hessian matrix to be positive definite. Under simple random sampling, A, B and C are positive because they

involve variance components, while the covariance structure of $(F_y, F_x \text{ and } F_z)$ is assumed to be positive definite. Consequently,

$$\nabla_1 > 0,$$

holds under the standard regularity assumptions used in multivariate survey sampling. Hence, the MSE function is strictly convex in the parameter vector

$$(\Omega_1, \Omega_2, \Omega_3)$$

and the stationary solution obtained by solving

$$\frac{\partial MSE}{\partial \Omega_1} = 0, \frac{\partial MSE}{\partial \Omega_2} = 0, \frac{\partial MSE}{\partial \Omega_3} = 0,$$

Accordingly:

$$\Omega_{1(opt)} = \frac{D(BC-F^2)}{\nabla_1}$$

$$\Omega_{2(opt)} = \frac{D(CE-FG)}{\nabla_1},$$

$$\Omega_{3(opt)} = \frac{D(BG-EF)}{\nabla_1},$$

are the unique optimal values minimizing the MSE of the proposed estimator. where

$$\begin{aligned} \nabla_1 &= ABC - AF^2 - CE^2 - BG^2 + 2EFG \\ \text{MSE}(\hat{F}_{Prop}) &= F_y^2 + \left(\frac{D^2 F^2 - BCD^2}{ABC - AF^2 - CE^2 - BG^2 + 2EFG} \right) \end{aligned} \quad (35)$$

where

$$A = F_y^2 + \lambda F_y^2 \left(C_{F_y}^2 + \frac{1}{4} C_{F_x}^2 + \frac{1}{4} C_{F_z}^2 \right)$$

$$B = F_x^2 \lambda C_{F_x}^2, C = F_z^2 \lambda C_{F_z}^2, D = F_y^2 + \lambda F_y^2 \frac{1}{8} (C_{F_x}^2 + C_{F_z}^2), E = F_y F_x \lambda \rho_{F_y F_x} C_{F_y} C_{F_x}, F = F_x F_z \lambda \rho_{F_x F_z} C_{F_x} C_{F_z}$$

$$G = F_y \lambda \rho_{F_y F_z} C_{F_y} C_{F_z}$$

□

5. Numerical Analysis

In this section we evaluate the effectiveness of the suggested and existing estimators, we used a real data set using situation-I and situation-II. We found MSE (Table 1 and Figure 1) and PRE (Table 2 and Figure 2) of all considered estimators.

Population-I: [Source: [25]]

Situation-I

$N = 923; n = 180; \bar{Y} = 436.4; \bar{X} = 333.2; \bar{Z} = 11,440, \lambda = 0.004472132; F_y = 0.2296858, F_x = 0.2708559; F_z = 0.2307692, C_{F_y} = 1.832324; C_{F_x} = 1.641621; C_{F_z} = 1.826732, \rho_{F_y F_x} = 0.8727393, \rho_{F_y F_z} = 0.8930209, \rho_{F_x F_z} = 0.8465885.$

Situation-II

$N = 923; n = 180; \bar{Y} = 394.5; \bar{X} = 367; \bar{Z} = 9975, \lambda = 0.004472132; F_y = 0.2502709, F_x = 0.2502709; F_z = 0.2502709, C_{F_y} = 1.731739; C_{F_x} = 1.731739; C_{F_z} = 1.731739, \rho_{F_y F_x} = 0.9133886, \rho_{F_y F_z} = 0.8902923, \rho_{F_x F_z} = 0.8614218.$

Table 1. Mean square error (MSE) using real data set.

Estimators	Situation-I $I(Y_i \leq 436.4)$ and $I(X_i \leq 333.2)$ $I(Z_i \leq 11,440)$	Situation-II $I(Y_i \leq 394.5)$ and $I(X_i \leq 367)$ $I(Z_i \leq 9975)$
$\hat{F}_{usual}$	0.0007921136	0.0008400402
$\hat{F}_{Ratio}$	0.0001892071	0.0001455141
$\hat{F}_{Product}$	0.002666644	0.003214647
$\hat{F}_{Reg}$	0.0001887813	0.0001392125
$\hat{F}_{ER}$	0.0003317073	0.0002827671
$\hat{F}_{EP}$	0.001570426	0.001817333
$\hat{F}_{K1}$	0.0006431316	0.0006596625
$\hat{F}_{K2}$	0.0009937937	0.001036924
$\hat{F}_{K3}$	0.001600968	0.001835229
$\hat{F}_{K4}$	0.00118032	0.0012347
$\hat{F}_{Upad}$	0.0006569431	0.0006990729
$\hat{F}_{MMPS}$	0.0001887813	0.0001392125
$\hat{F}_{Proposed}$	4.781164×10^{-5}	7.306818×10^{-5}

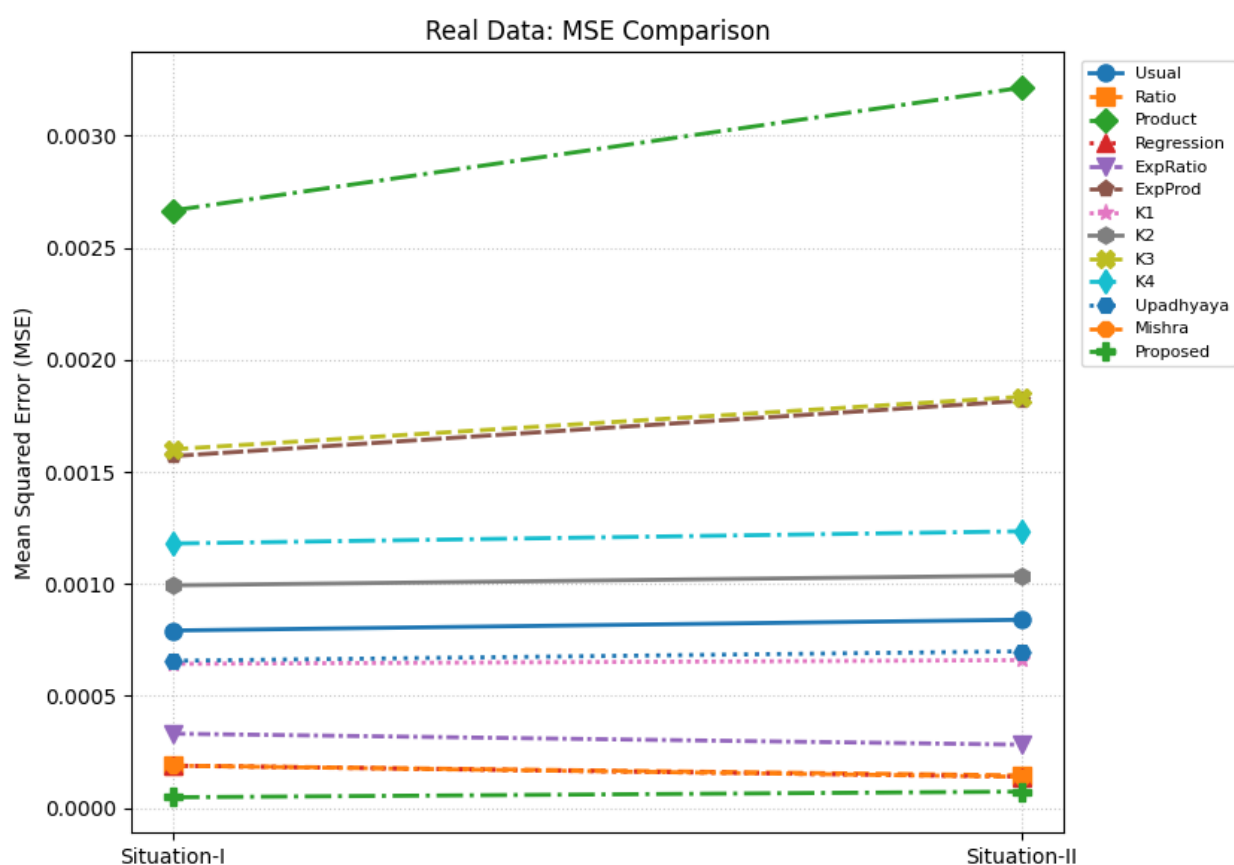
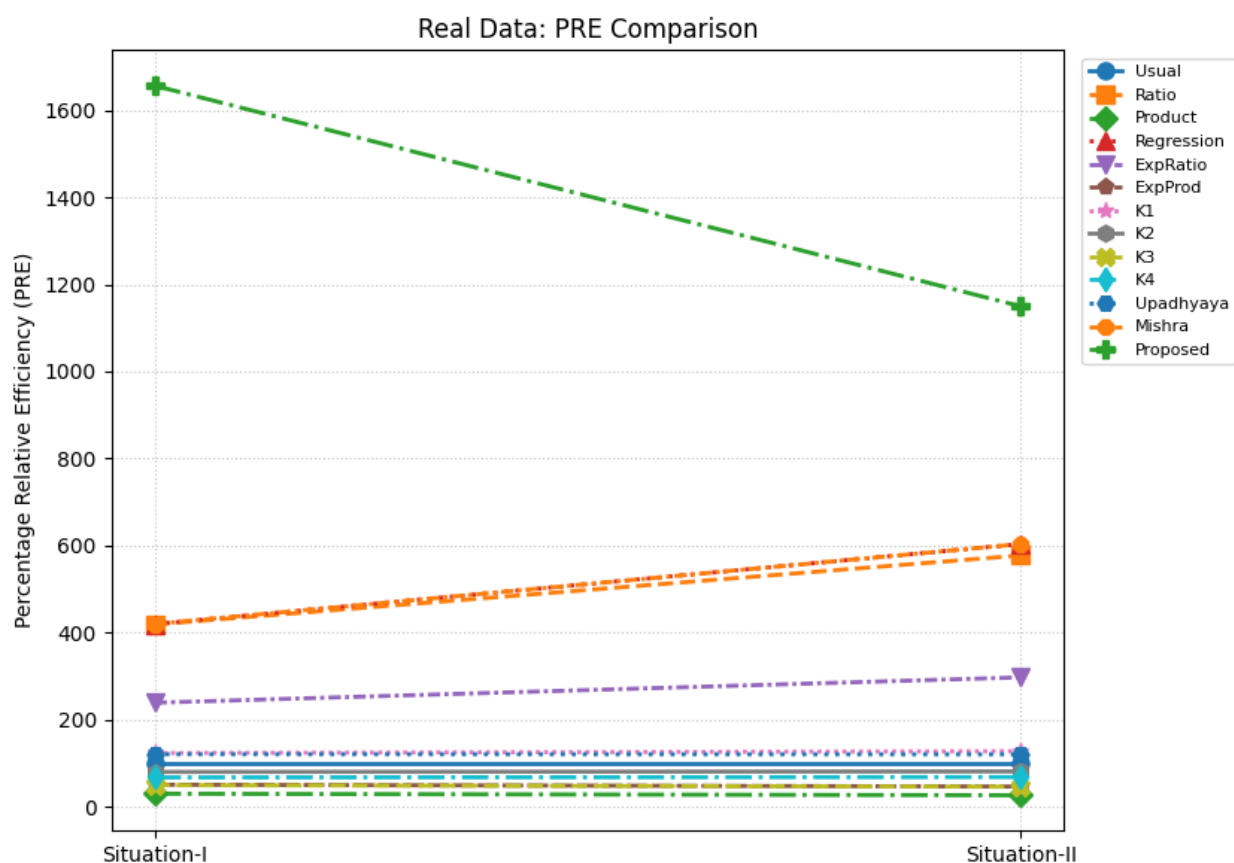
**Figure 1.** Showing MSEs of all considered estimators for both situations.

Table 2. PREs of all considered estimators using real data set.

Estimators	Situation-I $I(Y_i \leq 436.4)$ and $I(X_i \leq 333.2)$ $I(Z_i \leq 11,440)$	Situation-II $I(Y_i \leq 394.5)$ and $I(X_i \leq 367)$ $I(Z_i \leq 9975)$
$\hat{F}_{usual}$	100	100
$\hat{F}_{Ratio}$	418.649	577.2914
$\hat{F}_{Product}$	29.70451	26.13165
$\hat{F}_{Reg}$	419.5932	603.4231
$\hat{F}_{ER}$	238.799	297.0785
$\hat{F}_{EP}$	50.43941	46.22378
$\hat{F}_{K1}$	123.1651	127.3439
$\hat{F}_{K2}$	79.70603	81.01273
$\hat{F}_{K3}$	49.47717	45.77305
$\hat{F}_{K4}$	67.11009	68.03599
$\hat{F}_{Upad}$	120.5757	120.1649
$\hat{F}_{MMPS}$	419.5932	603.4231
$\hat{F}_{Proposed}$	1656.738	1149.666

**Figure 2.** Showing PREs of all considered estimators for both situations.

6. Simulation Analysis

In this section, a comprehensive and rigorous simulation study is performed to obtain empirical results for comparing the finite sample performance of the proposed es-

timator with that of the existing estimators. The main goal of this investigation will be to confirm the theoretical efficiency results obtained in the previous sections and to investigate whether the theoretical advantages presented under theoretical sampling conditions hold in real-life situations. The proposed estimators are evaluated in particular about their robustness, reliability and relative efficiency in case of different population structures, varying sample sizes and varying association between the study variable and the auxiliary variables. Two different artificial populations are produced with two different distributional features and data behaviors. These populations are similar to real-life examples typical of a survey sampling study, such as moderately to highly variable populations, with different levels of association between the study variable and the auxiliary variable.

In addition, multiple levels of correlation between the study variable and the auxiliary variable are included in the simulation design to explore the sensitivity of the estimators to changes in the auxiliary information. The study also systematically analyzes the auxiliary information used by each of these estimators by systematically varying the correlation coefficient to gain insight into the effectiveness of each estimator at utilizing auxiliary information to improve estimation accuracy. Mean squared error (MSE) and percentage relative efficiency (PRE) are commonly used to compare estimators, providing a quantitative, clear basis for comparison. Overall, the simulation study is intended not only to validate the theoretical results but also to illustrate the usefulness and performance of the suggested estimators across a variety of realistic scenarios.

The super population model used in the simulation study is given by

$$Y = 16 + \sqrt{1 - \rho_{xy}^2} Y^* + \rho_{xy} \left(\frac{\sigma_y}{\sigma_x} \right) X^*, X = 15 + X^*,$$

where X^* and Y^* are mutually independent random variables, generated from specified probability distributions. The parameter ρ_{xy} controls the strength of association between the study variable Y and the auxiliary variable X , thereby enabling the evaluation of estimator performance under weak, moderate, and strong correlation structures.

The simulation procedure is summarized as follows:

A rigorous and comprehensive simulation study is carried out to empirically investigate the performance of the proposed estimator in finite samples. The design of the simulation has been done on the basis of the super population design, as is commonly used in the literature on survey sampling design, and is described in Ref. [26]. Artificial populations with varying distributional types are created and the performance of the estimators is investigated for different sample sizes, different type and degree of correlation, and different population structures. Using the super population model defined above, two distinct finite populations of size $N = 1000$ are generated and take $n = 50, 100, 150, 200, 250$ and 300 , to represent different underlying distributional characteristics:

Gamma Population:

$X^* \sim \text{Gamma}(2.1, 0.51)$ and

$Y^* \sim \text{Gamma}(8.9, 0.11)$, generated under correlation coefficients

$$\rho_{xy} = 0.1, 0.3, 0.5, 0.7, 0.9.$$

Beta-I Population:

$X^* \sim \text{Beta}(2, 5)$ and

$Y^* \sim \text{Beta}(3, 4)$, generated under the same set of correlation coefficients

$$\rho_{xy} = 0.1, 0.3, 0.5, 0.7, 0.9.$$

Population size: $N = 1000$

Sample sizes: $n = 50, 100, 150, 200, 250, 300$ (SRSWOR)

Monte Carlo replications: $R = 20,000$

Distribution functions: For a fixed threshold $t_y = \text{median}(Y)$, $t_x = \text{median}(X)$, $t_z = \text{median}(Z)$, we define:

$$F_y = \frac{1}{N} \sum_{i=1}^N I(Y_i \leq t_y), F_x = \frac{1}{N} \sum_{i=1}^N I(X_i \leq t_x), F_z = \frac{1}{N} \sum_{i=1}^N I(Z_i \leq t_z).$$

The MSE and PRE are computed over R replications as:

$$\text{MSE}(\hat{F}) = \frac{1}{R} \sum_{r=1}^R (\hat{F}_r - F_y)^2,$$

$$\text{PRE}(\hat{F}) = \frac{\text{Var}(\hat{F}_{\text{usual}})}{\text{MSE}(\hat{F})} \times 100.$$

where

$$\hat{F} = (\hat{F}_{\text{usual}}, \hat{F}_{\text{Ratio}}, \hat{F}_{\text{Product}}, \hat{F}_{\text{Reg}}, \dots, \hat{F}_{\text{Proposed}}).$$

Note: For numerical results of MSEs and PREs based on Gamma Population see Appendix A (Tables A1–A6). We also visualize its numerical results See Figures A1–A6. And for numerical results of MSEs and PREs based on Beta-I population see Appendix B (Tables A7–A12). We also visualize its numerical results see Figures A7–A12.

7. Discussion

Both empirical and simulation studies show that the proposed estimator has the highest efficiency for estimating the population distribution function by making use of the auxiliary information. The estimator's aim is not just to reduce the sampling error, but to make an efficient use of the connection between the study and the auxiliary variables, to gain a significant improvement in estimation efficiency. This result is in accord with the theoretical results of the bias and mean squared error obtained in the previous sections, which indicates that this efficiency gain can be achieved if the auxiliary information relevant to the problem is used properly.

The proposed estimator has the least mean squared errors (MSEs) and the highest percentage relative efficiency (PRE) compared to all the other competing estimators for the real data analysis for both scenarios. The results suggest that the estimator provides more precise and stable estimates of population distribution function compared with the “conventional” estimators studied here. This is confirmed again in the empirical analysis, where the superiority is observed in the estimator's ability to successfully pick up the available auxiliary information, and to transfer that information into a reduction in estimation variability.

The relative performance of the existing estimators also offers interesting clues as to the effect of the correlation structure. The regression and MMPS estimators are relatively good since they effectively take advantage of the strong positive correlation between the study and auxiliary variables. Similarly, the ratio estimator is satisfactory when there is a positive correlation, as predicted by sampling theory. The product-type estimators, on the other hand, have relatively low performances since product-type estimators are more appropriate when the study and auxiliary variables are negatively correlated. As the data at hand are positively associated, the product-type estimators cannot take advantage of the auxiliary information very much in this case and thus have relatively high MSEs and low PREs.

The simulation study reinforces these results as it has been shown that the proposed estimator is superior for both the Gamma and the Beta-I super-population models and for a variety of correlation levels. The greater the relationship between the study variable and the auxiliary variables the more efficient the proposed estimator is, as evidenced by a

steady decrease in MSE and a steady rise in PRE. This behavior indicates the estimator's capacity to take advantage of the growing information content in auxiliary variables and yet to exhibit consistent performance across different population parameter distributions. The superiority of the results obtained from several distributions is that they indicate that the estimator is not sensitive to the model of a particular population, which shows good robustness.

Another practical lesson from the simulation results is that the proposed estimator becomes more competitive with the standard maximum likelihood-based estimator even for low values of the correlation, and becomes more beneficial as more and more accurate auxiliary information becomes available. Thus, the estimator is very appropriate for survey applications in which reliable auxiliary variables are available from administrative records or previous surveys. In cases like these, the estimator may obtain significant gains in precision without adding to the sample size and thus have higher precision and cost-effectiveness when estimating a survey.

The empirical and simulation studies corroborate the theoretical properties discussed in this paper generally. The results indicate that the proposed estimator is a trustworthy and efficient one for the estimation of the population distribution function as it shows a consistent decrease across all scenarios in MSE distance, with consistent improvement in PRE. The results suggest that the estimator is quite general and can be used by survey practitioners to provide more precise estimation when using auxiliary information.

8. Conclusions

Estimating the distribution of the population is a basic issue of survey sampling because it provides a thorough description of a study variable in a population. In this work, an attempt has been made to enhance the efficiency of estimating the population distribution function under simple random sampling without replacement by using two auxiliary variables. The rationale behind this strategy is that the inclusion of auxiliary information can in many ways improve the accuracy of estimators when used appropriately.

A modification of the traditional empirical distribution function to include dual auxiliary information in an appropriate functional form has been proposed as a new estimator. With the two auxiliary variables, the relationship between the study variable and auxiliary variables can be better exploited increasing the accuracy of the estimation. It is hoped that the proposed estimator will be as simple and practical as possible, and more efficient than the traditional estimators.

The statistical properties of the proposed estimator, such as bias and mean square error, have been obtained to first order. The theoretical outcomes of the results give a good insight into the behavior of the estimator and the conditions in which it works optimally. It has been noted that the efficiency of the proposed estimator is affected by the strength of the correlation between the study variable and the auxiliary variables and the variability of the auxiliary variables.

The proposed estimator has been compared to several existing estimators, including the empirical distribution function and estimators using only one auxiliary variable. Comparing the proposed estimator with the traditional estimators, based on a mean square error criterion, shows that the proposed estimator is more effective than the traditional estimators when both auxiliary variables are both strongly correlated with the variable of study. This shows the benefit of having dual auxiliary information in the estimation process.

Finally, this paper discusses the development of sampling theory as it provides an effective and efficient estimator that makes good use of two auxiliary variables. Results

highlight the value of auxiliary information for estimation accuracy of estimation and show that significant efficiency improvements can be achieved by using multiple auxiliary variables. In many spheres where it is necessary to estimate the distribution of a variable, the proposed estimator may be used.

In future studies, the given methodology can be applied to other types of sampling, including stratified or systematic sampling. Moreover, it is possible to perform extra work to generalize estimators or to include more than two auxiliary variables. These extensions would also increase the utility and strength of estimation methods in survey sampling.

9. Future Extension of the Study

There are several possible extensions that could be considered for improving the applicability and efficiency of the proposed estimator when estimating the population distribution function using dual auxiliary information.

- Several possible extensions could be further expanded upon to more complex sampling designs which include survey sampling, stratified random sampling, cluster sampling, multistage sampling, and systematic sampling. The estimators' performance under complex sampling schemes would make it more useful in practice, as is done in real-life surveys.
- Second, future studies might be able to use a non-response adjustment in the proposed estimator. In many applications of surveys, it is not possible to avoid missing observations or non-response, and these can come with a price of inaccurate estimation. Developing strong versions of the estimator under nonresponse would give the methodology greater plausibility and usefulness in large-scale survey investigations.
- The incorporation of powerful statistical methodologies for dealing with outliers and extreme observations is another key extension. If the data contain influential values, the current estimator may behave differently, so in cases where the data are skewed or contaminated, there may be greater benefit to using a robust estimator.
- Machine learning and artificial intelligence techniques can also be investigated for future research to derive suitable auxiliary variables and parameters. It is possible that a combination of traditional survey estimation techniques and modern computation methods could yield more adaptive and very efficient estimators.
- Furthermore, the reliability and robustness of the proposed estimator under various population structures, correlation levels, and sample sizes can be studied through simulation. These studies would help better understand the estimator's behavior in real-world settings.
- The method can also be further extended to estimate other important population parameters such as the population mean, variance, median, quantiles and finite population proportion. This would expand the scope of the proposed framework in the area of Statistics and survey sampling theory.

Finally, the proposed estimator could be used in other interdisciplinary areas like environmental sciences, medical research, agriculture, economics, and data analysis in engineering, where accurate estimation of population distribution functions is of vital importance in the process of decision making and policy making.

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Abbreviations

The following abbreviations are used in this manuscript:

MSE	Mean squared error
PRE	Percentage relative efficiency
SRS	Simple random sampling
DF	Distribution function

Appendix A

Table A1. Showing MSE and PRE results of all considered estimators using Gamma Population for $n = 50$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{usual}$	MSE = 0.002487 PRE = 100.0	MSE = 0.002491 PRE = 100.0	MSE = 0.002493 PRE = 100.0	MSE = 0.002489 PRE = 100.0	MSE = 0.002485 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.003872 PRE = 64.2	MSE = 0.003401 PRE = 73.2	MSE = 0.003147 PRE = 79.2	MSE = 0.002560 PRE = 97.2	MSE = 0.002101 PRE = 118.3
$\hat{F}_{Product}$	MSE = 0.004538 PRE = 54.8	MSE = 0.004599 PRE = 54.2	MSE = 0.004661 PRE = 53.5	MSE = 0.004745 PRE = 52.5	MSE = 0.004851 PRE = 51.2
$\hat{F}_{Reg}$	MSE = 0.002487 PRE = 100.0	MSE = 0.002480 PRE = 100.4	MSE = 0.002505 PRE = 99.5	MSE = 0.002589 PRE = 96.1	MSE = 0.002687 PRE = 92.5
$\hat{F}_{ER}$	MSE = 0.002796 PRE = 89.0	MSE = 0.002665 PRE = 93.5	MSE = 0.002535 PRE = 98.4	MSE = 0.002394 PRE = 104.0	MSE = 0.002257 PRE = 110.1
$\hat{F}_{EP}$	MSE = 0.003130 PRE = 79.5	MSE = 0.003164 PRE = 78.7	MSE = 0.003192 PRE = 78.1	MSE = 0.003236 PRE = 76.9	MSE = 0.003280 PRE = 75.8
$\hat{F}_{K1}$	MSE = 0.000892 PRE = 278.8	MSE = 0.000889 PRE = 280.2	MSE = 0.000891 PRE = 279.8	MSE = 0.000890 PRE = 279.7	MSE = 0.000889 PRE = 279.5
$\hat{F}_{K2}$	MSE = 0.002257 PRE = 110.2	MSE = 0.002250 PRE = 110.7	MSE = 0.002256 PRE = 110.5	MSE = 0.002252 PRE = 110.5	MSE = 0.002253 PRE = 110.3
$\hat{F}_{K3}$	MSE = 0.002257 PRE = 110.2	MSE = 0.002250 PRE = 110.7	MSE = 0.002256 PRE = 110.5	MSE = 0.002252 PRE = 110.5	MSE = 0.002253 PRE = 110.3
$\hat{F}_{K4}$	MSE = 0.002257 PRE = 110.2	MSE = 0.002250 PRE = 110.7	MSE = 0.002256 PRE = 110.5	MSE = 0.002252 PRE = 110.5	MSE = 0.002253 PRE = 110.3
$\hat{F}_{Upad}$	MSE = 0.002278 PRE = 110.190	MSE = 0.002277 PRE = 109.486	MSE = 0.002275 PRE = 108.755	MSE = 0.002273 PRE = 109.326	MSE = 0.002270 PRE = 109.471
$\hat{F}_{MMPS}$	MSE = 0.002460 PRE = 101.1	MSE = 0.002526 PRE = 98.6	MSE = 0.002652 PRE = 94.0	MSE = 0.002923 PRE = 85.2	MSE = 0.003252 PRE = 76.4
$\hat{F}_{Proposed}$	MSE = 0.000561 PRE = 443.3	MSE = 0.000444 PRE = 561.0	MSE = 0.000332 PRE = 750.9	MSE = 0.000216 PRE = 1152.3	MSE = 0.000108 PRE = 2300.9

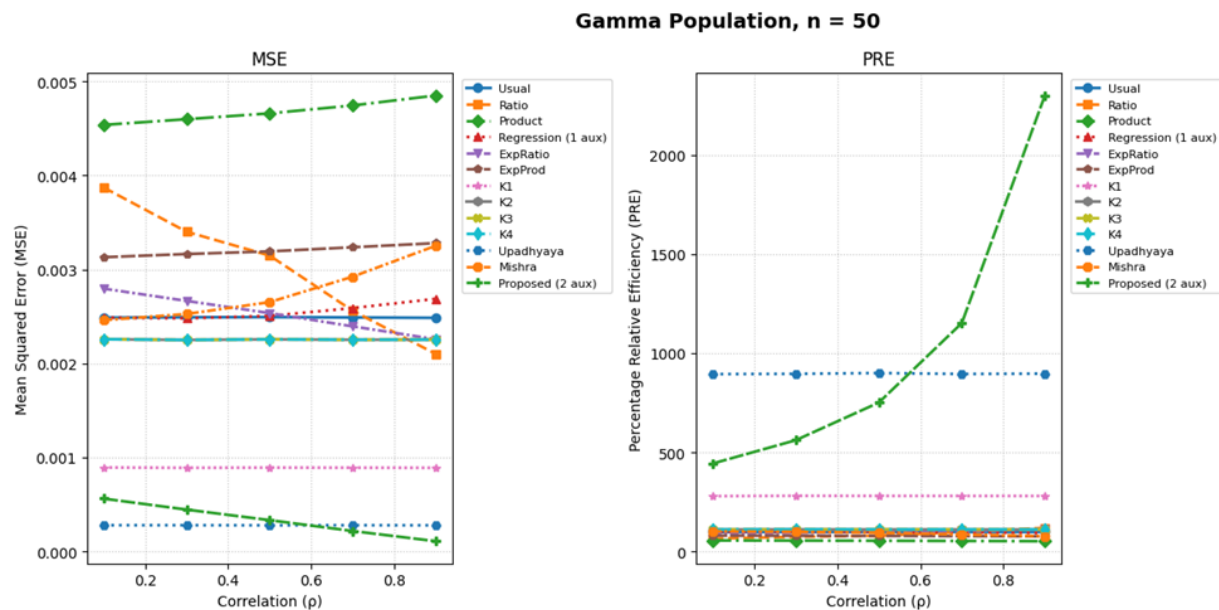


Figure A1. Showing MSEs and PREs using Gamma Population when $n = 50$.

Table A2. Showing MSE and PRE results of all considered estimators using Gamma Population for $n = 100$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{Usual}$	MSE = 0.001243 PRE = 100.0	MSE = 0.001246 PRE = 100.0	MSE = 0.001248 PRE = 100.0	MSE = 0.001245 PRE = 100.0	MSE = 0.001241 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.001872 PRE = 66.4	MSE = 0.001703 PRE = 73.2	MSE = 0.001547 PRE = 80.7	MSE = 0.001260 PRE = 98.8	MSE = 0.001001 PRE = 124.0
$\hat{F}_{Product}$	MSE = 0.002538 PRE = 49.0	MSE = 0.002599 PRE = 47.9	MSE = 0.002661 PRE = 46.9	MSE = 0.002745 PRE = 45.4	MSE = 0.002851 PRE = 43.5
$\hat{F}_{Reg}$	MSE = 0.001243 PRE = 100.0	MSE = 0.001240 PRE = 100.5	MSE = 0.001255 PRE = 99.4	MSE = 0.001289 PRE = 96.6	MSE = 0.001387 PRE = 89.5
$\hat{F}_{ER}$	MSE = 0.001396 PRE = 89.1	MSE = 0.001365 PRE = 91.3	MSE = 0.001335 PRE = 93.5	MSE = 0.001294 PRE = 96.2	MSE = 0.001257 PRE = 98.7
$\hat{F}_{EP}$	MSE = 0.001630 PRE = 76.3	MSE = 0.001664 PRE = 74.9	MSE = 0.001692 PRE = 73.8	MSE = 0.001736 PRE = 71.7	MSE = 0.001780 PRE = 69.7
$\hat{F}_{K1}$	MSE = 0.000445 PRE = 279.3	MSE = 0.000443 PRE = 281.3	MSE = 0.000444 PRE = 281.1	MSE = 0.000444 PRE = 280.4	MSE = 0.000443 PRE = 280.1
$\hat{F}_{K2}$	MSE = 0.001128 PRE = 110.2	MSE = 0.001125 PRE = 110.8	MSE = 0.001126 PRE = 110.8	MSE = 0.001126 PRE = 110.6	MSE = 0.001127 PRE = 110.1
$\hat{F}_{K3}$	MSE = 0.001128 PRE = 110.2	MSE = 0.001125 PRE = 110.8	MSE = 0.001126 PRE = 110.8	MSE = 0.001126 PRE = 110.6	MSE = 0.001127 PRE = 110.1
$\hat{F}_{K4}$	MSE = 0.001128 PRE = 110.2	MSE = 0.001125 PRE = 110.8	MSE = 0.001126 PRE = 110.8	MSE = 0.001126 PRE = 110.6	MSE = 0.001127 PRE = 110.1
$\hat{F}_{Upad}$	MSE = 0.001139 PRE = 109.130	MSE = 0.001136 PRE = 109.683	MSE = 0.001135 PRE = 109.955	MSE = 0.001133 PRE = 109.88	MSE = 0.001130 PRE = 109.823
$\hat{F}_{MMPS}$	MSE = 0.001230 PRE = 101.1	MSE = 0.001296 PRE = 96.1	MSE = 0.001422 PRE = 87.8	MSE = 0.001693 PRE = 73.5	MSE = 0.002022 PRE = 61.4
$\hat{F}_{Proposed}$	MSE = 0.000281 PRE = 442.3	MSE = 0.000222 PRE = 561.3	MSE = 0.000166 PRE = 751.8	MSE = 0.000108 PRE = 1152.8	MSE = 0.000054 PRE = 2298.1

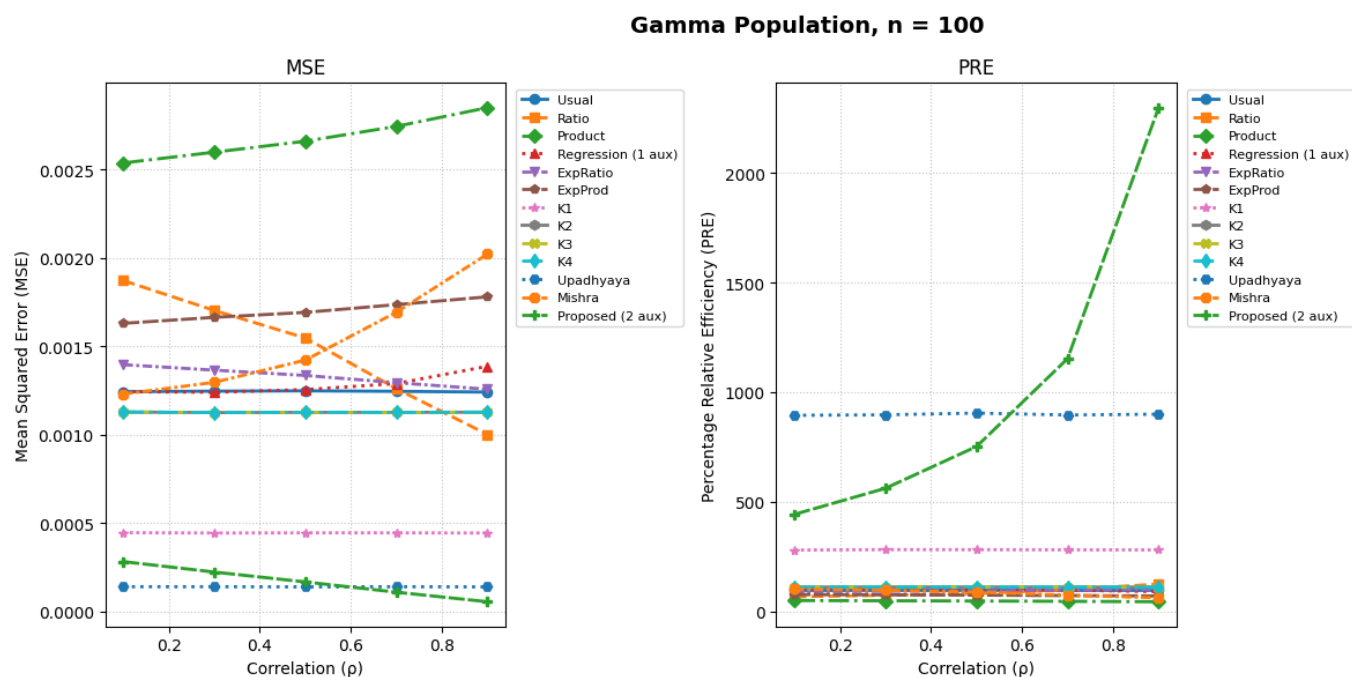


Figure A2. Showing MSEs and PREs using Gamma Population when $n = 100$.

Table A3. Showing MSE and PRE results of all considered estimators using Gamma Population for $n = 150$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{usual}$	MSE = 0.000829 PRE = 100.0	MSE = 0.000831 PRE = 100.0	MSE = 0.000832 PRE = 100.0	MSE = 0.000830 PRE = 100.0	MSE = 0.000828 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.001272 PRE = 65.2	MSE = 0.001143 PRE = 72.7	MSE = 0.001027 PRE = 81.0	MSE = 0.000840 PRE = 98.8	MSE = 0.000661 PRE = 125.3
$\hat{F}_{Product}$	MSE = 0.001738 PRE = 47.7	MSE = 0.001799 PRE = 46.2	MSE = 0.001861 PRE = 44.7	MSE = 0.001945 PRE = 42.7	MSE = 0.002051 PRE = 40.4
$\hat{F}_{Reg}$	MSE = 0.000829 PRE = 100.0	MSE = 0.000827 PRE = 100.5	MSE = 0.000837 PRE = 99.4	MSE = 0.000859 PRE = 96.6	MSE = 0.000925 PRE = 89.5
$\hat{F}_{ER}$	MSE = 0.000929 PRE = 89.2	MSE = 0.000910 PRE = 91.3	MSE = 0.000890 PRE = 93.5	MSE = 0.000863 PRE = 96.2	MSE = 0.000838 PRE = 98.8
$\hat{F}_{EP}$	MSE = 0.001087 PRE = 76.3	MSE = 0.001109 PRE = 74.9	MSE = 0.001128 PRE = 73.8	MSE = 0.001157 PRE = 71.7	MSE = 0.001187 PRE = 69.8
$\hat{F}_{K1}$	MSE = 0.000298 PRE = 278.2	MSE = 0.000297 PRE = 279.8	MSE = 0.000297 PRE = 280.1	MSE = 0.000297 PRE = 279.5	MSE = 0.000296 PRE = 279.7
$\hat{F}_{K2}$	MSE = 0.000752 PRE = 110.2	MSE = 0.000750 PRE = 110.8	MSE = 0.000751 PRE = 110.8	MSE = 0.000751 PRE = 110.5	MSE = 0.000751 PRE = 110.3
$\hat{F}_{K3}$	MSE = 0.000752 PRE = 110.2	MSE = 0.000750 PRE = 110.8	MSE = 0.000751 PRE = 110.8	MSE = 0.000751 PRE = 110.5	MSE = 0.000751 PRE = 110.3
$\hat{F}_{K4}$	MSE = 0.000752 PRE = 110.2	MSE = 0.000750 PRE = 110.8	MSE = 0.000751 PRE = 110.8	MSE = 0.000751 PRE = 110.5	MSE = 0.000751 PRE = 110.3
$\hat{F}_{Upad}$	MSE = 0.000793 PRE = 104.539	MSE = 0.000792 PRE = 104.924	MSE = 0.000790 PRE = 105.316	MSE = 0.000789 PRE = 105.196	MSE = 0.000785 PRE = 105.477
$\hat{F}_{MMPS}$	MSE = 0.000819 PRE = 101.2	MSE = 0.000866 PRE = 96.0	MSE = 0.000948 PRE = 87.8	MSE = 0.001129 PRE = 73.5	MSE = 0.001348 PRE = 61.4
$\hat{F}_{Proposed}$	MSE = 0.000187 PRE = 443.3	MSE = 0.000148 PRE = 561.5	MSE = 0.000111 PRE = 749.5	MSE = 0.000072 PRE = 1152.8	MSE = 0.000036 PRE = 2300.0

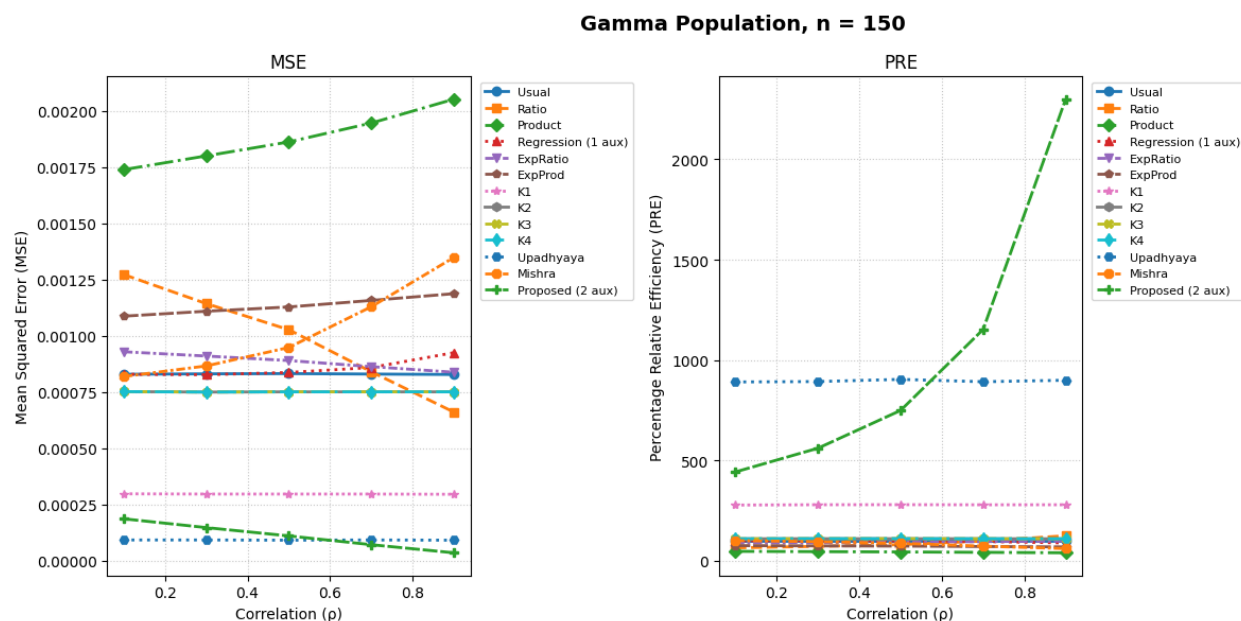


Figure A3. Showing MSEs and PREs using Gamma Population when $n = 150$.

Table A4. Showing MSEs and PREs results of all considered estimators using Gamma Population for $n = 200$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{Usual}$	MSE = 0.000621 PRE = 100.0	MSE = 0.000622 PRE = 100.0	MSE = 0.000623 PRE = 100.0	MSE = 0.000622 PRE = 100.0	MSE = 0.000620 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.000952 PRE = 65.2	MSE = 0.000858 PRE = 72.5	MSE = 0.000773 PRE = 80.6	MSE = 0.000630 PRE = 98.7	MSE = 0.000501 PRE = 123.8
$\hat{F}_{Product}$	MSE = 0.001308 PRE = 47.5	MSE = 0.001369 PRE = 45.4	MSE = 0.001431 PRE = 43.5	MSE = 0.001515 PRE = 41.1	MSE = 0.001621 PRE = 38.2
$\hat{F}_{Reg}$	MSE = 0.000621 PRE = 100.0	MSE = 0.000620 PRE = 100.3	MSE = 0.000627 PRE = 99.4	MSE = 0.000645 PRE = 96.4	MSE = 0.000693 PRE = 89.5
$\hat{F}_{ER}$	MSE = 0.000697 PRE = 89.1	MSE = 0.000682 PRE = 91.2	MSE = 0.000667 PRE = 93.4	MSE = 0.000647 PRE = 96.1	MSE = 0.000628 PRE = 98.7
$\hat{F}_{EP}$	MSE = 0.000815 PRE = 76.2	MSE = 0.000832 PRE = 74.8	MSE = 0.000846 PRE = 73.6	MSE = 0.000868 PRE = 71.7	MSE = 0.000890 PRE = 69.7
$\hat{F}_{K1}$	MSE = 0.000223 PRE = 278.5	MSE = 0.000222 PRE = 280.2	MSE = 0.000222 PRE = 280.6	MSE = 0.000222 PRE = 280.2	MSE = 0.000222 PRE = 279.3
$\hat{F}_{K2}$	MSE = 0.000564 PRE = 110.1	MSE = 0.000563 PRE = 110.5	MSE = 0.000563 PRE = 110.7	MSE = 0.000563 PRE = 110.5	MSE = 0.000563 PRE = 110.1
$\hat{F}_{K3}$	MSE = 0.000564 PRE = 110.1	MSE = 0.000563 PRE = 110.5	MSE = 0.000563 PRE = 110.7	MSE = 0.000563 PRE = 110.5	MSE = 0.000563 PRE = 110.1
$\hat{F}_{K4}$	MSE = 0.000564 PRE = 110.1	MSE = 0.000563 PRE = 110.5	MSE = 0.000563 PRE = 110.7	MSE = 0.000563 PRE = 110.5	MSE = 0.000563 PRE = 110.1
$\hat{F}_{Upad}$	MSE = 0.000569 PRE = 109.1388	MSE = 0.000566 PRE = 109.893	MSE = 0.000563 PRE = 110.657	MSE = 0.000560 PRE = 111.071	MSE = 0.000557 PRE = 111.310
$\hat{F}_{MMPS}$	MSE = 0.000614 PRE = 101.1	MSE = 0.000650 PRE = 95.7	MSE = 0.000711 PRE = 87.6	MSE = 0.000847 PRE = 73.4	MSE = 0.001011 PRE = 61.3
$\hat{F}_{Proposed}$	MSE = 0.000141 PRE = 440.4	MSE = 0.000111 PRE = 560.4	MSE = 0.000083 PRE = 750.6	MSE = 0.000054 PRE = 1151.9	MSE = 0.000027 PRE = 2296.3

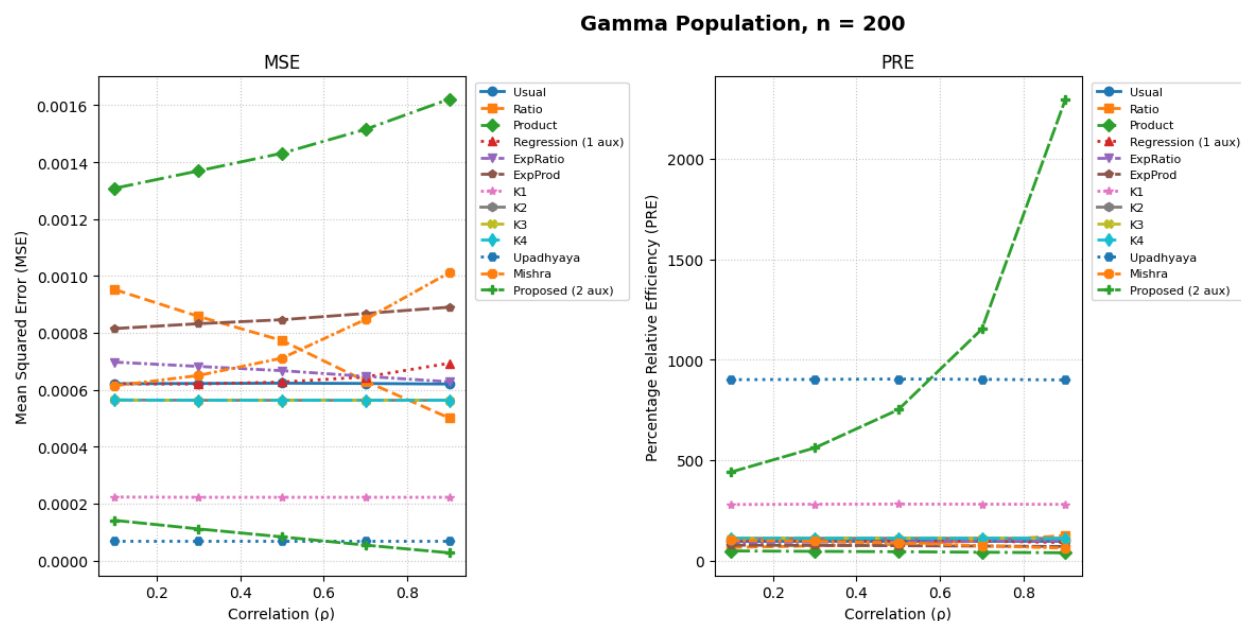


Figure A4. Showing MSEs and PREs using Gamma Population when $n = 200$.

Table A5. Showing MSEs and PREs results of all considered estimators using Gamma Population for $n = 250$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{usual}$	MSE = 0.000497 PRE = 100.0	MSE = 0.000499 PRE = 100.0	MSE = 0.000498 PRE = 100.0	MSE = 0.000497 PRE = 100.0	MSE = 0.000496 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.000762 PRE = 65.2	MSE = 0.000683 PRE = 73.1	MSE = 0.000613 PRE = 81.2	MSE = 0.000501 PRE = 99.2	MSE = 0.000397 PRE = 124.9
$\hat{F}_{Product}$	MSE = 0.001038 PRE = 47.9	MSE = 0.001099 PRE = 45.4	MSE = 0.001161 PRE = 42.9	MSE = 0.001245 PRE = 39.9	MSE = 0.001351 PRE = 36.7
$\hat{F}_{Reg}$	MSE = 0.000497 PRE = 100.0	MSE = 0.000496 PRE = 100.6	MSE = 0.000503 PRE = 99.0	MSE = 0.000516 PRE = 96.3	MSE = 0.000554 PRE = 89.5
$\hat{F}_{ER}$	MSE = 0.000557 PRE = 89.2	MSE = 0.000546 PRE = 91.4	MSE = 0.000534 PRE = 93.3	MSE = 0.000518 PRE = 96.0	MSE = 0.000503 PRE = 98.6
$\hat{F}_{EP}$	MSE = 0.000651 PRE = 76.3	MSE = 0.000665 PRE = 75.0	MSE = 0.000676 PRE = 73.7	MSE = 0.000694 PRE = 71.6	MSE = 0.000712 PRE = 69.7
$\hat{F}_{K1}$	MSE = 0.000178 PRE = 279.2	MSE = 0.000178 PRE = 280.3	MSE = 0.000178 PRE = 279.8	MSE = 0.000178 PRE = 279.2	MSE = 0.000178 PRE = 278.7
$\hat{F}_{K2}$	MSE = 0.000451 PRE = 110.2	MSE = 0.000450 PRE = 110.9	MSE = 0.000450 PRE = 110.7	MSE = 0.000450 PRE = 110.4	MSE = 0.000450 PRE = 110.2
$\hat{F}_{K3}$	MSE = 0.000451 PRE = 110.2	MSE = 0.000450 PRE = 110.9	MSE = 0.000450 PRE = 110.7	MSE = 0.000450 PRE = 110.4	MSE = 0.000450 PRE = 110.2
$\hat{F}_{K4}$	MSE = 0.000451 PRE = 110.2	MSE = 0.000450 PRE = 110.9	MSE = 0.000450 PRE = 110.7	MSE = 0.000450 PRE = 110.4	MSE = 0.000450 PRE = 110.2
$\hat{F}_{Upad}$	MSE = 0.000455 PRE = 109.230	MSE = 0.000454 PRE = 109.911	MSE = 0.000453 PRE = 109.933	MSE = 0.000450 PRE = 100.444	MSE = 0.000448 PRE = 110.714
$\hat{F}_{MMPS}$	MSE = 0.000492 PRE = 101.0	MSE = 0.000520 PRE = 96.0	MSE = 0.000569 PRE = 87.5	MSE = 0.000677 PRE = 73.4	MSE = 0.000809 PRE = 61.3
$\hat{F}_{Proposed}$	MSE = 0.000112 PRE = 443.8	MSE = 0.000088 PRE = 567.0	MSE = 0.000066 PRE = 754.5	MSE = 0.000043 PRE = 1155.8	MSE = 0.000022 PRE = 2254.5

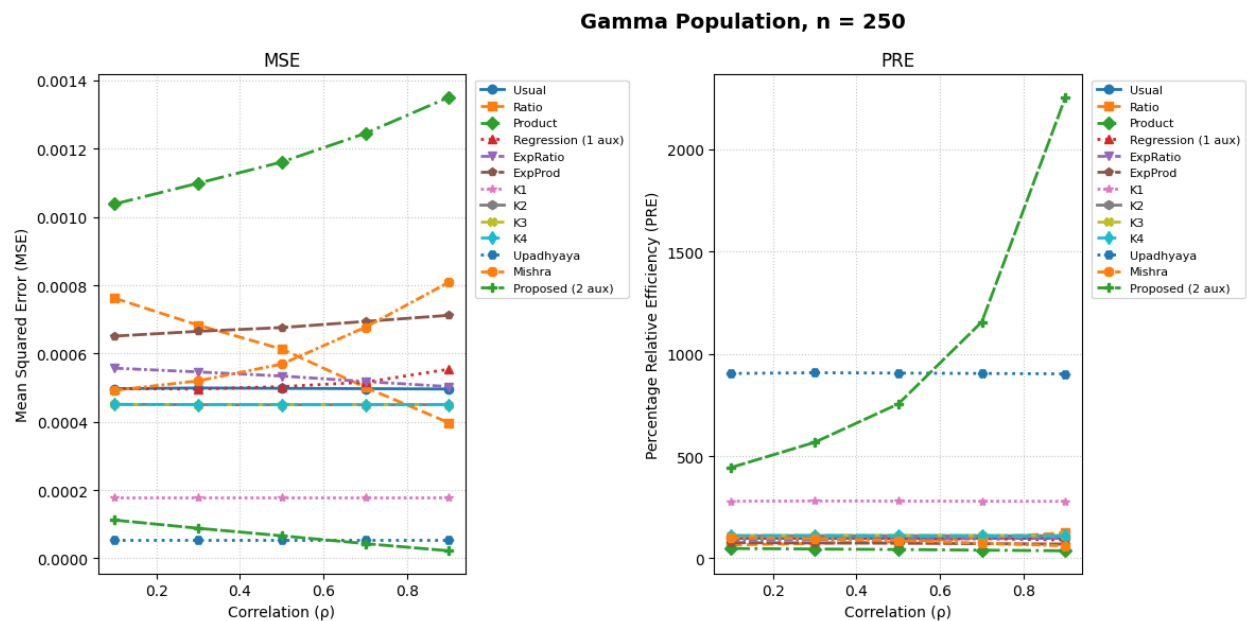


Figure A5. Showing MSEs and PREs using Gamma Population when $n = 250$.

Table A6. Showing MSE and PRE results of all considered estimators using Gamma Population for $n = 300$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{Usual}$	MSE = 0.000415 PRE = 100.0	MSE = 0.000416 PRE = 100.0	MSE = 0.000414 PRE = 100.0	MSE = 0.000415 PRE = 100.0	MSE = 0.000414 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.000632 PRE = 65.7	MSE = 0.000568 PRE = 73.2	MSE = 0.000514 PRE = 80.5	MSE = 0.000415 PRE = 100.0	MSE = 0.000331 PRE = 125.1
$\hat{F}_{Product}$	MSE = 0.000868 PRE = 47.8	MSE = 0.000929 PRE = 44.8	MSE = 0.000991 PRE = 41.8	MSE = 0.001075 PRE = 38.6	MSE = 0.001181 PRE = 35.1
$\hat{F}_{Reg}$	MSE = 0.000415 PRE = 100.0	MSE = 0.000414 PRE = 100.5	MSE = 0.000418 PRE = 99.0	MSE = 0.000430 PRE = 96.5	MSE = 0.000462 PRE = 89.6
$\hat{F}_{ER}$	MSE = 0.000465 PRE = 89.2	MSE = 0.000455 PRE = 91.4	MSE = 0.000445 PRE = 93.0	MSE = 0.000431 PRE = 96.3	MSE = 0.000419 PRE = 98.8
$\hat{F}_{EP}$	MSE = 0.000544 PRE = 76.3	MSE = 0.000554 PRE = 75.1	MSE = 0.000563 PRE = 73.5	MSE = 0.000578 PRE = 71.8	MSE = 0.000593 PRE = 69.8
$\hat{F}_{K1}$	MSE = 0.000149 PRE = 278.5	MSE = 0.000148 PRE = 281.1	MSE = 0.000148 PRE = 279.7	MSE = 0.000149 PRE = 278.5	MSE = 0.000148 PRE = 279.7
$\hat{F}_{K2}$	MSE = 0.000375 PRE = 110.7	MSE = 0.000375 PRE = 110.9	MSE = 0.000375 PRE = 110.4	MSE = 0.000375 PRE = 110.7	MSE = 0.000375 PRE = 110.4
$\hat{F}_{K3}$	MSE = 0.000375 PRE = 110.7	MSE = 0.000375 PRE = 110.9	MSE = 0.000375 PRE = 110.4	MSE = 0.000375 PRE = 110.7	MSE = 0.000375 PRE = 110.4
$\hat{F}_{K4}$	MSE = 0.000375 PRE = 110.7	MSE = 0.000375 PRE = 110.9	MSE = 0.000375 PRE = 110.4	MSE = 0.000375 PRE = 110.7	MSE = 0.000375 PRE = 110.4
$\hat{F}_{Upad}$	MSE = 0.000369 PRE = 112.466	MSE = 0.000367 PRE = 112.806	MSE = 0.000365 PRE = 113.424	MSE = 0.000364 PRE = 113.885	MSE = 0.000366 PRE = 113.114
$\hat{F}_{MMPS}$	MSE = 0.000410 PRE = 101.2	MSE = 0.000434 PRE = 95.9	MSE = 0.000474 PRE = 87.3	MSE = 0.000565 PRE = 73.5	MSE = 0.000675 PRE = 61.3
$\hat{F}_{Proposed}$	MSE = 0.000094 PRE = 441.5	MSE = 0.000074 PRE = 562.2	MSE = 0.000055 PRE = 752.7	MSE = 0.000036 PRE = 1152.8	MSE = 0.000018 PRE = 2300.0

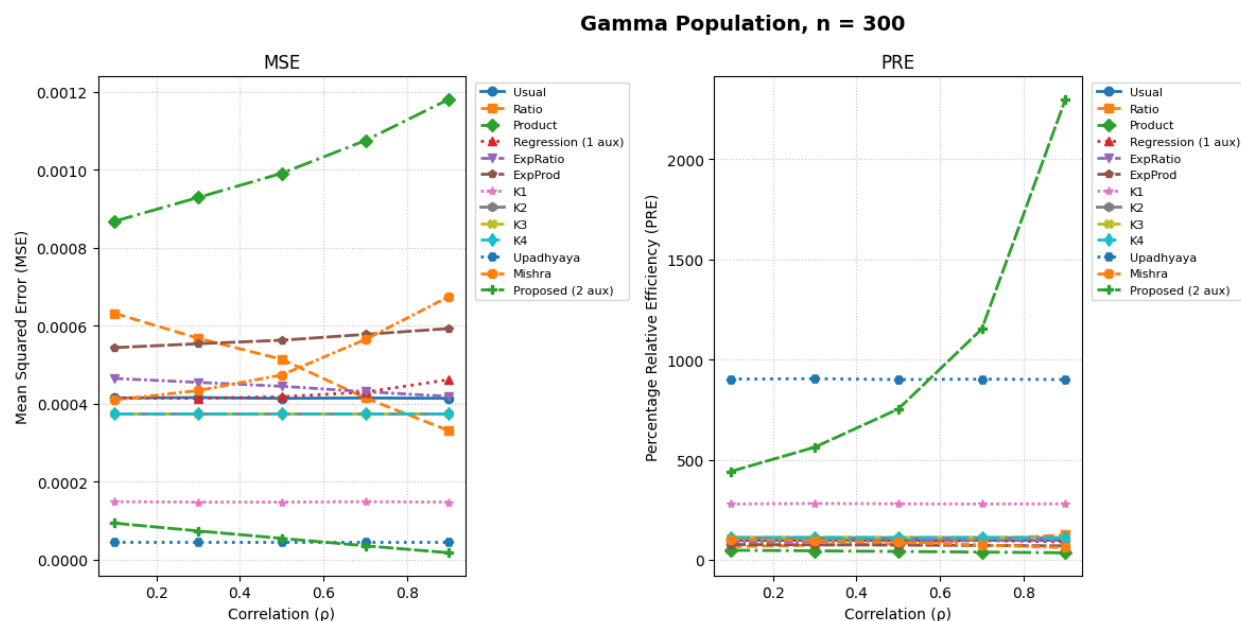


Figure A6. Showing MSEs and PREs using Gamma Population when $n = 300$.

Appendix B

Table A7. Showing MSE and PRE results of all considered estimators using Beta-I Population for $n = 50$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{usual}$	MSE = 0.002489 PRE = 100.0	MSE = 0.002488 PRE = 100.0	MSE = 0.002491 PRE = 100.0	MSE = 0.002487 PRE = 100.0	MSE = 0.002490 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.003854 PRE = 64.6	MSE = 0.003393 PRE = 73.3	MSE = 0.003137 PRE = 79.4	MSE = 0.002557 PRE = 97.3	MSE = 0.002093 PRE = 119.0
$\hat{F}_{Product}$	MSE = 0.004537 PRE = 54.9	MSE = 0.004598 PRE = 54.1	MSE = 0.004660 PRE = 53.5	MSE = 0.004745 PRE = 52.4	MSE = 0.004854 PRE = 51.3
$\hat{F}_{Reg}$	MSE = 0.002487 PRE = 100.1	MSE = 0.002481 PRE = 100.3	MSE = 0.002503 PRE = 99.5	MSE = 0.002588 PRE = 96.1	MSE = 0.002686 PRE = 92.7
$\hat{F}_{ER}$	MSE = 0.002787 PRE = 89.3	MSE = 0.002657 PRE = 93.6	MSE = 0.002528 PRE = 98.5	MSE = 0.002388 PRE = 104.1	MSE = 0.002246 PRE = 110.9
$\hat{F}_{EP}$	MSE = 0.003129 PRE = 79.5	MSE = 0.003161 PRE = 78.7	MSE = 0.003194 PRE = 78.0	MSE = 0.003236 PRE = 76.9	MSE = 0.003279 PRE = 75.9
$\hat{F}_{K1}$	MSE = 0.000890 PRE = 279.7	MSE = 0.000891 PRE = 279.2	MSE = 0.000889 PRE = 280.2	MSE = 0.000889 PRE = 279.7	MSE = 0.000888 PRE = 280.4
$\hat{F}_{K2}$	MSE = 0.002258 PRE = 110.2	MSE = 0.002251 PRE = 110.5	MSE = 0.002256 PRE = 110.4	MSE = 0.002252 PRE = 110.4	MSE = 0.002253 PRE = 110.5
$\hat{F}_{K3}$	MSE = 0.002258 PRE = 110.2	MSE = 0.002251 PRE = 110.5	MSE = 0.002256 PRE = 110.4	MSE = 0.002252 PRE = 110.4	MSE = 0.002253 PRE = 110.5
$\hat{F}_{K4}$	MSE = 0.002258 PRE = 110.2	MSE = 0.002251 PRE = 110.5	MSE = 0.002256 PRE = 110.4	MSE = 0.002252 PRE = 110.4	MSE = 0.002253 PRE = 110.5
$\hat{F}_{Upad}$	MSE = 0.00228 PRE = 109.166	MSE = 0.00228 PRE = 109.122	MSE = 0.00227 PRE = 109.735	MSE = 0.00228 PRE = 109.078	MSE = 0.00227 PRE = 109.691
$\hat{F}_{MMPS}$	MSE = 0.002461 PRE = 101.1	MSE = 0.002527 PRE = 98.5	MSE = 0.002652 PRE = 93.9	MSE = 0.002923 PRE = 85.1	MSE = 0.003251 PRE = 76.6
$\hat{F}_{Proposed}$	MSE = 0.000562 PRE = 442.9	MSE = 0.000443 PRE = 561.6	MSE = 0.000331 PRE = 752.6	MSE = 0.000215 PRE = 1156.7	MSE = 0.000108 PRE = 2305.6

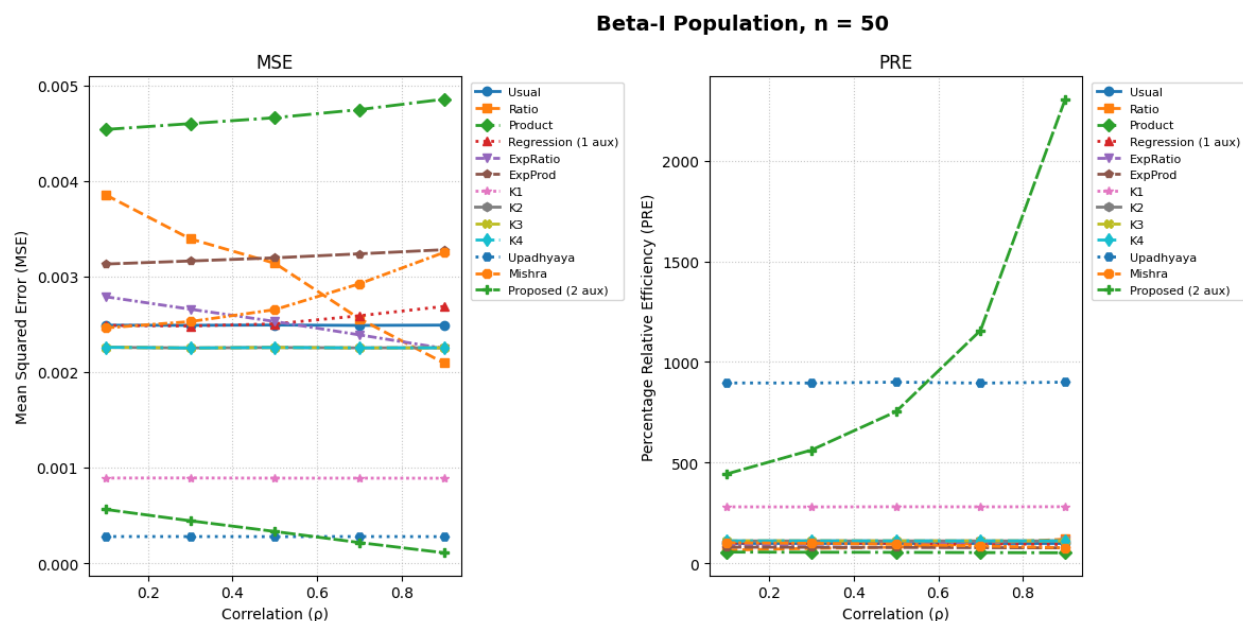


Figure A7. Showing MSEs and PREs using Beta-I Population when $n = 50$.

Table A8. Showing MSE and PRE results of all considered estimators using Beta-I Population for $n = 100$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{Usual}$	MSE = 0.001244 PRE = 100.0	MSE = 0.001243 PRE = 100.0	MSE = 0.001245 PRE = 100.0	MSE = 0.001244 PRE = 100.0	MSE = 0.001243 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.001871 PRE = 66.5	MSE = 0.001701 PRE = 73.1	MSE = 0.001546 PRE = 80.6	MSE = 0.001261 PRE = 98.7	MSE = 0.001001 PRE = 124.2
$\hat{F}_{Product}$	MSE = 0.002538 PRE = 49.0	MSE = 0.002599 PRE = 47.8	MSE = 0.002661 PRE = 46.8	MSE = 0.002745 PRE = 45.3	MSE = 0.002851 PRE = 43.6
$\hat{F}_{Reg}$	MSE = 0.001243 PRE = 100.1	MSE = 0.001240 PRE = 100.2	MSE = 0.001253 PRE = 99.4	MSE = 0.001289 PRE = 96.5	MSE = 0.001386 PRE = 89.7
$\hat{F}_{ER}$	MSE = 0.001397 PRE = 89.0	MSE = 0.001366 PRE = 91.0	MSE = 0.001334 PRE = 93.3	MSE = 0.001293 PRE = 96.2	MSE = 0.001256 PRE = 99.0
$\hat{F}_{EP}$	MSE = 0.001630 PRE = 76.3	MSE = 0.001663 PRE = 74.8	MSE = 0.001692 PRE = 73.6	MSE = 0.001736 PRE = 71.7	MSE = 0.001780 PRE = 69.8
$\hat{F}_{K1}$	MSE = 0.000446 PRE = 278.9	MSE = 0.000445 PRE = 279.3	MSE = 0.000445 PRE = 279.8	MSE = 0.000444 PRE = 280.2	MSE = 0.000444 PRE = 280.0
$\hat{F}_{K2}$	MSE = 0.001127 PRE = 110.4	MSE = 0.001125 PRE = 110.5	MSE = 0.001125 PRE = 110.7	MSE = 0.001126 PRE = 110.5	MSE = 0.001126 PRE = 110.4
$\hat{F}_{K3}$	MSE = 0.001127 PRE = 110.4	MSE = 0.001125 PRE = 110.5	MSE = 0.001125 PRE = 110.7	MSE = 0.001126 PRE = 110.5	MSE = 0.001126 PRE = 110.4
$\hat{F}_{K4}$	MSE = 0.001127 PRE = 110.4	MSE = 0.001125 PRE = 110.5	MSE = 0.001125 PRE = 110.7	MSE = 0.001126 PRE = 110.5	MSE = 0.001126 PRE = 110.4
$\hat{F}_{Upad}$	MSE = 0.001126 PRE = 110.479	MSE = 0.001123 PRE = 110.685	MSE = 0.00127 PRE = 98.031	MSE = 0.00122 PRE = 101.967	MSE = 0.001120 PRE = 110.982
$\hat{F}_{MMPS}$	MSE = 0.001231 PRE = 101.1	MSE = 0.001296 PRE = 95.9	MSE = 0.001423 PRE = 87.5	MSE = 0.001693 PRE = 73.5	MSE = 0.002021 PRE = 61.5
$\hat{F}_{Proposed}$	MSE = 0.000281 PRE = 442.7	MSE = 0.000222 PRE = 560.0	MSE = 0.000166 PRE = 750.0	MSE = 0.000108 PRE = 1151.9	MSE = 0.000054 PRE = 2301.9

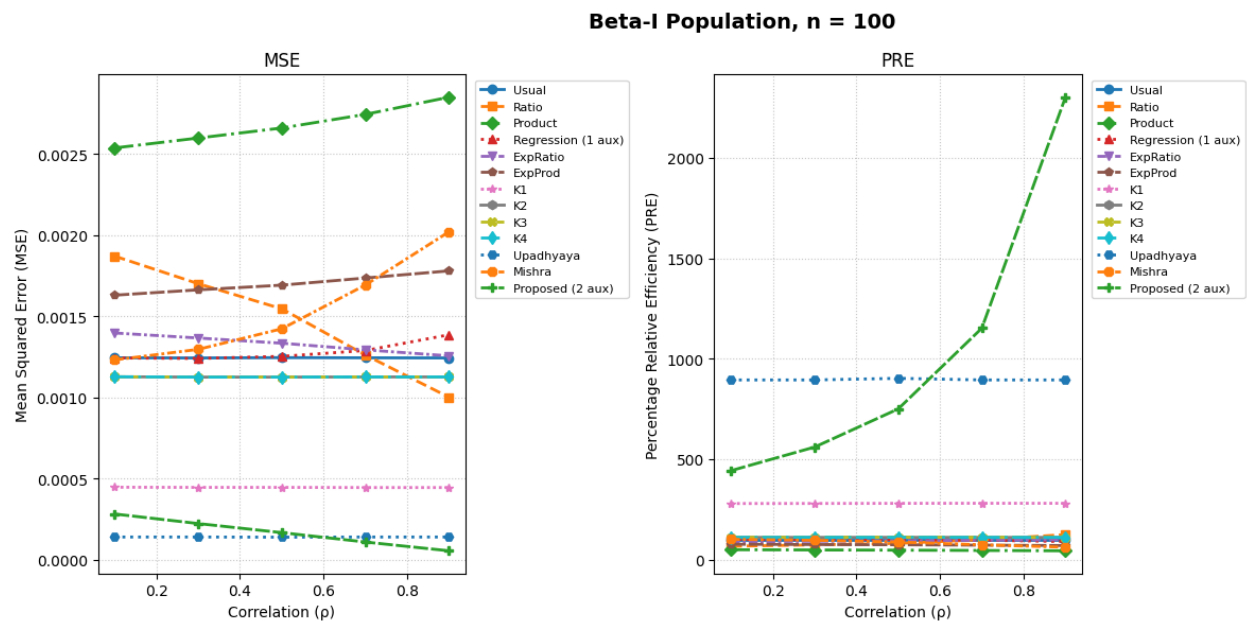


Figure A8. Showing MSEs and PREs using Beta-I Population when $n = 100$.

Table A9. Showing MSE and PRE results of all considered estimators using Beta-I Population for $n = 150$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{usual}$	MSE = 0.000830 PRE = 100.0	MSE = 0.000829 PRE = 100.0	MSE = 0.000831 PRE = 100.0	MSE = 0.000830 PRE = 100.0	MSE = 0.000829 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.001272 PRE = 65.3	MSE = 0.001142 PRE = 72.6	MSE = 0.001027 PRE = 80.9	MSE = 0.000840 PRE = 98.8	MSE = 0.000660 PRE = 125.6
$\hat{F}_{Product}$	MSE = 0.001739 PRE = 47.7	MSE = 0.001799 PRE = 46.1	MSE = 0.001861 PRE = 44.7	MSE = 0.001945 PRE = 42.7	MSE = 0.002051 PRE = 40.4
$\hat{F}_{Reg}$	MSE = 0.000830 PRE = 100.0	MSE = 0.000827 PRE = 100.2	MSE = 0.000837 PRE = 99.3	MSE = 0.000859 PRE = 96.6	MSE = 0.000924 PRE = 89.7
$\hat{F}_{ER}$	MSE = 0.000930 PRE = 89.2	MSE = 0.000909 PRE = 91.2	MSE = 0.000890 PRE = 93.4	MSE = 0.000862 PRE = 96.3	MSE = 0.000838 PRE = 98.9
$\hat{F}_{EP}$	MSE = 0.001088 PRE = 76.3	MSE = 0.001109 PRE = 74.8	MSE = 0.001128 PRE = 73.7	MSE = 0.001157 PRE = 71.7	MSE = 0.001186 PRE = 69.9
$\hat{F}_{K1}$	MSE = 0.000297 PRE = 279.5	MSE = 0.000297 PRE = 279.1	MSE = 0.000297 PRE = 279.8	MSE = 0.000296 PRE = 280.4	MSE = 0.000297 PRE = 279.1
$\hat{F}_{K2}$	MSE = 0.000753 PRE = 110.2	MSE = 0.000751 PRE = 110.4	MSE = 0.000752 PRE = 110.5	MSE = 0.000752 PRE = 110.4	MSE = 0.000751 PRE = 110.4
$\hat{F}_{K3}$	MSE = 0.000753 PRE = 110.2	MSE = 0.000751 PRE = 110.4	MSE = 0.000752 PRE = 110.5	MSE = 0.000752 PRE = 110.4	MSE = 0.000751 PRE = 110.4
$\hat{F}_{K4}$	MSE = 0.000753 PRE = 110.2	MSE = 0.000751 PRE = 110.4	MSE = 0.000752 PRE = 110.5	MSE = 0.000752 PRE = 110.4	MSE = 0.000751 PRE = 110.4
$\hat{F}_{Upad}$	MSE = 0.00093 PRE = 89.247	MSE = 0.00093 PRE = 89.139	MSE = 0.00092 PRE = 90.32	MSE = 0.00093 PRE = 89.247	MSE = 0.00092 PRE = 90.108
$\hat{F}_{MMPS}$	MSE = 0.000820 PRE = 101.2	MSE = 0.000866 PRE = 95.7	MSE = 0.000948 PRE = 87.7	MSE = 0.001129 PRE = 73.5	MSE = 0.001348 PRE = 61.5
$\hat{F}_{Proposed}$	MSE = 0.000187 PRE = 443.9	MSE = 0.000148 PRE = 560.1	MSE = 0.000111 PRE = 748.6	MSE = 0.000072 PRE = 1152.8	MSE = 0.000036 PRE = 2302.8

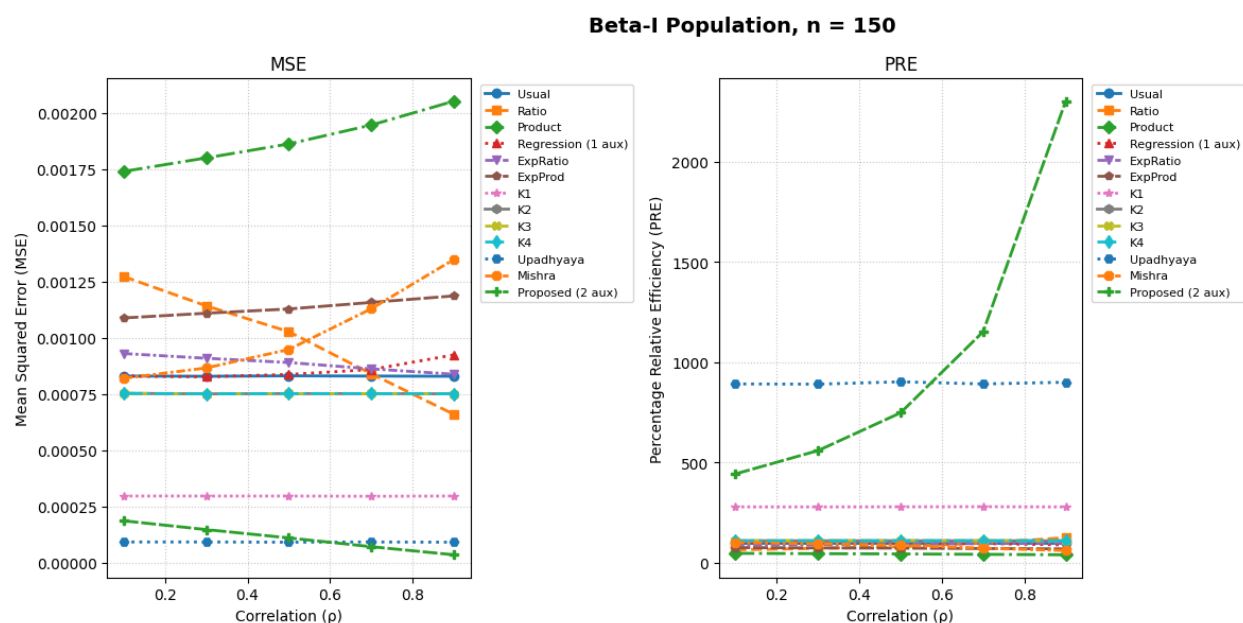


Figure A9. Showing MSEs and PREs using Beta-I Population when $n = 150$.

Table A10. Showing MSE and PRE results of all considered estimators using Beta-I Population for $n = 200$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{Usual}$	MSE = 0.000622 PRE = 100.0	MSE = 0.000621 PRE = 100.0	MSE = 0.000623 PRE = 100.0	MSE = 0.000622 PRE = 100.0	MSE = 0.000621 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.000954 PRE = 65.2	MSE = 0.000857 PRE = 72.5	MSE = 0.000773 PRE = 80.6	MSE = 0.000630 PRE = 98.7	MSE = 0.000501 PRE = 124.0
$\hat{F}_{Product}$	MSE = 0.001309 PRE = 47.5	MSE = 0.001369 PRE = 45.4	MSE = 0.001431 PRE = 43.5	MSE = 0.001515 PRE = 41.1	MSE = 0.001621 PRE = 38.3
$\hat{F}_{Reg}$	MSE = 0.000622 PRE = 100.0	MSE = 0.000620 PRE = 100.2	MSE = 0.000627 PRE = 99.4	MSE = 0.000645 PRE = 96.4	MSE = 0.000693 PRE = 89.6
$\hat{F}_{ER}$	MSE = 0.000697 PRE = 89.2	MSE = 0.000681 PRE = 91.2	MSE = 0.000667 PRE = 93.4	MSE = 0.000647 PRE = 96.1	MSE = 0.000628 PRE = 98.9
$\hat{F}_{EP}$	MSE = 0.000815 PRE = 76.3	MSE = 0.000832 PRE = 74.6	MSE = 0.000846 PRE = 73.6	MSE = 0.000868 PRE = 71.7	MSE = 0.000890 PRE = 69.8
$\hat{F}_{K1}$	MSE = 0.000223 PRE = 278.9	MSE = 0.000222 PRE = 279.7	MSE = 0.000223 PRE = 279.4	MSE = 0.000222 PRE = 280.2	MSE = 0.000222 PRE = 279.7
$\hat{F}_{K2}$	MSE = 0.000564 PRE = 110.3	MSE = 0.000563 PRE = 110.3	MSE = 0.000563 PRE = 110.7	MSE = 0.000563 PRE = 110.5	MSE = 0.000563 PRE = 110.3
$\hat{F}_{K3}$	MSE = 0.000564 PRE = 110.3	MSE = 0.000563 PRE = 110.3	MSE = 0.000563 PRE = 110.7	MSE = 0.000563 PRE = 110.5	MSE = 0.000563 PRE = 110.3
$\hat{F}_{K4}$	MSE = 0.000564 PRE = 110.3	MSE = 0.000563 PRE = 110.3	MSE = 0.000563 PRE = 110.7	MSE = 0.000563 PRE = 110.5	MSE = 0.000563 PRE = 110.3
$\hat{F}_{Upad}$	MSE = 0.00069 PRE = 90.144	MSE = 0.00068 PRE = 91.323	MSE = 0.00067 PRE = 92.985	MSE = 0.00064 PRE = 97.1875	MSE = 0.00065 PRE = 95.538
$\hat{F}_{MMPS}$	MSE = 0.000614 PRE = 101.3	MSE = 0.000650 PRE = 95.5	MSE = 0.000711 PRE = 87.6	MSE = 0.000847 PRE = 73.4	MSE = 0.001011 PRE = 61.4
$\hat{F}_{Proposed}$	MSE = 0.000141 PRE = 441.1	MSE = 0.000111 PRE = 559.5	MSE = 0.000083 PRE = 750.6	MSE = 0.000054 PRE = 1151.9	MSE = 0.000027 PRE = 2300.0

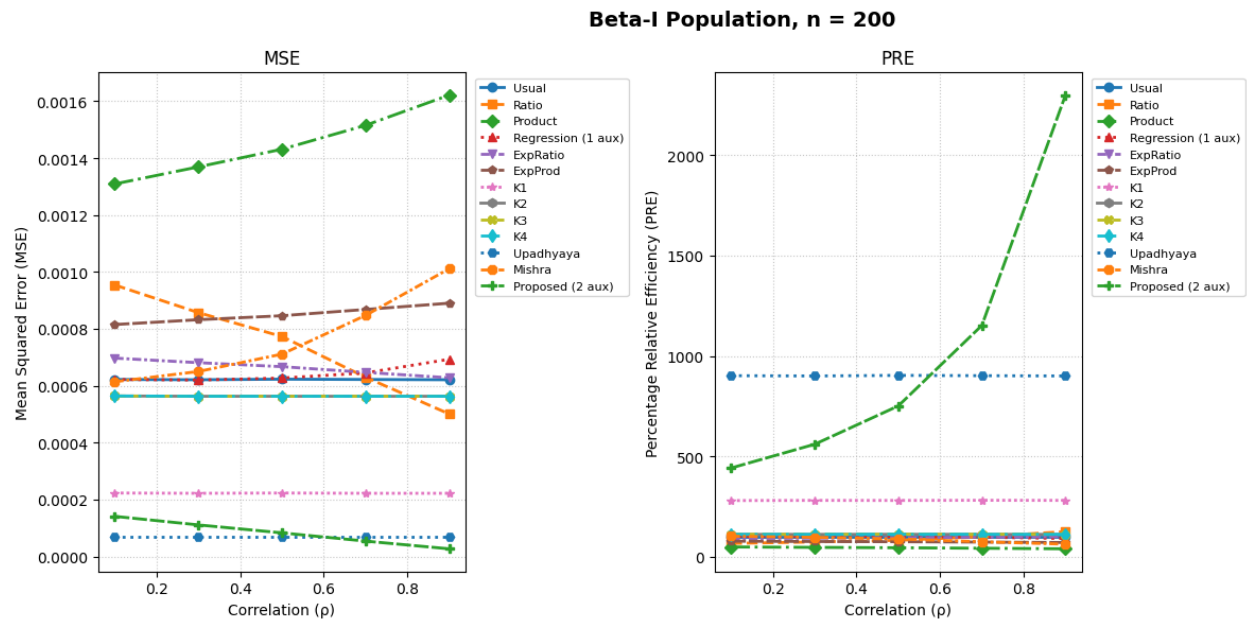


Figure A10. Showing MSEs and PREs using Beta-I Population when $n = 200$.

Table A11. Showing MSE and PRE results of all considered estimators using Beta-I Population for $n = 250$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{Usual}$	MSE = 0.000498 PRE = 100.0	MSE = 0.000497 PRE = 100.0	MSE = 0.000499 PRE = 100.0	MSE = 0.000497 PRE = 100.0	MSE = 0.000498 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.000763 PRE = 65.3	MSE = 0.000684 PRE = 72.7	MSE = 0.000614 PRE = 81.3	MSE = 0.000500 PRE = 99.4	MSE = 0.000398 PRE = 125.1
$\hat{F}_{Product}$	MSE = 0.001039 PRE = 47.9	MSE = 0.001100 PRE = 45.2	MSE = 0.001162 PRE = 42.9	MSE = 0.001246 PRE = 39.9	MSE = 0.001352 PRE = 36.8
$\hat{F}_{Reg}$	MSE = 0.000498 PRE = 100.0	MSE = 0.000496 PRE = 100.2	MSE = 0.000503 PRE = 99.2	MSE = 0.000516 PRE = 96.3	MSE = 0.000555 PRE = 89.7
$\hat{F}_{ER}$	MSE = 0.000558 PRE = 89.2	MSE = 0.000546 PRE = 91.0	MSE = 0.000534 PRE = 93.4	MSE = 0.000518 PRE = 96.0	MSE = 0.000503 PRE = 99.0
$\hat{F}_{EP}$	MSE = 0.000652 PRE = 76.4	MSE = 0.000666 PRE = 74.6	MSE = 0.000677 PRE = 73.7	MSE = 0.000695 PRE = 71.5	MSE = 0.000713 PRE = 69.8
$\hat{F}_{K1}$	MSE = 0.000178 PRE = 279.8	MSE = 0.000178 PRE = 279.2	MSE = 0.000178 PRE = 280.3	MSE = 0.000178 PRE = 279.2	MSE = 0.000178 PRE = 279.8
$\hat{F}_{K2}$	MSE = 0.000451 PRE = 110.4	MSE = 0.000451 PRE = 110.2	MSE = 0.000451 PRE = 110.6	MSE = 0.000451 PRE = 110.2	MSE = 0.000451 PRE = 110.4
$\hat{F}_{K3}$	MSE = 0.000451 PRE = 110.4	MSE = 0.000451 PRE = 110.2	MSE = 0.000451 PRE = 110.6	MSE = 0.000451 PRE = 110.2	MSE = 0.000451 PRE = 110.4
$\hat{F}_{K4}$	MSE = 0.000451 PRE = 110.4	MSE = 0.000451 PRE = 110.2	MSE = 0.000451 PRE = 110.6	MSE = 0.000451 PRE = 110.2	MSE = 0.000451 PRE = 110.4
$\hat{F}_{Upad}$	MSE = 0.00055 PRE = 90.545	MSE = 0.00054 PRE = 92.037	MSE = 0.00055 PRE = 90.727	MSE = 0.00055 PRE = 90.363	MSE = 0.00055 PRE = 90.545
$\hat{F}_{MMPS}$	MSE = 0.000493 PRE = 101.0	MSE = 0.000521 PRE = 95.4	MSE = 0.000570 PRE = 87.5	MSE = 0.000678 PRE = 73.3	MSE = 0.000810 PRE = 61.5
$\hat{F}_{Proposed}$	MSE = 0.000112 PRE = 444.6	MSE = 0.000088 PRE = 564.8	MSE = 0.000066 PRE = 756.1	MSE = 0.000043 PRE = 1155.8	MSE = 0.000022 PRE = 2263.6

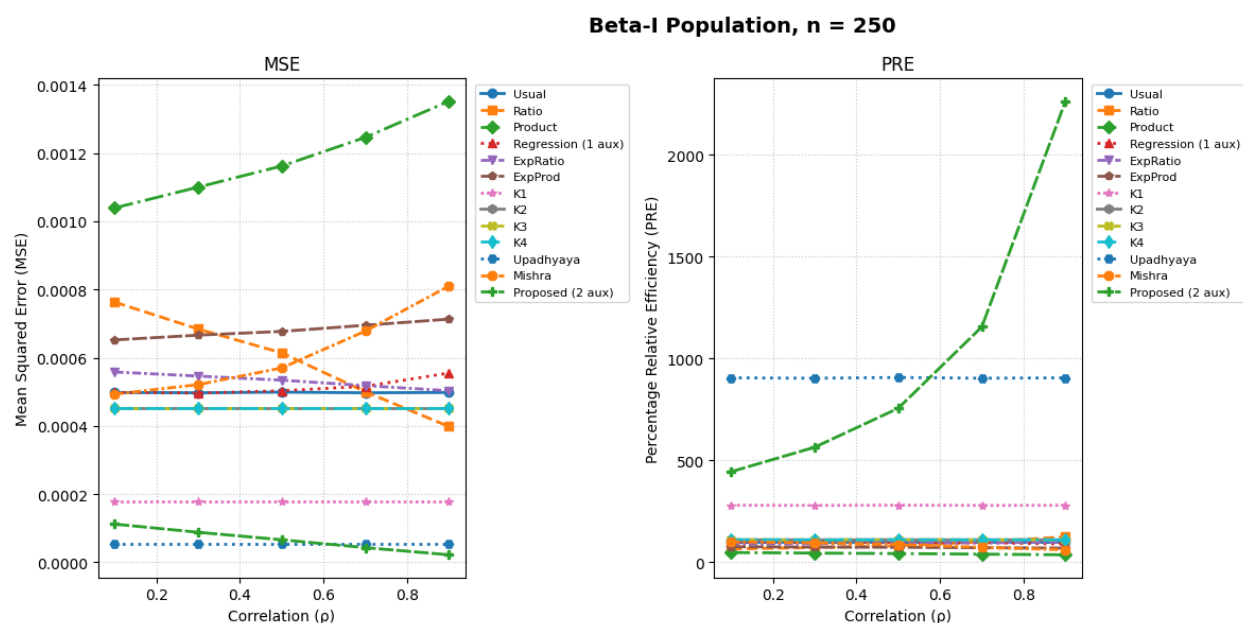


Figure A11. Showing MSEs and PREs using Beta-I Population when $n = 250$.

Table A12. Showing MSE and PRE results of all considered estimators using Beta-I Population for $n = 300$.

Estimator	$\rho = 0.1$	$\rho = 0.3$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\hat{F}_{Usual}$	MSE = 0.000416 PRE = 100.0	MSE = 0.000415 PRE = 100.0	MSE = 0.000416 PRE = 100.0	MSE = 0.000414 PRE = 100.0	MSE = 0.000415 PRE = 100.0
$\hat{F}_{Ratio}$	MSE = 0.000635 PRE = 65.5	MSE = 0.000568 PRE = 73.1	MSE = 0.000514 PRE = 80.9	MSE = 0.000414 PRE = 100.0	MSE = 0.000332 PRE = 125.0
$\hat{F}_{Product}$	MSE = 0.000870 PRE = 47.8	MSE = 0.000930 PRE = 44.6	MSE = 0.000992 PRE = 41.9	MSE = 0.001076 PRE = 38.5	MSE = 0.001182 PRE = 35.1
$\hat{F}_{Reg}$	MSE = 0.000416 PRE = 100.0	MSE = 0.000414 PRE = 100.2	MSE = 0.000419 PRE = 99.3	MSE = 0.000430 PRE = 96.3	MSE = 0.000463 PRE = 89.6
$\hat{F}_{ER}$	MSE = 0.000466 PRE = 89.3	MSE = 0.000455 PRE = 91.2	MSE = 0.000445 PRE = 93.5	MSE = 0.000431 PRE = 96.1	MSE = 0.000419 PRE = 99.0
$\hat{F}_{EP}$	MSE = 0.000545 PRE = 76.3	MSE = 0.000555 PRE = 74.8	MSE = 0.000564 PRE = 73.8	MSE = 0.000579 PRE = 71.5	MSE = 0.000594 PRE = 69.9
$\hat{F}_{K1}$	MSE = 0.000149 PRE = 279.2	MSE = 0.000149 PRE = 278.5	MSE = 0.000149 PRE = 279.2	MSE = 0.000148 PRE = 279.7	MSE = 0.000149 PRE = 278.5
$\hat{F}_{K2}$	MSE = 0.000375 PRE = 110.9	MSE = 0.000376 PRE = 110.4	MSE = 0.000375 PRE = 110.9	MSE = 0.000375 PRE = 110.4	MSE = 0.000375 PRE = 110.7
$\hat{F}_{K3}$	MSE = 0.000375 PRE = 110.9	MSE = 0.000376 PRE = 110.4	MSE = 0.000375 PRE = 110.9	MSE = 0.000375 PRE = 110.4	MSE = 0.000375 PRE = 110.7
$\hat{F}_{K4}$	MSE = 0.000375 PRE = 110.9	MSE = 0.000376 PRE = 110.4	MSE = 0.000375 PRE = 110.9	MSE = 0.000375 PRE = 110.4	MSE = 0.000375 PRE = 110.7
$\hat{F}_{Upad}$	MSE = 0.00046 PRE = 90.434	MSE = 0.00045 PRE = 92.22	MSE = 0.00043 PRE = 96.744	MSE = 0.00042 PRE = 98.571	MSE = 0.00044 PRE = 94.318
$\hat{F}_{MMPS}$	MSE = 0.000411 PRE = 101.2	MSE = 0.000435 PRE = 95.4	MSE = 0.000475 PRE = 87.6	MSE = 0.000566 PRE = 73.1	MSE = 0.000676 PRE = 61.4
$\hat{F}_{Proposed}$	MSE = 0.000094 PRE = 442.6	MSE = 0.000074 PRE = 560.8	MSE = 0.000055 PRE = 756.4	MSE = 0.000036 PRE = 1150.0	MSE = 0.000018 PRE = 2305.6

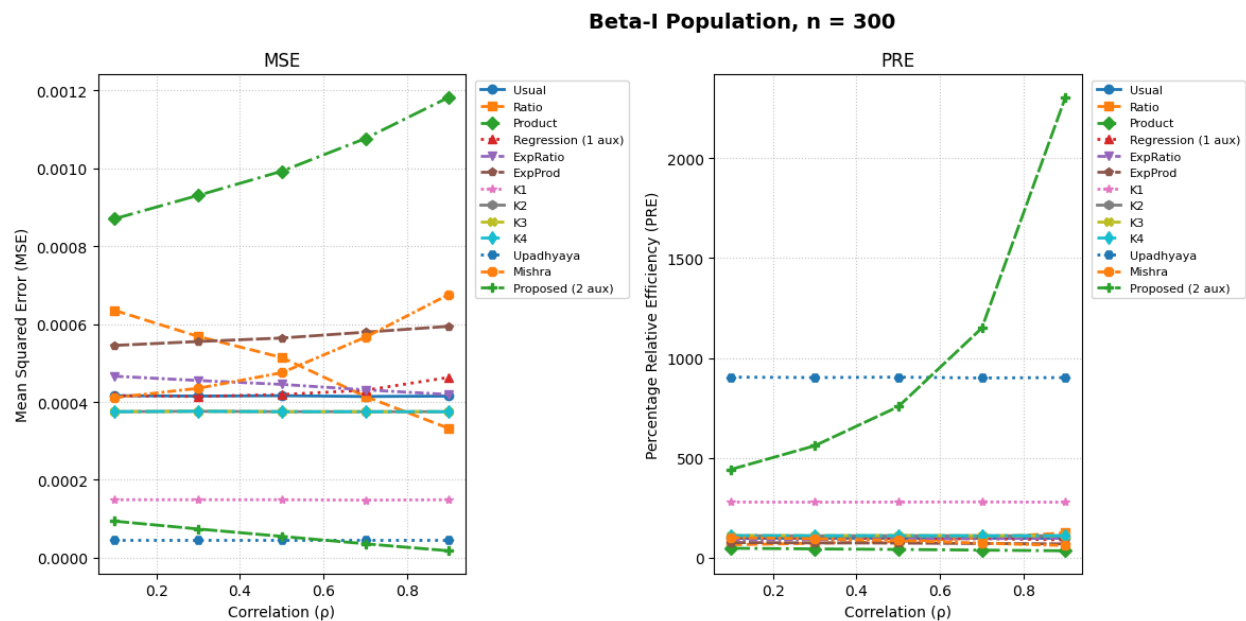


Figure A12. Showing MSEs and PREs using Beta-I Population when $n = 300$.

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