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Modeling Lifetime Data with a Novel Alpha-Power DUS Lindley Distribution

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Abstract

This paper introduces a new two-parameter probability distribution, called the alpha-power transformed DUS-Lindley (APT-DUSL) distribution. It is constructed by integrating the alpha-power transformation with the DUS-Lindley distribution. Various distributional and reliability properties are derived and investigated, including the hazard and survival functions, quantile function, moments, order statistics, and Rényi entropy. The APT-DUSL parameters are estimated using the maximum likelihood method based on complete samples and three progressive type-II censoring schemes. The estimator's performance is assessed in terms of bias and root mean square error for various sample sizes. The results indicate that the early censoring scheme yields estimates that are the closest to those obtained from the complete samples. The applicability of the proposed model is demonstrated through five real-life datasets. The APT-DUSL exhibits considerable flexibility and provides a better fit than several competing distributions, including the alpha-power Lindley, gamma, Lindley, exponential, and DUS-Lindley models.

Keywords: alpha-power transformation; DUS transformation; Lindley distribution; progressive type-II censoring; maximum likelihood estimation; Rényi entropy

MSC: 62F10; 62F30; 62N02; 60E05

1. Introduction

Modeling lifetime data is a central task in reliability and survival analysis, particularly when observations exhibit complex characteristics such as skewness, heavy-tailed behavior, and non-standard hazard-rate structures. In many practical situations, classical lifetime distributions, including the exponential, gamma, Weibull, and Lindley models, may not provide sufficient flexibility to describe such behaviors adequately. These limitations have motivated the development of extended and generalized distributions through the introduction of additional parameters and transformation-based approaches to improve model adaptability and achieve a more accurate representation of real data.

A common approach to extending existing models is to construct new families of classical distributions through innovative generating methods. These approaches improve the ability of probability distributions to accurately model complex data patterns.

Examples include the Beta-generated (Beta-G) family proposed by [1], the Kumaraswamy-generated (Kw-G) family introduced by [2], and the transformed transformer (T-X) family [3].

An important advancement in this area is the alpha-power transformation (APT) method, introduced by [4]. The APT provides a novel framework to expand the scope of



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distributions by introducing an additional parameter, thereby enhancing their modeling flexibility while preserving their core structure. This method enables the construction of new lifetime families that can capture asymmetry, skewness, and heavy-tailed behavior, resulting in more accurate models for diverse datasets. The cumulative distribution function (CDF) and the probability density function (PDF) are defined as follows:

$$F(x; \alpha) = \begin{cases} \frac{\alpha^{G(x)} - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 1, \\ G(x), & \alpha = 1, \end{cases} \quad (1)$$

$$f(x; \alpha) = \begin{cases} \frac{\log(\alpha)}{\alpha - 1} g(x) \alpha^{G(x)}, & \alpha > 0, \alpha \neq 1, \\ g(x), & \alpha = 1, \end{cases} \quad (2)$$

where $G(x)$ and $g(x)$ represent the CDF and PDF of the baseline distribution, respectively. The alpha-power (AP) method has gained significant attention in the development of new distributions. For example, Ref. [5] introduced the AP Lindley (AP-L) distribution. In addition, Ref. [6] merged the well-established techniques of AP and Kumaraswamy to formulate the AP Kumaraswamy–Burr III distribution. Further developments include the pseudo-Lindley AP distribution [7], the modified AP Weibull distribution [8], the novel AP beta distribution [9], and the AP inverse Weibull distribution [10]. More recently, [11] proposed the Kumaraswamy AP Lomax distribution.

Recent studies have focused on transformation-based techniques that generate flexible distributions without introducing additional parameters. The resulting models offer greater efficiency than existing ones, thereby enabling more accurate reliability estimates for complex data. The Dinesh-Umesh-Sanjay (DUS) transformation was first suggested by [12]. This innovative transformation method improves the accuracy of the classical distributions while also ensuring ease of computation and understanding. As a result, it yields a simplified distribution, with the CDF and PDF represented as follows:

$$F(x) = \frac{1}{e - 1} [e^{G(x)} - 1], \quad (3)$$

$$f(x) = \frac{1}{e - 1} g(x) e^{G(x)}. \quad (4)$$

Ref. [13] introduced a new lifetime distribution based on the generalized DUS transformation, using the Weibull distribution as the baseline. In addition, Kavya and Manoharan (KM) proposed the KM transformation [14], which generates new lifetime models without introducing additional parameters, thus improving flexibility and broadening applicability in lifetime modeling.

The DUS and KM transformations have been extensively applied to a variety of distributions. For instance, [15] introduced the DUS-transformed Weibull distribution, Ref. [16] introduced the KM–Burr X, [17] introduced the DUS-powered inverse Rayleigh, Ref. [18] presented the KM log-logistic, [19] developed the generalized DUS exponential, and Ref. [20] developed the KM generalized exponential. The modified Kumaraswamy-based model by [21] and the KM generalized inverse Kumaraswamy distribution proposed by [22] are two additional extensions.

In recent years, there have been numerous developments, including the power generalized DUS inverse Kumaraswamy distribution [23], the DUS Topp–Leone family of distributions [24], the DUS-Lindley distribution [25], the DUS-transformed generalized linear failure rate distribution [26], the generalized DUS-Bilal distribution [27], the power generalized KM exponential distribution [28], and the KM-DUS family [29].

The Lindley distribution (LD), introduced by [30], has garnered substantial attention in statistical research. Compared to the exponential distribution, LD has exhibited superior performance in modeling survival and reliability data. It is defined as a mixture of the exponential distribution and the gamma distribution with a single parameter. The LD's CDF and PDF are given by

$$F(x; \xi) = 1 - \left(1 + \frac{\xi x}{1 + \xi}\right) e^{-\xi x}, \quad x > 0, \xi > 0. \quad (5)$$

$$f(x; \xi) = \frac{\xi^2}{1 + \xi} (1 + x) e^{-\xi x}, \quad x > 0, \xi > 0. \quad (6)$$

The mathematical properties of the LD were developed by [31]. Numerous extensions have been developed in the literature to enhance the efficacy of LD, such as the three-parameter generalization of the LD introduced by [32], the power LD by [33], the one-parameter inverse LD proposed by [34], the two-parameter generalized inverse LD proposed by [35], and the inverse power LD introduced by [36]. Further advances have enhanced the LD, such as the exponentiated generalized power LD by [37], the APT power inverse LD by [38], the X-LD by [39], the Marshall–Olkin extended generalized LD by [40], and the AP extended power LD by [41].

Recent developments include the truncated Lindley-G family [42], the exponentiated X-Lindley [43], and the Lindley-X class [44]. Most recently, Ref. [25] introduced a combination of DUS transformation with the LD, resulting in the DUS-Lindley (DUSL) distribution. The CDF and PDF of the DUSL distribution are as follows:

$$F(x; \xi) = \left(\frac{1}{e-1}\right) \left(e^{\left(1 - \left(1 + \frac{\xi x}{1 + \xi}\right) e^{-\xi x}\right)} - 1\right), \quad x, \xi > 0. \quad (7)$$

$$f(x; \xi) = \left(\frac{1}{e-1}\right) \left(\frac{\xi^2}{1 + \xi}\right) (1 + x) e^{-\xi x} \left(e^{\left(1 - \left(1 + \frac{\xi x}{1 + \xi}\right) e^{-\xi x}\right)}\right), \quad x, \xi > 0. \quad (8)$$

This work develops a novel lifetime model, referred to as the alpha-power transformed DUS-Lindley (APT-DUSL) distribution, by integrating the APT technique with the DUS transformation to the LD. The APT-DUSL distribution will overcome the drawbacks of current Lindley-based generalizations, producing a flexible and generalized distribution capable of modeling a wide range of lifetime data. The main benefits of introducing the APT-DUSL can be summarized as follows:

- While existing models use a single transformation (either AP or DUS), the APT-DUSL applies a dual-action method. It combines the Alpha-Power Transformation's ability to adjust global skewness with the DUSL framework's mechanism for non-linear density re-weighting. This combination allows the model to process more complex data structures that neither of the underlying models can handle individually.
- The APT-DUSL offers improved parametric control, enabling the hazard rate to capture more accurately the non-monotonic hazard-rate behavior of the empirical data, thereby effectively overcoming the inflexibility of standard LD.
- The structure of the APT-DUSL model is particularly well-suited for handling data with significant variability. The model provides a more robust fit to data with extreme skewness or kurtosis by allowing the APT parameter to determine tail behavior and the DUSL structure to regulate density concentration, both of which are prevalent characteristics in many real-world reliability and actuarial data.
- The APT-DUSL improves the log-likelihood and goodness-of-fit (GOF) measures significantly. This outweighs the parsimonious penalty of adding an extra parameter.

The lower GOF values compared to AP-L or DUSL, observed across five datasets, demonstrate the strongest evidence of parsimonious superiority.

- Traditional transformation approaches tend to generate an “over-smoothed” fit that masks important information in the tails and near regions of mode-shifting. The APT-DUSL model overcomes this by offering a more detailed analytical framework that ensures the theoretical model has a precise, reliable alignment with the empirical density and hazard functions throughout the whole time frame.

Accordingly, the APT-DUSL is a useful extension of the theory of distribution, providing a flexible, robust, and better empirical framework for survival and reliability analysis.

This paper is organized as follows: Section 2 represents the derivation of the probability density and cumulative distribution functions of APT-DUSL. Sections 3 and 4 introduce APT-DUSL’s reliability analysis and statistical properties. The maximum likelihood method for complete and progressive type-II censored samples is discussed in Section 5. Simulation studies are conducted in Section 6 to estimate APT-DUSL’s parameters and evaluate the performance of the estimators. The effectiveness of the APT-DUSL distribution, in comparison with selected competing classical distributions, is demonstrated in Section 7 using five real lifetime data sets. Finally, Section 8 presents the concluding remarks.

2. The APT-DUSL-Lindley Distribution

The APT-DUSL distribution obtains its CDF from the DUSL distribution by substituting the DUSL distribution $G(x)$ in (7) into the general APT Equation (1) as follows:

$$F(x; \zeta, \alpha) = \frac{\alpha^{\left(\frac{1}{e-1}\right)} \left(e^{\left(1 - \left(1 + \frac{\zeta x}{1+\zeta}\right) e^{-\zeta x}\right)} - 1 \right) - 1}{\alpha - 1}, \quad x, \zeta, \alpha > 0. \quad (9)$$

The corresponding PDF for the APT-DUSL distribution is obtained by replacing $g(x)$ in (2) with (8) as follows:

$$f(x; \zeta, \alpha) = \frac{\log(\alpha)}{\alpha - 1} \left(\frac{1}{e - 1} \right) \left(\frac{\zeta^2}{1 + \zeta} \right) (1 + x) e^{-\zeta x} \left(e^{\left(1 - \left(1 + \frac{\zeta x}{1+\zeta}\right) e^{-\zeta x}\right)} \right)^{\alpha^{\left(\frac{1}{e-1}\right)} \left(e^{\left(1 - \left(1 + \frac{\zeta x}{1+\zeta}\right) e^{-\zeta x}\right)} - 1 \right)}, \quad (10)$$

where α controls the tail behavior and skewness of the distribution and ζ is the scale parameter that determines the acceleration of failure. Increasing ζ will shift the distribution towards the origin and enhance the hazard-rate intensity. In addition, α allows the model to handle heavy-tailed behavior by shifting probability mass away from the mean, thereby providing the essential parametric flexibility to fit datasets with kurtosis considerably larger than that of the typical DUSL distribution.

Special case

The proposed APT-DUSL distribution contains the DUSL distribution as a limiting special case. As in $\alpha \rightarrow 1$, we obtain

$$\lim_{\alpha \rightarrow 1} F(x; \zeta, \alpha) = G(x; \zeta),$$

and consequently

$$\lim_{\alpha \rightarrow 1} f(x; \zeta, \alpha) = g(x; \zeta),$$

where $G(x; \zeta)$ and $g(x; \zeta)$ are the CDF and PDF of DUSL given in Equations (7) and (8).

Figure 1 shows the PDF plots of APT-DUSL density, highlighting its flexibility in capturing different degrees of skewness and tail behavior. Specifically, the parameter ζ controls the skewness and modal shift, whereas α modifies the tail behavior and the overall shape of the distribution, providing additional modeling flexibility.

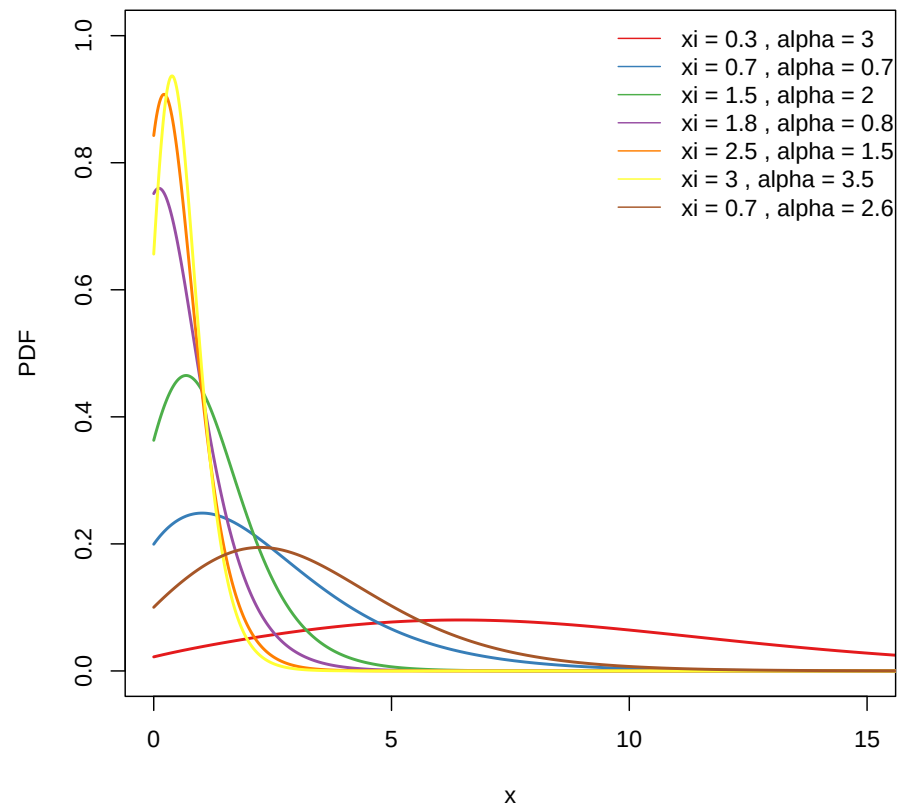


Figure 1. PDFs of the APT-DUSL distribution under different parameter combinations.

3. Reliability Analysis

3.1. Survival Function

The survival function for the APT-DUSL distribution can be expressed as

$$S(x) = 1 - \frac{\alpha \left(\frac{1}{e-1} \right) \left(e^{\left(1 - \left(1 + \frac{\zeta x}{1+\zeta} \right) e^{-\zeta x} \right) - 1} \right)}{\alpha - 1}.$$

3.2. Hazard-Rate Function

The hazard-rate function, $h(x)$, is expressed as

$$h(x) = \frac{\frac{\log(\alpha)}{\alpha-1} \left(\frac{1}{e-1} \right) \left(\frac{\zeta^2}{1+\zeta} \right) (1+x) e^{-\zeta x} \left(e^{\left(1 - \left(1 + \frac{\zeta x}{1+\zeta} \right) e^{-\zeta x} \right) - 1} \right) \alpha \left(\frac{1}{e-1} \right) \left(e^{\left(1 - \left(1 + \frac{\zeta x}{1+\zeta} \right) e^{-\zeta x} \right) - 1} \right)}{1 - \frac{\alpha \left(\frac{1}{e-1} \right) \left(e^{\left(1 - \left(1 + \frac{\zeta x}{1+\zeta} \right) e^{-\zeta x} \right) - 1} \right)}{\alpha-1}}.$$

Figures 2 show that the hazard-rate function can exhibit several behaviors depending on the parameter values. The plots are very flexible with increasing, unimodal (upside-down bathtub), and decreasing-to-stable shapes. The parameter ζ influences mostly the shape of the hazard function, while α primarily affects the level and long-term pattern. These results illustrate the flexibility of APT-DUSL in modeling different lifetime risk patterns.

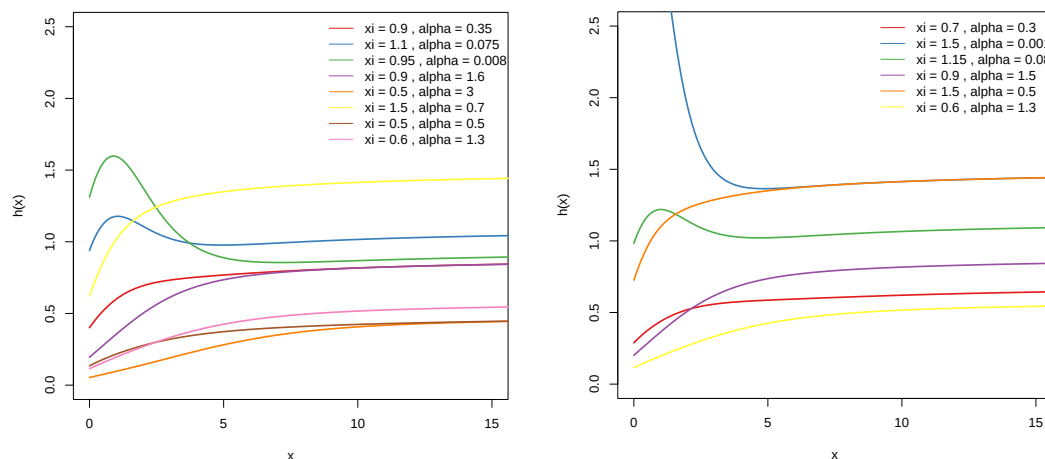


Figure 2. $h(x)$ plots of APT-DUSL distribution.

3.3. Reverse Hazard Function

The reverse hazard function of the APT-DUSL distribution is

$$H^T(x) = \frac{f(x; \xi, \alpha)}{F(x; \xi, \alpha)}$$

$$H^T(x) = \frac{\log(\alpha) \left(\frac{1}{e-1} \right) \left(\frac{\xi^2}{1+\xi} \right) (1+x) e^{-\xi x} \exp \left(1 - \left(1 + \frac{\xi x}{1+\xi} \right) e^{-\xi x} \right) \alpha^{\left(\frac{1}{e-1} \right) \left(e^{\left(1 - \left(1 + \frac{\xi x}{1+\xi} \right) e^{-\xi x} \right) - 1} \right)}}{\alpha^{\left(\frac{1}{e-1} \right) \left(e^{\left(1 - \left(1 + \frac{\xi x}{1+\xi} \right) e^{-\xi x} \right) - 1} \right)} - 1}. \quad (11)$$

3.4. Mills Ratio

The Mills ratio associated with the APT-DUSL distribution is given as follows:

$$M(x) = \frac{S(x)}{f(x)}.$$

Alternatively, it can be represented through the reverse hazard function as:

$$M(x) = \frac{1}{H^T(x)}.$$

$$\text{Mills ratio} = \frac{\alpha^{\left(\frac{1}{e-1} \right) \left(e^{\left(1 - \left(1 + \frac{\xi x}{1+\xi} \right) e^{-\xi x} \right) - 1} \right)} - 1}{\frac{\log(\alpha)}{\alpha-1} \left(\frac{1}{e-1} \right) \left(\frac{\xi^2}{1+\xi} \right) (1+x) e^{-\xi x} \left(e^{\left(1 - \left(1 + \frac{\xi x}{1+\xi} \right) e^{-\xi x} \right) - 1} \right) \alpha^{\left(\frac{1}{e-1} \right) \left(e^{\left(1 - \left(1 + \frac{\xi x}{1+\xi} \right) e^{-\xi x} \right) - 1} \right)}}.$$

4. Some Statistical Properties of APT-DUSL Distribution

In this section, we discuss some of the properties of our proposed distribution.

4.1. Quantile Function

The quantile function is obtained by inverting the CDF of the APT-DUSL distribution as follows:

$$F(Q_p; \xi, \alpha) = p, \quad 0 < p < 1,$$

By solving for Q_p , the quantile function is obtained as

$$Q_p = -\frac{1}{\xi} - 1 - \frac{1}{\xi} W_{-1} \left(e^{-(1+\xi)} (1 + \xi) \left[\log \left(\frac{(e-1) \log(p(\alpha-1)+1)}{\log(\alpha)} + 1 \right) - 1 \right] \right), \quad (12)$$

where $W(\cdot)$ denotes the Lambert W function, defined as the inverse of we^w [45]. The notation $W_{-1}(\cdot)$ represents its negative (lower) branch, which is used in deriving the quantile function.

The first and third quartiles are obtained by substituting $p = 0.25$ and $p = 0.75$, respectively.

Median

The median is obtained by substitution $p = 1/2$ into Equation (12), giving

$$Q_{0.5} = -\frac{1}{\xi} - 1 - \frac{1}{\xi} W_{-1} \left(e^{-(1+\xi)} (1 + \xi) \left[\log \left(\frac{(e-1) \log((0.5)(\alpha-1)+1)}{\log(\alpha)} + 1 \right) - 1 \right] \right).$$

Figure 3 presents the surface of the median $Q_{0.5}$. It can be seen that the median decreases as ξ increases. On the other hand, the median exhibits an increasing tendency with regard to the shape parameter α , meaning that larger values of α shift the center of the distribution to the right.

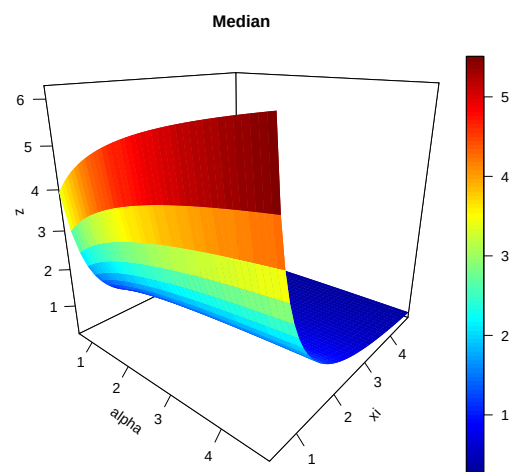


Figure 3. 3D visualization of the median for APT-DUSL distribution.

4.2. Quantile Measures

The skewness and kurtosis measures can be expressed in terms of the quantile function Q_p . Therefore, the Galton skewness (Sk_G) [46] and the Moors kurtosis (kur_M) [47] of the APT-DUSL are defined as follows:

$$Sk_G = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}, \quad (13)$$

$$kur_M = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}. \quad (14)$$

where Q_p is the quantile function given in Equation (12).

Figure 4 shows the plots of Sk_G and kur_M for the APT-DUSL distribution. In addition, Table 1 presents the median, Sk_G and kur_M at different values of α and ξ .

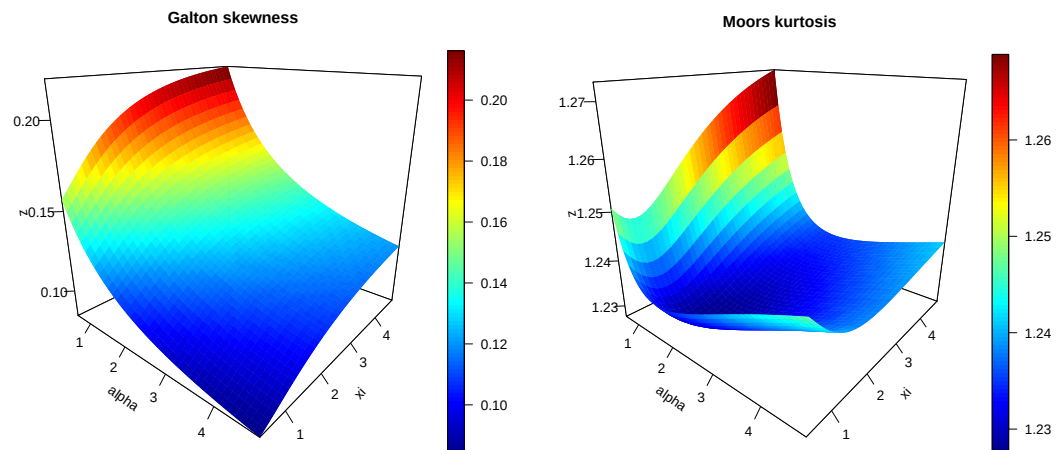


Figure 4. 3D plots of the Galton skewness and Moors kurtosis measures for the APT-DUSL distribution.

Table 1. Quantile-based measures of the APT-DUSL distribution.

Parameters		Quantile Measures		
α	ζ	Median	Sk_G	kur_M
0.8	0.5	3.2808	0.1487	1.2417
1.5	1.0	1.7145	0.1347	1.2288
2.5	2.0	0.8584	0.1290	1.2291
4.0	4.0	0.4294	0.1244	1.2376
4.5	3.0	0.6133	0.1156	1.2379
2.0	4.5	0.3154	0.1558	1.2354
0.5	3.0	0.3317	0.221	1.274
3.0	0.8	2.6229	0.1024	1.2345

The plot of Sk_G in Figure 4 shows that the skewness of the APT-DUSL distribution is consistently influenced by the interaction between the shape and scale parameters. The behavior of Sk_G varies greatly along the parameter space. This interaction demonstrates the high flexibility of the model to allow for the fine-tuning of right-skewness to fit datasets with various tail-weight characteristics. In comparison, the kur_M plot in Figure 4 remains nearly constant across both parameters, indicating that the tail behavior is not very sensitive to changes in α and ζ .

We clarify the APT-DUSL's asymmetry by examining the relation between the shape and the scale parameter. It can be seen in Figure 1 that the APT-DUSL distribution is always positively skewed and is confirmed by the quantile-based metrics in Table 1. Furthermore, Figure 4 demonstrates that right-skewness is not only monotonic in α but also influenced by the complex interaction between α and ζ . This complex parametric relationship reveals that the APT-DUSL model is not restricted to a specific skewness profile but provides the necessary parametric flexibility to accommodate varying degrees of right-tail stretching and thereby accurately describe the positive asymmetry generally observed in the empirical mechanical and survival data.

4.3. Moment

The r -th moment of a random variable X is a measure that provides information on the shape of a distribution and its central tendency and variations. Let X be a random variable with PDF in Equation (8), then the r -th moment is given by:

$$\mu_r = \mathbb{E}(X^r) = \int_0^{\infty} x^r f(x) dx.$$

Initially, we will express the PDF through a series expansion:

$$f(x; \xi, \alpha) = \sum_{k=0}^{\infty} \sum_{n=0}^k \sum_{m=0}^{\infty} \sum_{z=0}^m (-1)^{(n+m-k)} \binom{k}{n} \binom{m}{z} \frac{(\log(\alpha))^{k+1}}{(\alpha-1)k!} \frac{(n+1)^m}{m!} \\ \times \left(\frac{1}{e-1} \right)^{k+1} \left(\frac{\xi^{z+2}}{(1+\xi)^{z+1}} \right) e^{(n+1)(x^z + x^{z+1})} e^{-\xi x(m+1)}.$$

Consequently, the PDF can be expressed as

$$f(x; \xi, \alpha) = \sum_{m=0}^{\infty} \sum_{z=0}^m \delta_{m,z} (x^z + x^{z+1}) e^{-\xi x(m+1)}, \quad (15)$$

where

$$\delta_{m,z} = \sum_{k=0}^{\infty} \sum_{n=0}^k (-1)^{(n+m-k)} \binom{k}{n} \binom{m}{z} \frac{(\log(\alpha))^{k+1}}{(\alpha-1)k!} \frac{(n+1)^m}{m!} \left(\frac{1}{e-1} \right)^{k+1} \left(\frac{\xi^{z+2}}{(1+\xi)^{z+1}} \right) e^{(n+1)}. \quad (16)$$

Therefore,

$$\mu_r = \sum_{m=0}^{\infty} \sum_{z=0}^m \delta_{m,z} \int_0^{\infty} x^r (x^z + x^{z+1}) e^{-\xi x(m+1)} dx.$$

Then, the final form of μ_r is obtained as

$$\mu_r = \sum_{m=0}^{\infty} \sum_{z=0}^m \delta_{m,z} \left[\frac{\Gamma(r+z+1)(\xi(m+1)) + \Gamma(r+z+2)}{[\xi(m+1)]^{r+z+2}} \right], \quad (17)$$

where $\delta_{m,z}$ is given by Equation (16) and is absolutely convergent for all $\alpha > 0$ and $\xi > 0$.

The numerical evaluation of the first four central moments for the APT-DUSL for different parameter sets is presented in Table 2. The results illustrate the stability and finite values of the mean, variance, and higher-order moments and provide empirical support for the absolute convergence of the series representation. These computations show the existence of the moments for $(\alpha > 0, \xi > 0)$ and confirm the theoretical statement of the finite moment structure of APT-DUSL.

Table 2. Moment-based measures of the APT-DUSL distribution for selected parameter values.

Parameters		Moment Measures			
α	ξ	Mean	Variance	μ_3	μ_4
0.8	0.5	3.8985	8.5315	3.27×10^1	4.05×10^2
1.5	1.0	2.0128	2.2258	4.1221	2.61×10^1
2.5	2.0	1.0021	0.5426	0.4909	1.5481
4.0	4.0	0.4972	0.1280	0.0561	0.0871
4.5	3.0	0.7014	0.2409	0.1382	0.2971
2.0	4.5	0.3824	0.0909	0.0381	0.0487
0.5	3.0	0.4448	0.1654	0.1176	0.2057
3.0	0.8	2.9447	3.8666	8.1067	7.10×10^1

4.4. Moment-Generating Function

The moment-generating function (MGF) for the APT-DUSL distribution, referred to as $M(t)$, can be derived in the following manner:

$$M_x(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} f(x; \xi, \alpha) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \cdot \mu_r$$

Therefore,

$$M_x(t) = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{z=0}^m \delta_{m,z} \frac{t^r}{r!} \left[\frac{\Gamma(r+z+1)(\xi(m+1)) + \Gamma(r+z+2)}{[\xi(m+1)]^{r+z+2}} \right]. \quad (18)$$

4.5. Characteristic Function

The characteristic function $\phi_X(t)$ of a continuous random variable X with $f(x)$ is given by:

$$\begin{aligned} \phi_X(t) &= E[e^{itx}] = \int_0^{\infty} e^{itx} f(x; \xi, \alpha) dx = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \cdot \mu_r \\ \phi_X(t) &= \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{z=0}^m \delta_{m,z} \frac{(it)^r}{r!} \left[\frac{\Gamma(r+z+1)(\xi(m+1)) + \Gamma(r+z+2)}{[\xi(m+1)]^{r+z+2}} \right] \end{aligned} \quad (19)$$

4.6. Rényi Entropy and Shannon Entropy

The Rényi and Shannon entropy measurements are important in the content and uncertainty of lifetime processes. The Rényi entropy of order ν (where $\nu > 0$ and $\nu \neq 1$) serves as a broader form of the Shannon entropy and offers insights into the distribution's variability of information. Its definition is given by:

$$H_{\nu}(X) = \frac{1}{1-\nu} \log \left(\int_0^{\infty} f(x)^{\nu} dx \right)$$

Therefore, using the PDF of APT-DUSL distribution in (10), $f(x)^{\nu}$ is expressed as

$$\begin{aligned} f(x; \xi, \alpha)^{\nu} &= \left[\frac{\log(\alpha)}{\alpha - 1} \right]^{\nu} \left[\frac{1}{e - 1} \right]^{\nu} \left[\frac{\xi^2}{1 + \xi} \right]^{\nu} \left[(1 + x)e^{-\xi x} \right]^{\nu} \\ &\times \left[e^{(1 - (1 + \frac{\xi x}{1 + \xi}))e^{-\xi x}} \right]^{\nu} \left[\alpha^{\left(\frac{1}{e-1} \right) \left(e^{(1 - (1 + \frac{\xi x}{1 + \xi}))e^{-\xi x}} - 1 \right)} \right]^{\nu}. \end{aligned}$$

Following the implementation of several expansions, we obtain:

$$\begin{aligned} f(x; \xi, \alpha)^{\nu} &= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^k \sum_{n=0}^m \frac{(-1)^m \nu^i (\log(\alpha))^{\nu+i} (i + \nu)^k}{(\alpha - 1)^{\nu} (e - 1)^{\nu+i} i! k!} \binom{k}{m} \binom{m}{n} \\ &\times \frac{\xi^{2\nu+n}}{(1 + \xi)^{\nu+n}} e^{-\frac{\nu \log(\alpha)}{e-1}} (1 + x)^{\nu} x^n e^{-(m+\nu)\xi x}. \end{aligned}$$

For non-integer ν , using the formal generalized binomial expansion,

$$(1 + x)^{\nu} = \sum_{a=0}^{\infty} \binom{\nu}{a} x^a.$$

Thus,

$$\begin{aligned} f(x; \xi, \alpha)^{\nu} &= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^k \sum_{n=0}^m \sum_{a=0}^{\infty} \frac{(-1)^m \nu^i (\log(\alpha))^{\nu+i} (i + \nu)^k}{(\alpha - 1)^{\nu} (e - 1)^{\nu+i} i! k!} \binom{k}{m} \binom{m}{n} \binom{\nu}{a} \\ &\times \frac{\xi^{2\nu+n}}{(1 + \xi)^{\nu+n}} e^{-\frac{\nu \log(\alpha)}{e-1}} x^{n+a} e^{-(m+\nu)\xi x}. \end{aligned}$$

Consequently, by defining

$$\delta_{i,k,m,n,a}^* = \frac{(-1)^m \nu^i (\log \alpha)^i (i + \nu)^k}{(e - 1)^{\nu+i} i! k!} \binom{k}{m} \binom{m}{n} \binom{\nu}{a} \frac{\xi^{2\nu+n}}{(1 + \xi)^{\nu+n}} \frac{\Gamma(n + a + 1)}{[(m + \nu)\xi]^{n+a+1}}.$$

The Rényi entropy of the APT-DUSL distribution can be written as

$$H_\nu(X) = \frac{1}{1-\nu} \log \left[\frac{(\log \alpha)^\nu}{(\alpha-1)^\nu} e^{-\frac{\nu \log(\alpha)}{e-1}} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^k \sum_{n=0}^m \sum_{a=0}^{\infty} \delta_{i,k,m,n,a}^* \right]. \quad (20)$$

As a limiting case of the Rényi entropy, letting $\nu \rightarrow 1$ yields the Shannon entropy of APT-DUSL distribution as follows:

$$H(X) = - \int_0^\infty f(x; \xi, \alpha) \log f(x; \xi, \alpha) dx.$$

4.7. Order Statistics

Order statistics are essential in statistics and real-data applications for determining summary statistics such as the minimum, maximum, quartiles, and percentiles. In addition, in reliability and lifetime studies, order statistics can be used to model the lifetime of the strongest or weakest components.

Let $X_1, X_2, \dots, X_n$ be a random sample drawn from a continuous distribution with PDF and CDF. The corresponding order statistics are denoted by

$$X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n},$$

where $X_{i:n}$ represents the i^{th} smallest observation in the sample. The CDF of the r^{th} order statistic $X_{r:n}$ is given by

$$F_{r:n}(x) = P(X_{r:n} \leq x) = \sum_{j=r}^n \binom{n}{j} [F(x)]^j [1 - F(x)]^{n-j}.$$

Applying the binomial expansion to the term $[1 - F(x)]^{n-j}$ yields

$$F_{r:n}(x) = \sum_{j=r}^n \sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} (-1)^k [F(x)]^{j+k}.$$

Substituting the CDF of the APT-DUSL distribution into the above expression, and defining

$$\eta(x; \xi, \alpha) = \left(\frac{\alpha^{\left(\frac{1}{e-1}\right) \left(e^{\left(1 - \left(1 + \frac{\xi x}{1+\xi}\right) e^{-\xi x}} \right) - 1 \right)}}{\alpha - 1} - 1 \right).$$

The CDF of the r^{th} order statistic can be written as

$$F_{r:n}(x) = \sum_{j=r}^n \sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} (-1)^k \eta(x; \xi, \alpha)^{j+k}. \quad (21)$$

The PDF of the i^{th} order statistic $X_{i:n}$ is defined as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) [F(x)]^{i-1} [1 - F(x)]^{n-i}, \quad x \in \mathbb{R}. \quad (22)$$

Using the binomial expansion for $[1 - F(x)]^{n-i}$, Equation (22) can be rewritten as

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k n!}{(i-1)!(n-i)!} f(x) [F(x)]^{k+i-1},$$

where $F(x)$ and $f(x)$ are the CDF and PDF of the APT-DUSL given in Equations (9) and (10).

5. Parameter Estimation

In this section, we estimate the unknown parameters of the proposed APT-DUSL distribution model using the maximum likelihood estimation (MLE) method based on the complete sample and the progressive type-II censored sample:

5.1. Maximum Likelihood Estimation Under Complete Samples

Let $X_1, X_2, \dots, X_n$ be a random sample of size n taken from the APT-DUSL distribution. The likelihood function is defined as

$$L(\xi, \alpha) = \prod_{i=1}^n f(x_i; \xi, \alpha).$$

The log-likelihood function, denoted by $\ell(\xi, \alpha)$, is obtained as

$$\begin{aligned} \ell(\xi, \alpha) = & n \log\left(\frac{\log(\alpha)}{\alpha - 1}\right) + n \log\left(\frac{1}{e - 1}\right) + n \log\left(\frac{\xi^2}{1 + \xi}\right) + \sum_{i=1}^n \log(1 + x_i) - \xi \sum_{i=1}^n x_i \\ & + \sum_{i=1}^n \left[1 - \left(1 + \frac{\xi x_i}{1 + \xi}\right) e^{-\xi x_i}\right] + \frac{\log(\alpha)}{e - 1} \sum_{i=1}^n \left[e^{\left(1 - \left(1 + \frac{\xi x_i}{1 + \xi}\right) e^{-\xi x_i}\right)} - 1\right]. \end{aligned} \quad (23)$$

Differentiating (23) with respect to ξ and setting it equal to zero yields

$$\begin{aligned} \frac{\partial \ell}{\partial \xi} = & \frac{2n}{\xi} - \frac{n}{1 + \xi} - \sum_{i=1}^n x_i + \sum_{i=1}^n \left[\frac{x_i e^{-\xi x_i}}{1 + \xi} - \frac{x_i^2 e^{-\xi x_i}}{(1 + \xi)^2} + x_i \left(1 + \frac{\xi x_i}{1 + \xi}\right) e^{-\xi x_i} \right] \\ & + \frac{\log(\alpha)}{e - 1} \sum_{i=1}^n \left[\left(\frac{x_i e^{-\xi x_i}}{1 + \xi} - \frac{x_i^2 e^{-\xi x_i}}{(1 + \xi)^2} + x_i \left(1 + \frac{\xi x_i}{1 + \xi}\right) e^{-\xi x_i} \right) e^{\left(1 - \left(1 + \frac{\xi x_i}{1 + \xi}\right) e^{-\xi x_i}\right)} - 1 \right] = 0. \end{aligned} \quad (24)$$

Differentiating (23) with respect to α yields

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha \log(\alpha)} - \frac{n}{\alpha - 1} + \frac{1}{\alpha(e - 1)} \sum_{i=1}^n \left[e^{\left(1 - \left(1 + \frac{\xi x_i}{1 + \xi}\right) e^{-\xi x_i}\right)} - 1 \right] = 0. \quad (25)$$

Due to the non-linearity and complexity of Equations (24) and (25), a numerical solution is needed. Therefore, parameter estimates are obtained numerically by minimizing the $-\ell(\xi, \alpha)$ using the R statistical software version 4.4.2 (31 October 2024) and utilizing the “optim” function with L-BFGS-B algorithm.

Since the sampling distribution of the MLE of $\hat{\theta} = (\hat{\alpha}, \hat{\xi})$ cannot be obtained in closed form, the exact sampling distribution of the maximum likelihood estimator cannot be determined, and we cannot calculate a precise confidence interval for $\hat{\theta}$. Instead, relying on the regularity conditions of the APT-DUSL distribution, the maximum likelihood estimators $\hat{\theta}$ are asymptotically consistent and normally distributed, which allows for the construction of asymptotic confidence intervals,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, I^{-1}(\theta)),$$

where $I(\theta)$ is the Fisher information matrix, which is approximated by the observed information matrix $J(\hat{\theta})$ based on the negative Hessian matrix of the log-likelihood function at the MLEs. The parameters have asymptotic $(1 - \gamma)100\%$ confidence intervals such that

$$\hat{\theta} \pm z_{\gamma/2} \sqrt{\text{diag}(J^{-1}(\hat{\theta}))},$$

where $z_{\gamma/2}$ is the upper $(\gamma/2)$ percentile of the standard normal distribution.

5.2. Maximum Likelihood Estimation Under Progressive Type-II Censored Sample

Consider a life-testing experiment in which n units are placed on test and the experiment continues until m failures are observed, where $1 \leq m < n$. After the i th observed failure time $x_{i:m:n}$, R_i surviving units are randomly removed according to a predefined censoring scheme $(R_1, \dots, R_m)$ satisfying

$$n = m + \sum_{i=1}^m R_i.$$

Figure 5 represents the progressive Type-II censoring mechanism. The experiment starts with n units, and at each observed failure time $x_{i:n}$, a predetermined number R_i of the remaining units are removed. This process continues until m failures are observed, after which all remaining units are removed. The censored scheme $(R_1, R_2, \dots, R_m)$ controls the amount of information retained for estimation.

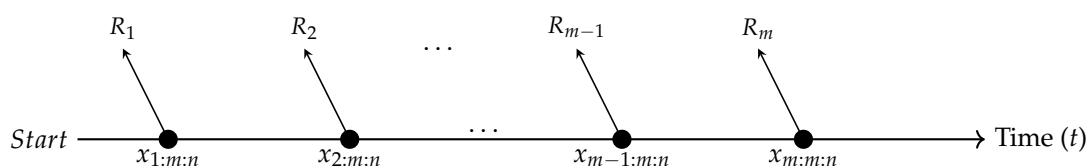


Figure 5. Schematic representation of progressively type-II censored.

Under this scheme, the likelihood function of the proposed distribution is

$$L(\xi, \alpha) = C \prod_{i=1}^m f(x_{i:m:n}; \xi, \alpha) [1 - F(x_{i:m:n}; \xi, \alpha)]^{R_i},$$

where C is a constant independent of the parameters, given by

$$C = n(n-1-R_1)(n-2-R_1-R_2) \dots \left(n-m+1 - \sum_{i=1}^{m-1} R_i \right).$$

Ignoring the constant term, we have

$$\ell(\xi, \alpha) = \left[\sum_{i=1}^m \log f(x_{i:m:n}; \xi, \alpha) + \sum_{i=1}^m R_i \log(1 - F(x_{i:m:n}; \xi, \alpha)) \right].$$

Substituting the CDF and PDF of the APT-DUSL distribution given in Equations (9) and (10) into the above expression, the $\ell(\xi, \alpha)$ becomes

$$\begin{aligned} \ell(\xi, \alpha) = & m \log \left(\frac{\log(\alpha)}{\alpha - 1} \right) + m \log \left(\frac{1}{e - 1} \right) + m \log \left(\frac{\xi^2}{1 + \xi} \right) + \sum_{i=1}^m \log(1 + x_{i:m:n}) - \xi \sum_{i=1}^m x_{i:m:n} \\ & + \sum_{i=1}^m \left[1 - \left(1 + \frac{\xi x_{i:m:n}}{1 + \xi} \right) e^{-\xi x_{i:m:n}} \right] + \frac{\log(\alpha)}{e - 1} \sum_{i=1}^m \left[e^{\left(1 - \left(1 + \frac{\xi x_{i:m:n}}{1 + \xi} \right) e^{-\xi x_{i:m:n}} \right)} - 1 \right] \\ & + \sum_{i=1}^m R_i \log \left(1 - \frac{\left(\frac{1}{e-1} \right) \left[e^{\left(1 - \left(1 + \frac{\xi x_{i:m:n}}{1 + \xi} \right) e^{-\xi x_{i:m:n}} \right)} - 1 \right]}{\alpha - 1} \right). \end{aligned} \quad (26)$$

The MLEs of ζ and α are the solutions of the likelihood equations, which can be obtained by differentiating Equation (26) with respect to the parameters and equating each derivative to zero. However, an iterative numerical procedure is used to estimate ζ and α . The numerical optimization is performed in R using the `optim()` function with the L-BFGS-B method.

6. Simulations Study

A Monte Carlo simulation study is conducted to evaluate the performance of MLE under different sample sizes ($n = 30, 50, 100, 150, 200, 250, 300$) and four cases of parameter values defined as follows:

- Case 1: ($\zeta = 2.5, \alpha = 1.5$),
- Case 2: ($\zeta = 0.7, \alpha = 0.7$),
- Case 3: ($\zeta = 1.5, \alpha = 2.0$),
- Case 4: ($\zeta = 0.3, \alpha = 3.0$).

For each case ($N = 1000$), independent samples are generated to ensure the stability and reliability of the simulation results. The performance of the estimators is evaluated using bias and root mean square error (RMSE) as primary accuracy measures, which provide insight into both systematic deviation and overall estimation variability, respectively. These measures are defined as follows:

$$\widehat{\text{Bias}}(\hat{\zeta}) = \frac{1}{N} \sum_{i=1}^N (\hat{\zeta}_i - \zeta), \quad \widehat{\text{Bias}}(\hat{\alpha}) = \frac{1}{N} \sum_{i=1}^N (\hat{\alpha}_i - \alpha). \quad (27)$$

$$\text{RMSE}(\hat{\zeta}) = \sqrt{\frac{1}{N} \sum_{i=1}^N (\hat{\zeta}_i - \zeta)^2}, \quad \text{RMSE}(\hat{\alpha}) = \sqrt{\frac{1}{N} \sum_{i=1}^N (\hat{\alpha}_i - \alpha)^2}. \quad (28)$$

The simulation experiment is conducted under both complete samples and progressive type-II censored samples to examine the impact of censoring schemes, in addition to sample size, on the accuracy and stability of the estimators. This comparison provides a clearer understanding of how information loss due to censoring affects estimation performance.

6.1. Simulation Under Complete Samples

Complete random samples are generated from the APT-DUSL distribution using the quantile function in (12). For each simulated sample, the unknown parameters are estimated by numerically solving the equations in (24) and (25). The performance of the estimators is assessed in terms of bias and RMSE as defined in Equations (27) and (28).

The results in Table 3 show that estimation accuracy improves as the sample size increases, as both bias and RMSE decrease across all cases. This behavior confirms the consistency of the MLEs. The parameter ζ is estimated with higher accuracy, as its bias is small and rapidly approaches zero with increasing n , and its RMSE decreases steadily. In contrast, the parameter α exhibits greater variability for small sample sizes, but its performance improves significantly as the sample size increases, approaching the true values. This problem is due to the high dimensionality of the parameter space and the intrinsic sensitivity of the AP transformation under limited data conditions. As the sample size decreases, the likelihood surface grows flatter around α , resulting in a tendency of the optimizer to fluctuate from the specific sample. However, the uniform convergence of the `optim` algorithm across all simulated replicates and the observed stabilization of estimates with increasing sample sizes reveal the computational robustness and stability of the APT-DUSL estimation procedure.

Table 3. Bias and RMSE of MLE under complete samples.

Case	<i>n</i>	$\hat{\xi}$	$\hat{\alpha}$	Bias $_{\xi}$	Bias $_{\alpha}$	RMSE $_{\xi}$	RMSE $_{\alpha}$
Case 1	30	2.6258	4.7995	0.1258	3.2995	0.6490	19.7436
	50	2.5433	2.5032	0.0433	1.0032	0.4987	3.1120
	70	2.5278	2.1215	0.0278	0.6215	0.4287	2.1641
	100	2.5319	1.9618	0.0319	0.4618	0.3427	1.4451
	150	2.5161	1.7646	0.0161	0.2646	0.2800	1.0606
	200	2.4992	1.6715	−0.0008	0.1715	0.2380	0.8186
	250	2.5079	1.6498	0.0079	0.1498	0.2121	0.7428
	300	2.5047	1.6171	0.0047	0.1171	0.1968	0.6686
Case 2	30	0.7585	2.3141	0.0585	1.6141	0.2153	4.8635
	50	0.7329	1.4856	0.0329	0.7856	0.1637	2.0492
	70	0.7069	1.1143	0.0069	0.4143	0.1476	1.2949
	100	0.7019	0.9818	0.0019	0.2818	0.1282	0.9473
	150	0.7103	0.9172	0.0103	0.2172	0.1029	0.6823
	200	0.6987	0.8078	−0.0013	0.1078	0.0903	0.4924
	250	0.7011	0.8132	0.0011	0.1132	0.0832	0.4671
	300	0.6967	0.7673	−0.0033	0.0673	0.0733	0.4051
Case 3	30	1.5591	5.4477	0.0591	3.4477	0.3547	11.1447
	50	1.5294	3.7423	0.0294	1.7423	0.2828	5.3878
	70	1.5108	2.9840	0.0108	0.9840	0.2299	3.0842
	100	1.5123	2.6223	0.0123	0.6223	0.1955	2.0013
	150	1.5141	2.3827	0.0141	0.3827	0.1500	1.3884
	200	1.5037	2.2301	0.0037	0.2301	0.1312	1.1226
	250	1.5021	2.1867	0.0021	0.1867	0.1150	0.9716
	300	1.5071	2.1901	0.0071	0.1901	0.1074	0.9097
Case 4	30	0.3053	8.8742	0.0053	5.8742	0.0638	24.1336
	50	0.3030	5.5843	0.0030	2.5843	0.0493	10.1106
	70	0.3035	4.6505	0.0035	1.6505	0.0415	4.8846
	100	0.3011	3.9470	0.0011	0.9470	0.0349	3.1811
	150	0.3016	3.6523	0.0016	0.6523	0.0279	2.4363
	200	0.3010	3.4807	0.0010	0.4807	0.0232	1.8649
	250	0.3010	3.3559	0.0010	0.3559	0.0198	1.4404
	300	0.3004	3.2757	0.0004	0.2757	0.0199	1.4072

6.2. Simulation Under Progressive Type-II Censoring

In the simulation study, a fixed percentage of 70% is used, where

$$m = 0.7 \times n.$$

The following censoring schemes are considered:

- Early scheme: $R_1 = n - m$, $R_2 = \dots = R_m = 0$.
- Middle scheme: $R_i = \frac{n-m}{m}$, $i = 1, \dots, m$.
- Last scheme: $R_1 = \dots = R_{m-1} = 0$, $R_m = n - m$.

Tables 4–6 compare the performance of the MLEs under different progressive censoring schemes in terms of bias and RMSE. As shown in Table 4, the early censoring scheme yields lower Bias and RMSE in most cases.

Table 5 shows a similar pattern, where the middle scheme provides results close to the early censoring scheme, especially for large sample sizes. In contrast, Table 6 shows that the last scheme yields larger Bias and RMSE values, indicating greater variability and lower estimation accuracy, particularly for the parameter (α).

Table 4. Bias and RMSE of MLE under the early censoring scheme.

Case	n	$\hat{\xi}$	$\hat{\alpha}$	Bias $_{\xi}$	Bias $_{\alpha}$	RMSE $_{\xi}$	RMSE $_{\alpha}$
Case 1	30	2.6726	4.9232	0.1726	3.4232	0.7451	10.4762
	50	2.6119	3.1779	0.1119	1.6779	0.5956	4.4251
	70	2.5353	2.4749	0.0353	0.9749	0.4863	3.0330
	100	2.5202	2.1168	0.0202	0.6168	0.4224	2.0253
	150	2.5153	1.8515	0.0153	0.3515	0.3454	1.3088
	200	2.5187	1.7756	0.0187	0.2756	0.2879	1.0497
	250	2.5107	1.7022	0.0107	0.2022	0.2534	0.9081
	300	2.5054	1.6618	0.0054	0.1618	0.2257	0.7937
Case 2	30	1.5987	6.9628	0.0987	4.9628	0.4102	15.3410
	50	1.5429	4.0498	0.0429	2.0498	0.3188	5.8865
	70	1.5457	3.4502	0.0457	1.4502	0.2806	3.9895
	100	1.5137	2.8030	0.0137	0.8030	0.2224	2.5636
	150	1.5134	2.5753	0.0134	0.5753	0.1821	1.7869
	200	1.5102	2.3915	0.0102	0.3915	0.1575	1.5098
	250	1.5063	2.3050	0.0063	0.3050	0.1398	1.2206
	300	1.4997	2.1938	−0.0003	0.1938	0.1281	1.0499
Case 3	30	0.3125	10.7614	0.0125	7.7614	0.0730	25.4021
	50	0.3043	6.2893	0.0043	3.2893	0.0580	9.8488
	70	0.3034	5.1651	0.0034	2.1651	0.0491	6.5519
	100	0.3039	4.4888	0.0039	1.4888	0.0398	4.1473
	150	0.3002	3.8490	0.0002	0.8490	0.0337	2.9827
	200	0.3011	3.5557	0.0011	0.5557	0.0282	2.2306
	250	0.3002	3.4276	0.0002	0.4276	0.0248	1.8564
	300	0.3022	3.4962	0.0022	0.4962	0.0225	1.7392
Case 4	30	0.7729	3.0855	0.0729	2.3855	0.2415	7.0181
	50	0.7429	1.7173	0.0429	1.0173	0.1877	2.7155
	70	0.7183	1.3411	0.0183	0.6411	0.1650	1.8635
	100	0.7100	1.1318	0.0100	0.4318	0.1427	1.2282
	150	0.7099	0.9921	0.0099	0.2921	0.1141	0.8776
	200	0.7035	0.8981	0.0035	0.1981	0.1057	0.6831
	250	0.7002	0.8448	0.0002	0.1448	0.0991	0.5658
	300	0.6976	0.8031	−0.0024	0.1031	0.0885	0.5041

Table 5. Bias and RMSE of MLE under the middle censoring scheme.

Case	n	$\hat{\xi}$	$\hat{\alpha}$	Bias $_{\xi}$	Bias $_{\alpha}$	RMSE $_{\xi}$	RMSE $_{\alpha}$
Case 1	30	2.7053	4.3109	0.2053	2.8109	0.7664	8.7821
	50	2.6284	3.0326	0.1284	1.5326	0.6381	4.0903
	70	2.5884	2.4397	0.0884	0.9397	0.5399	2.5348
	100	2.5359	2.0121	0.0359	0.5121	0.4607	1.6700
	150	2.5123	1.8561	0.0123	0.3561	0.3570	1.2901
	200	2.5113	1.7287	0.0113	0.2287	0.3028	0.9982
	250	2.4992	1.6497	−0.0008	0.1497	0.2769	0.8479
	300	2.5179	1.6868	0.0179	0.1868	0.2452	0.7756
Case 2	30	1.6065	7.3491	0.1065	5.3491	0.4613	21.7901
	50	1.5444	3.7717	0.0444	1.7717	0.3406	5.0276
	70	1.5312	3.2847	0.0312	1.2847	0.2912	3.9349
	100	1.5215	2.7138	0.0215	0.7138	0.2389	2.3414
	150	1.5081	2.4394	0.0081	0.4394	0.1956	1.6802
	200	1.5110	2.3564	0.0110	0.3564	0.1726	1.3071
	250	1.5111	2.2542	0.0111	0.2542	0.1438	1.0964
	300	1.5066	2.2212	0.0066	0.2212	0.1367	0.9971

Table 5. Cont.

Case	n	$\hat{\xi}$	$\hat{\alpha}$	Bias $_{\xi}$	Bias $_{\alpha}$	RMSE $_{\xi}$	RMSE $_{\alpha}$
Case 3	30	0.3119	10.3213	0.0119	7.3213	0.0767	33.8309
	50	0.3101	6.7671	0.0101	3.7671	0.0641	10.6478
	70	0.3059	4.9971	0.0059	1.9971	0.0510	5.9893
	100	0.3021	4.1453	0.0021	1.1453	0.0435	3.8530
	150	0.3008	3.7008	0.0008	0.7008	0.0350	2.5624
	200	0.3011	3.5578	0.0011	0.5578	0.0310	2.1696
	250	0.3005	3.3781	0.0005	0.3781	0.0262	1.7510
	300	0.3011	3.3502	0.0011	0.3502	0.0247	1.6222
Case 4	30	0.7747	2.8858	0.0747	2.1858	0.2613	6.4378
	50	0.7588	1.9412	0.0588	1.2412	0.2211	3.6146
	70	0.7188	1.2881	0.0188	0.5881	0.1751	1.6780
	100	0.7174	1.1307	0.0174	0.4307	0.1610	1.2639
	150	0.7129	0.9940	0.0129	0.2940	0.1334	0.8613
	200	0.6984	0.8728	−0.0016	0.1728	0.1145	0.6463
	250	0.7064	0.8668	0.0064	0.1668	0.1049	0.5800
	300	0.7045	0.8405	0.0045	0.1405	0.0967	0.5492

Table 6. Bias and RMSE of MLE under the last censoring scheme.

Case	n	$\hat{\xi}$	$\hat{\alpha}$	Bias $_{\xi}$	Bias $_{\alpha}$	RMSE $_{\xi}$	RMSE $_{\alpha}$
Case 1	30	2.7194	6.3811	0.2194	4.8811	0.9517	17.2227
	50	2.6236	3.6012	0.1236	2.1012	0.8061	5.5637
	70	2.5419	2.6810	0.0419	1.1810	0.6718	3.3047
	100	2.5127	2.2024	0.0127	0.7024	0.5599	2.1086
	150	2.4811	1.9140	−0.0189	0.4140	0.4797	1.5186
	200	2.5080	1.8539	0.0080	0.3539	0.4004	1.2719
	250	2.4914	1.7379	−0.0086	0.2379	0.3622	1.0742
	300	2.5086	1.7171	0.0086	0.2171	0.3262	0.9307
Case 2	30	1.5990	7.5469	0.0990	5.5469	0.5003	17.9356
	50	1.5531	4.8906	0.0531	2.8906	0.4029	8.8573
	70	1.5290	3.4857	0.0290	1.4857	0.3510	4.4453
	100	1.5158	3.0902	0.0158	1.0902	0.3052	3.1173
	150	1.5134	2.6149	0.0134	0.6149	0.2408	1.9981
	200	1.4993	2.3790	−0.0007	0.3790	0.2047	1.4695
	250	1.5053	2.3358	0.0053	0.3358	0.1790	1.3167
	300	1.5056	2.3055	0.0056	0.3055	0.1678	1.2256
Case 3	30	0.3142	13.9578	0.0142	10.9578	0.0874	39.3901
	50	0.3091	7.1980	0.0091	4.1980	0.0677	11.4876
	70	0.3018	5.4906	0.0018	2.4906	0.0628	7.0553
	100	0.3025	4.4923	0.0025	1.4923	0.0505	4.2821
	150	0.3005	3.9140	0.0005	0.9140	0.0444	3.1011
	200	0.2987	3.5666	−0.0013	0.5666	0.0364	2.4132
	250	0.2999	3.4689	−0.0001	0.4689	0.0313	1.9920
	300	0.3007	3.4666	0.0007	0.4666	0.0286	1.7674
Case 4	30	0.8157	4.7132	0.1157	4.0132	0.3111	16.4107
	50	0.7670	2.3151	0.0670	1.6151	0.2548	4.0623
	70	0.7456	1.7308	0.0456	1.0308	0.2170	2.5642
	100	0.7207	1.3214	0.0207	0.6214	0.1891	1.5956
	150	0.7025	1.0642	0.0025	0.3642	0.1678	1.1426
	200	0.7027	0.9964	0.0027	0.2964	0.1520	0.8770
	250	0.6934	0.9354	−0.0066	0.2354	0.1442	0.8100
	300	0.6928	0.8792	−0.0072	0.1792	0.1349	0.6925

These findings are further confirmed by Figure 6, where the RMSE values under the early and middle censoring schemes remain relatively close to those of the complete samples across different sample sizes. Overall, the results suggest that the early censoring scheme provides a good balance between estimation efficiency and reducing experimental time and cost while preserving estimation accuracy close to that of the complete samples.

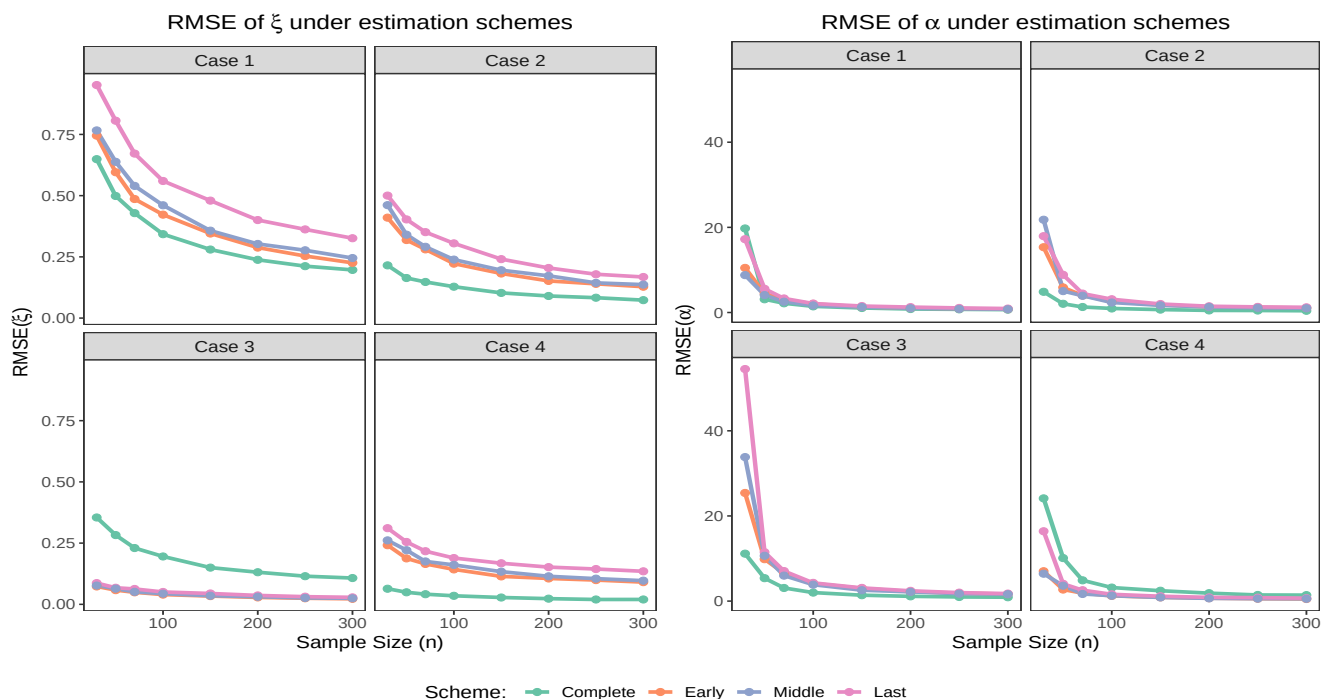


Figure 6. RMSE of the estimators of ξ and α under the complete samples and different progressive censoring schemes.

7. Data Analysis

In this section, five real-world datasets obtained from complete samples are analyzed to assess the applicability and effectiveness of the APT-DUSL distribution.

Data 1: Breaking stress data

The actual data, sourced from [48], contains 100 values about the (breaking) stress of carbon fibers (in Gba).

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90
3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67	2.93	3.22
3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85	2.56	3.56
3.15	2.35	2.55	2.59	2.38	2.81	2.77	2.17	2.83	1.92
1.41	3.68	2.97	1.36	0.98	2.76	4.91	3.68	1.84	1.59
3.19	1.57	0.81	5.56	1.73	1.59	2.00	1.22	1.12	1.71
2.17	1.17	5.08	2.48	1.18	3.51	2.17	1.69	1.25	4.38
1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03	1.80
1.57	1.08	2.03	1.61	2.12	1.89	2.88	2.82	2.05	3.65

Data 2: Aircraft Windshield Failure times

The actual data, sourced from [49], consists of 84 observations. The details of the data are as follows:

0.040	1.866	2.385	3.443	0.301	1.876	2.481	3.467	0.309	1.899	2.610
3.478	0.557	1.911	2.625	3.578	0.943	1.912	2.632	3.595	1.070	1.914
2.646	3.699	1.124	1.981	2.661	3.779	1.248	2.010	2.688	3.924	1.281
2.038	2.823	4.035	1.281	2.085	2.890	4.121	1.303	2.089	2.902	4.167
1.432	2.097	2.934	4.240	1.480	2.135	2.962	4.255	1.505	2.154	2.964
4.278	1.506	2.190	3.000	4.305	1.568	2.194	3.103	4.376	1.615	2.223
3.114	4.449	1.619	2.224	3.117	4.485	1.652	2.229	3.166	4.570	1.652
2.300	3.344	4.602	1.757	2.324	3.376	4.663				

Data 3: Service Times of Aircraft Windshield:

The actual data, sourced from [49], comprise 63 observations. The details of the data are as follows:

0.046	1.436	2.592	0.140	1.492	2.600	0.150	1.580
2.670	0.248	1.719	2.717	0.280	1.794	2.819	0.313
1.915	2.820	0.389	1.920	2.878	0.487	1.963	2.950
0.622	1.978	3.003	0.900	2.053	3.102	0.952	2.065
3.304	0.996	2.117	3.483	1.003	2.137	3.500	1.010
2.141	3.622	1.085	2.163	3.665	1.092	2.183	3.695
1.152	2.240	4.015	1.183	2.341	4.628	1.244	2.435
4.806	1.249	2.464	4.881	1.262	2.543	5.140	

Data 4: Fatigue Fracture of Kevlar 373/epoxy

The actual data, sourced from [50], illustrate the fatigue fracture lifespan of Kevlar 373/epoxy, exposed to steady pressure at a 90% stress level until complete failure occurred.

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671	0.6566
0.6748	0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113
0.9120	0.9836	1.0483	1.0596	1.0773	1.1733	1.2570	1.2766	1.2985	1.3211
1.3503	1.3551	1.4595	1.4880	1.5728	1.5733	1.7083	1.7263	1.7460	1.7630
1.7746	1.8275	1.8375	1.8503	1.8808	1.8878	1.8881	1.9316	1.9558	2.0048
2.0408	2.0903	2.1093	2.1330	2.2100	2.2460	2.2878	2.3203	2.3470	2.3513
2.4951	2.5260	2.9911	3.0256	3.2678	3.4045				

Data 5: Turkish Meteorological Services for Sinop data

The actual data, sourced from [51], represent a random sample of the average daily wind speed data for March 2015, collected by the Turkish Meteorological Services for Sinop, Turkey. The data are recorded as follows:

2.8	1.8	3.2	5.0	2.4	4.8	2.9	2.9	2.3	3.2
2.3	2.0	1.9	3.3	4.4	6.7	4.3	1.9	2.2	3.3
2.1	4.0	2.0	3.1	3.8	3.1	3.2	3.4	2.8	2.1
3.1									

The descriptive statistics for the five datasets are reported in Table 7. These statistics are supported by presenting some basic descriptive graphs of the datasets, such as box plots, cumulative hazard, and total time on test (TTT) plots, which establish the hazard-rate shape of the data. Figures 7–11 display the graphical representation of the five datasets.

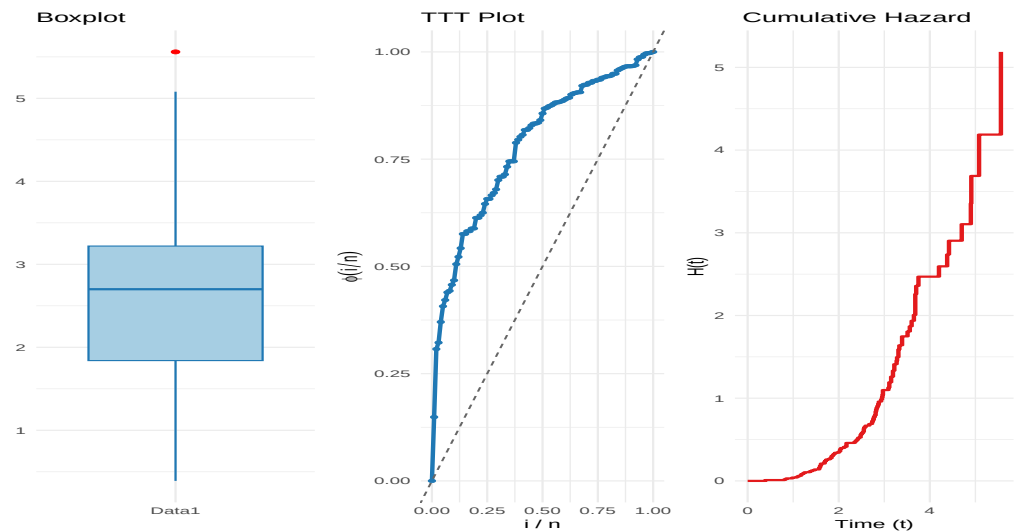


Figure 7. A graphical representation of the Breaking stress data.

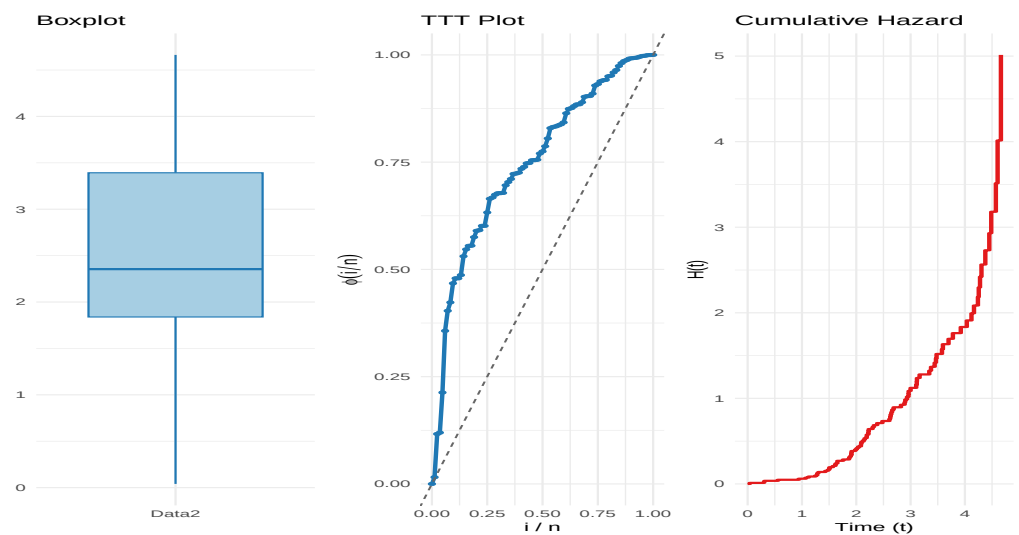


Figure 8. A graphical representation of the Aircraft Windshield Failure times data.

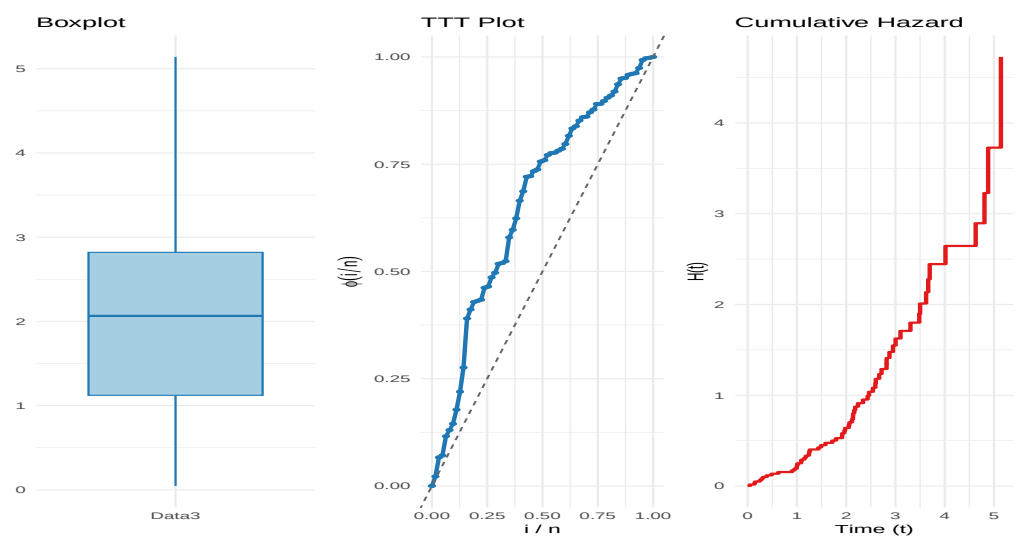


Figure 9. A graphical representation of the service times of Aircraft Windshield data.

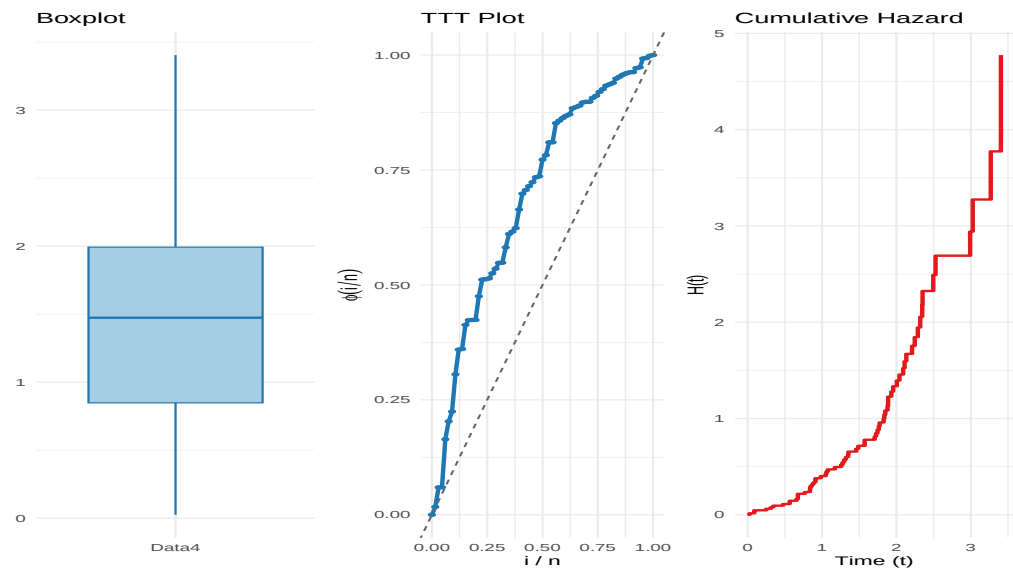


Figure 10. A graphical representation of the fatigue fracture lifespan of Kevlar 373/epoxy data.

Table 7. Summary of descriptive statistics for the datasets.

Data	Size	Mean	Variance	Median	Min	Max	First Quartile	Third Quartile
Data 1	100	2.6214	1.0280	2.7000	0.3900	5.5600	1.8400	3.2200
Data 2	84	2.5575	1.2518	2.3545	0.0400	4.663	1.8388	3.3928
Data 3	63	2.0853	1.5506	2.065	0.046	5.14	1.1220	2.8195
Data 4	66	1.4725	0.6271	1.4738	0.0251	3.4045	0.8480	1.9926
Data 5	31	3.1065	1.1973	3.1000	1.8000	6.7000	2.2500	3.3500

For the five datasets in Figures 7–11, we observe a consistent pattern of right-skewness and increasing failure rates. However, the degree of tail elongation varies across different datasets. Breaking stress data shows an extreme outlier (red dot) and extreme right tail, with clear deviation from the TTT diagonal. The fifth dataset exhibits a significantly more complex structure with high positive skewness and substantial right-tail behavior with two extreme outliers. The sharply concave TTT plots for Datasets 1 and 5 imply fast accumulation of the hazard rate, which presents a serious problem for lifetime models. Datasets 2, 3, and 4, however, are more moderately skewed, with hazard functions that increase at more moderate rates. These results show that the probability of failure increases with stress or time across all five datasets, with Datasets 1 and 5 showing the most heavy-tailed distributions. Therefore, fitting the APT-DUSL distribution to these datasets will illustrate its unique advantage. The inherent parametric flexibility of the APT-DUSL distribution enables it to adapt to these various tail-weight behaviors; thus, it provides a robust tool for modeling extreme failure-time data where simpler, less flexible models often fail.

Subsequently, the proposed model is compared with several competing distributions using a variety of GOF criteria, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), Hannan–Quinn Information Criterion (HQIC), the Kolmogorov–Smirnov statistic (KS) and its corresponding p-value, as well as the log-likelihood value, $\ell(\xi, \alpha)$, evaluated at the MLEs of the parameters.

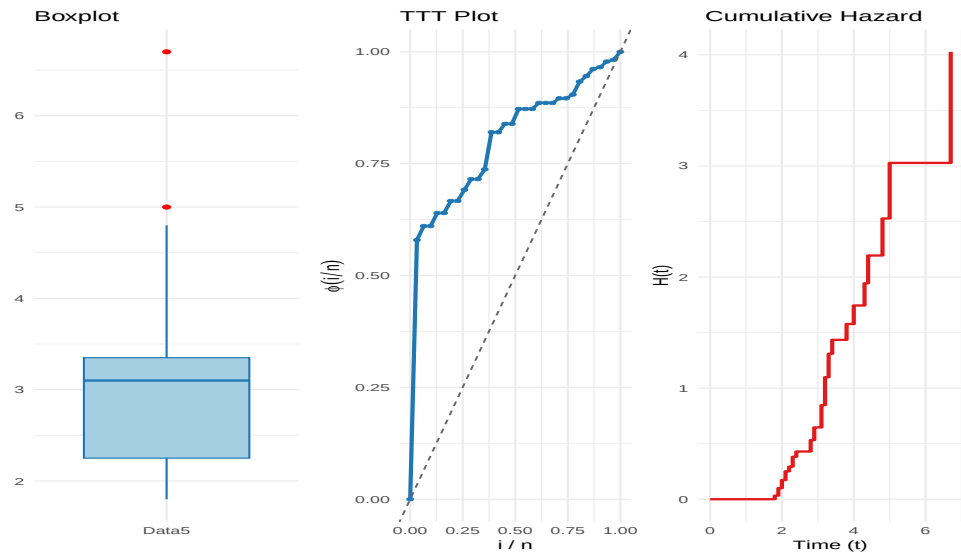


Figure 11. A graphical representation of the Turkish Meteorological Services for Sinop data.

The best-fitting model is determined by the highest value of $\ell(\zeta, \alpha)$ and the lowest values of AIC, BIC, CAIC, HQIC, and KS, together with the largest KS p-value. The performance of the APT-DUSL distribution, given in Equation (9), is evaluated against the Lindley, DUSL, Exponential, AP-L, and Gamma distributions defined in Equations (5), (7), (29), (30) and (31), respectively. For completeness, the CDFs of exponential, AP-L, and Gamma distributions are presented below.

$$F(x; \lambda) = 1 - e^{-\lambda x}, \quad x > 0, \lambda > 0. \quad (29)$$

$$F(x; \theta, \alpha) = \frac{\alpha^{1 - (1 + \frac{\theta x}{\theta + 1})e^{-\theta x}} - 1}{\alpha - 1}, \quad x > 0, \theta > 0, \alpha > 0, \alpha \neq 1. \quad (30)$$

$$F(x; \alpha, \beta) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}, \quad x > 0, \alpha > 0, \beta > 0. \quad (31)$$

The MLEs of the model parameters, along with their corresponding standard errors (SEs), and GOF criteria for the five datasets are reported in Tables 8–17. According to the GOF criteria in Tables 9, 11, 13, 15 and 17, the proposed APT-DUSL distribution outperforms competing models. This conclusion is further supported by the graphical analysis in Figures 12–16. The fitted PDF closely follows the shape of the empirical histogram, capturing the main features of the data such as skewness and tail behavior. The APT-DUSL distribution closely fits the real datasets, confirming the superior performance of the proposed model.

Table 8. MLEs and SEs for the breaking stress data.

Distribution	Parameter	Estimate	SE
Exponential	λ	0.3815	0.0381
LD	θ	0.6173	0.0453
DUSL	ξ	0.7667	0.0490
AP-L	θ	1.3618	0.0512
	α	4575.7794	5.50×10^{-7}
Gamma	α	5.9526	0.8193
	β	2.2708	0.3261
APT-DUSL	ξ	1.4316	0.0526
	α	738.4864	4.40×10^{-6}

Table 9. GOF criteria for breaking stress data.

Distribution	ℓ	AIC	BIC	CAIC	HQIC	KS	p -Value
APT-DUSL	−142.6957	289.3915	294.6018	296.6018	291.5002	0.0864	0.4445
AP-L	−143.1689	290.3378	295.5481	297.5481	292.4465	0.0903	0.3890
Gamma	−143.2336	290.4673	295.6776	297.6776	292.5759	0.0934	0.3471
LD	−181.7535	365.5069	368.1121	369.1121	366.5613	0.2633	0.0000
Exponential	−196.3709	394.7417	397.3469	398.3469	395.7961	0.3206	0.0000
DUSL	−172.9813	347.9626	350.5677	351.5677	349.0169	0.2432	0.0000

Table 10. MLEs and SEs for the aircraft windshield failure times data.

Distribution	Parameter	Estimate	SE
Exponential	λ	0.3910	0.0427
LD	θ	0.6308	0.0506
DUSL	ξ	0.7780	0.0545
AP-L	θ	1.2214	0.0885
	α	294.8065	3.00×10^2
Gamma	α	3.4923	0.5152
	β	1.3655	0.2167
APT-DUSL	ξ	1.3090	0.0920
	α	104.4883	9.29×10^1

Table 11. GOF criteria for aircraft windshield failure times data.

Distribution	ℓ	AIC	BIC	CAIC	HQIC	KS	p -Value
APT-DUSL	−128.9663	261.9326	266.7942	268.7942	263.8869	0.0584	0.9365
AP-L	−129.9675	263.9349	268.7965	270.7965	265.8892	0.0661	0.8566
Gamma	−136.9368	277.8737	282.7353	284.7353	279.8280	0.1034	0.3299
LD	−152.0513	306.1025	308.5333	309.5333	307.0797	0.2419	0.0001
Exponential	−162.8770	327.7539	330.1848	331.1848	328.7311	0.3028	0.0000
DUSL	−145.5708	293.1416	295.5724	296.5724	294.1187	0.2156	0.0008

Table 12. MLEs and SEs for the service times of aircraft windshield data.

Distribution	Parameter	Estimate	SE
Exponential	λ	0.4796	0.0604
LD	θ	0.7531	0.0704
DUSL	ξ	0.9131	0.0756

Table 12. *Cont.*

Distribution	Parameter	Estimate	SE
AP-L	θ	1.1363	0.1215
	α	14.5799	11.4836
Gamma	α	1.9085	0.3148
	β	0.9152	0.1725
APT-DUSL	ξ	1.2077	0.1310
	α	6.8673	5.2587

Table 13. GOF criteria for service times of aircraft windshields data.

Distribution	ℓ	AIC	BIC	CAIC	HQIC	KS	p -Value
APT-DUSL	−99.1071	202.2142	206.5004	208.5004	203.9000	0.0859	0.7085
AP-L	−99.5403	203.0806	207.3669	209.3669	204.7664	0.0948	0.5899
Gamma	−102.8326	209.6652	213.9515	215.9515	211.3510	0.1387	0.1613
LD	−104.5778	211.1557	213.2988	214.2988	211.9986	0.1564	0.0821
Exponential	−109.2986	220.5972	222.7403	223.7403	221.4401	0.2078	0.0073
DUSL	−101.7664	205.5329	207.6760	208.6760	206.3758	0.1392	0.1583

Table 14. MLEs and SEs for the Kevlar 373/epoxy data.

Distribution	Parameter	Estimate	SE
Exponential	λ	0.6791	0.0836
LD	θ	1.0160	0.0946
DUSL	ξ	1.2451	0.1020
AP-L	θ	1.7094	0.1611
	α	39.0347	3.28×10^1
Gamma	α	2.2334	0.3635
	β	1.5167	0.2767
APT-DUSL	ξ	1.8240	0.1703
	α	17.0139	13.2363

Table 15. GOF criteria for the fatigue fracture lifespan of Kevlar 373/epoxy data.

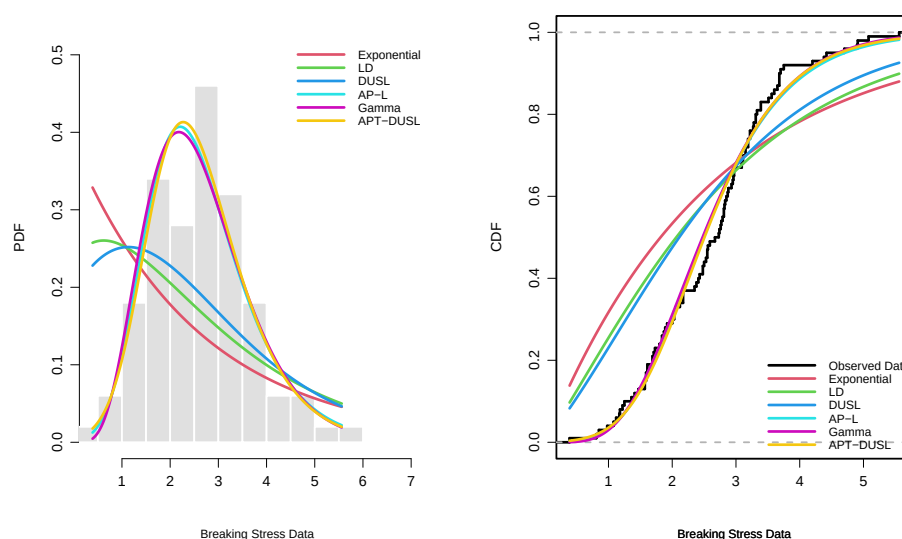
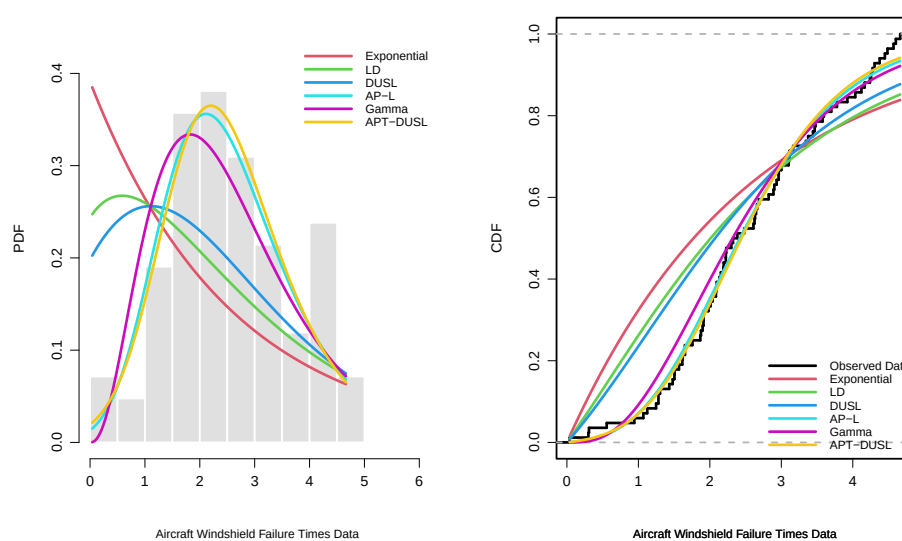
Distribution	ℓ	AIC	BIC	CAIC	HQIC	KS	p -Value
APT-DUSL	−76.5773	157.1547	161.5340	163.5340	158.8852	0.1048	0.4332
AP-L	−77.2168	158.4336	162.8129	164.8129	160.1641	0.1090	0.3858
Gamma	−81.6788	167.3577	171.7369	173.7370	169.0881	0.1281	0.2098
LD	−86.7794	175.5589	177.7486	178.7486	176.4241	0.1806	0.0235
Exponential	−91.5386	185.0772	187.2668	188.2668	185.9424	0.2234	0.0022
DUSL	−82.7006	167.4011	169.5908	170.5908	168.2664	0.1500	0.0923

Table 16. MLEs and SEs for the Turkish Meteorological Services for Sinop data.

Distribution	Parameter	Estimate	SE
Exponential	λ	0.3219	0.0578
LD	θ	0.5320	0.0697
DUSL	ξ	0.6649	0.0756
AP-L	θ	1.3544	0.0761
	α	1085258	1.86×10^{-9}
Gamma	α	9.6097	2.3997
	β	3.0935	0.7930
APT-DUSL	ξ	1.5345	0.0774
	α	1929514	1.08×10^{-9}

Table 17. GOF criteria for the Turkish Meteorological Services for Sinop data.

Distribution	ℓ	AIC	BIC	CAIC	HQIC	KS	p -Value
APT-DUSL	−42.2581	88.5162	91.3842	93.3842	89.4511	0.1106	0.8427
AP-L	−42.5482	89.0964	91.9644	93.9644	90.0313	0.1115	0.8358
Gamma	−42.9483	89.8967	92.7646	94.7646	90.8316	0.1254	0.7147
LD	−60.7385	123.4770	124.9110	125.9110	123.9444	0.3763	0.0003
Exponential	−66.1379	134.2758	135.7098	136.7098	134.7433	0.4398	0.0000
DUSL	−57.7848	117.5697	119.0037	120.0037	118.0371	0.3592	0.0007

**Figure 12.** Fitted PDF and CDF for the breaking stress data.**Figure 13.** Fitted PDF and CDF for the aircraft windshield failure times data.

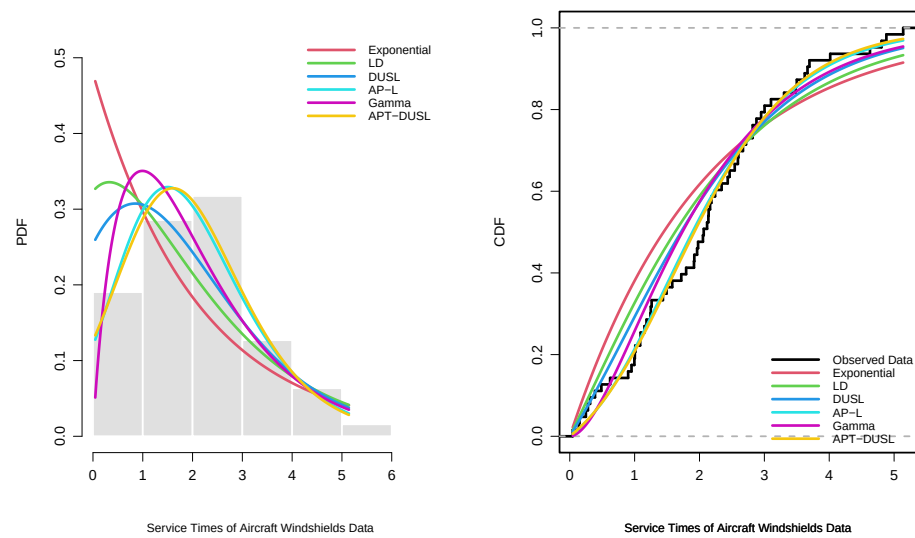


Figure 14. Fitted PDF and CDF for the service times of Aircraft Windshield data.

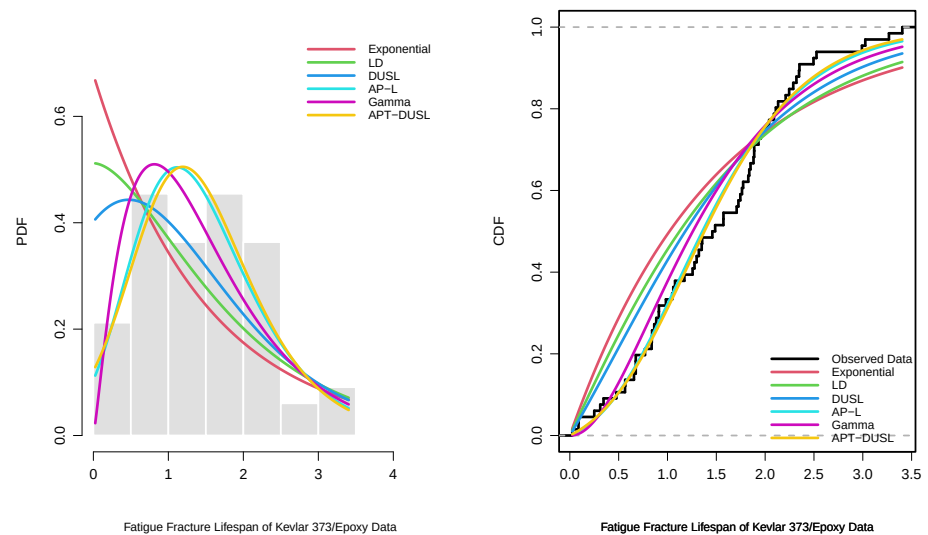


Figure 15. Fitted PDF and CDF for the fatigue fracture lifespan of Kevlar 373/epoxy data.

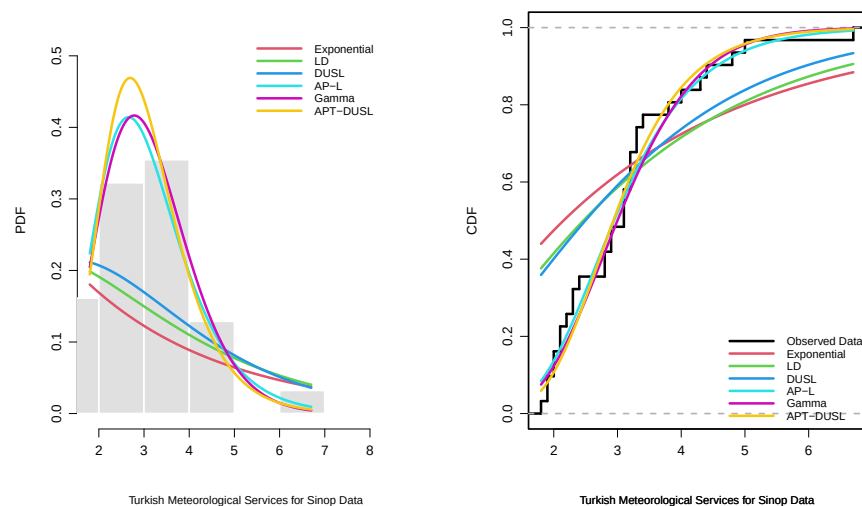


Figure 16. Fitted PDF and CDF for the Turkish Meteorological Services for Sinop data.

8. Conclusions

A new two-parameter lifetime distribution, called the APT-DUSL distribution, is introduced by combining the APT and DUS transformations. This model offers a flexible framework for modeling lifetime data and accommodates a variety of distributional shapes and hazard-rate behaviors. Several key statistical properties are derived in closed form, including the PDF, CDF, quantile function, moments, MGF, order statistics, and Rényi entropy. Additionally, important reliability measures such as the survival function, hazard function, reverse hazard function, and Mills ratio are established.

The flexibility of the APT-DUSL distribution is illustrated through graphical analyses. The PDF displays a wide range of shapes, with the parameter ξ primarily influencing skewness and the modal location, while α adjusts tail behavior and the overall shape. The hazard function also exhibits distinct patterns, with ξ determining the curve's structure and α controlling the overall level. Additionally, quantile-based metrics such as Galton's skewness and Moors' kurtosis are used to map APT-DUSL's behavior across different parameter values to validate its capabilities. The 3D surface plots show that the APT-DUSL can simultaneously fine-tune skewness and kurtosis via the interaction between α and ξ . The ability to adapt to the empirical tail weight and hazard behavior is the primary practical benefit and added value of APT-DUSL in survival analysis.

Parameter estimation was performed using the MLE method. A Monte Carlo simulation study was conducted with complete samples and various progressive type-II censoring schemes, including early, middle, and last censoring schemes. The aim was to evaluate the efficiency and stability of the estimators under different levels of information loss. The results indicate that estimation accuracy improves with increasing sample size, as reflected in decreasing bias and RMSE. Among the considered censoring schemes, the early censoring scheme consistently yields estimates closest to those from the complete samples, with lower bias and RMSE in most cases. The middle scheme also performs well, particularly for large sample sizes, whereas the last scheme shows greater variability and lower estimation accuracy. These findings demonstrate that progressive type-II censoring, especially under the early censoring scheme, can substantially reduce experimental time and cost while still providing estimates close to those of the complete samples. The practical usefulness of the proposed model was further illustrated using five real datasets. The results indicate that the APT-DUSL distribution is a promising model for the data sets included in the present study, providing a superior or comparable fit compared with other models considered.

In general, the APT-DUSL distribution provides a flexible and competitive model for analyzing lifetime data. Future research could focus on extending the proposed model, investigating alternative estimation methods such as Bayesian approaches, and developing new families of distributions based on the same transformation structure.

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