

Article

A Two-Parameter Geometric Model for Volume Estimation and Grade Classification of ISA Brown Eggs

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Abstract

Commercial egg grading remains anchored to weight-based thresholds, a single metric that incompletely represents internal composition and is automated only through industrial graders too costly for small and medium-sized enterprises. Existing parametric egg models treat volume in isolation from grading and rely on parameters that are difficult to extract from simple measurements. We propose a two-parameter geometric framework adapting Smart's parametric egg profile to the ISA Brown breed, in which the egg asymmetry is encoded through a single dimensionless shape parameter and the volume is obtained in closed form as a solid of revolution from only the maximum length and maximum width. The model is validated on 120 ISA Brown eggs against fluid-displacement ground truth, attaining a Pearson correlation of 0.894 and a mean absolute percentage error of 6.02 %; a single closed-form correction factor, derived from the model's own b^2 -dependence, reduces the error to 3.09 %, below that of an empirically fitted formula. A closed-form linear discriminant on the same two measurements recovers the commercial grade with 92.0 % cross-validated accuracy, and the product $H \cdot W^2$ alone classifies 93.3 % by a single arithmetic rule. The shape parameter and whole-egg density are essentially invariant across grades, establishing a geometric self-similarity that reduces the framework to breed-level constants.

Keywords: parametric egg profile; closed-form volume; solid of revolution; statistical validation; linear discriminant analysis

MSC: 65D30; 51M25; 62P10; 62H30



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1. Introduction

Eggs are among the most widely consumed and affordable sources of high-quality animal protein worldwide [1]. Rich in essential amino acids, vitamins, and micronutrients, they play a central role in human nutrition across all age groups [2]. For patients managing specific dietary requirements—such as protein-controlled regimens following kidney disease, post-surgical recovery, or clinical nutrition programmes—knowing the precise protein content of a consumed egg carries genuine medical significance. Yet the nutrient content of an egg is governed by its edible volume, not its mass alone, and no simple relationship connects the two for an irregularly shaped biological solid. Because the geometry of the

egg varies with breed, age of the laying hen, and size grade, weight alone cannot serve as a reliable proxy for internal nutritional content [3,4].

At the commercial level, eggs are classified into size categories using weight-based thresholds defined by national food standards. This convention is administratively convenient but physically imprecise: because eggs deviate from any standard geometric form, two eggs of equal mass may differ substantially in internal volume and hence in protein, albumen, and yolk content. Industrial grading machines that combine weight sensing with optical geometry measurement partially address this limitation, but their capital cost places them far beyond the budget of small and medium-sized enterprises (SMEs). A small poultry farm or independent retailer that cannot acquire dedicated grading hardware is left with manual weighing as its only option, sacrificing both geometric accuracy and any possibility of nutrient-level reporting. This economic barrier disadvantages SMEs in the market and leaves health-conscious consumers and patients without the nutritional detail they need. The grading standard adopted in this study is the Thai national standard (TAS 6702-2010) [5], which defines six size categories (0–5) by mass, from jumbo (> 70 g) to peewee (45–50 g); the four largest categories (0–3), which account for the majority of commercial production, are the focus of this work.

The geometric modelling of egg shape has a well-established theoretical foundation. Seminal work by Romanoff and Romanoff [1] established the basic morphological description of avian eggs, and Preston [6] subsequently proposed polynomial deformations of the ellipse to capture the diversity of avian egg profiles. Smart [7] introduced a parametric model grounded in the oviduct physiology of egg formation, deriving an analytical description of the egg profile from the taper angle of the oviduct wall. These foundational works motivated a succession of quantitative models aimed at calculating egg volume, surface area, and related parameters from simple linear measurements [8–10]. More recently, Hügelschäffer’s ovoid construction has been applied to describe the asymmetric egg profile with high accuracy [11–13], and two-dimensional digital imaging has been coupled with these geometric models to enable non-destructive volume and surface area estimation without physical contact [14,15].

The closest prior work to the present study is that of Weng et al. [16], who derived an analytical egg volume equation directly from Smart’s parametric model. Their approach achieves a maximum volume error of 1.91 % across 47 eggs and demonstrates that only the egg length, maximum breadth, and the offset distance x_m from the longitudinal centre to the plane of maximum breadth are required for volume calculation. However, the parameter x_m requires precise localisation of the widest cross-section along the long axis and varies substantially between individual eggs, providing no mechanism for predicting it from breed or grade information alone. The present work removes this limitation by replacing x_m with a breed-level shape parameter that characterises the asymmetry of an entire breed without requiring per-egg measurement.

The present work makes three integrated contributions. First, we derive an improved mathematical model by adapting Smart’s parametric egg profile specifically for the ISA Brown breed. First, we derive an improved mathematical model by adapting Smart’s parametric egg profile specifically for the ISA Brown breed. The asymmetry of the profile is encoded through a single dimensionless breed-level shape parameter $g \equiv \theta$, the oviduct taper angle, which enters the substitution constant $k = -2 \tan g / b \approx -2(g + \frac{1}{3}g^3)/b$. Through the substitution $u = 1 + ky$, this yields an exact closed-form expression for the egg volume V_{egg} as a solid of revolution from only two caliper measurements—maximum length H and maximum width W . The surface area is obtained by composite trapezoidal integration with $n = 100$ sub-intervals, a value fixed by a convergence study showing discretisation error below 0.1 % across all four grades, together with an endpoint analysis that

resolves the $0 \times \infty$ indeterminacy of the arc-length integrand at the poles. This eliminates the difficult-to-measure offset parameter x_m required by Weng et al. [16].

Second, we provide a statistical validation of the framework on 120 ISA Brown eggs, comparing the deterministic model volume against fluid-displacement ground truth and quantifying agreement, bias, and limits of agreement. The systematic bias is traced analytically to a geometric approximation and removed by a single closed-form correction factor, reducing the mean absolute percentage error below that of an empirically fitted comparison formula while preserving the exact analytical form.

Third, we demonstrate that the same two measurements (H, W) that drive the volume model are sufficient for commercial grade classification, using a closed-form linear discriminant whose assumptions are tested directly [17], and for per-egg nutrient estimation through a density-based mass-equivalent conversion—establishing a unified two-measurement framework with multiple analytical outputs.

Together, these contributions address the cost barrier faced by SMEs and the nutritional information gap faced by health-conscious consumers and patients. By providing a mathematically principled, statistically validated alternative grounded in breed-specific geometry, the framework makes accurate, science-based egg assessment accessible to any producer or consumer with a Vernier caliper. Extension of the two dimensions to image-based measurement is a natural direction for subsequent work.

The remainder of this paper is organised as follows. Section 2 derives the mathematical model: the adaptation of Smart's egg profile to the ISA Brown breed, the closed-form volume expression, the geometric construction of the shape parameter g , and the trapezoidal surface-area approximation with its convergence study. Section 3 describes the materials and methods, covering the egg specimens, the caliper-based geometric and gravimetric measurements, the fluid-displacement volume reference, and the statistical and classification procedures. Section 4 presents and discusses the results: descriptive statistics and ANOVA, validation of the volume model against fluid-displacement ground truth, commercial grade classification from the two geometric measurements, nutrient content estimation, and a ranking of grade indicators that identifies a practical single-number grading rule. Finally, Section 5 summarises the contributions and outlines directions for future work.

2. Mathematical Modelling

2.1. Adaptation of Smart's Egg Model for ISA Brown Eggs

The mathematical model for the ISA Brown chicken egg was developed based on Smart's egg model [7]. To align with the standard practice of storing eggs with the acute (pointed) end downwards, the model was adapted to this orientation. The egg profile is described by the parametric equation:

$$\frac{x^2(1+ky)}{b^2} + \frac{y^2}{a^2} = 1, \quad (1)$$

where the geometric quantities, illustrated in Figure 1, are defined as follows:

- a is the semi-major axis, equal to half the maximum length H of the egg ($2a = H$), measured along the vertical axis between the acute pole G (at $y = -a$) and the blunt pole P_b (at $y = +a$);
- $b_{\max}$ is the maximum width of the egg (segment $\overline{AC}$ in Figure 1), located above the geometric centre of the egg;
- b is the semi-minor axis (segment $\overline{EF}$ in Figure 1), defined as the half-width of the egg profile at the geometric centre ($y = 0$). For ISA Brown eggs, $b \approx b_{\max}/2 = W/2$, where W is the maximum width measured by Vernier caliper;

- t is the distance from the line containing $b_{\max}$ to the blunt pole P_b , characterising the longitudinal asymmetry of the egg;
- k is a substitution constant with units of length^{-1} , derived from the dimensionless shape parameter g (defined in Section 2.2);
- y is the coordinate along the vertical axis, with the acute pole at $y = -a$ and the blunt pole at $y = +a$.

Rearranging (1) gives the explicit expression for the radial coordinate x at height y :

$$x^2 = \frac{b^2}{1 + ky} \left(1 - \frac{y^2}{a^2} \right). \quad (2)$$

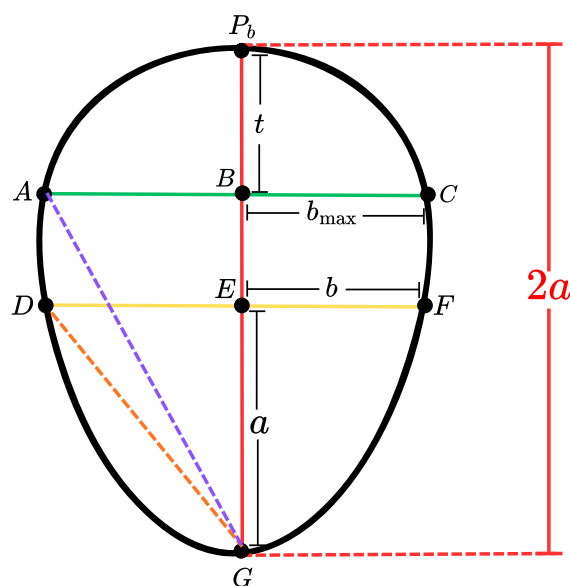


Figure 1. Geometric description of the ISA Brown egg profile, oriented with the acute pole G downward and the blunt pole P_b upward. The semi-major axis a and total length $2a$ are indicated; the maximum width $b_{\max}$ lies along segment $\overline{AC}$ above the geometric centre, while the semi-minor axis b is the half-width at the geometric centre ($\overline{EF}$). The parameter t measures the distance from the $b_{\max}$ line to the blunt pole.

2.2. Derivation of the Shape Parameter g

The shape parameter g characterises the differential curvature of the eggshell between the acute and blunt poles. Geometrically, it arises from the construction illustrated in Figure 2: a reference circle of diameter $2a$ is inscribed around the egg, and a circumscribing square of side length $2a$ is drawn. A tangent line is constructed at the outermost point of the maximum width $b_{\max}$, and a second line is drawn from corner C of the square so as to intersect the tangent at exactly that point. The angle θ formed at this intersection, satisfying $\tan \theta = \overline{AB}/2a$, quantifies the deviation of the eggshell curvature from a perfect sphere of radius a and is used as the structural basis for g .

The shape parameter is identified with the taper angle itself, $g \equiv \theta$, a dimensionless quantity (measured in radians) that is independent of egg size. The substitution constant k in (1) is defined through the tangent of this angle,

$$k = -\frac{2 \tan g}{b}, \quad (3)$$

which controls the fore-aft asymmetry of the profile. Expanding the tangent about $g = 0$,

$$\tan g = g + \frac{1}{3}g^3 + \frac{2}{15}g^5 + O(g^7), \quad (4)$$

and retaining terms through third order gives the working form

$$k = -\frac{2\left(g + \frac{1}{3}g^3\right)}{b}. \quad (5)$$

For the ISA Brown breed the taper angle is small ($g = \theta \approx 0.05$; Section 2.1), so the cubic term contributes a relative correction of order $\frac{1}{3}g^2 \approx 8 \times 10^{-4}$ to k , and the linear approximation $k \approx -2g/b$ is accurate to within 0.1 % (Table 1). The dimensionless factor $2(g + \frac{1}{3}g^3)$ divided by the semi-minor axis b gives k the required units of length^{-1} , so that ky in (1) and (2) is dimensionless.

Table 1. Accuracy of the linear approximation $\tan g \approx g$ used in the substitution constant k . Retaining only the linear term introduces a relative error below 1.4 % across the full plausible range of taper angles, and below 0.1 % at the operating value $g = 0.05$.

g (rad)	$\tan g$	Relative Error of $\tan g \approx g$ (%)
0.02	0.0200	0.01
0.05	0.0500	0.08
0.10	0.1003	0.33
0.15	0.1511	0.75
0.20	0.2027	1.34

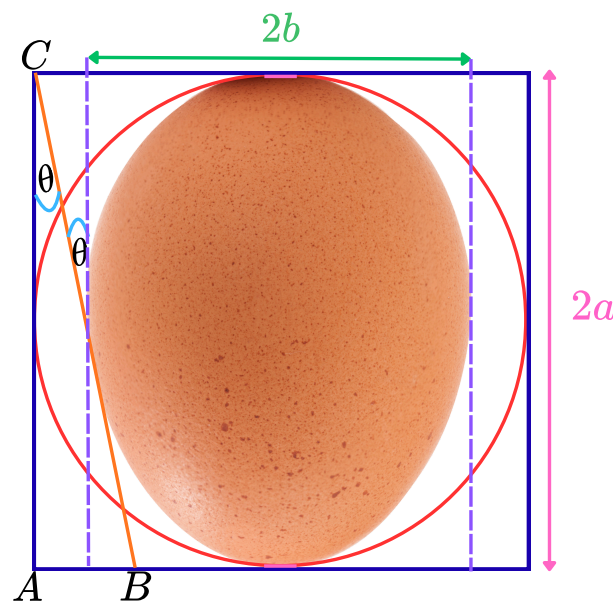


Figure 2. Geometric construction of the taper angle $g \equiv \theta$ for an ISA Brown egg. A reference circle (red) of diameter $2a$, equal to the egg length, is inscribed in a square (blue) of side $2a$ with base $\overline{AB'}$ and upper-left corner C . The purple vertical through B marks the horizontal offset of the egg's blunt end from the acute pole. The line $\overline{CB}$ (orange), drawn from the corner C to the base point B , forms the angle θ with the left edge $\overline{CA}$ of the square, so that $\tan \theta = \overline{AB}/2a$. Because $\overline{CA}$ is parallel to the purple vertical, the same angle θ reappears where $\overline{CB}$ crosses that vertical (both marked). This angle quantifies the fore–aft asymmetry of the egg relative to the reference sphere and defines the taper angle $g \equiv \theta$ used in the substitution constant (3).

The value $g = 0.05$ is therefore adopted throughout. To verify that the volume model is not sensitive to small uncertainties in this choice, all 120 egg volumes were recomputed for $g \in \{0, 0.02, 0.05, 0.08, 0.10\}$ and compared against the fluid-displacement reference. As summarised in Table 2, doubling g from 0.05 to 0.10 increases the MAPE by only about one percentage point (6.02 % to 7.00 %), and the systematic bias changes by less than

0.6 mL across the entire tested range. This confirms that the residual model error originates predominantly in the geometric approximation $b \approx b_{\max}/2$ (Section 4.2.2) rather than in the chosen value of g .

Table 2. Sensitivity of the model volume to the shape parameter g , evaluated on the full 120-egg dataset against the fluid-displacement reference. Volume estimates are nearly independent of g in the small-angle regime, confirming that the residual error originates in the approximation $b \approx b_{\max}/2$ rather than in the value of g .

g	$\tilde{V}_{\text{egg}}$ (mL)	Bias (mL)	MAPE (%)
0.00	60.11	+3.03	5.72
0.02	60.15	+3.06	5.77
0.05	60.31	+3.23	6.02
0.08	60.63	+3.54	6.52
0.10	60.92	+3.84	7.00

2.3. Analytical Calculation of Egg Volume

The volume V_{egg} is computed as a solid of revolution of the profile (2) about the vertical y -axis:

$$V_{\text{egg}} = \pi \int_{-a}^a x^2 dy = \pi b^2 \int_{-a}^a \frac{1 - y^2/a^2}{1 + ky} dy. \quad (6)$$

Applying the substitution $u = 1 + ky$ (so $y = (u - 1)/k$, $dy = du/k$) with limits $u = 1 \mp ka$ transforms (6) into a standard rational integral. Evaluating term by term yields the closed-form result:

$$V_{\text{egg}} = \frac{\pi b^2}{k} \left[\frac{2}{ka} + \left(1 - \frac{1}{k^2 a^2} \right) \ln \left(\frac{1 + ka}{1 - ka} \right) \right]. \quad (7)$$

2.4. Numerical Approximation of Egg Surface Area

The surface area S_{egg} is computed from the arc-length integral for a surface of revolution about the y -axis:

$$S_{\text{egg}} = 2\pi \int_{-a}^a x \sqrt{1 + \left(\frac{dx}{dy} \right)^2} dy. \quad (8)$$

2.4.1. Derivative of the Profile

The derivative dx/dy is obtained by implicit differentiation of (1). Differentiating with respect to y :

$$\frac{d}{dy} \left[\frac{x^2(1 + ky)}{b^2} \right] + \frac{2y}{a^2} = 0, \quad (9)$$

which expands to:

$$\frac{2x(1 + ky)}{b^2} \frac{dx}{dy} + \frac{kx^2}{b^2} + \frac{2y}{a^2} = 0. \quad (10)$$

Solving for dx/dy and substituting x^2 from (2) to eliminate the remaining x^2 term yields:

$$\frac{dx}{dy} = -\frac{b^2}{2a^2} \cdot \frac{k(a^2 + y^2) + 2y}{x(1 + ky)^2}. \quad (11)$$

2.4.2. Non-Existence of a Closed-Form Antiderivative

Substituting (2) and (11) into (8), the integrand becomes proportional to the square root of a degree-7 rational function in y . This belongs to the class of elliptic inte-

grals, which possess no antiderivative in elementary functions. Numerical integration is therefore required.

2.4.3. Endpoint Analysis

At the two poles of the egg, $y_0 = -a$ and $y_n = +a$, the profile closes to a single point. This means that $x \rightarrow 0$ at the endpoints, while the slope dx/dy becomes infinite as the curve turns vertically back toward the axis. The integrand

$$f(y) = 2\pi x \sqrt{1 + (dx/dy)^2} \quad (12)$$

Therefore involves a product of a quantity going to zero and one going to infinity, so its value at the endpoints cannot be read off directly: it must be obtained as a limit. Taking the limit of the product $x |dx/dy|$ as y approaches $\mp a$, and applying L'Hôpital's rule, yields the finite values:

$$f(y_0) = \lim_{y \rightarrow -a} 2\pi x \left| \frac{dx}{dy} \right| = \frac{2\pi b^2}{a(1 - ka)}, \quad (13)$$

$$f(y_n) = \lim_{y \rightarrow +a} 2\pi x \left| \frac{dx}{dy} \right| = \frac{2\pi b^2}{a(1 + ka)}. \quad (14)$$

These analytical limits must be used in place of the naive substitution $f = 0$ at the endpoints, which would otherwise introduce a systematic underestimation of S_{egg} .

2.4.4. Convergence Study and Choice of n

The number of sub-intervals n in the composite trapezoidal rule was selected by a convergence study. The surface area S_{egg} was computed for $n \in \{5, 10, 20, 50, 100, 200, 500, 1000\}$ using representative dimensions for each of the four commercial size grades, with the $n = 1000$ result taken as a high-resolution reference. Grade 0 constitutes the worst case owing to its largest absolute dimensions; its full convergence sequence is shown in Table 3.

Table 3. Convergence of the trapezoidal approximation for S_{egg} with the number of sub-intervals n (worst-case Grade 0; $a = 29.9$ mm, $b = 23.3$ mm). Error is relative to the $n = 1000$ reference.

n	S_{egg} (mm ²)	Change (%)	Error (%)
5	7368.66	—	9.72
10	7923.11	7.52	2.93
20	8092.34	2.14	0.85
50	8148.87	0.70	0.16
100	8158.42	0.12	0.04
200	8161.10	0.03	0.01
500	8161.94	0.01	0.00
1000	8162.08	0.00	—

At $n = 5$, the error relative to the converged result reaches 9.72% for Grade 0, rendering it unsuitable for accurate surface-area estimation. The error falls below 0.1% at $n = 100$ for all four grades (Grade 0: 0.04%; Grade 1: 0.04%; Grade 2: 0.04%; Grade 3: 0.04%), whereas $n = 50$ leaves a residual error above 0.1%. The value $n = 100$ was therefore adopted throughout, providing sub-0.1% discretisation accuracy at negligible computational cost.

2.4.5. Composite Trapezoidal Formula

The surface area is approximated with step size $h = 2a/100$ as:

$$S_{\text{egg}} \approx \frac{a}{100} \left[f(y_0) + 2 \sum_{i=1}^{99} f(y_i) + f(y_{100}) \right], \quad (15)$$

where the equally spaced nodes are:

$$y_i = -a + i \cdot \frac{2a}{100}, \quad i = 0, 1, \dots, 100, \quad (16)$$

the endpoint values $f(y_0)$ and $f(y_{100})$ are given analytically by (13)–(14), and all interior node values $f(y_1), \dots, f(y_{99})$ are evaluated numerically via (11) and (2).

2.5. Conversion of Estimated Egg Volume to Equivalent Mass

The estimated egg volume V_{egg} obtained from the geometric model (7) can be converted to an equivalent egg mass M_{egg} using the whole-egg density:

$$M_{\text{egg}} = V_{\text{egg}} \cdot D_{\text{egg}}, \quad (17)$$

where M_{egg} is the equivalent mass of the egg in grams, V_{egg} is the estimated egg volume from (7) in cm^3 , and D_{egg} is the density of the whole egg in g cm^{-3} . The grade-level mean density obtained from the experimental measurements (Table 4) is used as D_{egg} . This single-step conversion provides the equivalent mass directly from two caliper-measured parameters (H, W) via the volume model, without any weighing.

Table 4. Descriptive statistics of physical measurements by grade (mean $\pm$ SD; $n = 30$ per grade).

Parameter	Grade 0	Grade 1	Grade 2	Grade 3
H (mm)	59.8 ± 1.8	57.8 ± 1.7	56.3 ± 1.6	54.0 ± 1.0
W (mm)	46.6 ± 0.9	45.2 ± 0.6	44.2 ± 0.6	43.4 ± 0.4
Mass M (g)	73.4 ± 4.0	67.5 ± 1.5	63.1 ± 1.3	58.0 ± 1.4
V_{ref} (mL)	63.5 ± 3.7	59.7 ± 2.2	54.7 ± 1.6	50.4 ± 2.0
D_{egg} (g cm^{-3})	1.156 ± 0.037	1.132 ± 0.036	1.153 ± 0.029	1.152 ± 0.035
Shape index ($W/H \times 100$)	78.0 ± 3.0	78.2 ± 3.1	78.6 ± 3.0	80.4 ± 1.8

For the mass and nutrient conversions, the bias-corrected volume $V_{\text{egg}} = \kappa^2 V_{\text{egg}}^{(\text{raw})}$ with $\kappa = 0.97$ (Section 4.2.3) is used, so that the equivalent mass is not inflated by the systematic volume bias.

ISA Brown chicken eggs contain protein, lipid, vitamins, minerals, cholesterol, and metabolic energy. Reference macronutrient, cholesterol, and energy contents per 100 g of edible portion are summarised in Table 5; vitamin and mineral contents are summarised in Table 6. Both sets of values are taken from the Thai food composition database for chicken eggs [18].

Table 5. Reference macronutrient, cholesterol, and energy contents per 100 g of edible portion of raw ISA brown chicken egg (mean $\pm$ SD) [18].

Protein (g)	Lipid (g)	Cholesterol (mg)	Energy (kcal)
13.1 ± 0.3	8.4 ± 0.9	392 ± 41	133 ± 5

Table 6. Reference vitamin and mineral contents per 100 g of edible portion of raw ISA Brown chicken egg (mean $\pm$ SD) [18].

Vit. A (μ g)	Folate (μ g)	Calcium (mg)	Phosphorus (mg)	Magnesium (mg)
235 $\pm$ 33	76 $\pm$ 12	49 $\pm$ 1	167 $\pm$ 7	9.3 $\pm$ 0.4
Iron (mg)	Sodium (mg)	Potassium (mg)	Copper (mg)	Zinc (mg)
1.3 $\pm$ 0.1	137 $\pm$ 10	130 $\pm$ 12	0.03 [†]	0.9 $\pm$ 0.1

[†] Reported as a single value in the source; no standard deviation available.

The absolute amount of each nutrient in a single egg is obtained by scaling the reference 100 g concentration by the equivalent egg mass M_{egg} from (17):

$$\text{Nutrient}_{\text{egg}} = \frac{K \cdot M_{\text{egg}}}{100}, \quad (18)$$

where $\text{Nutrient}_{\text{egg}}$ is the calculated quantity of the nutrient (protein, lipid, vitamin, mineral, cholesterol, or energy) in a single egg, K is the reference content of that nutrient per 100 g of edible portion (Tables 5 and 6), and M_{egg} is the equivalent egg mass in grams obtained from (17).

Worked Example

For a representative Grade 0 egg with κ -corrected model volume $V_{\text{egg}} = 64.1 \text{ cm}^3$ and grade-level density $D_{\text{egg}} = 1.156 \text{ g cm}^{-3}$ (Table 4), the equivalent mass is

$$M_{\text{egg}} = 64.1 \times 1.156 \approx 74.1 \text{ g}, \quad (19)$$

in close agreement with the measured Grade 0 mean mass of 73.4 g. The estimated protein content is therefore

$$\text{Protein}_{\text{egg}} = \frac{13.1 \times 74.1}{100} \approx 9.71 \text{ g per egg}. \quad (20)$$

The same procedure applies to every nutrient in Tables 5 and 6: substituting the appropriate K value yields the lipid, vitamin, mineral, cholesterol, or energy content per egg. The pipeline from two physical measurements (H, W) to per-egg nutrient quantities is thus fully analytical, with no weighing or chemical assay required at the point of grading.

3. Materials and Methods

3.1. Egg Specimens

A total of 120 ISA Brown chicken eggs were collected from NK Farm (operated by Narong Narinya, 123 Moo 16, Tambon Tha Song Khon, Mueang District, Maha Sarakham Province, Thailand; 16.18° N, 103.21° E), comprising 30 eggs per commercial size grade (0–3) as defined by the Thai national standard for hen eggs [5] (As noted above, the Thai standard TAS 6702-2010 defines six size categories (0–5) by mass. Size 4 (small, > 50 to 55 g) and size 5 (peewee, > 45 to 50 g) were not included in the present study, which focuses on the four most commercially prevalent grades.).

Eggs were collected directly from the laying house at approximately 10:00 each day and transported to the laboratory within 30 min. All geometric measurements were performed on the day of collection in a temperature-controlled room maintained at approximately 25 °C, reducing thermal variation between specimens. The short collection-to-measurement interval minimises any quality deterioration that might alter geometric parameters [19,20].

Grade assignment was verified independently by mass according to the Thai national standard [5]: Grade 0 (jumbo) eggs exceeded 70 g, Grade 1 (extra large) fell in the range 65–70 g, Grade 2 (large) in 60–65 g, and Grade 3 (medium) in 55–60 g. All 120 eggs in the dataset satisfied these mass thresholds. The grade assignment is also consistent with the ISA Brown breed characterisation reported in the literature [3,4].

3.2. Physical Measurements

3.2.1. Geometric Dimensions

Maximum length H and maximum width W of each egg were measured using a custom fixture built from a 95 mm aluminium angle clamp (PUMPKIN PTT-95ACM). The egg was seated against the fixed jaw of the clamp, and the sliding jaw was advanced until it contacted the opposite pole of the egg and then locked in place. The egg was removed and the locked gap measured with a Vernier caliper (precision ± 0.02 mm). This fixture-based procedure ensures that the caliper records the true extremal dimension along a well-defined axis, eliminating the operator error associated with locating the pole-to-pole and maximum-width contact points by hand. The same procedure was applied for both the length and the width axes, and each measurement was repeated three times per egg; the mean of the three readings was recorded.

In the notation of Figure 1, H is the pole-to-pole distance along the long axis ($2a = H$), and W is the maximum transverse diameter perpendicular to H , corresponding to $b_{\max}$. The model's semi-minor axis b is approximated as $b \approx W/2 = b_{\max}/2$, an approximation addressed quantitatively in Section 4.2.2.

3.2.2. Mass

Egg mass was recorded on an analytical balance (resolution 0.001 g) with the egg at rest on a flat pan. Three successive readings were taken and averaged.

3.2.3. Volume by Fluid Displacement

True egg volume was measured by the water displacement method, following the procedure of Weng et al. [16]. A 500 mL graduated cylinder was filled with distilled water to the 400 mL mark. Each egg was fully submerged; water droplets adhering to the shell surface were removed before the egg was extracted to prevent overestimation. The analytical balance was tared, and the volume of water required to restore the level to 400 mL was weighed and converted to volume using the density of water (0.998 g cm^{-3} at 20°C). The procedure was repeated twice per egg and the mean recorded. These fluid-displacement volumes serve as the ground-truth reference for validating the model (7).

3.2.4. Internal Component Volumes

Albumen and yolk were separated manually following standard protocols [1]. The volume of each component was measured individually using a 50 mL graduated cylinder following the same displacement procedure described in Section 3.2.3. Shell volume was estimated as the difference between the total egg volume and the sum of albumen and yolk volumes. Whole-egg density D_{egg} was calculated as the ratio of egg mass to fluid-displacement volume and used as the grade-level density constant in the mass-conversion model (17).

3.3. Statistical Analysis and Classification

To establish both that the four commercial grades are geometrically distinct and that the two model inputs (H, W) suffice to recover them, three analyses were performed. Throughout, results are reported as continuous measures of compatibility between data

and model, with emphasis on effect sizes and confidence intervals rather than dichotomous significance thresholds [21].

3.3.1. Between-Grade Comparison

For each measured parameter, one-way analysis of variance quantifies between-grade separation through the F -statistic. The assumptions of the F -test were examined directly—residual normality by the Shapiro–Wilk test and homogeneity of variance by Levene’s test—with the distribution-free Kruskal–Wallis test as a robustness check (Section 4.1).

3.3.2. Pairwise Separability Metric

For the two model inputs, the standardised separation ratio between adjacent grade pairs was computed:

$$d_{ij}^{(x)} = \frac{|\bar{x}_i - \bar{x}_j|}{\frac{1}{2}(\hat{\sigma}_i + \hat{\sigma}_j)}, \quad (21)$$

where $\bar{x}_i$ and $\hat{\sigma}_i$ are the sample mean and standard deviation of parameter x for grade i . This ratio is the discrete analogue of Cohen’s d and measures separation in units of within-grade dispersion.

3.3.3. Grade Classification

Commercial grade was classified from the feature pair (H, W) using linear discriminant analysis (LDA) [17], chosen for its closed-form decision rule and because its assumptions—within-grade normality and a common covariance—coincide with those examined in Section 4.1. The discriminant assigns each egg to the grade maximising a linear score derived from the class means and the pooled within-grade covariance; the classifier introduces no tuning parameters (Section 4.3).

Classification accuracy was estimated by stratified five-fold cross-validation with feature standardisation fitted within each training fold only, so that no statistic derived from a validation sample influences the model classifying it. Because a single five-fold partition of $n = 120$ eggs is subject to partition-dependent variability, the accuracy is reported as the mean over 200 independent stratified partitions. A percentile bootstrap over 2000 stratified resamples provides a non-parametric 95 % confidence interval for the accuracy. Independently, a stratified 25 % hold-out set ($n = 32$), withheld before any model fitting, was used once to obtain the confusion matrix.

For reference, three further classifiers were evaluated on the same features— k -nearest neighbours ($k = 5$) [22], a radial-basis-function support vector machine [23], and a depth-five decision tree [24]. All analyses, including the closed-form discriminant, the statistical tests (implemented through the incomplete beta and gamma functions), and the bootstrap, were carried out in base MATLAB R2023a without additional toolboxes.

Given the modest sample size ($n = 120$), the accuracies are reported with their associated uncertainty and interpreted as evidence that the geometric features (H, W) carry sufficient information to recover commercial grade, rather than as a claim of production-ready classifier performance; external validation on larger, multi-farm datasets is identified as future work (Section 5).

4. Results and Discussion

4.1. Descriptive Statistics and ANOVA

Descriptive statistics for all physical measurements are presented in Table 4. All geometric and gravimetric parameters decrease monotonically from Grade 0 to Grade 3, confirming that grade assignment is consistent with physical egg size and that the dataset spans the four most commercially prevalent grades (0–3) without gap or overlap at the grade

level. Grade 0 exhibits the largest within-grade variability ($\hat{\sigma}_{\text{mass}} = 3.99 \text{ g}$, $\hat{\sigma}_H = 1.80 \text{ mm}$), while Grade 3 is the most tightly clustered ($\hat{\sigma}_W = 0.39 \text{ mm}$, the smallest standard deviation across all grade–parameter combinations).

Whole-egg density, computed as the ratio of measured mass to fluid-displacement volume, is nearly constant across the four commercial grades. The grade means span only 1.132 to 1.156 g cm^{-3} , a total spread of 0.024 g cm^{-3} or approximately 2.1 % of the mean, in sharp contrast to the 27 % variation in mass across the same grades. The 95 % confidence intervals of the four grade means overlap substantially—Grade 0: [1.143, 1.170]; Grade 1: [1.118, 1.145]; Grade 2: [1.142, 1.163]; Grade 3: [1.139, 1.165] g cm^{-3} —and the largest pairwise difference between grade means is smaller than the within-grade standard deviation. Although this residual variation is statistically detectable by one-way ANOVA ($F(3, 116) = 3.16$), its magnitude is negligible for the mass-conversion model (17), so density is treated as a single breed-level constant throughout, consistent with the geometric self-similarity established in Section 4.4.3. The mean value obtained here, 1.148 g cm^{-3} , lies slightly above the range commonly reported for fresh hen eggs; this offset is attributable to the fluid-displacement protocol and, because density enters the framework only as a grade-independent multiplicative constant, does not affect any of the comparative conclusions.

One-way ANOVA shows strong between-grade separation for all four size variables, with large F -statistics relative to the within-grade spread (H : $F(3, 116) = 75.4$; W : $F = 127.2$; mass: $F = 238.8$; V_{ref} : $F = 153.9$). These values place the between-grade variance one to two orders of magnitude above the within-grade variance, confirming that the four grades are physically distinct and that the dataset is suitable for both volume-model validation and classifier training.

Assumption Checks for the ANOVA

The ANOVA F -test relies on the assumptions of residual normality and homogeneity of variance across groups. Both assumptions were formally examined. Shapiro–Wilk tests on the within-grade residuals indicate that H , density, and the shape index satisfy the normality assumption (all $p > 0.4$), while W , mass, and V_{ref} depart from normality ($p \leq 10^{-3}$)—a consequence of the mass-threshold-based grade definition, which truncates the within-grade mass and volume distributions by construction. Levene’s test for homoscedasticity indicates homogeneous variances for density ($p = 0.56$), borderline homogeneity for H , W , and the shape index ($p \in [0.05, 0.08]$), and heterogeneity for mass and V_{ref} ($p < 0.05$), the latter again driven by the open-ended nature of Grade 0 ($M > 70 \text{ g}$).

To confirm that these departures from the classical ANOVA assumptions do not affect the substantive conclusions, the Kruskal–Wallis non-parametric test was applied as a robust alternative. The qualitative findings are unchanged: all variables shown as differing strongly across grades in the parametric ANOVA (H , W , mass, V_{ref}) are also strongly separated by the Kruskal–Wallis test ($p < 10^{-17}$ in every case), and density remains weakly separated by both tests (Kruskal–Wallis $p = 0.02$; small effect). The agreement between the parametric and non-parametric analyses confirms that the conclusions of Section 4.1 are robust to the choice of test.

4.2. Validation of the Egg Volume Model

4.2.1. Accuracy Against Fluid-Displacement Ground Truth

The model volume V_{egg} of (7) is a deterministic function of the two measurements (H, W) through $a = H/2$, $b = W/2$, and the fixed breed-level shape parameter $g = 0.05$; no information from the reference volume enters its computation. For all 120 eggs, the substitution constant satisfies $|ka| < 0.14$, so the logarithmic term in (7) is real and well-conditioned throughout the sample. The predicted volumes were compared against the

fluid-displacement reference V_{ref} ; per-grade error metrics are reported in Table 7, and the scatter of predicted against reference volume is shown in Figure 3.

Table 7. Volume-model validation: deterministic model prediction V_{egg} from (7) ($g = 0.05$) against the fluid-displacement reference ($n = 30$ per grade).

Grade	$\bar{V}_{\text{ref}}$ (mL)	$\bar{V}_{\text{egg}}$ (mL)	Bias (mL)	RMSE (mL)	MAPE (%)
0	63.55	68.13	+4.58	6.13	8.24
1	59.67	61.93	+2.27	2.97	4.11
2	54.73	57.83	+3.10	3.36	5.71
3	50.38	53.34	+2.97	3.37	6.03
All	57.08	60.31	+3.23	4.16	6.02

The model predictions are strongly correlated with the reference volumes (Pearson $r = 0.894$), and the overall mean absolute percentage error is 6.02 %. The two-parameter geometric model therefore recovers egg volume to within commercially useful precision from length and width alone, using a single breed-level constant and no calibration against the reference.

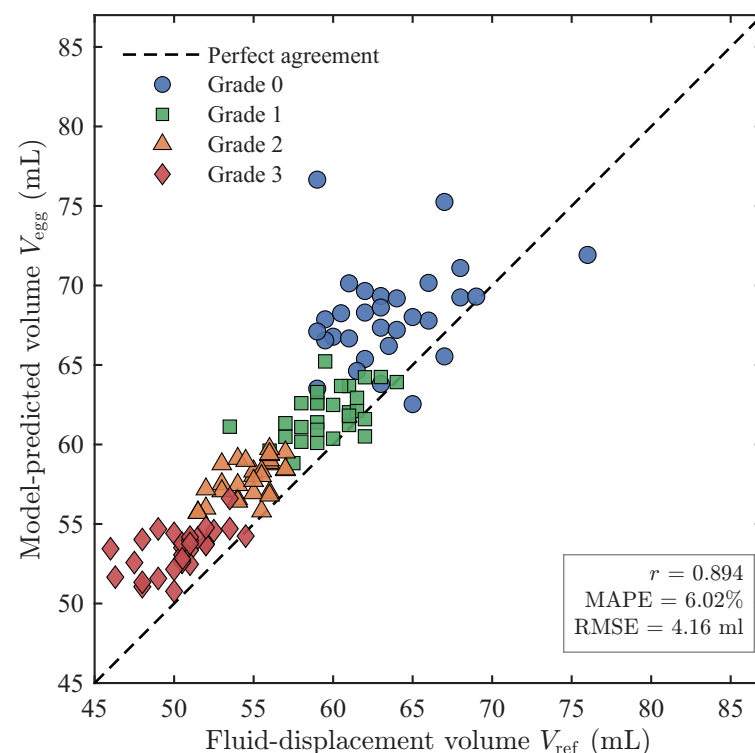


Figure 3. Model-predicted versus fluid-displacement egg volume for all 120 eggs. The dashed line denotes perfect agreement; markers consistently above it indicate the systematic positive bias quantified in Section 4.2.2. Marker shape and colour indicate commercial grade.

4.2.2. Systematic Bias Analysis

A paired-sample analysis reveals a systematic positive bias of +3.23 mL (95 % CI [+2.75, +3.70] mL; paired $t(119) = 13.47$), indicating that the model consistently overestimates volume. Bland–Altman analysis (Figure 4) gives the same bias with 95 % limits of agreement [−1.92, +8.38] mL. The bias magnitude relative to the mean egg volume of 57 mL is 5.7 %, and the confidence interval lies entirely on the positive side, confirming a genuine directional offset rather than random scatter.

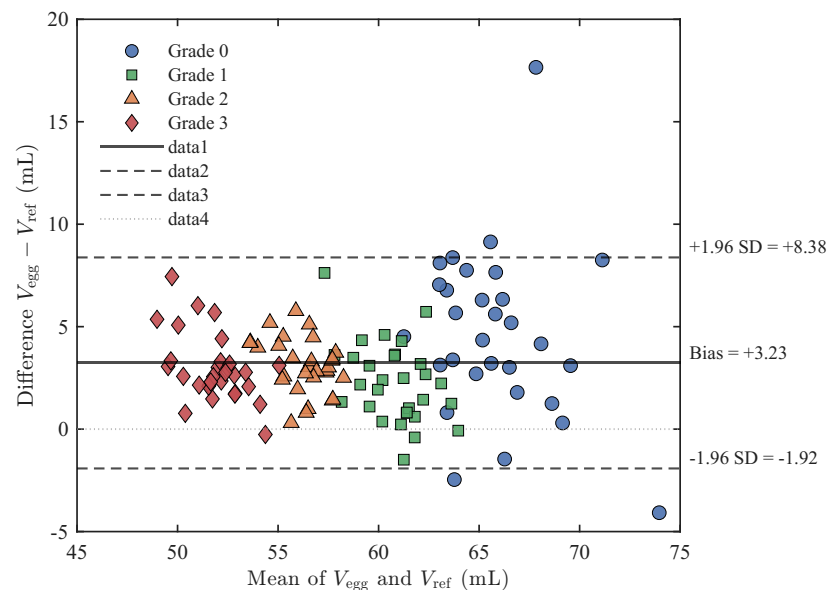


Figure 4. Bland–Altman agreement plot for the volume model. The solid line is the mean bias (+3.23 mL); the dashed lines are the 95 % limits of agreement (−1.92 to +8.38 mL); the dotted line marks zero difference. Marker shape and colour indicate commercial grade.

The origin of this bias follows directly from the analytical structure of (7). To leading order in the small asymmetry parameter, V_{egg} scales as b^2 , where b is the semi-minor axis at the geometric centre. In the operational model, this is taken as $b = W/2 = b_{\text{max}}/2$, using the maximum width W . For ISA Brown eggs, however, the maximum width occurs above the geometric centre by an offset t (Figure 1), so the true centre-line semi-minor axis b_{true} is smaller than $b_{\text{max}}/2$. Because volume scales as b^2 , substituting $W/2$ inflates the prediction by a factor of approximately $(b_{\text{max}}/2b_{\text{true}})^2$. This explains both the consistently positive sign of the bias across all grades and its larger magnitude for Grade 0, whose greater absolute size amplifies the squared-radius error.

4.2.3. Bias-Corrected Model: The κ Correction

The analytical bias structure suggests a single-parameter correction. Writing the effective semi-minor axis as

$$b = \kappa \frac{b_{\text{max}}}{2} = \kappa \frac{W}{2}, \quad \kappa \leq 1, \quad (22)$$

and substituting into (7) preserves the closed-form structure of the model. To leading order, the volume scales as b^2 : writing $V_{\text{egg}} = \frac{4}{3}\pi ab^2[1 + O(ka)]$ and noting that $|ka| \lesssim 0.14$ throughout the sample, the rescaling $b \mapsto \kappa b$ multiplies the volume by κ^2 up to a relative correction below 0.02 % for the ISA Brown parameters. Matching the mean corrected volume to the mean reference volume therefore gives the near-closed-form estimate.

$$\kappa \approx \sqrt{\bar{V}_{\text{ref}}/\bar{V}_{\text{egg}}} = 0.973, \quad (23)$$

which coincides with the exact zero-bias value ($\kappa = 0.9728$, obtained numerically) to four decimal places. Table 8 reports the validation metrics over a range of κ .

The correction reduces the MAPE below that of the empirically fitted formula of Narushin et al. [15] (4.54 %; Section 4.2.4) while retaining the analytical form derived from the parametric profile. Both g and κ are breed-level constants estimated once from the sample; neither is fitted per egg.

Table 8. Validation of the κ -corrected model, $b = \kappa b_{\max}/2$ in (7). The value $\kappa = 0.97$ reduces the mean bias to -0.32 mL and the MAPE from 6.02 % to 3.09 %.

κ	Bias (mL)	RMSE (mL)	MAPE (%)
1.00 (uncorrected)	+3.23	4.16	6.02
0.99	+2.03	3.29	4.34
0.98	+0.85	2.69	3.23
0.97	-0.32	2.55	3.09
0.96	-1.48	2.92	3.94
0.95	-2.63	3.63	5.24

4.2.4. Comparison with Prior Volume Models

The same 120 eggs were evaluated using two related published methods: the linear formula $V = 0.5163 LB^2$ of Narushin et al. [15] and the ellipsoidal approximation $V = \frac{4}{3}\pi ab^2$ ($k_v = 0.5236$). Results are summarised in Table 9.

Table 9. Comparison of volume models on the 120-egg ISA Brown dataset. The proposed model is derived from first principles; the comparison formulae are empirically fitted.

Model	Inputs	RMSE (mL)	MAPE (%)
Proposed, uncorrected	H, W	4.16	6.02
Proposed, κ -corrected	H, W	2.55	3.09
Narushin et al. [15]	H, W	3.40	4.54
Ellipsoid ($k_v = 0.5236$)	H, W	4.00	5.72

The uncorrected first-principles model (6.02 %) is initially less accurate than the empirically tuned Narushin formula (4.54 %), reflecting the cost of deriving the model from geometry rather than fitting it to data. The single-parameter κ correction, itself grounded in the leading-order b^2 -dependence of (7), reverses this ordering: the corrected model attains 3.09 %, the lowest MAPE of the four, while remaining an exact closed-form expression rather than a regression fit.

We note that Weng et al. [16] report substantially smaller errors for both their Equation (maximum 1.91 %) and the Narushin formula (maximum 1.59 %) on their own 47-egg sample. Two factors account for the difference. First, their model uses the per-egg measured offset x_m , whereas the present comparison uses only (H, W) with breed-level constants, trading per-egg accuracy for a two-measurement protocol. Second, the error metrics differ: they report the maximum per-egg percentage error, while Table 9 reports the mean absolute percentage error (MAPE) over 120 ISA Brown eggs. The two studies are therefore not directly comparable; the present figures characterise the (H, W) -only regime relevant to low-cost deployment.

Notably, Weng et al. [16] attribute the residual error of their formulation to power-series truncation of the shape terms and to approximations in recovering $\tan \theta$ and the semi-minor axis. The present framework addresses both sources explicitly: the tangent expansion is retained to third order with its truncation error quantified (Table 1), and the semi-minor-axis approximation is corrected analytically through the factor κ introduced above.

4.3. Classification of Commercial Grade from (H, W)

Having established the volume model, we now show that the same two measurements (H, W) are sufficient to recover the commercial size grade. The classifier is chosen to match the analytical character of the model: rather than a kernel method with tuned hyperparameters, we use linear discriminant analysis (LDA) [17], whose decision rule

is available in closed form and whose distributional assumptions were tested directly in Section 4.1.

4.3.1. Linear Discriminant Classifier

Under LDA, an egg with feature vector $\mathbf{z} = (H, W)^\top$ is assigned to the grade c that maximises the linear discriminant score

$$\delta_c(\mathbf{z}) = \mathbf{z}^\top \Sigma^{-1} \boldsymbol{\mu}_c - \frac{1}{2} \boldsymbol{\mu}_c^\top \Sigma^{-1} \boldsymbol{\mu}_c + \ln \pi_c, \quad (24)$$

where $\boldsymbol{\mu}_c$ is the mean feature vector of grade c , π_c its prior proportion, and Σ the pooled within-grade covariance matrix, estimated as

$$\Sigma = \frac{1}{N - C} \sum_{c=1}^C \sum_{i \in c} (\mathbf{z}_i - \boldsymbol{\mu}_c)(\mathbf{z}_i - \boldsymbol{\mu}_c)^\top, \quad (25)$$

with $C = 4$ grades and $N = 120$ eggs. The classifier is therefore fully specified by the class means and a single shared covariance matrix; it introduces no tuning parameters.

The two assumptions underlying (24)—approximate within-grade normality and a common covariance across grades—were examined in Section 4.1. The geometric features H and W show no strong departure from normality, and Levene's test indicates that the width variance is homogeneous across grades; the length variance is borderline. These conditions are sufficiently well met for the pooled-covariance model to apply, and the robustness of the resulting classification is confirmed empirically below.

4.3.2. Measured Pairwise Separability

Standardised separation ratios (21) between adjacent grade pairs are given in Table 10. Width W separates adjacent grades more strongly ($d = 1.51$ – 1.86) than length H ($d = 0.93$ – 1.81). Neither feature alone is sufficient: a single-feature LDA reaches 71.7 % on W and 65.0 % on H , both well below the two-feature result. The pair (H, W) is required for reliable separation.

Table 10. Standardised separation ratio d between adjacent grade pairs for the two geometric features.

Grade Pair	d_H	d_W
0 vs. 1	1.13	1.86
1 vs. 2	0.93	1.51
2 vs. 3	1.81	1.65

4.3.3. Cross-Validated Accuracy

Under stratified five-fold cross-validation, with feature standardisation fitted within each training fold only, the linear discriminant classifier attains an accuracy of 92.5 ± 3.1 %. A bootstrap over 2000 stratified resamples gives a 95 % confidence interval of $[85.8, 95.8]$ %, whose lower bound lies far above the 25 % chance level for four balanced classes. The interval width reflects the modest sample size ($n = 120$); it indicates that the two geometric features carry clearly sufficient grade information, while a larger multi-farm dataset would be needed to narrow the estimate for a production classifier.

For comparison, a radial-basis-function support vector machine reaches a nominally higher cross-validated accuracy (94.2 %), but its value depends on the kernel-scale heuristic and varied by more than two percentage points between software implementations; its bootstrap interval ($[87.5, 97.5]$ %) overlaps almost entirely with that of the closed-form discriminant. We therefore report LDA as the primary classifier: it achieves statistically indistinguishable accuracy while remaining fully specified, assumption-checkable, and free of tuning.

4.3.4. Hold-Out Test Performance

On a stratified 25 % hold-out set ($n = 32$) held out before any model fitting, the LDA classifier attains 87.5 % accuracy. All errors occur between adjacent grades; no egg is misclassified by more than one grade step (Table 11). Grades 2 and 3 achieve perfect recall, reflecting their tight geometric clustering.

Table 11. Confusion matrix for the LDA classifier on the stratified 25 % hold-out set ($n = 32$).

Actual	Predicted				Recall (%)
	G0	G1	G2	G3	
Grade 0	8	0	0	0	100.0
Grade 1	0	7	1	0	87.5
Grade 2	0	0	8	0	100.0
Grade 3	0	0	1	7	87.5

The feature pair (H, W)—identical to the two inputs of the volume model (7)—thus serves a dual role: it drives both the closed-form volume computation and, through the closed-form discriminant (24), the recovery of commercial grade.

4.4. Nutrient Content Estimation

4.4.1. Estimated Nutrient Quantities per Grade

Using the κ -corrected model volumes V_{egg} (Section 4.2.3) and the grade-level densities from Table 4, the equivalent egg mass M_{egg} was computed via (17), and the per-egg nutrient content was estimated via (18) using the reference 100 g values of Tables 5 and 6. Estimated per-egg macronutrient quantities are summarised in Table 12.

Table 12. Estimated per-egg macronutrient and energy content by grade. The equivalent mass is $M_{\text{egg}} = V_{\text{egg}}D_{\text{egg}}$ (Equation (17)) using the κ -corrected model volume and the grade-level density; each nutrient is $KM_{\text{egg}}/100$ (Equation (18)).

Grade	V_{egg} (cm^3)	D_{egg} (g cm^{-3})	M_{egg} (g)	Protein (g)	Lipid (g)	Energy (kcal)
0	64.1	1.156	74.1	9.71	6.23	98.6
1	58.3	1.132	66.0	8.64	5.54	87.7
2	54.4	1.153	62.7	8.22	5.27	83.4
3	50.2	1.152	57.8	7.58	4.86	76.9

4.4.2. Clinical Relevance

The estimated protein content varies from 9.71 g per Grade 0 egg to 7.58 g per Grade 3 egg—a difference of 2.14 g per egg, or approximately 22 %. For a 60 kg adult on the WHO-recommended 0.8 g kg^{-1} daily protein intake [2] of 48 g, this represents 4.5 % of the daily target per egg. A patient consuming three eggs per day therefore receives a protein intake that varies by up to 6.4 g—equivalent to 13.4 % of the daily target—depending solely on the size grade. This is a clinically non-trivial difference that a weight-only label does not convey.

4.4.3. Ranking of Grade Indicators

To identify which physical and derived quantities most strongly indicate commercial grade, every measured and computed parameter was evaluated as a single-feature linear discriminant under identical stratified five-fold cross-validation.

The shape parameter g is excluded from this ranking because it is held fixed at the breed level (Section 2) and therefore carries no per-egg discriminative information by

construction—a direct consequence of the geometric self-similarity of ISA Brown eggs across grades.

Three findings emerge from this analysis. First, mass remains the strongest single discriminator at 96.7 %, confirming that the existing commercial grading standard, which uses mass thresholds, is statistically well-founded. Second, the model-derived volume V_{egg} achieves 93.3 % accuracy from only two non-destructive measurements (H, W) , recovering nearly the same discriminative information as direct weighing without requiring a balance. The geometric product $H \cdot W^2$, which scales as volume, achieves 93.3 %—matching the closed-form volume model and within 3.4 percentage points of mass—making it a strong candidate for practical use (Section 4.4.4).

Third, three measured quantities are grade-invariant in ISA Brown eggs, namely, whole-egg density, the shape index W/H , and the absolute yolk volume, each classifying near the 25 % chance level for four balanced classes. The model shape parameter g is likewise grade-independent, being fixed at the breed level by construction. Although density is weakly separated in a statistical sense ($F(3, 116) = 3.2$), its between-grade variation of about 2 % is negligible compared to the 27 % variation in mass. This is a positive scientific finding: ISA Brown eggs are geometrically self-similar across commercial size grades, so grade differences are driven by overall size rather than by changes in shape, composition, or density. A single breed-level density and a single breed-level shape parameter therefore suffice throughout the framework, simplifying its practical deployment.

The model volume V_{egg} and the geometric product $H \cdot W^2$ share the same F -statistic (317.4): because V_{egg} is, to the model's working order, a strictly monotone function of $H \cdot W^2$, and the one-way F -statistic is invariant under strictly monotone affine rescaling of the feature, the two quantities are statistically equivalent as grade discriminators.

4.4.4. Practical Single-Number Grade Indicator

While Table 13 ranks every measured and derived quantity by discriminative power, deployment in an SME setting requires a single indicator that is both highly discriminative and directly computable in the field. The geometric product

$$I = H \cdot W^2 \quad (26)$$

satisfies both criteria. It achieves a single-feature cross-validated accuracy of 93.3 % (Table 13)—matching the model-derived volume and within 3.4 percentage points of direct mass weighing—while requiring, unlike the closed-form volume model (7) or the linear discriminant (24), no logarithm, no numerical integration, no feature standardisation, and no matrix inversion: a pocket calculator suffices.

The grade-mean values of I observed in this study, and the corresponding decision thresholds taken as midpoints between adjacent grade means, are summarised in Table 14. A user who has measured H and W in millimetres computes $I = H \cdot W^2$ once and assigns the egg to its commercial grade by direct threshold comparison. Applied to the 120-egg dataset, this lookup procedure correctly classifies 93.3 % of eggs—comparable to the hold-out performance of the full (H, W) discriminant (93.8 %, Section 4.3).

This threshold-based rule is a minimal grading procedure consistent with the geometric framework: two caliper measurements, one multiplication, and one comparison against three numerical thresholds. It distils the central result of this section into operational form—that the discriminative information contained jointly in (H, W) is sufficient to recover commercial egg grade without specialised hardware, statistical software, or the closed-form volume formula. For applications requiring the additional outputs of the full framework (volume, surface area, mass-equivalent, nutrient content), the analytical model (7) remains available; for grade assignment alone, (26) is sufficient.

The midpoint choice of threshold is not arbitrary. For two adjacent grades with approximately equal within-grade variance in I and equal prior proportions, the linear discriminant (24) reduces in one dimension to the rule $I \geq \frac{1}{2}(\bar{I}_c + \bar{I}_{c+1})$: the decision boundary lies exactly at the midpoint of the two grade means. The threshold rule of Table 14 is therefore the one-dimensional specialisation of the closed-form discriminant applied to the single feature I , rather than a separate heuristic. This explains why its accuracy (93.3 %) is statistically indistinguishable from that of the full two-feature discriminant.

Table 13. Ranking of grade indicators by single-feature LDA classification accuracy ($n = 120$, stratified five-fold cross-validation). The ANOVA F -statistic measures between-grade separation; indicators are ordered by classification accuracy.

Indicator	ANOVA F	Accuracy (%)
Strong discriminators		
Mass M (g)	238.8	96.7
V_{egg} (model)	317.4	93.3
$H \cdot W^2$	317.4	93.3
$H \cdot W$	255.0	82.5
$H + W$	220.6	80.8
V_{ref} (fluid)	153.9	76.7
Albumen volume	119.8	74.2
Width W (mm)	127.2	70.8
Height H (mm)	75.5	65.8
Weak or non-discriminators		
Shape index W/H	4.6	33.3
Density (g cm^{-3})	3.2	31.7
Yolk volume (mL)	5.0	30.0

Table 14. Practical grade-assignment thresholds using the indicator $I = H \cdot W^2$. Each threshold is the midpoint of the adjacent grade means; the resulting lookup rule classifies 93.3 % of the 120-egg dataset correctly.

Grade	Mean I (mm^3)	Assignment Rule (mm^3)
0	129,696	$I \geq 123,795$
1	117,894	$113,992 \leq I < 123,795$
2	110,089	$105,825 \leq I < 113,992$
3	101,562	$I < 105,825$

4.5. Limitations and Scope of Validity

The results established in this work hold under a specific and clearly bounded set of conditions, and the following limitations delineate the domain over which the model has been validated.

First, the geometric model and all reported error metrics are validated for a single breed, ISA Brown, and for the four commercial size grades (0–3) defined in Section 3.1. The breed-level shape parameter $g = 0.05$ and the correction factor $\kappa = 0.97$ are estimated from the present sample and should be regarded as breed-specific constants. The closed-form volume expression (7) is itself breed-independent—it follows from the geometry of the parametric profile alone—but the particular numerical values of g and κ that make it accurate are not. Extension to other breeds, to the smaller grades 4 and 5, or to eggs of other avian species would require re-estimation of these two constants, and the model's accuracy on such data remains to be established.

Second, the invariance of the shape parameter and of whole-egg density across grades is demonstrated only within the sampling frame of the present study: eggs collected from a single farm, over a limited collection window, from hens of a comparable age. Whether g remains constant across hen age, across production seasons, and across genetic lines within the ISA Brown population is not tested here. The geometric self-similarity reported in Section 4.4.3 should therefore be read as a property of this sample rather than as an established biological invariant; confirming its stability across production conditions is a natural direction for further study.

Third, the classification results rest on a dataset of $n = 120$ eggs, while the cross-validated and hold-out accuracies are consistent and are reported with their associated uncertainty (Section 4.3), this sample size is modest by the standards of statistical learning. The appropriate reading of these results is that the two geometric measurements (H, W) carry sufficient information to recover commercial grade, rather than that the trained classifier is ready for deployment at scale. External validation on larger, multi-farm datasets would be required before any operational grading claim could be made.

Fourth, the systematic positive bias analysed in Section 4.2.2 originates in the geometric approximation $b \approx b_{\max}/2$. The correction $b = \kappa b_{\max}/2$, with κ obtained in closed form by matching the mean predicted volume to the mean reference volume, reduces this bias to a negligible level on the present data. Like the breed-level density constant, κ is estimated from the present sample; its transferability to independent samples has not been tested and would require confirmation on external data.

These limitations do not affect the internal validity of the mathematical results—the closed-form volume expression, its convergence properties, and the statistical comparisons reported here—but they bound the range of practical application. Accordingly, the framework is presented as a validated proof of concept for low-cost geometric grading of ISA Brown eggs, and claims of broader commercial applicability are deferred pending the external validation described above.

5. Conclusions

This paper presented a breed-specific adaptation of Smart's parametric egg model for the ISA Brown chicken egg, validated on 120 eggs across the four most commercially prevalent size grades. Three contributions were made. First, a closed-form analytical volume expression was derived as a solid of revolution from only two measurements (H, W), eliminating the difficult-to-measure offset parameter x_m of the closest prior work; the shape parameter enters through a single breed-level constant $g \equiv \theta$, and the surface-area companion is obtained by trapezoidal integration with an endpoint analysis that resolves the $0 \times \infty$ indeterminacy at the poles. Second, the model was validated against fluid-displacement ground truth, yielding strong agreement (Pearson $r = 0.894$, MAPE 6.02 %) with a systematic positive bias of +3.23 mL; the bias was traced analytically to the approximation $b \approx b_{\max}/2$, and a single closed-form correction factor $\kappa = 0.97$ —itself a consequence of the leading-order b^2 -dependence of the volume—reduced the MAPE to 3.09 %, below that of the empirically fitted comparison formula while preserving the analytical form. Third, the same two measurements were shown to be sufficient for both commercial grade classification, using a closed-form linear discriminant (92.0 % cross-validated accuracy, 95 % CI [85.8, 95.8] %), and per-egg nutrient estimation (protein content 7.58–9.71 g across grades, a 13.4 % in the daily intake of a patient consuming three eggs).

A systematic ranking of grade indicators confirmed that ISA Brown eggs are geometrically self-similar across commercial grades: whole-egg density, the shape index, and yolk volume each classify near the 25 % chance level, and the shape parameter g is grade-independent by construction. Grade differences are therefore driven by overall size rather

than by changes in shape, composition, or density, so a single breed-level value of each constant suffices throughout the framework. For low-resource deployment, the geometric product $I = H \cdot W^2$ —the one-dimensional reduction of the linear discriminant—recovers 93.3 % grade-assignment accuracy through one multiplication and three threshold comparisons, requiring no software.

The framework is presented as a proof of concept validated for ISA Brown eggs, grades 0–3; the shape parameter g and the correction factor κ are breed-level constants, and broader application to other breeds, ages, and production conditions awaits external validation. A natural extension is the derivation of the two dimensions from calibrated images rather than caliper readings, which would remove the need for physical instruments while preserving the analytical framework developed here.

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