

Article

Bayesian Identification of Double-Seasonal Vector Autoregressive Models

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Abstract

Identifying the autoregressive (AR) orders of a multivariate time series is a foundational step whose accuracy governs every downstream modelling and forecasting task. When the series exhibits two simultaneously operating seasonal periodicities, as is routinely observed in hourly electricity loads and intraday financial prices, no principled Bayesian identification framework currently exists. This paper addresses this gap by introducing a complete Bayesian order identification procedure for double-seasonal vector autoregressive (DSVAR) models. These DSVAR models are multivariate processes governed by a multiplicative triple autoregressive operator that jointly captures regular, first-seasonal, and second-seasonal dynamics. We treat the three order indices as unknown discrete parameters and derive closed-form expressions for the joint posterior probability mass function of the order triple. Two complementary prior specifications are considered: a conjugate matrix normal-Wishart prior and Jeffreys' non-informative prior. The analysis is carried out under the assumption of symmetric, normally distributed errors, which ensures analytical tractability and allows for the posterior probabilities to be evaluated exactly for every admissible combination of orders. Specifically, under each prior, the posterior mass function reduces to explicit determinantal expressions that can be evaluated by a straightforward three-dimensional grid search. Monte Carlo experiments on various DSVAR processes confirm that the proposed technique achieves high identification accuracy even at moderate sample sizes across a range of parameter configurations and prior choices. The proposed Bayesian procedure is benchmarked against the standard Bayesian information criterion (BIC), consistently achieving higher correct identification rates across all cases. Empirical applications to hourly electricity load data from the Czech Republic and Germany, as well as to hourly solar radiation in Najran, Saudi Arabia, demonstrate the practical applicability of the Bayesian identification method.



Academic Editor: Raymond Lee

Received: 2 June 2026

Revised: 4 July 2026

Accepted: 7 July 2026

Published: 9 July 2026

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Keywords: double seasonality; VAR models; Bayesian identification; posterior mass function; matrix normal-Wishart prior; electricity load data; solar radiation

MSC: 62M10; 62F15

1. Introduction

Seasonal multivariate time series arise in settings where several interdependent variables are observed at a sufficiently high frequency for recurrent temporal patterns to become evident. Illustrative examples include hourly electricity demand across interconnected

power systems, quarter-hourly call centre arrivals disaggregated by region, and sub-daily meteorological measurements collected over spatial monitoring networks. A defining feature of such high-frequency data is the existence of two nested seasonal cycles, namely, for hourly time series, an intraday cycle comprising 24 observations and an intraweek cycle comprising 168 observations. Accurate representation of both cycles is essential for rigorous modelling and forecasting [1–3].

The univariate analysis of double seasonality has a long and well-established tradition. Taylor [1] introduced the double-seasonal exponential smoothing methods together with their ARMA analogue. Subsequent developments extended this framework to triple seasonality [4,5]. Within the Bayesian literature, several important contributions address univariate double-seasonal time series. Amin [6] proposed Bayesian estimation methods for double-seasonal AR (DSAR) and double-seasonal ARMA models. Subsequent contributions employed the Gibbs sampler to obtain full Bayesian inference for DSAR models [3]. Most recently, Amin [7] provided a Bayesian treatment of triple seasonal AR time series models using the Gibbs sampler.

Progress in the multivariate domain has been considerably slower. The vector autoregressive (VAR) framework of Sims [8] remains the standard approach for modelling the joint dynamics of multiple related time series, providing a flexible, theory-agnostic representation of interdependencies [9]. In its conventional form, however, the VAR does not incorporate any explicit mechanism for seasonal structure. Bayesian VAR modelling began with the Minnesota prior of Litterman [10] and Doan et al. [11], which shrinks coefficients toward a random-walk benchmark to curb dimensionality. Kadiyala and Karlsson [12] later placed this prior within a normal-Wishart conjugate framework, yielding closed-form posterior and predictive results. Scaled variants of the Minnesota prior were subsequently shown by Bańbura et al. [13] to enable reliable estimation of high-dimensional VARs.

Accurately determining the orders of a seasonal AR model is widely regarded as the most consequential phase of time series analysis because every subsequent step, namely, estimation, diagnostic checking, and out-of-sample prediction, inherits the consequences of any misspecification committed at identification [9,14,15]. Two broad paradigms have been used for this task. The classical approach of Box et al. [14] matches sample partial autocorrelation and cross-correlation functions against their theoretical counterparts. While intuitive, its inherently subjective nature resists automation and is ill-suited to the several structured lag groups that arise in double-seasonal models. Information criterion methods such as the Akaike information criterion (AIC) [16] and the Bayesian information criterion (BIC) [17] provide a fully automatic alternative but offer no explicit probabilistic statement about the selected orders, and their consistency properties in the high-dimensional seasonal multivariate setting remain incompletely understood.

The Bayesian paradigm affords a fully probabilistic treatment. The unknown orders are assigned a prior probability mass function, and inference proceeds via the posterior mass function of the order vector conditional on the observations. Bayesian order identification for univariate AR models was first pioneered by Broemeling and Shaarawy [18], and later extended to single-seasonal AR processes by Shaarawy and Ali [19], to double-seasonal AR models by Amin [15], and to triple seasonal AR formulations by Amin and Alghamdi [20]. For the non-seasonal multivariate setting, Shaarawy and Ali [21] derived exact closed-form posterior mass functions for the orders of a standard VAR process, and Shaarawy et al. [22] addressed the identification of moving-average models. The single-seasonal multivariate generalisation was developed by Shaarawy and Ali [23], who obtained the joint posterior mass function for the order pair of the seasonal VAR (SVAR) models in a tractable closed form.

Despite these advances, no Bayesian order identification framework exists for a *multivariate* process carrying *two* simultaneous seasonal periodicities. This gap is practically important because the energy, finance, and environmental applications that motivate double-seasonal modelling almost invariably involve several series measured jointly. This paper addresses this gap and advances the literature through the following contributions:

- (i) We formally define the DSVAR model through its multiplicative triple operator, derive its fully expanded form, and construct the order-dependent matrix regression representation in which all matrices are written as explicit functions of the DSVAR order.
- (ii) Under the approximate conditional likelihood and a conjugate matrix normal-Wishart prior, we derive in closed form the joint posterior mass function of the DSVAR order. The parallel result under Jeffreys' non-informative prior is established.
- (iii) Both posterior mass functions are explicit. Evaluating each candidate triple requires only the computation of two matrix determinants and a product of gamma function values, enabling an efficient three-dimensional grid search for the maximum a posteriori (MAP) orders.
- (iv) Monte Carlo experiments across several parameter configurations and three prior choices quantify identification accuracy, and the procedure is benchmarked against the BIC. Applications to hourly electricity load data from the Czech Republic and Germany, together with hourly solar radiation in Najran, Saudi Arabia, illustrate the framework in realistic double-seasonal multivariate settings.

This paper is organised as follows. Section 2 introduces the DSVAR model, its expansion, and the order-dependent matrix regression form. Section 3 presents the Bayesian concepts for the DSVAR models, derives the posterior mass function under each prior, and describes the identification algorithm. Sections 4 and 5 report the simulation study and the empirical applications. Section 6 concludes the paper.

2. Double-Seasonal VAR (DSVAR) Models

2.1. DSVAR Model Definition

Let $\{\mathbf{y}(t)\}_{t=1}^n$ be a k -dimensional ($k \geq 2$) observable time series. The backshift operator B satisfies $B^r \mathbf{y}(t) = \mathbf{y}(t - r)$ for every positive integer r . Two seasonal periods, s_1 and s_2 , with $1 < s_1 < s_2$, are treated as fixed and known. Let p, P_1, P_2 denote non-negative integers representing the regular autoregressive order and the first- and second-seasonal autoregressive orders, respectively.

The three matrix lag polynomial operators are:

$$\boldsymbol{\phi}_p(B) = \mathbf{I}_k - \sum_{i=1}^p \boldsymbol{\phi}_i B^i, \quad (1)$$

$$\boldsymbol{\Psi}_{P_1}(B^{s_1}) = \mathbf{I}_k - \sum_{j=1}^{P_1} \boldsymbol{\Psi}_j B^{js_1}, \quad (2)$$

$$\boldsymbol{\Gamma}_{P_2}(B^{s_2}) = \mathbf{I}_k - \sum_{\ell=1}^{P_2} \boldsymbol{\Gamma}_\ell B^{\ell s_2}, \quad (3)$$

where $\boldsymbol{\phi}_i, \boldsymbol{\Psi}_j, \boldsymbol{\Gamma}_\ell \in \mathbb{R}^{k \times k}$ are unknown $k \times k$ coefficient matrices and $\mathbf{I}_k$ is the $k \times k$ identity.

Definition 1 (DSVAR model). *The double-seasonal vector autoregressive process of regular order p , first-seasonal order P_1 at period s_1 , and second-seasonal order P_2 at period s_2 , with dimension k , denoted as $\text{DSVAR}_k(p, P_1, P_2)_{s_1, s_2}$, is governed by the multiplicative triple operator equation:*

$$\phi_p(B) \Psi_{P_1}(B^{s_1}) \Gamma_{P_2}(B^{s_2}) \mathbf{y}(t) = \varepsilon(t), \quad (4)$$

where $\{\varepsilon(t)\}$ is a sequence of independent k -dimensional white-noise vectors with $\varepsilon(t) \sim \mathcal{N}_k(\mathbf{0}, \mathbf{T}^{-1})$, and $\mathbf{T}$ is an unknown $k \times k$ positive definite precision matrix. The process is covariance-stationary if and only if all roots of the three determinantal polynomial equations

$$|\phi_p(z)| = 0, \quad |\Psi_{P_1}(z^{s_1})| = 0, \quad |\Gamma_{P_2}(z^{s_2})| = 0$$

lie strictly outside the complex unit circle $|z| > 1$.

Equation (4) extends the single-seasonal operator $\phi_p(B) \Phi_P(B^s) \mathbf{y}(t) = \varepsilon(t)$ of Shaarawy and Ali [23] by appending the additional factor $\Gamma_{P_2}(B^{s_2})$, analogously to how the double-seasonal univariate AR (DSAR) extends its single-seasonal counterpart [1,3].

Remark 1 (special cases). Setting $P_1 = P_2 = 0$ recovers the standard $\text{VAR}_k(p)$ [8,9]. Setting $P_2 = 0$ yields the single-seasonal $\text{SVAR}_k(p, P_1)_{s_1}$ of Shaarawy and Ali [23]. Setting $k = 1$ limits the process to the univariate DSAR process studied in Amin [3].

We can expand the triple product (4) in two steps.

Step 1. Multiply the first two factors:

$$\phi_p(B) \Psi_{P_1}(B^{s_1}) = \mathbf{I}_k - \sum_{i=1}^p \phi_i B^i - \sum_{j=1}^{P_1} \Psi_j B^{js_1} + \sum_{i=1}^p \sum_{j=1}^{P_1} \phi_i \Psi_j B^{i+js_1}. \quad (5)$$

Step 2. Multiply (5) by $\Gamma_{P_2}(B^{s_2}) = \mathbf{I}_k - \sum_{\ell=1}^{P_2} \Gamma_\ell B^{\ell s_2}$ and rearrange. Substituting into (4) yields the fully expanded representation:

$$\begin{aligned} \mathbf{y}(t) = & \underbrace{\sum_{i=1}^p \phi_i \mathbf{y}(t-i)}_{\text{Group 1}} + \underbrace{\sum_{j=1}^{P_1} \Psi_j \mathbf{y}(t-js_1)}_{\text{Group 2}} - \underbrace{\sum_{i=1}^p \sum_{j=1}^{P_1} \phi_i \Psi_j \mathbf{y}(t-i-js_1)}_{\text{Group 3}} \\ & + \underbrace{\sum_{\ell=1}^{P_2} \Gamma_\ell \mathbf{y}(t-\ell s_2)}_{\text{Group 4}} - \underbrace{\sum_{i=1}^p \sum_{\ell=1}^{P_2} \phi_i \Gamma_\ell \mathbf{y}(t-i-\ell s_2)}_{\text{Group 5}} \\ & - \underbrace{\sum_{j=1}^{P_1} \sum_{\ell=1}^{P_2} \Psi_j \Gamma_\ell \mathbf{y}(t-js_1-\ell s_2)}_{\text{Group 6}} + \underbrace{\sum_{i=1}^p \sum_{j=1}^{P_1} \sum_{\ell=1}^{P_2} \phi_i \Psi_j \Gamma_\ell \mathbf{y}(t-i-js_1-\ell s_2)}_{\text{Group 7}} + \varepsilon(t). \quad (6) \end{aligned}$$

The seven groups and their properties are summarised in Table 1. The signs of Groups 3, 5, and 6 (negative) and Group 7 (positive) arise entirely from the algebra of the multiplicative expansion; they are not free parameters. The total number of lag groups per variable is:

$$h(p, P_1, P_2) = p + P_1 + pP_1 + P_2 + pP_2 + P_1P_2 + pP_1P_2 = (1+p)(1+P_1)(1+P_2) - 1, \quad (7)$$

which reduces to $h = p + P_1 + pP_1$ when $P_2 = 0$, confirming consistency with the $\text{SVAR}_k(p, P_1)_{s_1}$ model of Shaarawy and Ali [23].

Table 1. The seven structured regressor groups arising in the $\text{DSVAR}_k(p, P_1, P_2)_{s_1, s_2}$ expansion (6).

Group	Type	Lag Set	Sign	Count
1	Regular AR	$\{i : 1 \leq i \leq p\}$	+	p
2	First-seasonal	$\{js_1 : 1 \leq j \leq P_1\}$	+	P_1
3	Regular $\times$ first-seasonal	$\{i + js_1\}$	−	pP_1
4	Second-seasonal	$\{\ell s_2 : 1 \leq \ell \leq P_2\}$	+	P_2
5	Regular $\times$ second-seasonal	$\{i + \ell s_2\}$	−	pP_2
6	First $\times$ second-seasonal	$\{js_1 + \ell s_2\}$	−	P_1P_2
7	Triple cross	$\{i + js_1 + \ell s_2\}$	+	pP_1P_2
Total per variable:				$h(p, P_1, P_2)$

2.2. Order-Dependent Matrix Regression Form

In the exact model (6), the coefficient matrices of Groups 3, 5, 6, and 7 are nonlinear functions of the primary coefficients $\{\phi_i\}$, $\{\Psi_j\}$, and $\{\Gamma_\ell\}$. To obtain an analytically tractable likelihood and posterior, we adopt the free-parameter approximation [18,23], where the coefficient matrices of all seven groups are treated as unrestricted, independent free parameters.

Under this device, the model takes the standard multivariate linear regression form:

$$Y(p, P_1, P_2) = X(p, P_1, P_2) B(p, P_1, P_2) + U(p, P_1, P_2). \quad (8)$$

The dimensions and roles of the four matrices are given in Table 2.

Table 2. Matrices in the $\text{DSVAR}_k(p, P_1, P_2)_{s_1, s_2}$ regression representation (8). Here, $n^* = n - p - P_1s_1 - P_2s_2$ and $h = (1 + p)(1 + P_1)(1 + P_2) - 1$.

Matrix	Dimension	Description
$Y(p, P_1, P_2)$	$n^* \times k$	Stacked observation vectors
$X(p, P_1, P_2)$	$n^* \times kh$	Stacked regressor vectors
$B(p, P_1, P_2)$	$kh \times k$	Full free-parameter coefficient matrix
$U(p, P_1, P_2)$	$n^* \times k$	White-noise matrix

The t -th row of $X(p, P_1, P_2)$ is the $1 \times kh$ vector $\mathbf{x}_{p, P_1, P_2}^\top(t - 1)$ formed from the seven groups of lagged observations following the sign convention in Table 1. The coefficient matrix $B(p, P_1, P_2)$ partitions into seven conformable blocks $[\beta_1^\top, \dots, \beta_7^\top]^\top$, with $\beta_1 = [\phi_1^\top, \dots, \phi_p^\top]^\top$, $\beta_2 = [\Psi_1^\top, \dots, \Psi_{P_1}^\top]^\top$, $\beta_4 = [\Gamma_1^\top, \dots, \Gamma_{P_2}^\top]^\top$, and $\beta_3, \beta_5, \beta_6$, and β_7 being the free cross-product blocks. The noise matrix $U(p, P_1, P_2)$ has rows $\varepsilon(t)^\top \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}_k(\mathbf{0}, \mathbf{T}^{-1})$.

This representation is the direct three-dimensional extension of the SVAR regression form. The scalar period s and order P are replaced by the pair (s_1, P_1) and (s_2, P_2) , and every matrix dimension is correspondingly enlarged to accommodate the four additional regressor groups.

3. Posterior Analysis and Proposed Order Identification Procedure

3.1. Bayesian Concepts for DSVAR Models

3.1.1. Likelihood Function

Conditioning on the first $p + P_1s_1 + P_2s_2$ observations and using the regression form (8), the approximate conditional likelihood of $(B(p, P_1, P_2), T)$, given the data $S_n = [y(1), \dots, y(n)]^\top$, is:

$$\mathcal{L}(B, T \mid S_n, p, P_1, P_2) \propto |T|^{n^*/2} \exp\left(-\frac{1}{2} \text{tr}\left[(Y - XB)^\top (Y - XB) T\right]\right), \quad (9)$$

where we write Y , X , and B for $Y(p, P_1, P_2)$, $X(p, P_1, P_2)$, and $B(p, P_1, P_2)$. Completing the square in B decomposes the exponent as:

$$\mathcal{L} \propto |T|^{n^*/2} \exp\left(-\frac{1}{2} \text{tr}\left[(B - \hat{B})^\top (X^\top X)(B - \hat{B}) T + S_e T\right]\right), \quad (10)$$

where the OLS estimator and residual sum-of-squares matrix are:

$$\hat{B}(p, P_1, P_2) = (X^\top X)^{-1} X^\top Y, \quad (11)$$

$$S_e(p, P_1, P_2) = Y^\top Y - Y^\top X (X^\top X)^{-1} X^\top Y. \quad (12)$$

The Bayesian approach treats (p, P_1, P_2) as unknown discrete parameters with a prior probability mass, and (B, T) as continuous unknowns with a prior density conditional on the orders.

3.1.2. Prior Distributions

Let $\zeta_{p,P_1,P_2} = \Pr[p = p_0, P_1 = P_{10}, P_2 = P_{20}]$ denote the prior probability assigned to a specific order triple, with $\sum_{p,P_1,P_2} \zeta_{p,P_1,P_2} = 1$. We employ three prior mass functions ζ_{p,P_1,P_2} :

- **Prior 1 (uniform):** Equal probability for every candidate triple:

$$\zeta_{p,P_1,P_2} \propto \frac{1}{(p^* + 1)(P_1^* + 1)(P_2^* + 1)}, \quad 0 \leq p \leq p^*, 0 \leq P_1 \leq P_1^*, 0 \leq P_2 \leq P_2^* \quad (13)$$

- **Prior 2 (exponentially decreasing):**

$$\zeta_{p,P_1,P_2} \propto (0.5)^{p+P_1+P_2}, \quad 0 \leq p \leq p^*, 0 \leq P_1 \leq P_1^*, 0 \leq P_2 \leq P_2^* \quad (14)$$

favouring lower-order models.

- **Prior 3 (linearly decreasing):** ζ_{p,P_1,P_2} proportional to weights that decrease by a constant step as the total order $p + P_1 + P_2$ increases:

$$\zeta_{p,P_1,P_2} \propto \frac{q - (p + P_1 + P_2)}{q}, \quad 0 \leq p \leq p^*, 0 \leq P_1 \leq P_1^*, 0 \leq P_2 \leq P_2^* \quad (15)$$

where $q = p^* + P_1^* + P_2^* + 1$

Conditional on the DSVAR orders (p, P_1, P_2) and on the precision matrix T , a matrix normal prior is placed on B :

$$f_1(B | T, p, P_1, P_2) \propto |Q|^{k/2} |T|^{kh/2} \exp\left(-\frac{1}{2} \text{tr}[(B - D)^\top Q(B - D) T]\right), \quad (16)$$

where $D(p, P_1, P_2) \in \mathbb{R}^{kh \times k}$ is the prior mean matrix, and $Q(p, P_1, P_2)$ is a $kh \times kh$ symmetric positive definite prior precision matrix. The precision matrix T independently follows a Wishart prior:

$$f_2(T) \propto |T|^{[a-(k+1)]/2} \exp\left(-\frac{1}{2} \text{tr}[\Psi T]\right), \quad (17)$$

with degrees-of-freedom $a > 0$ and $k \times k$ positive definite scale matrix Ψ . The resulting joint prior belongs to the matrix normal-Wishart class [12,23,24]:

$$f_3(B, T | p, P_1, P_2) \propto |Q|^{k/2} |T|^{[kh+a-(k+1)]/2} \exp\left(-\frac{1}{2} \text{tr}[(B - D)^\top Q(B - D) + \Psi] T\right). \quad (18)$$

When no reliable prior knowledge is available, Jeffreys' vague prior provides a principled default [24–26]:

$$g_1(\mathbf{B}, \mathbf{T} \mid p, P_1, P_2) \propto |\mathbf{T}|^{-(k+1)/2}, \quad \mathbf{B} \in \mathbb{R}^{kh \times k}, \quad \mathbf{T} > 0. \quad (19)$$

This is recovered from (18) by taking $\mathbf{Q} \rightarrow \mathbf{0}_{kh}$, $\mathbf{D} \rightarrow \mathbf{0}$, $a \rightarrow 0$, $\mathbf{\Psi} \rightarrow \mathbf{0}_k$.

3.1.3. Joint Posterior Distribution

The joint posterior of $(\mathbf{B}, \mathbf{T}, p, P_1, P_2)$ is proportional to the product of the likelihood (9), the prior density (18) or (19), and the prior mass ζ_{p, P_1, P_2} . The joint posterior of $(\mathbf{B}, \mathbf{T}, p, P_1, P_2)$ under the normal-Wishart prior is given as:

$$g(\mathbf{B}, \mathbf{T}, p, P_1, P_2 \mid \mathbf{S}_n) \propto \zeta_{p, P_1, P_2} |\mathbf{Q}|^{k/2} |\mathbf{T}|^{(n^* + kh + a - k - 1)/2} \exp\left(-\frac{1}{2} \text{tr}\left\{\left(\mathbf{Y} - \mathbf{XB}\right)^\top \left(\mathbf{Y} - \mathbf{XB}\right) + \left(\mathbf{B} - \mathbf{D}\right)^\top \mathbf{Q} \left(\mathbf{B} - \mathbf{D}\right) + \mathbf{\Psi}\right\} \mathbf{T}\right), \quad (20)$$

and the joint posterior of $(\mathbf{B}, \mathbf{T}, p, P_1, P_2)$ under Jeffreys' prior is given as:

$$g(\mathbf{B}, \mathbf{T}, p, P_1, P_2 \mid \mathbf{S}_n) \propto \zeta_{p, P_1, P_2} |\mathbf{T}|^{(n^* - k - 1)/2} \exp\left(-\frac{1}{2} \text{tr}\left[(\mathbf{Y} - \mathbf{XB})^\top (\mathbf{Y} - \mathbf{XB}) \mathbf{T}\right]\right), \quad (21)$$

3.2. Posterior Mass Function of DSVAR Order

Before deriving the posterior mass function of the DSVAR order, we define three order-dependent matrices that summarise the combination of prior and data information:

$$\mathbf{A}(p, P_1, P_2) = \mathbf{Q}(p, P_1, P_2) + \mathbf{X}(p, P_1, P_2)^\top \mathbf{X}(p, P_1, P_2), \quad (22)$$

$$\mathbf{M}(p, P_1, P_2) = \mathbf{Q}(p, P_1, P_2) \mathbf{D}(p, P_1, P_2) + \mathbf{X}(p, P_1, P_2)^\top \mathbf{Y}(p, P_1, P_2), \quad (23)$$

$$\mathbf{\Omega}(p, P_1, P_2) = \mathbf{D}^\top \mathbf{Q} \mathbf{D} + \mathbf{\Psi} + \mathbf{Y}^\top \mathbf{Y} - \mathbf{M}^\top \mathbf{A}^{-1} \mathbf{M}, \quad (24)$$

where arguments (p, P_1, P_2) are suppressed on the right side of (24) for compactness. Here, $\mathbf{A}$ is the updated precision matrix combining prior and data, $\tilde{\mathbf{B}} = \mathbf{A}^{-1} \mathbf{M}$ is the generalised ridge posterior mode of $\mathbf{B}$, and $\mathbf{\Omega}$ serves as the posterior scale matrix.

Theorem 1 (posterior mass function under normal-Wishart prior). *Let $\mathbf{S}_n = [\mathbf{y}(1), \dots, \mathbf{y}(n)]^\top$ denote n observations from a $\text{DSVAR}_k(p_0, P_{10}, P_{20})_{s_1, s_2}$ process. Under the approximate conditional likelihood (9), the conjugate prior (18), and the prior mass ζ_{p, P_1, P_2} , the joint posterior mass function of the order triple is:*

$$h(p, P_1, P_2 \mid \mathbf{S}_n) \propto \zeta_{p, P_1, P_2} \pi^{k(p + P_1 s_1 + P_2 s_2)/2} |\mathbf{Q}|^{k/2} |\mathbf{A}|^{-k/2} |\mathbf{\Omega}|^{-(n^* + a)/2} \prod_{m=1}^k \Gamma\left(\frac{n^* + a + m - k}{2}\right), \quad (25)$$

valid for $n^* + a > k - 1$, i.e., $n > p + P_1 s_1 + P_2 s_2 + k - 1 - a$, where $\mathbf{A}$, $\mathbf{M}$, $\mathbf{\Omega}$ are as in (22)–(24).

Proof. By Bayes' theorem, the joint posterior of $(\mathbf{B}, \mathbf{T}, p, P_1, P_2)$ is proportional to the product of the likelihood (9), the prior density (18), and the prior mass ζ_{p, P_1, P_2} :

$$g(\mathbf{B}, \mathbf{T}, p, P_1, P_2 \mid \mathbf{S}_n) \propto \zeta_{p, P_1, P_2} |\mathbf{Q}|^{k/2} |\mathbf{T}|^{(n^* + kh + a - k - 1)/2} \exp\left(-\frac{1}{2} \text{tr}[\mathbf{Q} \mathbf{B} \mathbf{T}]\right), \quad (26)$$

where $Q_B = (Y - XB)^\top (Y - XB) + (B - D)^\top Q(B - D) + \Psi$. Completing the square in B by expanding Q_B :

$$Q_B = B^\top AB - 2B^\top M + Y^\top Y + D^\top QD + \Psi.$$

Since A is positive definite, the square completes to:

$$Q_B = (B - \tilde{B})^\top A(B - \tilde{B}) + \Omega, \quad (27)$$

where $\tilde{B} = A^{-1}M$ and Ω is as in (24). Substituting (27) into (26) and integrating out B via the matrix normal normalising constant, which contributes a factor $|T|^{-kh/2}|A|^{-k/2}$, gives the following:

$$g_1(p, P_1, P_2, T \mid S_n) \propto \zeta_{p, P_1, P_2} |Q|^{k/2} |A|^{-k/2} |T|^{(n^* + a - k - 1)/2} \exp\left(-\frac{1}{2} \text{tr}[\Omega T]\right). \quad (28)$$

The expression (28) is proportional to the kernel of a $\text{Wishart}_k(n^* + a, \Omega)$ density in T . Integrating using the Wishart normalising constant [24] gives the following:

$$\int |T|^{(n^* + a - k - 1)/2} \exp\left(-\frac{1}{2} \text{tr}[\Omega T]\right) dT = 2^{k(n^* + a)/2} \pi^{k(k-1)/4} |\Omega|^{-(n^* + a)/2} \prod_{m=1}^k \Gamma\left(\frac{n^* + a + m - k}{2}\right), \quad (29)$$

valid when $n^* + a > k - 1$. Incorporating the factor $\pi^{k(p + P_1 s_1 + P_2 s_2)/2}$ that arises from the initial-value conditioning, absorbing all terms independent of (p, P_1, P_2) into the proportionality constant, and then collecting, yields (25). $\square$

Corollary 1 (posterior mass function under Jeffreys' prior). *Under Jeffreys' prior (19) and prior mass ζ_{p, P_1, P_2} , the joint posterior mass function of the order triple is:*

$$h_1(p, P_1, P_2 \mid S_n) \propto \zeta_{p, P_1, P_2} \pi^{k[kh(p, P_1, P_2) + p + P_1 s_1 + P_2 s_2]/2} |X^\top X|^{-k/2} |S_e|^{-(n^* - kh)/2} \prod_{m=1}^k \Gamma\left(\frac{n^* - k(h + 1) + m}{2}\right), \quad (30)$$

valid for $n^* > k(h + 1)$, i.e., $n > p + P_1 s_1 + P_2 s_2 + k(h + 1)$, where $h \equiv h(p, P_1, P_2)$.

Proof. Set $Q \rightarrow 0_{kh}$, $D \rightarrow 0$, $a \rightarrow 0$, $\Psi \rightarrow 0_k$ in Theorem 1. The summary matrices simplify to $A \rightarrow X^\top X$, $M \rightarrow X^\top Y$, $\tilde{B} \rightarrow \hat{B}$, and $\Omega \rightarrow S_e$. The ratio $|Q|^{k/2}/|A|^{k/2}$ reduces to $|X^\top X|^{-k/2}$, and substituting into (25) yields (30). $\square$

Remark 2 (Jeffreys' prior as a data-driven criterion). *Under a uniform prior $\zeta_{p, P_1, P_2} = \text{const.}$, the log of (30) contains $-\frac{n^* - kh}{2} \log |S_e|$ (rewarding goodness of fit) and $-\frac{k}{2} \log |X^\top X|$ (penalising model complexity, since $|X^\top X|$ decreases as the regressor matrix becomes more collinear at higher orders). This dual structure mirrors the form of information criteria such as the AIC [16] and BIC [17], but is derived from a fully probabilistic prior specification rather than ad hoc penalty design.*

3.3. Proposed Order Identification Procedure

3.3.1. Maximum A Posteriori (MAP) Identification

The Bayesian identified order triple is the maximiser of the posterior mass function over all candidate triples in the search grid:

$$(\hat{p}, \hat{P}_1, \hat{P}_2) = \arg \max_{0 \leq p \leq p^*, 0 \leq P_1 \leq P_1^*, 0 \leq P_2 \leq P_2^*} \log h(p, P_1, P_2 \mid S_n), \quad (31)$$

where the log scale is used for numerical stability. Under the normal-Wishart prior, $\log h(p, P_1, P_2 \mid S_n)$ is obtained from (25) as:

$$\log h(p, P_1, P_2 \mid S_n) = \log \zeta_{p, P_1, P_2} + \frac{k}{2} \log |\mathbf{Q}| + \frac{k}{2} (p + P_1 s_1 + P_2 s_2) \log \pi - \frac{k}{2} \log |\mathbf{A}| \\ - \frac{n^* + a}{2} \log |\mathbf{\Omega}| + \sum_{m=1}^k \log \Gamma\left(\frac{n^* + a + m - k}{2}\right) + \text{const.}, \quad (32)$$

and it can be obtained in the same way under Jeffreys' prior from (30).

3.3.2. Identification Algorithm

- Step 1. Specify the search bounds:** Choose upper bounds (p^*, P_1^*, P_2^*) and a prior mass function ζ_{p, P_1, P_2} .
- Step 2. Construct the candidate grid:** Enumerate all $(p^* + 1)(P_1^* + 1)(P_2^* + 1)$ triples (p, P_1, P_2) with $0 \leq p \leq p^*, 0 \leq P_1 \leq P_1^*, 0 \leq P_2 \leq P_2^*$.
- Step 3. Assemble order-dependent matrices:** For each candidate triple, form $\mathbf{X}(p, P_1, P_2)$, $\mathbf{Y}(p, P_1, P_2)$, and compute $\mathbf{A}$, $\mathbf{M}$, $\mathbf{\Omega}$ via (22)–(24).
- Step 4. Evaluate the log-posterior mass:** Compute $\log h(p, P_1, P_2 \mid S_n)$ from (32) at every grid point.
- Step 5. Select the MAP triple:** Report $(\hat{p}, \hat{P}_1, \hat{P}_2)$ as the triple attaining the maximum log-posterior mass.

The dominant computational cost per grid point is $\mathcal{O}(k^3 h^3)$ for matrix inversion and determinant evaluation, giving a total search cost of $\mathcal{O}(k^3 h_{\max}^3 p^* P_1^* P_2^*)$, which is manageable for the moderate order bounds typical in practice. For example, with $k = 2$ and $(p^*, P_1^*, P_2^*) = (3, 3, 3)$, only 64 grid evaluations are needed, each involving a 14×14 linear system. Thus, for moderate k and order bounds, the full grid search remains feasible and shows the scalability of the procedure. For very large k , sparse or structured approximations to $\mathbf{X}^\top \mathbf{X}$ can reduce this cost substantially, though such extensions are beyond the scope of the present paper.

Stationarity of each candidate $\text{DSVAR}_k(p, P_1, P_2)_{s_1, s_2}$ model is verified during the grid search by checking that all roots of the three determinantal polynomials $|\phi_p(z)| = 0$, $|\Psi_{P_1}(z^{s_1})| = 0$, and $|\Gamma_{P_2}(z^{s_2})| = 0$ lie strictly outside the unit circle. Candidate triples that fail this check are assigned $-\infty$ log-posterior mass and are excluded from the MAP comparison.

Remark 3 (Full posterior distribution over orders). *Beyond the MAP triple, the unnormalised log-posterior masses $\log h(p, P_1, P_2 \mid S_n)$ evaluated at every grid point can be normalised to yield the full posterior probability mass function $\Pr[(p, P_1, P_2) \mid S_n]$. These probabilities provide a richer characterisation of model-selection uncertainty: a sharply peaked distribution signals a decisive identification, while a flat or bimodal distribution indicates non-negligible mass on competing order triples.*

As noted earlier, the proposed identification procedure relies on the free-parameter approximation [18,23], which is supported by both theoretical arguments and empirical evidence. First, the coefficient matrix $\mathbf{B}(p, P_1, P_2)$ is intentionally treated as unrestricted, with the cross-product blocks regarded as free parameters rather than constrained nonlinear products implied by the multiplicative operator (4). This linearisation preserves conjugacy and enables an exact closed-form posterior mass function, whereas enforcing the nonlinear restrictions would eliminate tractability. Second, in the context of order identification, the posterior mass function (25) integrates out all continuous parameters $(\mathbf{B}(p, P_1, P_2), \mathbf{T})$, so any bias in individual coefficient estimates cancels in the marginal distribution of interest.

Third, expanding the parameter space under the free-parameter formulation amplifies the penalty induced by the determinantal factors $|A|^{-k/2}$ and $|\Omega|^{-(n^*+a)/2}$, producing a conservative shrinkage toward lower-order models that effectively mitigates over-identification. Finally, this approximation is widely adopted in Bayesian order selection for multiplicative seasonal time-series models [18,19,23], and the Monte Carlo experiments in the following section confirm that it delivers high correct identification rates even at moderate sample sizes, providing direct evidence that order inference is not materially distorted.

4. Simulation Study

4.1. Experimental Design

A Monte Carlo investigation is conducted to quantify identification accuracy for DSVAR processes. Data are generated from several DSVAR parameter configurations, namely, $\text{DSVAR}_2(1, 1, 1)_{5,25}$, $\text{DSVAR}_2(2, 1, 1)_{6,30}$, and $\text{DSVAR}_3(1, 1, 1)_{6,30}$, whose true coefficients and precision matrices are reported in Table 3.

Table 3. True parameters for the simulation configurations.

Config	ϕ_1	ϕ_2	Ψ_1	Γ_1	T
I	$\begin{bmatrix} 0.3 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$	—	$\begin{bmatrix} 0.4 & 0.3 \\ -0.3 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 0.3 & -0.2 \\ -0.2 & 0.4 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0.3 \\ 0.3 & 1.0 \end{bmatrix}$
II	$\begin{bmatrix} 0.2 & -0.3 \\ -0.1 & 0.3 \end{bmatrix}$	—	$\begin{bmatrix} 0.5 & 0.3 \\ -0.3 & 0.4 \end{bmatrix}$	$\begin{bmatrix} 0.4 & -0.2 \\ -0.3 & 0.4 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0.25 \\ 0.25 & 1.0 \end{bmatrix}$
III	$\begin{bmatrix} 0.2 & 0.2 \\ -0.2 & 0.3 \end{bmatrix}$	—	$\begin{bmatrix} 0.2 & 0.3 \\ -0.2 & 0.2 \end{bmatrix}$	$\begin{bmatrix} 0.3 & -0.2 \\ -0.3 & 0.2 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0.15 \\ 0.15 & 1.0 \end{bmatrix}$
IV	$\begin{bmatrix} 0.3 & -0.2 \\ -0.2 & 0.4 \end{bmatrix}$	$\begin{bmatrix} -0.3 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$	$\begin{bmatrix} 0.4 & 0.3 \\ -0.3 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.3 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0.15 \\ 0.15 & 1.0 \end{bmatrix}$
V	$\begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.1 & 0.2 & -0.2 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$	—	$\begin{bmatrix} 0.5 & 0.1 & -0.2 \\ -0.1 & -0.2 & 0.2 \\ 0.1 & -0.2 & 0.3 \end{bmatrix}$	$\begin{bmatrix} 0.3 & -0.2 & 0.1 \\ 0.2 & -0.3 & 0.2 \\ 0.1 & 0.2 & -0.4 \end{bmatrix}$	$\begin{bmatrix} 1.0 & -0.2 & -0.3 \\ -0.2 & 1.0 & -0.1 \\ -0.3 & -0.1 & 1.0 \end{bmatrix}$

All configurations satisfy the stationarity conditions, verified by checking that the roots of all three determinantal polynomials lie strictly outside the unit circle. Configurations I and II generated from $\text{DSVAR}_2(1, 1, 1)_{5,25}$ exhibit moderate contemporaneous correlations, with off-diagonal elements of T^{-1} around -0.30 and -0.27 , respectively. Configuration III generated from $\text{DSVAR}_2(1, 1, 1)_{5,25}$ and Configuration IV generated from $\text{DSVAR}_2(2, 1, 1)_{6,30}$ show weak correlations, while Configuration V generated from $\text{DSVAR}_3(1, 1, 1)_{6,30}$ ranges from weak to moderate levels. To benchmark the proposed procedure, results are compared with the classical BIC [17]. For each DSVAR candidate with triple (p, P_1, P_2) and k^* parameters, the BIC is computed as:

$$\text{BIC}(p, P_1, P_2) = -2 \log \mathcal{L}_{\max} + k^* \log n^*, \quad (33)$$

where $\log \mathcal{L}_{\max} = -\frac{n^*k}{2} \log(2\pi) + \frac{n^*}{2} \log |\hat{T}| - \frac{n^*k}{2}$, with $\hat{T} = n^* S_e^{-1}(p, P_1, P_2)$, and $S_e(p, P_1, P_2)$ is the residual sum-of-squares matrix (12). The order triple is selected as the minimiser of the BIC over the same candidate grid used for MAP identification.

For each configuration, $R = 500$ independent time series are generated at five sample sizes $n \in \{200, 250, 300, 350, 400\}$. Identification is performed using Jeffreys' prior for (B, T)

combined with each of three prior mass functions ζ_{p,P_1,P_2} introduced in Section 3.1. The adoption of Jeffreys' prior in this simulation study provides an objective, theoretically rigorous benchmark that isolates the framework's inherent identification capacity. By exerting zero subjective pull on the AR coefficients, it guarantees that the high rates of correct identification are driven exclusively by the data. Furthermore, this prior inherently imposes an implicit penalty on model complexity through the $|X^\top X|^{-k/2}$ factor, which heavily restricts over-parameterised models without requiring heuristically derived penalty terms. Finally, because it is derived from the Fisher Information matrix, the resulting posterior mass function is entirely invariant to scale and reparameterisation, while its closed-form reduction to standard least-squares quantities ensures the computational tractability required for high-dimensional grid searches. Two maximum-order settings are examined: $(p^*, P_1^*, P_2^*) = (3, 3, 3)$ (64 candidates) and $(4, 4, 4)$ (125 candidates). Identification accuracy is measured by the percentage of correct identification (PCI), i.e., the proportion of replications in which the estimated orders $(\hat{p}, \hat{P}_1, \hat{P}_2)$ coincide with the true data-generating orders.

4.2. Results and Discussion

Tables 4 and 5 report PCI values across all configurations, three prior specifications, and five sample sizes, under maximum orders $(3, 3, 3)$ and $(4, 4, 4)$, respectively, together with the corresponding PCI results based on BIC. The results are rich and varied, and we frame our discussion around six central themes.

Table 4. Percentage of correct identification (%) for the DSVAR configurations, three prior mass functions, BIC, and five sample sizes. Maximum assumed orders: $(p^*, P_1^*, P_2^*) = (3, 3, 3)$.

	n	Config I	Config II	Config III	Config IV	Config V
Prior I	200	51.8	47.8	30.2	55.4	26.8
	250	73.6	68.2	54.6	72.2	59.2
	300	86.6	86.8	76.6	86.4	87.4
	350	96.2	95.2	86.2	96.6	95.4
	400	98.4	97.8	95.4	99.2	99.2
Prior II	200	52.4	47.2	28.4	52.8	21.2
	250	70.8	69.2	50.2	71.4	53.6
	300	87.0	83.2	69.6	88.2	85.4
	350	94.4	93.4	84.2	93.0	96.6
	400	98.6	97.8	96.4	97.6	99.6
Prior III	200	49.0	45.2	25.2	52.2	23.4
	250	72.2	69.6	52.2	73.0	59.4
	300	87.8	85.4	73.8	88.4	84.6
	350	96.0	94.0	91.4	95.4	95.2
	400	98.6	98.4	96.4	98.4	98.4
BIC	200	6.8	6.2	1.8	12.4	0.8
	250	24.0	23.2	4.6	25.8	1.6
	300	43.6	41.8	11.0	46.8	6.6
	350	66.6	61.4	28.8	62.8	28.2
	400	80.0	76.4	52.8	81.6	57.0

Table 5. Percentage of correct identification (%) for the DSVAR configurations, three prior mass functions, BIC, and five sample sizes. Maximum assumed orders: $(p^*, P_1^*, P_2^*) = (4, 4, 4)$.

	n	Config I	Config II	Config III	Config IV	Config V
Prior I	200	54.6	51.2	33.4	54.2	22.2
	250	73.2	69.0	52.0	70.2	59.0
	300	89.6	86.6	76.4	87.6	83.2
	350	96.4	93.6	84.2	95.0	94.2
	400	97.8	97.0	95.8	98.8	98.2
Prior II	200	48.8	44.0	27.4	49.4	20.8
	250	69.8	62.6	48.2	67.4	53.4
	300	87.8	83.4	71.6	86.0	83.0
	350	96.0	94.4	88.0	93.0	94.0
	400	97.8	96.2	95.4	97.6	98.8
Prior III	200	49.8	48.6	28.6	50.4	23.2
	250	77.2	69.6	53.6	75.2	60.0
	300	87.0	84.2	76.6	87.6	83.6
	350	96.0	94.2	89.6	94.0	95.4
	400	99.2	98.4	96.6	98.4	99.4
BIC	200	9.4	8.0	0.8	12.4	0.6
	250	26.0	25.2	1.8	21.6	1.0
	300	42.6	39.2	14.8	45.6	8.8
	350	65.0	60.0	29.2	67.4	29.8
	400	81.2	77.6	52.8	83.4	61.0

(a) Monotone improvement with sample size.

Across all combinations of configuration, prior, and maximum-order setting, the PCI increases consistently with n . At $n = 200$, correct identification rates range from 20.8% to 54.6%; by $n = 350$, they exceed 84% in every case; and by $n = 400$, all rates surpass 95%. This monotone pattern is the hallmark of a consistent Bayesian identification procedure in which the posterior concentrates progressively on the true order triple as the data accumulate [15,23,27].

(b) Prior ordering and robustness.

The three priors deliver an essentially identical ranking across all sample sizes and configurations. Although Prior 3 imposes the strongest penalisation on high-order models, followed by Prior 2 and Prior 1, these differences shrink rapidly as n increases. By $n = 400$, the discrepancies are negligible, indicating that the identification outcome is effectively *prior-invariant* in large samples.

(c) Effect of the assumed maximum order.

Under the $(4, 4, 4)$ and $(3, 3, 3)$ grids, the two tables deliver nearly identical PCI rates, demonstrating that the procedure is relatively insensitive to the choice of upper bounds, especially at moderate and large sample sizes.

(d) Cross-configuration robustness.

The four configurations vary in their off-diagonal coefficients and inter-series correlations. At any fixed n and prior, PCIs differ across configurations, reflecting the fact that identification accuracy depends on the configuration's proximity to the stationarity boundary within the stable region. For example, under Prior I and $n = 200$, Configuration III

achieves only 30.2% correct identification, compared with 51.8% for Configuration I and 47.8% for Configuration II. A similar pattern holds for Priors II and III, where Configuration III again yields the lowest PCIs at 28.4% and 25.2%, respectively. Even at moderate sample sizes such as $n = 250$, Configuration III remains substantially harder to identify (e.g., 54.6% under Prior I versus 73.6% for Configuration I). These systematic gaps across configurations confirm that identification accuracy deteriorates as the true parameters approach the stationarity boundary.

(e) Higher-dimensional scalability.

Results of the Configuration V under the (4, 4, 4) and (3, 3, 3) grids confirm that the Bayesian MAP procedure scales effectively to higher-dimensional systems. The PCI increases monotonically from about 21% at $n = 200$ to 98.0% at $n = 400$.

(f) Comparison with the standard Bayesian information criterion (BIC).

Under the (4, 4, 4) and (3, 3, 3) grids, the proposed Bayesian procedure outperforms the BIC at all sample sizes and configurations. The BIC performs worse when the true parameters lie close to the stationarity boundary, i.e., Configuration III, and in higher-dimensional systems, i.e., Configuration V. These comparisons indicate that the proposed Bayesian framework offers a practical identification advantage over the standard BIC, particularly in small-to-moderate samples.

5. Empirical Applications

To demonstrate the practical applicability of the proposed Bayesian DSVAR identification framework, we use the hourly electricity load data for two European countries, the Czech Republic and Germany, and the hourly solar radiation in Najran, Saudi Arabia.

5.1. Application to Hourly Electricity Loads in the Czech Republic and Germany

The proposed identification framework is illustrated using hourly electricity load data for the Czech Republic and Germany obtained from the ENTSO-E Transparency Platform (<https://www.entsoe.eu/>). The dataset covers nine consecutive weeks beginning Monday, 20 May 2024, providing $n = 1512$ hourly observations per series, as displayed in Figure 1, alongside the corresponding autocorrelation and cross-correlation functions.

The bivariate observation vector is:

$$\mathbf{y}(t) = (y_{\text{CZ}}(t), y_{\text{DE}}(t))^{\top}, \quad (34)$$

where $y_{\text{CZ}}(t)$ and $y_{\text{DE}}(t)$ denote Czech and German electricity loads in GW at hour t . Hourly electricity loads from the Czech Republic and Germany provide a natural bivariate setting for assessing double-seasonal behaviour. Both series display clear daily ($s_1 = 24$) and weekly ($s_2 = 168$) periodicities, consistent with well-documented patterns in short-term electricity demand [2,28]. In addition, the two systems are linked through the Central European high-voltage transmission network, creating contemporaneous interactions in their load profiles. The DSVAR specification exploits this interconnectedness by modelling cross-series dynamics at both seasonal frequencies, a feature that univariate approaches cannot capture, as reflected in the cross-correlation structure shown in Figure 1. Moreover, the large sample size ($n = 1512$) comfortably satisfies the validity conditions of both Theorem 1 and Corollary 1 for all candidate triples in the search grid. Table 6 presents the main descriptive statistics for both hourly load series.

To characterise the double-seasonal structure prior to DSVAR model identification, we examine the sample autocorrelation function (ACF) of each series and the sample cross-correlation function (CCF) between the two series, as displayed in Figure 1. Both ACFs exhibit pronounced, slowly decaying peaks at multiples of 24 (daily cycle) and a

secondary peak structure repeating every 168 lags (weekly cycle). This pattern is the empirical signature of double seasonality and motivates the choice ($s_1 = 24$; $s_2 = 168$) in the DSVAR specification. The augmented Dickey–Fuller test applied to the differenced series rejects the unit-root null at the 1% level for both series, confirming stationarity. The CCF in Figure 1 shows a strong contemporaneous correlation between the Czech and German loads together with cross-correlations, indicating that the two series share both seasonal cycles and justifying a multivariate treatment.

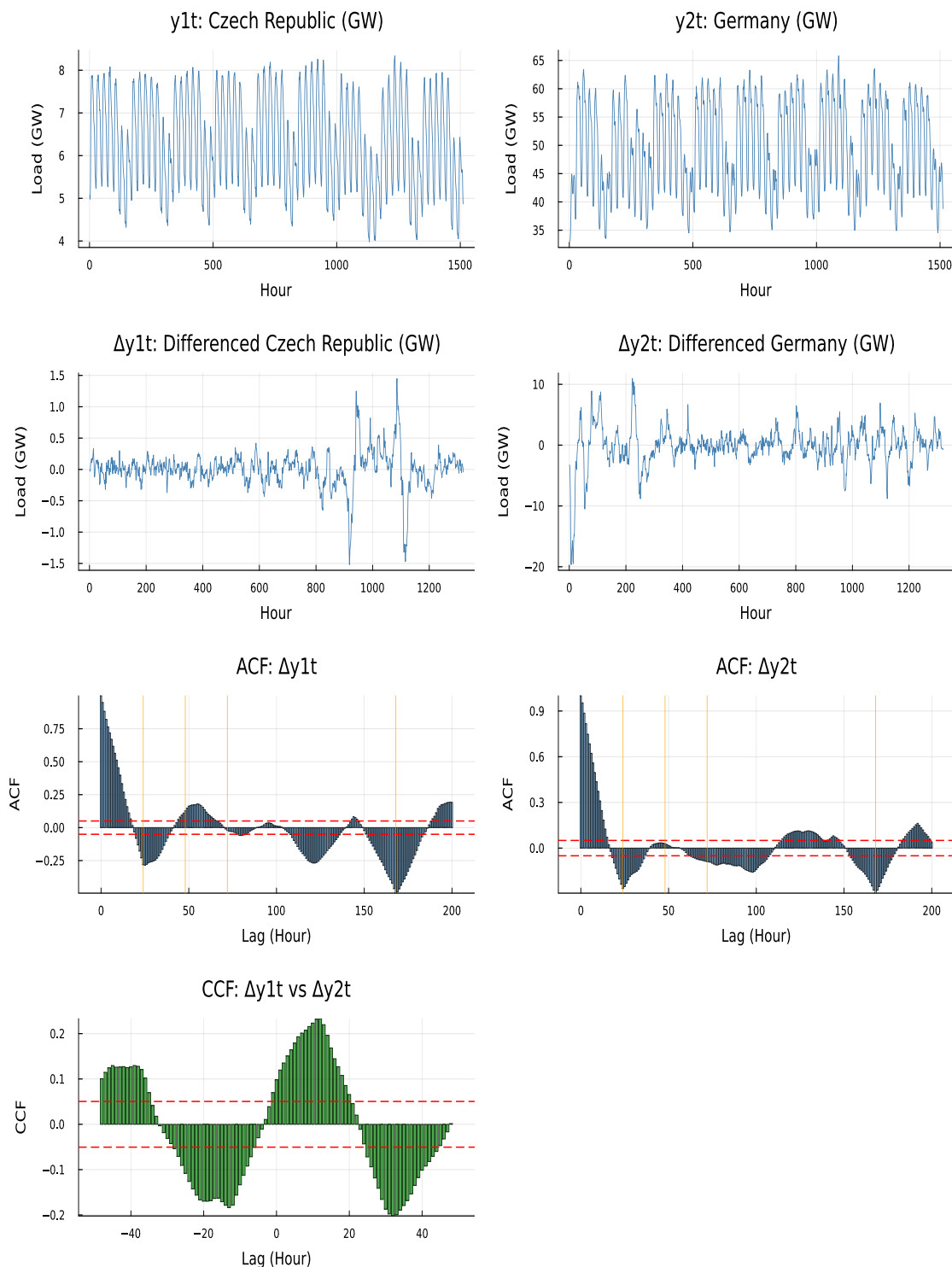


Figure 1. Hourly electricity load series for the Czech Republic and Germany, together with their autocorrelation (ACF) and cross-correlation functions (CCFs).

Table 6. Summary statistics of the Czech and German electricity loads.

Series	Mean	Median	Std	Min	Max
Czech Republic (GW)	6.30	6.29	1.08	3.98	8.34
Germany (GW)	49.51	48.70	7.91	32.94	65.84

The three prior mass functions are used in combination with Jeffreys' prior for the continuous parameters. The search grid is $(p^*, P_1^*, P_2^*) = (4, 4, 4)$, yielding $5^3 = 125$ candidate triples. Table 7 reports the identified orders under each prior, and Table 8 presents the ten highest-ranked candidates by log-posterior mass under Prior 1.

Table 7. Identified DSVAR orders for hourly Czech Republic and Germany electricity loads under three prior mass functions.

Prior	$\hat{p}$	$\hat{P}_1$	$\hat{P}_2$
Prior 1 (uniform)	1	2	2
Prior 2 (exponentially decreasing)	1	2	2
Prior 3 (linearly decreasing)	1	2	2

All three priors select $(\hat{p}, \hat{P}_1, \hat{P}_2) = (1, 2, 2)$, thereby identifying the $\text{DSVAR}_2(1, 2, 2)_{24,168}$ specification as the preferred model, and the BIC reaches the same conclusion on the same grid. This consensus is consistent with the simulation evidence that, in large samples, the posterior distribution concentrates sharply on the true orders irrespective of the prior employed. For comparison, the BIC selects the same $\text{DSVAR}_2(1, 2, 2)_{24,168}$ specification on the same grid.

The log-mass profile in Table 8 confirms the robustness of the selection. The next-best model, $(1, 1, 3)$, trails by $\Delta \log h = -13.657$, which implies a posterior odds ratio exceeding $e^{13.6} \approx 8 \times 10^5$ in favour of the MAP model, confirming near-certainty, i.e., posterior probability ≈ 1.0 , in the selection. The models with $P_1 = 3$ or $P_2 = 3$ each suffer penalisation of approximately 14–23 log units, indicating that two lags at each seasonal period are sufficient to characterise the joint Czech–German load dynamics. We computed the in-sample residuals and inspected their ACFs; the lack of remaining serial autocorrelation verifies that the model successfully accounts for the marginal dynamics of the series.

Table 8. Top-ten candidate triples ranked by log-posterior mass under Prior 1 and Jeffreys' prior for the Czech–Germany electricity load data. Differences ($\Delta \log h$) are measured relative to the MAP model $(1, 2, 2)$, set to zero. *Prob.* denotes the normalised posterior probabilities.

Rank	p	P_1	P_2	$\Delta \log h$	<i>Prob.</i>
1	1	2	2	0.000	0.999999
2	1	1	3	−13.657	1.17×10^{-6}
3	1	1	4	−15.160	2.61×10^{-7}
4	1	1	2	−17.564	2.36×10^{-8}
5	1	3	2	−19.907	2.26×10^{-9}
6	1	2	3	−22.756	1.31×10^{-10}
7	1	2	1	−30.515	5.59×10^{-14}
8	2	1	2	−33.271	3.55×10^{-15}
9	1	2	4	−33.672	2.38×10^{-15}
10	1	3	1	−35.578	3.54×10^{-16}

The identified $\text{DSVAR}_2(1, 2, 2)_{24,168}$ model is consistent with the physical structure of the two electricity systems: a single AR lag captures short-term momentum in consumption, the two 24 h seasonal lags encode the persistent daily activity cycle, and the two 168 h lags link each observation to the corresponding hour of the previous two weeks, capturing the weekday/weekend distinction. These seasonal drivers are well documented in the electricity load literature [2,29], and the identified orders are in agreement with the specifications employed in related univariate and Bayesian analyses [15].

5.2. Application to Hourly Solar Radiation in Najran, Saudi Arabia

The second application brings the proposed framework to bear on a meteorological dataset collected in Najran, a city in the arid southwestern interior of Saudi Arabia. The region receives among the highest solar irradiance levels in the Arabian Peninsula owing to its low latitude, predominantly cloud-free skies, and low atmospheric humidity, making it scientifically valuable for the study of high-frequency radiometric time series and for solar-energy resource assessment. Specifically, the southern part of Najran, located at about 17.5° N latitude, experiences persistently high direct-normal irradiance and occupies a geographically strategic position for advancing Saudi Arabia's Vision 2030 renewable energy objectives [30].

The dataset records three atmospheric quantities at hourly resolution over 80 consecutive daytime periods, yielding $n = 960$ hourly observations per variable. The three components that form the trivariate observation vector

$$\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t))^T \quad (35)$$

are described as follows.

- **Global Horizontal Irradiance (GHI)** $[y_1(t)]$: The total shortwave energy incident on a horizontal surface at ground level per unit area, integrating both the direct beam and the diffuse sky components, measured in Wh/m^2 by a pyranometer at the Najran meteorological station. GHI is the principal quantity governing the energy yield of a photovoltaic installation and therefore the standard target variable in solar-energy forecasting [31,32].
- **Clear-Sky GHI** $[y_2(t)]$: The GHI that would prevail under a hypothetically cloudless atmosphere, obtained from a radiative-transfer model parameterised by solar geometry (latitude, hour angle, Earth–Sun distance) and a standard atmospheric column (aerosol optical depth, column water vapour, ozone) [32,33]. Clear-sky GHI encodes the slowly evolving astronomical envelope of available solar energy; actual GHI fluctuates around this envelope in response to transient cloud cover and aerosol loading, so the two series are tightly coupled but not identical.
- **Ambient Temperature** $[y_3(t)]$: The near-surface air temperature in Kelvin (K), recorded at the same station. Temperature is partly driven by accumulated solar heating but also modulates local atmospheric dynamics, including boundary-layer turbulence and aerosol-particle suspension, thereby feeding back indirectly onto GHI through changes in atmospheric optical depth.

Figure 2 plots all three series together with their sample autocorrelation (ACF) and cross-correlation (CCF) functions, and Table 9 summarises the main descriptive statistics.

Before identifying any DSVAR model, we inspect the temporal dependence structure of the three series through their sample ACFs and pairwise CCFs (Figure 2). All three ACFs reveal a recurring peak pattern at lags that are multiples of 24, reflecting the dominant intraday solar cycle, superimposed on a slower-decaying envelope that produces a secondary concentration of autocorrelation mass near lag 168. This weekly recurrence in solar data

reflects the quasi-periodic variation in boundary-layer aerosol concentrations driven by weekly cycles in local traffic and industrial activity, as well as the tendency for synoptic weather systems to exhibit sub-weekly periodicities. These two nested periodicities jointly motivate setting $s_1 = 24$ and $s_2 = 168$ in the DSVAR operator. The stationarity of all three variables after appropriate differencing is confirmed by augmented Dickey–Fuller tests, which reject the unit-root null at the 1% significance level. The pairwise CCFs reveal strong instantaneous co-movement among GHI, Clear-Sky GHI, and temperature, as well as lagged cross-dependencies at the seasonal frequencies, providing direct empirical support for a joint multivariate rather than a series-by-series univariate treatment.

Table 9. Summary statistics of the hourly Najran solar radiation series.

Series	Mean	Median	Std	Min	Max
GHI (Wh/m ²)	511.3	554.0	316.6	6.0	1021.0
Clear-Sky GHI (Wh/m ²)	530.8	570.0	316.1	9.0	1034.0
Temperature (K)	307.5	310.0	6.4	291.9	316.3

The identification procedure is applied using all three prior mass functions ζ_{p,P_1,P_2} in combination with Jeffreys' prior for the continuous parameters (B, T) . A search grid of $(p^*, P_1^*, P_2^*) = (3, 3, 3)$, covering $4^3 = 64$ candidate triples, is adopted. Table 10 records the MAP order triple selected by each prior, and Table 11 ranks the ten best-scoring candidates by their log-posterior mass under Prior 1.

Table 10. Identified DSVAR orders for hourly Najran solar radiation series under three prior mass functions.

Prior	$\hat{p}$	$\hat{P}_1$	$\hat{P}_2$
Prior 1 (uniform)	1	1	1
Prior 2 (exponentially decreasing)	1	1	1
Prior 3 (linearly decreasing)	1	1	1

Table 11. Top-ten candidate triples ranked by log-posterior mass under Prior 1 and Jeffreys' prior for the hourly Najran solar radiation series. Differences ($\Delta \log h$) are measured relative to the MAP model $(1, 1, 1)$, set to zero. Prob. denotes the normalised posterior probabilities.

Rank	p	P_1	P_2	$\Delta \log h$	Prob.
1	1	1	1	0.000	0.999987
2	1	1	2	−11.2442	1.31×10^{-5}
3	2	1	1	−16.7616	5.25×10^{-8}
4	2	0	1	−20.5518	1.19×10^{-9}
5	1	2	1	−21.7215	3.69×10^{-10}
6	2	0	2	−23.0506	9.75×10^{-11}
7	2	0	3	−23.6668	5.27×10^{-11}
8	1	1	3	−30.8438	4.02×10^{-14}
9	1	0	3	−33.4849	2.87×10^{-15}
10	1	0	2	−37.4087	5.67×10^{-17}

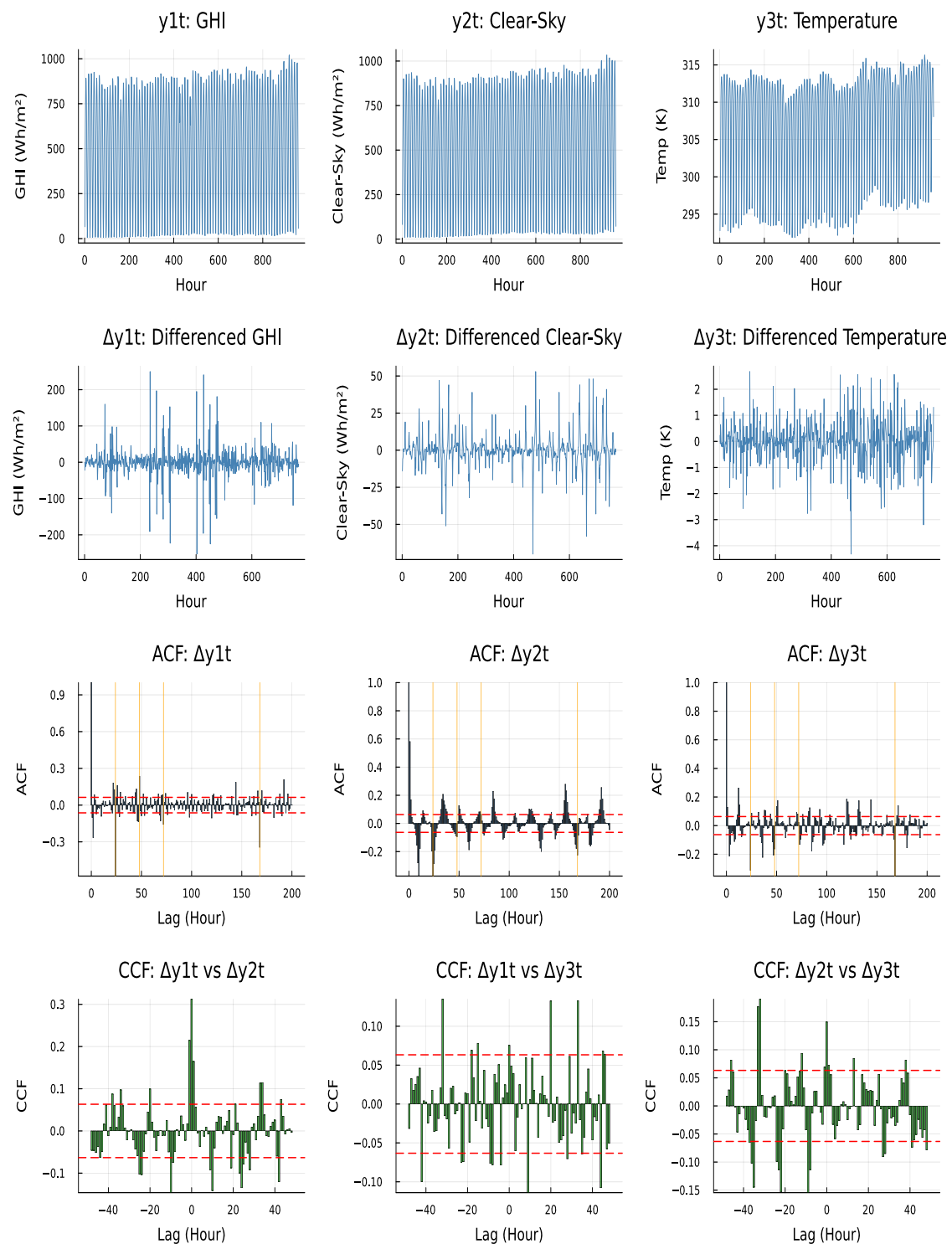


Figure 2. Hourly Najran solar radiation series, together with the autocorrelation (ACF) and cross-correlation (CCF) functions.

The MAP order $(\hat{p}, \hat{P}_1, \hat{P}_2) = (1, 1, 1)$, corresponding to the $\text{DSVAR}_3(1, 1, 1)_{24,168}$ specification, is unanimously selected by all three prior mass functions, with posterior probability close to 1.0. The BIC, evaluated on the same grid, identifies the same specification. This consensus across both Bayesian and frequentist criteria reinforces the robustness of the selection and is consistent with the simulation finding that, at sufficiently large sample sizes, the posterior mass concentrates decisively on the true order triple regardless of which prior is assigned to the discrete parameters.

The robustness of the selection is further quantified by the log-mass gap in Table 11. The runner-up triple $(1, 1, 2)$ lags the MAP by $\Delta \log h = -11.244$, a gap that corresponds to a Bayes factor exceeding $e^{11.2} \approx 7.4 \times 10^4$ in favour of $(1, 1, 1)$ [27], providing overwhelming posterior evidence against the second-seasonal lag at period $s_2 = 168$. Candidates featuring $p = 2$ or $P_1 \geq 2$ are penalised by at least 16.8 log units, and those with $P_2 \geq 2$ by at least 11.2 log units, demonstrating that a single lag at each of the regular, daily seasonal, and weekly seasonal frequencies is sufficient to account for the joint temporal dependence of the three radiometric variables. In-sample residuals are computed, and examination of their ACFs reveals no remaining serial autocorrelation, indicating that the model successfully captures the marginal dynamics of the series.

The identified $\text{DSVAR}_3(1, 1, 1)_{24,168}$ structure admits a clear physical interpretation. The regular AR component at lag 1 captures the inertial continuity of the solar flux and near-surface temperature from one hour to the next, a consequence of the gradual movement of the sun across the sky and the thermal mass of the ground surface. The single first-seasonal lag at $s_1 = 24$ h ties each observation to its counterpart at the same clock hour on the preceding day, encoding the deterministic astronomical recurrence of solar geometry that governs the amplitude envelope of daily irradiance. The single second-seasonal lag at $s_2 = 168$ h links each measurement to the value recorded exactly one week earlier, accounting for the systematic day-of-week pattern in local human activity and its downstream effect on aerosol loading and atmospheric turbidity. Together, these three lags provide a parsimonious yet physically grounded representation of the radiometric system at Najran.

6. Conclusions

This paper developed a principled Bayesian procedure for identifying the order triple of DSVAR models, a multivariate time series framework designed for processes simultaneously exhibiting two seasonal periodicities. These models were defined through a multiplicative triple autoregressive operator, whose expansion into seven structured regressor groups was used to cast the model as a multivariate linear regression with all matrices written as explicit functions of the unknown orders. The central theoretical contributions are two closed-form posterior mass functions. We derived the joint posterior mass of DSVAR orders under a conjugate matrix normal-Wishart prior, expressing it as a product of determinantal ratios and gamma function values that can be evaluated by a three-dimensional grid search. We provided the parallel result under Jeffreys' non-informative prior. Monte Carlo experiments across various parameter configurations, three prior mass functions, and two maximum-order settings confirmed that the proposed methodology achieves high correct identification rates. The performance proved robust to the choice of prior and to the specified upper bound on the order grid. Comparisons with the BIC showed that the Bayesian MAP procedure attains higher correct identification rates throughout, with the largest gains at small-to-moderate sample sizes. Applications to hourly electricity loads in the Czech Republic and Germany and to hourly Najran solar radiation data demonstrated the practical utility of the proposed DSVAR identification procedure. The free-parameter approximation underlying the regression representation was shown, both analytically and empirically, to impose a conservative penalty against over-identification rather than to bias the procedure toward spurious higher-order models. Future work may explore time-varying coefficient formulations that allow seasonal patterns to evolve over time, an extension of practical relevance in energy and financial applications where structural shifts are common, and extensions to higher-dimensional systems.

Author Contributions: Conceptualisation, A.A.A.; Methodology, A.A.A.; Software, A.A.A.; Validation, A.A.A. and F.E.A.; Writing—original draft, A.A.A.; Writing—review & editing, F.E.A.; Project administration, F.E.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU263711].

Data Availability Statement: The original data presented in the study are openly available at <https://www.entsoe.eu/>.

Acknowledgments: The authors are thankful to the Deanship of Graduate Studies and Scientific Research at Najran University for funding this work under the Consortium Funding Program grant code (NU/CPL/SERC/14/4440-3).

Conflicts of Interest: The authors declare no conflicts of interest.

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