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Moving-Boundary Fluctuation Analysis: Premium Drift, Ladder Structure, and Ruin in Phase-Type Cumulative Shock Models

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Abstract

We extend the phase-tagged fluctuation framework for cumulative shock models from a fixed failure threshold to a linearly moving boundary $u_0 + c\tau_n$, the premium drift regime that underlies insurance ruin theory. The moving boundary turns the first-passage problem from a fixed-level crossing into a crossing of a drifting walk, for which the partial sum truncation of the fixed-threshold theory no longer applies. We resolve this with a drift-aware ladder apparatus: a rank-one *drift ladder matrix* $G^\uparrow(c) = g(c)\mathbf{e}\mathbf{a}$ whose scalar $g(c)$ solves a discrete Lundberg equation $a(z) = z^c$, a closed-form ascending ladder-height law obtained from the roots of the Lundberg polynomial, and a compound-geometric (Pollaczek–Khinchine) representation of the ruin probability. Building on these, we derive in closed form the full five-variable moving-boundary reliability functional $\Phi_v^{\text{ruin}}(\xi, u, v, \theta, \theta)$, the drift analogue of the fixed-threshold phase-tagged functional, and show it lies in the same two-dimensional matrix subspace $\text{span}\{H(\theta), H(\omega)\}$: the premium drift deforms the scalar coefficients through a first-passage transform while leaving the matrix structure invariant. As the principal application, we obtain the phase-resolved Gerber–Shiu expected-discounted-penalty function, with the joint transform of the time of ruin, the deficit at ruin, and the surplus prior to ruin. The classical scalar Gerber–Shiu function and the fixed-threshold functional are recovered as projections and as the $c \rightarrow 0$ limit, respectively. All closed forms reduce to root-finding on a single polynomial for rational model primitives, and every structural result is verified against Monte Carlo simulation and exact recursion.

Keywords: ruin theory; Gerber–Shiu function; premium drift; moving boundary; phase-type distribution; ladder heights; Lundberg equation; fluctuation analysis; matrix-analytic methods; phase tagging

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1. Introduction

Reliability and insurance share a common mathematical core: a system accumulates random increments until it first crosses a critical level. In reliability, damage from intermittent shocks accumulates until it exceeds a sustainability threshold and the system fails. In insurance, claims accumulate against a reserve that is continuously replenished by premium income, and the insurer is ruined when net claims first exceed the reserve. The two problems differ in one structural respect that turns out to be decisive: the reliability threshold is *fixed*, whereas the insurance boundary *moves*, growing linearly with time as premium accrues.

1.1. From Fixed to Moving Thresholds

A phase-tagged fluctuation theory for the fixed-threshold problem was developed in [1] and its pre-failure extension [2]. That framework produces, in closed form, the joint transform of the failure index, the pre-failure and failure damage, the pre-failure and failure times, and the operational phase at failure, for cumulative shock models with phase-type inter-shock times. Its analytic engine is the Dshalalow $\mathfrak{D}$ -operator, which encodes the fixed-threshold level indicator $\mathbf{1}_{\{S_n \leq p_1\}}$ as a partial sum of generating function coefficients, together with a Sherman–Morrison reduction that resums the resulting matrix-geometric series and confines the functional to a low-dimensional matrix subspace generated by the identity and two matrix Laplace–Stieltjes transforms.

The present paper asks what survives, and what must be rebuilt, when the fixed threshold p_1 is replaced by the linearly moving boundary $u_0 + c n$ of the premium drift (insurance) regime. The answer is sharply two-sided. On one hand, the partial sum truncation at the heart of the fixed-threshold theory *fails outright*: the moving boundary is tested against a different level at every step, so no single $\mathfrak{D}$ -operator with a constant truncation order can encode it, and the telescoping identity that drives the fixed-threshold resummation breaks because premium income lets the surplus recover between claims, destroying the monotonicity of the crossing. On the other hand, the *matrix* structure proves remarkably robust: the rank-one phase-reset mechanism that gives the fixed-threshold functional its low-dimensional span survives the drift unchanged, and the functional remains confined to the same two-dimensional matrix subspace. The premium drift deforms the scalar coefficients but leaves the matrix geometry invariant.

1.2. Contributions

We replace the broken partial sum machinery with a drift-aware ladder apparatus and use it to build the moving-boundary theory in closed form. The specific contributions are as follows.

- (i) *The drift ladder matrix* (Section 3.2). We introduce a rank-one first-passage matrix $\mathbf{G}^\dagger(c) = g(c) \mathbf{e} \boldsymbol{\alpha}$ for the moving boundary, where $\mathbf{e}$ is the column vector of ones, $\boldsymbol{\alpha}$ is the phase-type initial-distribution row vector (so that $\mathbf{e} \boldsymbol{\alpha}$ is the rank-one phase-reset matrix), and the scalar $g(c)$ is the reciprocal of the dominant root of a discrete Lundberg equation $a(z) = z^c$. The rank-one structure is inherited from the phase-reset hypothesis, exactly as in the fixed-threshold matrices of [1,2], and the matrix generator is invariant under both premium drift and time discounting.
- (ii) *The ladder-height law in closed form* (Section 4). We show the ascending ladder-height distribution is finitely supported and given explicitly by the roots of the Lundberg polynomial $z^c - a(z)$ lying outside the unit disk, yielding a rational generating function for the maximum of the net-claim walk.
- (iii) *The Gerber–Shiu function* (Section 5). We obtain the ruin probability as a compound-geometric (Pollaczek–Khinchine) tail with an exact Lundberg decay rate, and the phase-resolved Gerber–Shiu expected-discounted-penalty function as the joint transform of the time of ruin, the deficit at ruin, and the surplus prior to ruin.
- (iv) *The moving-boundary central formula* (Section 7). We derive in closed form the full five-variable functional Φ_v^{ruin} , the drift analogue of the reliability functional, and prove it lies in the matrix subspace $\text{span}\{H(\theta), H(\omega)\}$. The fixed-threshold Sherman–Morrison resummation is replaced by a first-passage transform Ω_r governed by the ladder structure; the two coincide in the $c \rightarrow 0$ limit.
- (v) *Consistency and computation*. The classical scalar Gerber–Shiu function and the fixed-threshold functional of [2] are recovered as projections and as the $c \rightarrow 0$ limit. For rational model primitives, the entire apparatus reduces to root-finding

on a single polynomial, with no matrix iteration. Every structural result is verified against Monte Carlo simulation and exact recursion.

1.3. Relation to Classical Ruin Theory

The ladder-height and compound-geometric structure we use is classical [3,4]; what is new is the phase-resolved, joint closed form obtained by combining it with the phase-tagging of the matrix-analytic tradition [5,6] and the fluctuation-theoretic functional of the Dshalalow lineage [7,8]. We deliberately route the analysis through ladder heights rather than the storage/queue duality [3], because duality recovers only the scalar ruin probability as a stationary tail and discards the phase-resolved joint first-passage law that is the object of interest here.

Where classical ruin theory typically delivers the scalar ruin probability and the marginal Gerber–Shiu function, the present framework delivers the full joint transform with the operational phase at ruin, in closed form, for phase-type inter-claim times. The contemporary Gerber–Shiu literature spans several complementary directions and a brief positioning is warranted. The structural-properties programme of [9] systematically organizes which ruin-related quantities can be unified inside a generalized Gerber–Shiu penalty in (dependent) Sparre–Andersen models, without committing to a specific phase-type structure; our results are complementary in that they fix the inter-shock distribution as phase-type (so the rank-one phase-reset structure is available) and exploit it to obtain explicit closed forms for the resulting penalty functions. The matrix-analytic ruin literature surveyed in [10] approaches phase-type ruin via the fluid-queue embedding of the surplus process, delivering scale-matrix-based closed forms for the marginal Gerber–Shiu function and dividend-barrier quantities; our framework, by contrast, operates directly on the discrete-time random walk of the cumulative net-claim process and obtains the joint phase-resolved law of the five-variable ruin observable without an embedding step. Most directly parallel to our setting, ref. [11] develop a Gerber–Shiu theory for *discrete-time* Markov-modulated risk processes using discrete scale matrices $\mathcal{W}_v$ and $\mathcal{Z}_v$ that play the role of fluctuation-theoretic potentials. Their setting is upward (or downward) skip-free with general Markovian regime-switching; ours is non-skip-free (claim sizes integer-valued with arbitrary support) but with a phase-reset structure that collapses the ladder analysis to a one-dimensional Lundberg root extraction. The two frameworks address different limiting regimes within the same broad discrete-time Markov-additive umbrella, and the rank-one phase-reset assumption is what makes the closed forms of the present paper possible: under a fully Markov-modulated structure, as in [11], the scale-matrix machinery is the appropriate tool, whereas under the phase-reset hypothesis, the rank-one factorization yields the Pollaczek–Khinchine matrix-geometric in scalar form. Recent work has also approached the Gerber–Shiu function from a complementary, statistical direction, estimating it from observed surplus data via series expansions [12]; our contribution is analytic, providing closed-form phase-resolved expressions rather than estimators.

2. Model and Preliminaries

This section fixes notation and recalls the phase-type and operator machinery on which the moving-boundary analysis is built. The setup follows [1,2]; we recall only what is needed and refer there for fuller discussion. For the convenience of readers without access to [1], a self-contained statement of the specific results imported from that paper into the present analysis is collected in Appendix A.

2.1. The Cumulative Shock Model with Premium Drift

Shocks arrive at epochs $\tau_0 < \tau_1 < \tau_2 < \dots$ with inter-shock times $T_n = \tau_n - \tau_{n-1}$. The delay $T_0 \sim \text{PH}(\alpha_0, S_0)$ has phase-type distribution with initial vector α_0 and sub-generator S_0 on a phase space of size m_0 ; the post-delay inter-shock times $T_1, T_2, \dots$ are i.i.d. $\text{PH}(\alpha, S)$ with initial vector α and sub-generator S on $\{1, \dots, m\}$, with absorption-rate vectors $\mathbf{s}_0 = -S_0 \mathbf{e}$ and $\mathbf{s} = -S \mathbf{e}$, where $\mathbf{e}$ is the all-ones column. The phase process $\{J(t)\}_{t \geq \tau_0}$ governs the post-delay inter-shock dynamics, with $J(\tau_n) \sim \alpha$ for each $n \geq 0$.

The n -th shock deposits non-negative integer damage X_n . The delay claim X_0 has probability generating function (PGF) $a_0(u) = \mathbb{E}[u^{X_0}]$; the post-delay claims $X_1, X_2, \dots$ are i.i.d. with PGF $a(u) = \mathbb{E}[u^{X_1}] = \sum_{k \geq 0} p_k u^k$, $p_k = \mathbb{P}(X_1 = k)$. The damage sequence is independent of the inter-shock times and of the phase process. The cumulative damage is $S_n^X = X_0 + X_1 + \dots + X_n$, with $S_{-1}^X := 0$.

In the reliability reading, the system fails when S_n^X first exceeds a fixed threshold [1,2]. In the present *premium drift* reading, an insurer holds initial reserve u_0 and collects premium at integer rate c per period; the surplus after n claims is $u_0 - Z_n$ with net-claim walk $Z_n = S_n^X - cn$, and ruin is the first net-claim crossing of u_0 , that is, the first crossing of the linearly moving boundary $u_0 + cn$ by the cumulative claims. The ruin index ν and standing hypotheses are stated in Section 3.1.

2.2. Phase-Type Transforms

Definition 1 (Laplace–Stieltjes transforms). The scalar LST of $T_n \sim \text{PH}(\alpha, S)$ ($n \geq 1$) is $h(\theta) = \mathbb{E}[e^{-\theta T_1}] = \alpha(\theta I - S)^{-1} \mathbf{s}$ for $\text{Re}(\theta) \geq 0$, and the matrix LST is

$$H(\theta) = (\theta I - S)^{-1} \mathbf{s} \alpha = \mathbf{v}(\theta) \alpha, \quad \mathbf{v}(\theta) := (\theta I - S)^{-1} \mathbf{s}. \quad (1)$$

The delay transforms $h_0(\theta)$ and $H_0(\theta)$ are defined analogously from (α_0, S_0) . The matrix LST encodes the joint law of one inter-shock duration and the phase at the next epoch: $[H(\theta)]_{ij} = \mathbb{E}[e^{-\theta T_1} \mathbf{1}_{\{J(\tau_1)=j\}} \mid J(\tau_0^-) = i]$.

Proposition 1 (Properties of the matrix LST). For $\text{Re}(\theta) \geq 0$: (i) $h(\theta) = \alpha H(\theta) \mathbf{e}$; (ii) $H(\theta) = \mathbf{v}(\theta) \alpha$ has rank-one; (iii) idempotent identity $H(\theta)^2 = h(\theta) H(\theta)$, hence $H(\theta)^k = h(\theta)^{k-1} H(\theta)$ for $k \geq 1$; (iv) $H(0) = \mathbf{e} \alpha$; (v) Sherman–Morrison [13]: for $|c| |h(\theta)| < 1$, $(I - cH(\theta))^{-1} = I + \frac{c}{1 - c h(\theta)} H(\theta)$.

The proof is standard ([2], Proposition 3.4); properties (iii)–(v) are the engine of the closed forms throughout.

2.3. The Threshold-Truncation Operator

Definition 2 (The $\mathfrak{D}$ -operator). For a formal power series $f(x) = \sum_{j \geq 0} c_j x^j$ and an integer $k \geq 0$, the Dshalalow operator returns the partial sum of Taylor coefficients,

$$\mathfrak{D}_x^k \{f\} = \sum_{j=0}^k c_j, \quad (2)$$

so that $\mathbf{1}_{\{S_n^X \leq k\}} = \mathfrak{D}_x^k \{x^{S_n^X}\}$ encodes a fixed-threshold level indicator. For rational f , the operator is computed by an $O(k)$ coefficient recurrence.

The $\mathfrak{D}$ -operator is the device that linearizes the *fixed*-threshold first-excess problem [1,7,8]. A central theme of this paper is that the *moving* boundary $u_0 + cn$ cannot be encoded by a single $\mathfrak{D}_x^k$ with constant k ; the threshold grows with the step index, and the partial sum truncation must be replaced by the ladder first-passage apparatus developed in Sections 3.2

and 4. The $\mathfrak{D}$ -operator nonetheless persists in the delay claim initialization and in the fixed-threshold ($c \rightarrow 0$) limit.

2.4. The Fixed-Threshold Functionals

For reference, we recall the two fixed-threshold objects the present work generalizes. The phase-tagged first-excess functional of [1] is $\Psi(\zeta, \theta, u)_{ij} = \mathbb{E}[\zeta^v e^{-\theta \tau_v} u^{S_v} \mathbf{1}_{\{J(\tau_v)=j\}} \mid J(\tau_0)=i]$, tracking failure time observables and the phase at failure. Its pre-failure-enriched extension, the phase-tagged reliability functional of [2], is

$$\Phi_v(\zeta, u, v, \vartheta, \theta)_{ij} = \mathbb{E}[\zeta^v u^{S_{v-1}} v^{S_v} e^{-\vartheta \tau_{v-1} - \theta \tau_v} \mathbf{1}_{\{J(\tau_v)=j\}} \mid J(\tau_0)=i], \quad (3)$$

derived in closed form via the $\mathfrak{D}$ -operator and Sherman–Morrison reduction, lying in $\text{span}\{I, H(\theta), H(\omega)\}$ with $\omega = \vartheta + \theta$, and recovering the scalar Dshalalow–White functional under the projection $\alpha \Phi_v \mathbf{e}$ (the Dshalalow–White recovery theorem of [2]) and Ψ when pre-failure tracking is suppressed. The closed-form expression for (3), the catalogue of derived reliability indices, and the trajectory decomposition lemmas referenced below are all established in [2]; the present paper establishes their moving-boundary analogues. Throughout, references to the central formula, the recovery theorem, and the reliability indices of [2] point to those results.

3. The Drift Ladder Matrix

3.1. The Premium Drift Model and the Moving Boundary

We retain the phase-type inter-shock machinery of Section 2 and reinterpret the cumulative-damage process as the *net-claim* process of an insurance surplus. Claims arrive at the shock epochs $\tau_0 < \tau_1 < \tau_2 < \dots$, the n -th claim being the non-negative integer X_n with common probability generating function (PGF) $a(u) = \mathbb{E}[u^{X_1}] = \sum_{k \geq 0} p_k u^k$, $p_k = \mathbb{P}(X_1 = k)$. Premium income accrues at the constant integer rate $c \in \mathbb{N}$ per inter-shock period. The *net-claim walk* is

$$Z_n = S_n^X - c n, \quad S_n^X = \sum_{k=0}^n X_k, \quad Z_0 = X_0, \quad (4)$$

and, for an initial surplus $u_0 \in \mathbb{Z}_+$, *ruin* is the first epoch at which the walk crosses the level u_0 :

$$\nu = \inf\{n \geq 1 : Z_n > u_0\} = \inf\{n \geq 1 : S_n^X > u_0 + c n\}. \quad (5)$$

The right-hand form of (5) exhibits the structural departure from [1,2] and the fixed-threshold functional of the central formula of [2]: the cumulative claims are tested not against a fixed integer p_1 but against the *linearly moving boundary* $u_0 + cn$. We refer to c as the *premium drift*. The classical fixed-threshold theory is recovered at $c = 0$.

Remark 1 (Premium accrual per claim cycle versus per unit time). *The premium term cn in (5) accrues c units per claim cycle, not per unit of calendar time. Since the inter-claim times T_n are random and phase-type, premium income in calendar time is therefore indexed to claim occurrences rather than to elapsed clock time. A long claim-free interval accrues no additional premium under this convention. This departs from the standard continuous-time Cramér–Lundberg model [3], in which premium accrues continuously at a fixed rate per unit time. The per-cycle convention is the natural discrete analogue of the cycle-driven shock-and-payment models of the fluctuation-theoretic tradition [1,2,8], and it is the formulation under which the matrix-analytic and Lundberg-polynomial closed forms of the present paper become available. Operationally, it models arrangements in which premium accrual is tied to claim-bearing events (e.g., unit-priced policies billed per filed claim, or event-driven reinsurance treaties); the calendar time analogue is recovered,*

on average, by setting $c \mathbb{E}[T_1]$ equal to the calendar time premium rate. Section 9 comments on the continuous-time extension.

Remark 2 (The premium drift c enters nonlinearly). The downward increment of Z_n at each step is $X_n - c$, suggesting at first reading that the premium c merely shifts the claim distribution to $X - c$. This intuition is misleading. Ruin depends not on a single increment but on the running maximum of the cumulative net-claim walk (4), so the premium c enters every ruin observable nonlinearly, through the Lundberg equation $a(z) = z^c$ and its dominant root $z^*(c)$. The ladder scalar $g(c) = 1/z^*(c)$ and the geometric decay rate $R(c) = \log z^*(c)$ that govern the closed-form ruin probability are therefore non-trivial functions of c , and no closed-form expression for them in terms of the increment law $X - c$ alone exists. This nonlinearity is the conceptual content of the moving-boundary problem: subtracting a constant drift from a random walk and analyzing the running maximum are not the same operation, and the moving-boundary problem does not reduce to a fixed-threshold problem with a shifted claim law.

Remark 3 (Integer premium versus continuous premium). The standing assumption $c \in \mathbb{N}$ (cf. Assumption 1(D1) below) keeps the boundary $u_0 + cn$ on the integer lattice. This is not merely a numerical convenience: integrality renders the Lundberg equation $a(z) = z^c$ algebraic, so its roots can be computed by polynomial root-finding, and renders the ladder-height law finitely supported (Proposition 2). A continuous premium rate $c \in \mathbb{R}_+$ preserves the rank-one matrix structure of the ladder generator (Theorem 1), but replaces the polynomial Lundberg equation with a transcendental one and the finite ladder-height law with a continuous overshoot distribution; the closed forms then change character rather than continuing to hold verbatim. Section 9 discusses this extension.

Remark 4 (Strict crossing and the ladder convention). Ruin in (5) is the strict crossing $Z_n > u_0$, equivalently the surplus $u_0 - Z_n$ becoming strictly negative; the boundary-touching case $Z_n = u_0$ is survival. On the integer lattice, this is the strict ascending ladder convention, and it matches the strict failure definition $S_n^X > p_1$ of [1] (where $p_1 = M - 1$ converts $S_n^X \geq M$ into a strict inequality). The distinction between strict and weak ladders is immaterial for continuous-state walks but genuine on a lattice; we adopt the strict convention throughout, so that every first-passage recursion in the sequel assigns the event $\{X - c = r\}$ to continuation rather than to crossing, and the ascending ladder-height law of Proposition 2 is the strict-ladder law. The weak-ladder variant $Z_n \geq u_0$ is obtained by the boundary shift $u_0 \mapsto u_0 - 1$ and yields the same closed forms with that shift.

Assumption 1 (Standing hypotheses for the drift model). Throughout this section:

- (D1) The premium drift c is a fixed positive integer, so that the downward increments of Z_n are bounded below by $-c$;
- (D2) The net-profit condition $\mathbb{E}[X_1] < c$ holds, so that $Z_n \rightarrow -\infty$ almost surely and ruin is not certain;
- (D3) The phase process resets at each claim epoch, i.e., the inter-shock time $T_n \sim \text{PH}(\alpha, S)$ ($n \geq 1$) is initialized from α independently of the past, as in Section 2.2.

Remark 5 (Why the boundary must move on the lattice). Hypothesis (D1) keeps the moving boundary $u_0 + cn$ on the integer lattice and bounds the one-period downward jump of Z_n by c . Both are essential to the closed form below: integrality lets the threshold-truncation operator $\mathfrak{D}_x^k$ of Definition 2 continue to act, and the bounded downward jump truncates the ladder recursion of Theorem 1 at finitely many (c) terms. A continuous premium rate replaces the polynomial Lundberg equation of Theorem 1 by a transcendental one; the rank-one matrix structure survives unchanged, but the scalar fixed point is no longer algebraic. We treat the lattice case here and defer the continuous case to Section 9.

3.2. Definition and Closed Form

Ruin in (5) is a crossing of the moving boundary by Z_n , equivalently a crossing of the fixed level u_0 by the running maximum $\bar{Z}_n = \max_{0 \leq k \leq n} Z_k$. Because premium income lets the surplus recover between claims, the crossing is *non-monotone* and the telescoping indicator identity of the fixed-threshold theory [1] no longer applies. We therefore organize the analysis around the *ascending ladder epochs* of Z_n , the indices at which the walk attains a strict new maximum, and the phase carried at those epochs.

Definition 3 (Drift ladder matrix). *The drift ladder matrix $\mathbf{G}^\uparrow(c)$ is the $m \times m$ matrix that governs the matrix-geometric structure of the running maximum $M := \sup_{n \geq 0} Z_n$ of the net-claim walk Z_n . Concretely, it is the rank-one matrix whose iterates generate the Pollaczek–Khinchine matrix-geometric representation of the maximum’s law: there exists a non-negative sequence $\{v_n(u_0)\}_{n \geq 1}$ depending only on the ascending ladder-height law (derived explicitly in Proposition 2) such that*

$$\mathbb{P}(M > u_0 \mid J(\tau_0) = i) = \sum_{n \geq 1} [\mathbf{G}^\uparrow(c)^n]_i \cdot \mathbf{e} v_n(u_0). \quad (6)$$

Equivalently, $\mathbf{G}^\uparrow(c)$ is characterized by: (a) rank-one structure with left factor $\mathbf{e}\alpha$, inherited from the phase-reset hypothesis (D3); and (b) non-zero eigenvalue $g(c)$ equal to the per-level geometric decay rate of $\mathbb{P}(M > u_0 \mid J(\tau_0) = i)$ as $u_0 \rightarrow \infty$. Its time-discounted version, deforming the spectral scalar $g(c)$ to $g(c, \theta)$ through the inter-shock LST, is denoted $\mathbf{G}^\uparrow(c, \theta)$.

Remark 6 (On the spectral characterization). *The characterization of $\mathbf{G}^\uparrow(c)$ via (6) fixes $\mathbf{G}^\uparrow(c)$ as a spectral object: its non-zero eigenvalue is the geometric decay rate $g(c)$ governing the tail of the maximum, and its left-eigenvector structure is the phase-reset matrix $\mathbf{e}\alpha$ of Proposition 1(iv). This characterization is not the literal defective transition matrix of one strict ascending ladder step (whose row sum is the distinct quantity $p_L = \mathbb{P}(M > 0) \neq g(c)$ in general; see Remark 8), but rather the per-level rate at which the maximum decays. The two scalars coincide in the skip-free case $J = 1$ and differ for $J \geq 2$; the precise relationship between them is given by the polynomial factorization of Proposition 2. The Pollaczek–Khinchine formula (6) characterizes $\mathbf{G}^\uparrow(c)$ uniquely as the operator whose iterates generate the maximum’s distribution; the closed-form Theorem 1 identifies it explicitly.*

The matrix $\mathbf{G}^\uparrow(c)$ plays for the moving-boundary problem the role that Neuts’s first-passage matrix $\mathbf{G}$ plays for the fixed-level QBD problem [5,6], and that the boundary evaluation of the phase-tagged functional plays in ([1], Theorem 5.1). The next theorem gives it in closed form and shows it inherits the rank-one phase-reset structure that runs through the entire framework.

Theorem 1 (Closed form of the drift ladder matrix). *Under Assumption 1, the drift ladder matrix is rank-one,*

$$\mathbf{G}^\uparrow(c) = g(c) \mathbf{e}\alpha, \quad (7)$$

where the scalar ladder height $g(c) \in (0, 1]$ is the smallest root in $(0, 1]$ given by $g(c) = 1/z^$, where $z^* > 1$ is the dominant root of the discrete Lundberg (adjustment) equation*

$$a(z) = z^c, \quad \text{equivalently} \quad \mathbb{E}[z^{X_1 - c}] = 1. \quad (8)$$

The associated adjustment coefficient is $R = \log z^ = -\log g(c)$. Under the net-profit condition (D2), $\mathbb{E}[X_1] < c$, the moment-generating function $a(z)$ crosses z^c from below at a unique $z^* > 1$, so $g(c) = 1/z^* \in (0, 1)$; when $\mathbb{E}[X_1] \geq c$ the only root in $[1, \infty)$ is $z^* = 1$, giving $g(c) = 1$ and certain ruin.*

Remark 7 (On the form of the Lundberg equation). Equation (8) is the discrete-time, integer-premium analogue of the classical Cramér–Lundberg adjustment equation [3,14]: the net per-period increment is $X_1 - c$ with negative mean under (D2), and z^* is the point at which this increment has unit exponential moment. The scalar $g(c) = 1/z^*$ is the per-level ladder attenuation; the geometric decay $g(c)^{u_0}$ of the ruin probability (Corollary 4) is the exact Lundberg estimate, not merely a bound. When the claim PGF a is rational, (8) is a polynomial equation in z of degree $\max(\deg a, c)$, solvable by standard root-finding; no matrix iteration is required.

Proof. We organize the argument in three parts: (i) the phase-reset structure forces $G^\uparrow(c)$ to be rank-one with the left factor $\mathbf{e}\alpha$; (ii) the exponential martingale $M_n^{(z)} = z^{Z_n}$ identifies the Lundberg equation $a(z) = z^c$ as the characteristic equation for the geometric decay of the maximum; and (iii) the Pollaczek–Khinchine matrix-geometric structure (6) together with the polynomial factorization of Proposition 2 identifies $g(c) = 1/z^*$ as the unique non-zero eigenvalue of $G^\uparrow(c)$.

(i) *Rank-one structure.* By hypothesis (D3), the phase resets to α at every claim epoch, including every ascending ladder epoch, and this reset is independent of the trajectory leading to that epoch. The operator $G^\uparrow(c)$ acts on the row of phase masses carried at one ascending epoch and returns the row carried at the next: each application projects onto α in the codomain. The image of every input is therefore proportional to α , so $G^\uparrow(c)$ has rank at most one and the unique non-trivial eigenvector structure $G^\uparrow(c) = g(c) \mathbf{e}\alpha$ for some scalar $g(c)$.

(ii) *The exponential martingale and the Lundberg equation.* The per-step increments of the net-claim walk are $\Delta_n = Z_n - Z_{n-1} = X_n - c$, an i.i.d. integer-valued sequence with $\mathbb{E}[\Delta_n] = \mathbb{E}[X_1] - c < 0$ under (D2). For $z \in (0, \infty)$, consider $M_n^{(z)} := z^{Z_n}$; then

$$\mathbb{E}[M_n^{(z)} \mid M_0^{(z)}, \dots, M_{n-1}^{(z)}] = M_{n-1}^{(z)} \cdot \mathbb{E}[z^{\Delta_n}] = M_{n-1}^{(z)} \cdot z^{-c} a(z),$$

so $\{M_n^{(z)}\}$ is a non-negative martingale if and only if $z^{-c} a(z) = 1$, that is

$$a(z) = z^c, \quad (9)$$

which is (8).

Equation (9) admits a unique root $z^* > 1$ under (D2). To see this, set $\varphi(z) := a(z) - z^c$ for $z > 0$: $\varphi(1) = 0$, $\varphi'(1) = \mathbb{E}[X_1] - c < 0$, so φ is negative on $(1, 1 + \delta)$; and under (D2), the maximum claim $\max \text{supp}(a)$ exceeds c (else $X_1 \leq c$ would force $\mathbb{E}[X_1] \leq c$ with equality only when $X_1 = c$ a.s., contradicting (D2)), so the leading term in $a(z)$ has degree $d = \max \text{supp}(a) > c$ and $\varphi(z) \rightarrow +\infty$ as $z \rightarrow \infty$. Strict convexity of φ on $(1, \infty)$ (both a and $-z^c$ contribute non-negatively to φ'' on $(1, \infty)$ and a'' is not identically zero) forces exactly one crossing $\varphi(z^*) = 0$ with $z^* > 1$.

(iii) *Identification of $g(c)$ via the Pollaczek–Khinchine matrix-geometric.* Under the rank-one structure of part (i), the matrix iterates of $G^\uparrow(c)$ collapse to scalar iterates: $G^\uparrow(c)^n = g(c)^n \mathbf{e}\alpha$, so (6) becomes

$$\mathbb{P}(M > u_0 \mid J(\tau_0) = i) = \sum_{n \geq 1} g(c)^n (\mathbf{e})_i \alpha \mathbf{e} \nu_n(u_0) = \sum_{n \geq 1} g(c)^n \nu_n(u_0), \quad (10)$$

using $\alpha \mathbf{e} = 1$, so the tail of the maximum has matrix-free representation. Proposition 2 (derived in Section 4 from the polynomial factorization of $Q(z)$) gives the conditional ladder-height law χ in closed form together with the defective continuation probability p_L (see also Remark 8), and identifies the maximum's PGF as $\mathbb{E}[z^M] = (1 - p_L)/(1 - p_L \chi(z))$, a rational function whose poles in $|z| > 1$ are exactly the outside roots $\{z_1, \dots, z_I\}$ of the Lundberg polynomial $Q(z) = z^c - a(z)$ with $z^* = \min_i |z_i|$.

By the standard tail asymptotics of rational PGFs ([3], Ch. IV.7), the smallest-modulus pole controls the geometric decay rate of the coefficients: $\mathbb{P}(M > u_0) \sim C(z^*)^{-u_0}$ as $u_0 \rightarrow \infty$, with $C > 0$ a constant determined by p_L , χ , and the residue at z^* . Compare with (10): the left-hand side decays at the rate $(z^*)^{-1}$ per level, while the right-hand side decays at the rate $g(c)$ per iterate. Since these must match, and since each iterate corresponds to one ascending step which advances the running maximum by one or more units, the spectral identification is

$$g(c) = \frac{1}{z^*}, \quad (11)$$

the reciprocal of the dominant Lundberg root. Note that $g(c)$ here is the *per-level* decay scalar of the maximum's distribution, not the single-step ladder mass; the latter is the distinct quantity $p_L = \mathbb{P}(M > 0)$ of Remark 8.

Three consistency checks: (a) the exponential martingale of part (ii) is a true martingale at $z = z^*$ but *ceases to be* for $z < z^*$, confirming $1/z^*$ as the boundary scalar at which the PK geometric series in (10) retains finite radius of convergence; (b) at $c = 0$, the Lundberg equation reduces to $a(z) = 1$ whose only solution on $|z| \geq 1$ is $z = 1$, so $z^* \rightarrow 1$ and $g(0) = 1$, recovering $\mathbf{G}^\dagger(0) = \mathbf{e}\alpha = H(0)$ of Corollary 1; (c) in the skip-free case $J = 1$, the Lundberg polynomial has a single outside root z^* and the relations $1 - p_L = 1 - 1/z^*$ together with (11) give $g(c) = p_L = 1/z^*$, so the two scalars coincide (Remark 8). $\square$

Remark 8 (The two scalars: $g(c)$ versus p_L). The scalar $g(c) = 1/z^*$ of Theorem 1 is the *per-level* geometric decay rate of the maximum, characterized by $\mathbb{P}(M > u_0) \sim Cg(c)^{u_0}$ as $u_0 \rightarrow \infty$; it is the dominant eigenvalue of $\mathbf{G}^\dagger(c)$, as identified in the proof. A different scalar $p_L := \mathbb{P}(M > 0) = 1 - \prod_{i=1}^J (1 - 1/z_i)$ (Proposition 2) is the defective continuation probability of the first strict ascending ladder, equal to the row sum of the literal first-ladder transition matrix. The two coincide in the skip-free case $J = 1$ (then the single outside root is z^* and $p_L = 1 - (1 - 1/z^*) = 1/z^* = g(c)$), but differ in general for $J \geq 2$: at the parameters of Section 8.1, one has $z^* \approx 2.090$ giving $g(c) \approx 0.4785$, while the second outside root $z_2 \approx -4.077$ gives $p_L = 1 - (1 - 1/z^*)(1 - 1/z_2) \approx 0.3505 \neq g(c)$. The Pollaczek–Khinchine matrix-geometric of (6) uses the *per-level* geometric decay scalar $g(c)$ as the eigenvalue of $\mathbf{G}^\dagger(c)$; the factorization $1 - p_L\chi(z) = \prod(1 - z/z_i)$ then expresses the maximum's PGF in terms of p_L and the conditional ladder-height law χ . Both quantities are needed: $g(c)$ governs the tail decay, and p_L together with χ determines the finite-reserve constants.

Remark 9 (Why the naive level-recovery count yields the wrong scalar). It is tempting to derive a fixed point for $g(c)$ by conditioning on the first claim and counting unit levels of recovery, in analogy with the level-decrement argument for Neuts's $\mathbf{G}$ ([1], Theorem 5.2). That argument is well adapted to the fixed-threshold/skip-free setting, where each unit level is crossed in a single step and the matrix $\mathbf{G}$ is the literal one-step transition matrix. It does not transport to the moving-boundary setting: each “step” of the walk Z_n moves by $X_n - c \in \mathbb{Z}$, possibly by several units at once, and the relevant object for ruin is the running maximum, not the position after a single step. A naive identification of $g(c)$ with a defective single-step probability would yield the first-ladder mass p_L of Remark 8, which differs from $g(c) = 1/z^*$ in general (coinciding only when $J = 1$). The Pollaczek–Khinchine matrix-geometric of (6) requires the *per-level* decay rate $g(c)$, identified with the dominant eigenvalue of $\mathbf{G}^\dagger(c)$; the first-ladder mass p_L enters separately, through the ladder-height law χ of Proposition 2.

Remark 10 (Powers of the ladder matrix are free). Because $\mathbf{G}^\dagger(c) = g(c)\mathbf{e}\alpha$ is rank-one, its powers are immediate: $\mathbf{G}^\dagger(c)^n = g(c)^n\mathbf{e}\alpha$ for $n \geq 1$, by the same idempotency $(\mathbf{e}\alpha)^n = \mathbf{e}\alpha$. First passage of the running maximum to the level u_0 is thus governed by $g(c)^{u_0}$, giving the phase-resolved ruin probability in closed form without iterating the ladder recursion, the $O(m^2)$ -

versus- $O(m^3 \log(1/\varepsilon))$ saving of ([1], Section 7), transplanted to the moving-boundary problem. The geometric decay $g(c)^{u_0}$ is the exact, phase-resolved form of the classical Lundberg bound.

3.3. Recovery of the Fixed-Threshold Matrix at Zero Drift

The drift ladder matrix specializes, at zero premium drift, to the rank-one phase-reset matrix that generates Neuts's first-passage matrix $\mathbf{G}$ in the fixed-threshold theory of [1]. This is the precise sense in which the present moving-boundary framework contains the fixed-threshold framework as its $c = 0$ boundary.

Corollary 1 (Zero-drift limit). At $c = 0$,

$$\mathbf{G}^\dagger(0) = \mathbf{e}\boldsymbol{\alpha}, \quad (12)$$

the rank-one phase-reset matrix $H(0) = \mathbf{e}\boldsymbol{\alpha}$ of Proposition 1(iv). More generally, $g(c) \nearrow 1$ as $c \rightarrow 0^+$, and the defect $1 - g(c)$ equals the survival (non-ruin) probability of the drift model.

Proof. At $c = 0$, the adjustment Equation (8) reads $a(z) = z^0 = 1$; since a is a PGF with $a(1) = 1$ and a strictly increasing on $[1, \infty)$, the only root in $[1, \infty)$ is $z^* = 1$, whence $g(0) = 1/z^* = 1$ and $\mathbf{G}^\dagger(0) = 1 \cdot \mathbf{e}\boldsymbol{\alpha} = \mathbf{e}\boldsymbol{\alpha}$ by (7). For monotonicity, the dominant root $z^*(c)$ of $a(z) = z^c$ increases with c (raising the right-hand exponent pushes the crossing point outward), so $g(c) = 1/z^*(c)$ decreases from 1 as c grows beyond $\mathbb{E}[X_1]$; equivalently, $g(c) \nearrow 1$ as $c \rightarrow 0^+$. The interpretation of the defect as survival probability is immediate from Remark 10, since the ruin probability is proportional to $g(c)^{u_0} \rightarrow 1$ as $g(c) \rightarrow 1$. $\square$

Remark 11 (The rank-one structure invariant across the framework). Corollary 1 identifies the rank-one matrix $\mathbf{e}\boldsymbol{\alpha}$ as the common generator across the three settings of the framework: it produces Neuts's matrices $\mathbf{G}$ and $\mathbf{R}$ in the fixed-threshold matrix-analytic bridge of [1]; it generates, through the matrix LST $H(\theta) = \mathbf{v}(\theta)\boldsymbol{\alpha}$, the low-dimensional span of the phase-tagged reliability functional $\boldsymbol{\Phi}_v$ of [2]; and it generates, through the discrete Lundberg deformation $g(c)\mathbf{e}\boldsymbol{\alpha}$, the moving-boundary ladder matrix of Theorem 1. In all three cases, the rank-one structure is a direct consequence of the phase-reset hypothesis (D3): when the phase resets to $\boldsymbol{\alpha}$ at each renewal, every first-passage operator is a scalar multiple of $\mathbf{e}\boldsymbol{\alpha}$. The premium drift c enters only through the scalar Lundberg deformation $1 \mapsto g(c)$ and leaves the matrix structure invariant. The structure degrades, ceasing to be rank-one, precisely when the phase-reset hypothesis fails, that is, for Markovian-arrival (MAP) claims, exactly as the rank-one form of Neuts's $\mathbf{R}$ degrades for MAP[X] processes in [1].

3.4. The Discounted Drift Ladder Matrix

For the Gerber–Shiu analysis of Section 5, we require the time-discounted ladder matrix. The discount is carried entirely by the scalar ladder height through the discounted Lundberg root; the matrix generator remains the pure phase-reset matrix $\mathbf{e}\boldsymbol{\alpha}$, unchanged from the undiscounted case.

Theorem 2 (Closed form of the discounted ladder matrix). Under Assumption 1, for $\text{Re}(\theta) \geq 0$, the discounted drift ladder matrix is rank-one,

$$\mathbf{G}^\dagger(c, \theta) = g(c, \theta) \mathbf{e}\boldsymbol{\alpha} \in \text{span}\{\mathbf{e}\boldsymbol{\alpha}\}, \quad (13)$$

with the same matrix generator $\mathbf{e}\boldsymbol{\alpha} = H(0)$ as the undiscounted matrix (7), where the discounted ladder scalar $g(c, \theta) = 1/z^*(\theta) \in (0, 1]$ is determined by $z^*(\theta) > 1$, the dominant root of the discounted Lundberg equation

$$a(z)h(\theta) = z^c, \quad \text{equivalently} \quad \mathbb{E}[z^{X_1-c}]h(\theta) = 1. \quad (14)$$

The discount factor $h(\theta)$ enters as a single multiplicative weight on the per-period claim transform, the discrete-time discounting of one inter-shock interval; at $\theta = 0$, $h(0) = 1$ and (14) reduces to (8).

Proof. The proof parallels Theorem 1 with the discounted Lundberg equation in place of the undiscounted one. Part (i): by the phase-reset argument of Theorem 1(i), $G^\uparrow(c, \theta)$ is rank-one with left factor $\mathbf{e}\alpha$, so $G^\uparrow(c, \theta) = g(c, \theta) \mathbf{e}\alpha$ for a scalar $g(c, \theta)$ to be identified. Part (ii): the discounted exponential process $\tilde{M}_n^{(z, \theta)} := z^{Z_n} h(\theta)^n$ is a martingale if and only if $z^{-c} a(z) h(\theta) = 1$, that is, $a(z) h(\theta) = z^c$, the discounted Lundberg equation (14). For $\operatorname{Re}(\theta) \geq 0$, $|h(\theta)| \leq 1$ shifts the dominant root outward, $z^*(\theta) \geq z^*(0)$, and uniqueness/existence follow from the convexity argument of Theorem 1(ii) applied to $\tilde{q}_\theta(z) := a(z) h(\theta) - z^c$. Part (iii): the discounted maximum's PGF inherits the rational structure of the undiscounted case with $z^*(\theta)$ in place of z^* ; the per-level geometric decay rate is $g(c, \theta) = 1/z^*(\theta)$ by the same Pollaczek–Khinchine identification. The discount thereby carries entirely on the scalar $g(c, \theta)$; the matrix factor $\mathbf{e}\alpha$ remains θ -invariant. Finally, $g(c, \theta) \leq g(c) \leq 1$, with equality at $\theta = 0$, since $z^*(\theta) \geq z^*(0)$. $\square$

Remark 12 (The matrix generator is discount-invariant). The discounted drift ladder matrix (13) carries the same rank-one generator $\mathbf{e}\alpha$ as its undiscounted counterpart (7); only the scalar deforms with θ through the Lundberg root. This is in pointed contrast to the discounted Neuts matrix $G(\theta) = h_0(\theta)[(1 - p_{00})I + \kappa_0(\theta)H(\theta)]$ of ([1], Theorem 5.5), which lives in the two-dimensional span $\operatorname{span}\{I, H(\theta)\}$ and whose matrix content genuinely moves with θ . Two structural facts combine here: the moving boundary admits no phase-preserving transition (every increment is a claim epoch resetting the phase to α by (D3)), so there is no identity term; and the discount attaches to the scalar ladder height rather than to the matrix, so the generator is θ -invariant. The discounted ruin problem is therefore governed by a single one-parameter family of scalars $g(c, \theta)$ riding on the fixed rank-one matrix $\mathbf{e}\alpha$.

Corollary 2 (Consistency of the discounted ladder scalar). The discounted ladder scalar satisfies $g(c, 0) = g(c)$ of (8), since $h(0) = 1$ reduces (14) to (8). At $c = 0$, the equation $a(z)h(\theta) = z^0 = 1$ has its relevant root at the boundary $z^* \rightarrow 1$ as $\theta \rightarrow 0$ (using $a(1) = 1$), giving $g(0, 0) = 1$ and $G^\uparrow(0, 0) = H(0) = \mathbf{e}\alpha$, recovering Corollary 1.

4. The Ascending Ladder-Height Law

The compound-geometric representation of the ruin probability (Theorem 3) involves the ascending ladder-height distribution χ of the net-claim walk. Under the bounded-jump hypothesis (D1), this law is finitely supported and admits a closed form in terms of the roots of the Lundberg polynomial, with no iteration required.

Definition 4 (Lundberg polynomial). The Lundberg polynomial of the drift model is

$$Q(z) := z^c - a(z), \quad \deg Q = N := \max\{c, \deg a\}. \quad (15)$$

Its root $z = 1$ is always present (since $a(1) = 1$); under the net-profit condition (D2), it has a unique root $z^* > 1$, the reciprocal of the ladder scalar, $z^* = 1/g(c)$ of Theorem 1.

Proposition 2 (Closed-form ladder-height law). Let $J := \max \operatorname{supp}(a) - c$ be the maximal up-jump of the net-claim walk, and assume $J \geq 1$. The Lundberg polynomial Q has exactly J roots

$z_1, \dots, z_J$ outside the closed unit disk (counted with multiplicity), and the ascending ladder-height generating function $\chi(z) = \sum_{i=1}^J \chi_i z^i$, supported on $\{1, \dots, J\}$, is given by

$$1 - p_L \chi(z) = \prod_{i=1}^J \left(1 - \frac{z}{z_i}\right), \quad p_L = 1 - \prod_{i=1}^J \left(1 - \frac{1}{z_i}\right), \quad (16)$$

where $p_L = \mathbb{P}(M > 0)$ is the ladder continuation probability of Theorem 3. Equivalently, the maximum of the net-claim walk has the rational generating function

$$\mathbb{E}[z^M] = \frac{1 - p_L}{1 - p_L \chi(z)} = \frac{\prod_{i=1}^J (1 - 1/z_i)}{\prod_{i=1}^J (1 - z/z_i)}, \quad (17)$$

analytic on $|z| < z^* = \min_i |z_i|$ with a simple pole at $z^* = 1/g(c)$, reproducing the geometric tail decay $g(c)$ of Corollary 4.

Proof. The maximum $M = \sup_n Z_n$ of the net-claim walk with negative drift has, by the Pollaczek–Khinchine decomposition (Theorem 3), the compound-geometric generating function $\mathbb{E}[z^M] = (1 - p_L)/(1 - p_L \chi(z))$, a rational function whose denominator $1 - p_L \chi(z)$ has degree $J = \deg \chi$. Since $\chi(0) = 0$, the denominator equals 1 at $z = 0$, so it factors as $\prod_i (1 - z/r_i)$ over its reciprocal roots r_i . The generating function $\mathbb{E}[z^M]$ is analytic precisely where the walk’s one-step transform admits the Wiener–Hopf factorization, and its poles are the roots of $Q(z) = z^c - a(z)$ lying outside the unit disk: writing the walk’s increment transform and applying the factorization [3,4], the denominator of $\mathbb{E}[z^M]$ is the “outside” factor of Q . Under (D1), the up-jumps are bounded by J , so Q has exactly J roots outside the closed unit disk, giving the product (16); the normalization p_L follows from $\chi(1) = 1$. The smallest outside root is $z^* = 1/g(c)$ by definition of the Lundberg root, yielding the stated pole and tail decay. $\square$

Corollary 3 (Skip-free upward case). *If $J = 1$ (maximal up-jump equal to one, i.e., $\max \text{supp}(a) = c + 1$), then $\chi(z) = z$ is degenerate at 1, Q has a single outside root $z^* = 1/g(c)$, and*

$$\mathbb{E}[z^M] = \frac{1 - g(c)}{1 - g(c)z}, \quad \psi(u_0) = g(c)^{u_0+1}, \quad (18)$$

the pure-geometric maximum and ruin probability of Corollary 4.

Proof. With $J = 1$, the product (16) has one factor, $1 - p_L \chi(z) = 1 - z/z^* = 1 - g(c)z$, forcing $\chi(z) = z$ and $p_L = 1 - (1 - 1/z^*) = 1/z^* = g(c)$. Substituting into (17) gives the geometric PGF, whose tail is $\mathbb{P}(M > u_0) = g(c)^{u_0+1}$. $\square$

Remark 13 (Computational cost). *The ladder-height law (16) requires only the roots of the single degree- N polynomial $Q(z) = z^c - a(z)$, obtained by standard root-finding in $O(N^3)$ operations (or $O(N^2)$ by companion-matrix methods), independent of the reserve u_0 and of the phase space dimension m . The full ruin probability $\psi(u_0)$ and its discounted and phase-resolved refinements then follow by evaluating the rational function (17), with no matrix iteration at any stage. When $\deg a$ is small, as for the geometric, shifted-geometric, and low-order mixed claim laws common in applications, N is small and the entire ladder structure is available in closed algebraic form.*

5. The Gerber–Shiu Function

The drift ladder matrix of Section 3.2 delivers, as its principal application, a phase-resolved closed form for the Gerber–Shiu expected-discounted-penalty function of the associated risk model. We first fix the risk-theoretic dictionary, then define the matrix-

valued Gerber–Shiu functional and exhibit it as a specialization of the moving-boundary functional. The discounted ruin probability, the penalty-free case, is obtained in full from the proven ladder theorem; the general penalty case is recorded as the projection that recovers the classical scalar Gerber–Shiu function.

5.1. The Risk Model and Its First-Passage Dictionary

Reinterpret the net-claim walk (4) as the claim-epoch embedding of an insurance surplus process with initial reserve u_0 , premium rate c , and claim sizes X_n . The surplus immediately after the n -th claim is $u_0 - Z_n$, and ruin (5) is the first claim index ν at which the surplus turns negative. The three classical Gerber–Shiu observables [14] translate into the pre-failure/failure pair of [2] as follows.

Definition 5 (Gerber–Shiu observables). *At ruin index ν ,*

$$\text{time of ruin} : \tau_\nu, \quad (19)$$

$$\text{surplus prior to ruin} : U^- := u_0 - Z_{\nu-1} = u_0 - S_{\nu-1}^X + c(\nu - 1) \geq 0, \quad (20)$$

$$\text{deficit at ruin} : U^+ := Z_\nu - u_0 = S_\nu^X - c\nu - u_0 > 0. \quad (21)$$

Both U^- and U^+ are integer linear combinations of the six variables $(\nu, \tau_{\nu-1}, \tau_\nu, S_{\nu-1}^X, S_\nu^X, J(\tau_\nu))$ tracked by the phase-tagged pre-failure/failure functional of [2]. The surplus prior to ruin is built from the *pre-failure* damage $S_{\nu-1}^X$ and index $\nu - 1$; the deficit is built from the *failure* damage S_ν^X and index ν . The Gerber–Shiu function is therefore not a new object but a re-weighting of the moving-boundary functional, and this is the structural content of the present section.

5.2. The Matrix Gerber–Shiu Functional

Definition 6 (Matrix Gerber–Shiu functional). *For a penalty encoded through PGF variables y_-, y_+ on the pre-ruin surplus and deficit, and a discount variable θ on the time of ruin, the matrix Gerber–Shiu functional is the $m \times m$ matrix*

$$\mathbf{M}(\theta, y_-, y_+)_{ij} = \mathbb{E}\left[e^{-\theta\tau_\nu} y_-^{U^-} y_+^{U^+} \mathbf{1}_{\{\nu < \infty\}} \mathbf{1}_{\{J(\tau_\nu)=j\}} \mid J(\tau_0) = i\right]. \quad (22)$$

The classical scalar Gerber–Shiu function with penalty w and force of interest δ is recovered by the projection

$$m_w(u_0) = \alpha \widehat{\mathbf{M}}_w(\delta) \mathbf{e}, \quad (23)$$

where $\widehat{\mathbf{M}}_w$ denotes the inverse transform of $\mathbf{M}$ against w in the pair (y_-, y_+) evaluated at $\theta = \delta$.

Proposition 3 (Gerber–Shiu as a specialization of the moving-boundary functional). *Let $\Phi_\nu^{\text{ruin}}(\xi, u, v, \vartheta, \theta) = \Phi^{(0)} + \Phi^{(+)}$ denote the moving-boundary analogue of the phase-tagged pre-failure/failure functional of [2], split into its immediate ($\nu = 0$) and delayed ($\nu \geq 1$) parts, defined by replacing the fixed threshold p_1 by the moving boundary $u_0 + cn$ in (5). Then the matrix Gerber–Shiu functional is recovered term by term:*

$$\mathbf{M}(\theta, y_-, y_+) = y_-^{u_0} y_+^{-u_0} \Phi^{(0)}(v = y_+, \theta) + y_-^{u_0-c} y_+^{-u_0} \Phi^{(+)}(\xi = (y_-/y_+)^c, u = y_-^{-1}, v = y_+, \vartheta = 0, \theta). \quad (24)$$

The delayed term carries the prefactor $y_-^{u_0-c}$, one power of y_-^c lower than the immediate term, reflecting the constant $-c$ in the pre-failure exponent $c(\nu - 1)$.

Proof. On $\{\nu \geq 1\}$, substitute (20) and (21) into the integrand of (22) and collect powers. Using $Z_{\nu-1} = S_{\nu-1}^X - c(\nu - 1)$ and $Z_\nu = S_\nu^X - c\nu$,

$$y_-^{U^-} = y_-^{u_0-c} (y_-^c)^\nu (y_-^{-1})^{S_{\nu-1}^X}, \quad y_+^{U^+} = y_+^{-u_0} (y_+^{-c})^\nu y_+^{S_\nu^X},$$

where the constant $-c$ from $c(\nu - 1) = c\nu - c$ contributes the prefactor power y_-^{-c} . The product carries the index ν with weight $(y_-^c y_+^{-c})^\nu = \xi^\nu$, the pre-failure damage $S_{\nu-1}^X$ with $u = y_-^{-1}$, and the failure damage S_ν^X with $v = y_+$, giving the delayed term with prefactor $y_-^{u_0-c} y_+^{-u_0}$. On $\{\nu = 0\}$, ruin occurs on the delay claim with $Z_{-1} = 0$ (physical convention: the full reserve is intact before the first claim), so $U^- = u_0$ and $U^+ = X_0 - u_0$, contributing the immediate term with prefactor $y_-^{u_0} y_+^{-u_0}$ and only the failure variable $v = y_+$. The two boundary conventions differ by exactly y_-^{-c} , which is why the specialization must be applied term by term. $\square$

Remark 14 (Analytic continuation outside the unit polydisc). The substitution (24) evaluates Φ_v^{ruin} at arguments such as $u = y_-^{-1}$ and $\xi = (y_-/y_+)^c$ that need not lie in the unit polydisc, since penalty transforms probe reciprocal arguments. This is the familiar analytic continuation underlying Gerber–Shiu theory [14]: when the model primitives a_0, a, h are rational, the closed forms of the reliability-index formulas of [2] are rational functions that continue analytically beyond the probabilistic domain, so (24) is well defined wherever the relevant denominators do not vanish. The ladder-scalar fixed points $g(c), g(c, \theta)$ of Theorems 1 and 2 provide the location of such a dominant singularity, exactly as the adjustment coefficient does in the classical theory.

5.3. The Discounted Ruin Probability

The penalty-free case $y_- = y_+ = 1, w \equiv 1$, is the discounted (Laplace-transformed-in-time) ruin probability. It is governed by the ladder theorem through the compound-geometric (Pollaczek–Khinchine) representation of the maximum of the net-claim walk.

Theorem 3 (Phase-resolved discounted ruin probability). Under Assumption 1, write $M = \sup_{n \geq 0} Z_n$ for the all-time maximum of the net-claim walk (4); ruin from initial reserve u_0 occurs iff $M > u_0$. The maximum admits the compound-geometric decomposition

$$M = \sum_{i=1}^K Y_i, \quad \mathbb{P}(K = k) = (1 - p_L) p_L^k \quad (k \geq 0), \quad p_L := \mathbb{P}(M > 0), \quad (25)$$

where p_L is the ascending-ladder continuation probability, K is the number of ascending ladder steps, and the $Y_i \geq 1$ are i.i.d. ascending ladder heights with common PGF $\chi(\cdot)$ supported on $\{1, \dots, \max \text{supp}(a) - c\}$. Consequently, the discounted ruin probability has the closed form

$$\alpha \Psi(u_0, \theta) \mathbf{e} = \mathbb{P}_\theta(M > u_0) = \sum_{j > u_0} \rho_\theta(j), \quad (26)$$

where ρ_θ is the (defective, θ -discounted) compound-geometric mass of M obtained from (25). As $u_0 \rightarrow \infty$,

$$\alpha \Psi(u_0, \theta) \mathbf{e} \sim C(c, \theta) g(c, \theta)^{u_0}, \quad (27)$$

the exact Cramér–Lundberg geometric decay at rate $g(c, \theta)$ (the ladder scalar of Theorem 2), with $C(c, \theta)$ the Cramér–Lundberg constant determined by p_L and the ladder-height law χ .

Remark 15 (Two distinct ladder quantities). The decomposition involves two scalars that must not be conflated. The continuation probability $p_L = \mathbb{P}(M > 0)$ governs the geometric number of ladder steps and hence the mass at the origin, $\mathbb{P}(M = 0) = 1 - p_L$. The decay rate $g(c, \theta) = 1/z^*(\theta)$ of Theorems 1 and 2 governs the geometric tail (27). The two coincide,

$p_L = g(c)$, precisely in the pure-geometric case of Corollary 4 (maximal up-jump equal to one), where every ladder height is 1; in general, $p_L \neq g(c)$.

Proof. That ruin is the event $\{M > u_0\}$ and that M has the compound-geometric form (25), which is the Pollaczek–Khinchine decomposition of the maximum of a random walk with negative drift [3,4]: the walk makes a geometric number K of ascending ladder steps before drifting to $-\infty$, with continuation probability $p_L = \mathbb{P}(M > 0)$, and the heights between records are i.i.d. by the phase-reset hypothesis (D3), which renders successive ladder epochs independent. Discounting attaches through the discounted ladder scalar $g(c, \theta)$ of Theorem 2 via the projection identity $\alpha G^\dagger(c, \theta)^k \mathbf{e} = g(c, \theta)^k$ established there. Summing the compound-geometric mass above u_0 gives (26). The asymptotic (27) is the standard renewal-theoretic tail of a compound geometric whose increment law has exponential moment governed by the Lundberg root $z^*(\theta) = 1/g(c, \theta)$ [3]. $\square$

Corollary 4 (Undiscounted ruin probability and the Lundberg estimate). *Setting $\theta = 0$,*

$$\psi(u_0) = \alpha \Psi(u_0, 0) \mathbf{e} = \mathbb{P}(M > u_0) \sim C(c) g(c)^{u_0}, \quad (28)$$

with adjustment coefficient $R = \log z^ = -\log g(c)$. When the maximal up-jump $\max \text{supp}(a) - c$ equals 1, the ladder heights are identically 1, $\chi(z) = z$, the continuation probability and decay rate coincide ($p_L = g(c)$), the compound geometric collapses to a pure geometric, and (28) is exact for every u_0 :*

$$\psi(u_0) = g(c)^{u_0+1}. \quad (29)$$

Remark 16 (The ladder-height law and the Cramér–Lundberg constant). *The decay rate $g(c, \theta)$ is determined entirely by the Lundberg equation (8) and is exact, both undiscounted and discounted; this is the content of Theorems 1 and 2 and requires no information about the ladder-height law. The continuation probability p_L and the constant $C(c, \theta)$ depend on the ascending ladder-height distribution χ , the overshoot law of the net-claim walk at its records. Under the bounded-jump hypothesis (D1), χ is supported on $\{1, \dots, \max \text{supp}(a) - c\}$ and is computable in closed form from a by the renewal-overshoot construction [4]; in the maximal-up-jump-one case, it is degenerate at 1, giving the pure-geometric ruin probability (29). The running maximum advances by random ladder heights, not by unit levels; only the asymptotic rate is a clean power of g .*

5.4. The General Penalty Case and Scalar Recovery

The general Gerber–Shiu function, carrying a non-trivial penalty on the deficit and pre-ruin surplus, follows from Proposition 3 once the moving-boundary functional Φ_v^{ruin} is available in closed form. We state the recovery relation that this closed form must satisfy, the moving-boundary counterpart of the Dshalalow–White scalar recovery theorem of [2].

Theorem 4 (Recovery of the classical scalar Gerber–Shiu function). *Let $m_w(u_0)$ be the classical scalar Gerber–Shiu function [14] of the discretized risk model with claim PGF a , premium rate c , force of interest δ , and penalty $w(U^-, U^+)$. Then*

$$m_w(u_0) = \sum_{x \geq 0} \sum_{y \geq 1} w(x, y) [\alpha \mathbf{M}(\delta, \cdot, \cdot) \mathbf{e}]_{x,y}, \quad (30)$$

where $[\alpha \mathbf{M} \mathbf{e}]_{x,y}$ is the joint defective distribution of (U^-, U^+) at discounted ruin extracted from the projected matrix functional, and $\mathbf{M}$ is given by (24). In particular, the projection $\alpha \mathbf{M} \mathbf{e}$ discards the phase tag and reproduces exactly the scalar Gerber–Shiu function, while $\mathbf{M}$ itself refines it by the operational phase $J(\tau_v)$ at ruin.

Proof. By Proposition 3, M is the substitution-image of Φ_v^{ruin} ; its $\alpha(\cdot)\mathbf{e}$ projection coincides with the scalar moving-boundary functional by the scalar-recovery argument of the recovery theorem of [2] applied to the moving boundary. Extracting the joint (U^-, U^+) distribution by coefficient extraction in (y_-, y_+) and pairing against the penalty w gives (30). The full proof requires the closed form of Φ_v^{ruin} (Section 6) and is completed there. $\square$

Remark 17 (Status of the general case). *Theorem 3 and Corollary 4 are established in full from the proven ladder theorems Theorems 1 and 2. Proposition 3 is an exact change of variables and is also complete. Theorem 4 and the explicit deficit distribution depend on the moving-boundary central formula Φ_v^{ruin} , the drift analogue of the central formula of [2], whose derivation is the subject of Section 6. We have therefore reduced the general phase-resolved Gerber–Shiu function to a single closed-form computation of the moving-boundary functional, with the penalty-free ruin probability already in hand.*

6. The Penalty-Free Central Formula and Ruin Observables

We now assemble the phase-tagged matrix Gerber–Shiu functional $M(\theta, y_-, y_+)$ of Definition 6 in closed form. The derivation follows the trajectory decomposition of [2]: the delay claim X_0 contributes an immediate-ruin term, and the post-delay walk contributes a ladder term governed by the closed-form ladder-height law of Proposition 2. Throughout, a_0 is the PGF of the delay claim X_0 and a that of the generic claim, as in Section 2.1.

Lemma 1 (Delay decomposition of the ruin functional). *For the matrix Gerber–Shiu functional of Definition 6,*

$$M(\theta, y_-, y_+) = M^{(0)}(\theta, y_+) + M^{(+)}(\theta, y_-, y_+), \quad (31)$$

where the immediate-ruin term ($v = 0$, ruin on the delay claim) is

$$M^{(0)}(\theta, y_+) = H_0(\theta) \sum_{x > u_0} \mathbb{P}(X_0 = x) y_+^{x-u_0} y_-^{u_0} = y_-^{u_0} H_0(\theta) [a_0^{>u_0}(y_+)] y_+^{-u_0}, \quad (32)$$

with $a_0^{>u_0}(y_+) := \sum_{x > u_0} \mathbb{P}(X_0 = x) y_+^x$ the upper tail of the delay claim PGF, and the delayed-ruin term ($v \geq 1$, survive the delay then cross) is

$$M^{(+)}(\theta, y_-, y_+) = H_0(\theta) \sum_{x=0}^{u_0} \mathbb{P}(X_0 = x) W(\theta, y_-, y_+; u_0 - x), \quad (33)$$

where $W(\theta, y_-, y_+; r)$ is the post-delay walk Gerber–Shiu functional at reduced reserve r .

Proof. The events $\{v = 0\} = \{X_0 > u_0\}$ and $\{v \geq 1\} = \{X_0 \leq u_0\}$ partition the sample space, exactly as in ([2], Lemma (trajectory)). The delay interval contributes the matrix factor $H_0(\theta)$, carrying its duration and the phase reset. On $\{v = 0\}$, the surplus prior to ruin is $U^- = u_0 - Z_{-1} = u_0$ and the deficit is $U^+ = X_0 - u_0$, giving (32). On $\{v \geq 1\}$, conditioning on $X_0 = x \leq u_0$ leaves the post-delay walk starting from net level x , equivalently a fresh walk that must cross the reduced reserve $u_0 - x$; its phase-tagged discounted Gerber–Shiu functional is $W(\theta, y_-, y_+; u_0 - x)$, giving (33). $\square$

The post-delay walk functional W is where the ladder structure enters. Because the phase resets at every claim epoch (D3), W is rank-one with the phase-reset generator, and its scalar content is the compound-geometric overshoot transform of Proposition 2.

Theorem 5 (Closed form of the post-delay walk functional). *Under Assumption 1, the post-delay walk Gerber–Shiu functional is rank-one,*

$$W(\theta, y_-, y_+; r) = w(\theta, y_-, y_+; r) \mathbf{e}\alpha, \quad (34)$$

with the matrix factor the phase-reset matrix $\mathbf{e}\alpha$. The scalar w is the joint first-passage transform of the undershoot $U^- = r - Z_{v-1}$ and overshoot $U^+ = Z_v - r$ of the post-delay walk at level r . Writing $J = \max \text{supp}(a) - c$ for the maximal up-jump and $a^{>m}(y) := \sum_{X>m} \mathbb{P}(X=X) y^X$ for the upper-tail claim transform, the joint law factorizes as

$$\mathbb{P}(U^- = \alpha, U^+ = k, \text{ruin}) = \nu_r(\alpha) \mathbb{P}(X = c + \alpha + k), \quad 0 \leq \alpha \leq J-1, k \geq 1, \quad (35)$$

where $\nu_r(\alpha)$ is the pre-ruin undershoot weight, so that

$$w(0, y_-, y_+; r) = \sum_{\alpha=0}^{J-1} \nu_r(\alpha) y_-^\alpha y_+^{-(c+\alpha)} a^{>c+\alpha}(y_+). \quad (36)$$

The undershoot weights satisfy the first-passage recursion

$$w(0, y_-, y_+; r) = \sum_{X: X-c > r} \mathbb{P}(X=X) y_-^r y_+^{X-c-r} + \sum_{X: X-c \leq r} \mathbb{P}(X=X) w(0, y_-, y_+; r - (X - c)), \quad (37)$$

a finite linear system in the levels $\{r - (X - c)\}$ (with the convention $w(0, \cdot, \cdot; r') =$ the immediate-crossing value for $r' < 0$). The discounted functional carries one inter-shock transform $h(\theta)$ per step of the recursion,

$$w(\theta, y_-, y_+; r) = h(\theta) \left[\sum_{X: X-c > r} \mathbb{P}(X=X) y_-^r y_+^{X-c-r} + \sum_{X: X-c \leq r} \mathbb{P}(X=X) w(\theta, y_-, y_+; r - (X - c)) \right], \quad (38)$$

so that a first passage in ℓ steps accrues the discount $h(\theta)^\ell$, the discrete-time Laplace transform of the ruin time τ_v . In the penalty-free case $y_- = y_+ = 1$, $\theta = 0$, (36) reduces to $w(0, 1, 1; r) = \mathbb{P}(M_{\text{walk}} > r)$, the compound-geometric tail of the post-delay walk maximum, with generating function $\mathbb{E}[z^{M_{\text{walk}}}] = (1 - p_L) / (1 - p_L \chi(z))$ of (17); its geometric decay rate is the discounted ladder scalar $g(c, \theta)$ of Theorem 2.

Proof. The phase at ruin is the phase from which the ruinous claim's inter-shock interval is absorbed; by (D3), this is the reset distribution, so every contribution carries the rank-one factor $\mathbf{e}\alpha$ and $W = w \mathbf{e}\alpha$. For the scalar, ruin is the first passage of the walk Z_n over r ; the ruinous step takes the walk from $Z_{v-1} \leq r$ to $Z_v > r$ by the fatal claim X_v , so the fatal net increment is $X_v - c = U^- + U^+$, giving $X_v = c + U^- + U^+$ and the factorization (35): conditional on the pre-ruin position $Z_{v-1} = r - \alpha$, the deficit k is fixed by the fatal claim size $X_v = c + \alpha + k$, whose probability is $\mathbb{P}(X = c + \alpha + k)$, independent of the pre-ruin history; $\nu_r(\alpha)$ collects the probability that the pre-ruin position is $r - \alpha$. Since $X_v \leq \max \text{supp}(a)$, the undershoot is bounded, $\alpha \leq J-1$. Summing (35) against $y_-^\alpha y_+^k$ over $k \geq 1$ yields the upper-tail factor $y_+^{-(c+\alpha)} a^{>c+\alpha}(y_+)$ and (36). The recursion (37) follows by conditioning on the first post-delay step $X - c$: an immediate crossing ($X - c > r$) contributes the boundary term with undershoot r and overshoot $X - c - r$, while a non-crossing step ($X - c \leq r$) shifts the level to $r - (X - c)$, preserving the undershoot by spatial homogeneity. For the discounted recursion (38), each step of the recursion corresponds to one post-delay inter-shock interval; conditioning on its duration and using the scalar inter-shock transform $\alpha H(\theta) \mathbf{e} = h(\theta)$ inserts one factor $h(\theta)$ per step, whence a first passage in ℓ steps carries $h(\theta)^\ell = \mathbb{E}[e^{-\theta \tau_v} \mid \ell \text{ steps}]$. The penalty-free reduction is immediate from $a^{>c+\alpha}(1) = \mathbb{P}(X > c + \alpha)$ and $\sum_\alpha \nu_r(\alpha) \mathbb{P}(X > c + \alpha) = \mathbb{P}(M_{\text{walk}} > r)$. $\square$

Theorem 6 (Phase-tagged matrix Gerber–Shiu functional). *Under Assumption 1, the matrix Gerber–Shiu functional is*

$$\mathbf{M}(\theta, y_-, y_+) = y_-^{u_0} y_+^{-u_0} a_0^{>u_0}(y_+) H_0(\theta) + \left[\sum_{x=0}^{u_0} \mathbb{P}(X_0 = x) w(\theta, y_-, y_+; u_0 - x) \right] H_0(\theta) \mathbf{e} \boldsymbol{\alpha}, \quad (39)$$

lying in the span $\text{span}\{H_0(\theta), H_0(\theta) \mathbf{e} \boldsymbol{\alpha}\}$, with w the scalar of Theorem 5. The classical scalar Gerber–Shiu function is recovered by the projection $\boldsymbol{\alpha}(\cdot) \mathbf{e}$ (Theorem 4), and the penalty-free projection reproduces the ruin probability decomposition

$$\boldsymbol{\alpha} \mathbf{M}(0, 1, 1) \mathbf{e} = \mathbb{P}(X_0 > u_0) + \sum_{x=0}^{u_0} \mathbb{P}(X_0 = x) \mathbb{P}(M_{\text{walk}} > u_0 - x), \quad (40)$$

the delay decomposition of Theorem 3.

Proof. Substitute (34) into (33) and combine with (32) via Lemma 1, using $H_0(\theta) \mathbf{e} \boldsymbol{\alpha}$ as the rank-one delayed generator. The span statement is immediate from the two matrix terms. For the projection, $\boldsymbol{\alpha} H_0(\theta) \mathbf{e} = h_0(\theta)$ and $\boldsymbol{\alpha} H_0(\theta) \mathbf{e} \boldsymbol{\alpha} \mathbf{e} = h_0(\theta)$; at $\theta = 0$, $h_0(0) = 1$, $a_0^{>u_0}(1) = \mathbb{P}(X_0 > u_0)$, and $w(0, 1, 1; r) = \mathbb{P}(M_{\text{walk}} > r)$ by the penalty-free reduction above, yielding (40). $\square$

Remark 18 (Validation and the $c \rightarrow 0$ relationship). *The penalty-free projection (40) has been verified against direct simulation and an exact defective-renewal recursion: the delay decomposition reproduces the ruin probability to within Monte Carlo error, and the post-delay tail $\mathbb{P}(M_{\text{walk}} > r)$ is the closed-form compound-geometric of Proposition 2. The functional (39) is the moving-boundary, penalty-free specialization of the full five-variable functional Φ_v^{ruin} ; unlike the fixed-threshold central formula of [2], the $c \rightarrow 0$ limit here is not a reduction of (39) but a change of regime, since at $c = 0$, the moving boundary becomes fixed and the ladder scalar degenerates to $g(0) = 1$ (ruin certain, $z^* \rightarrow 1$); the appropriate fixed-threshold comparison is with the first-excess functional of [2] rather than with the $c = 0$ value of the ruin functional.*

7. The Moving-Boundary Reliability Functional

We now present the full five-variable analogue of the phase-tagged reliability functional of [2] for the moving boundary $u_0 + cn$. This is the central object of the paper: a single matrix-valued transform encoding the joint law of the ruin index, the pre-failure and failure damage, the pre-failure and failure times, and the operational phase at ruin. The penalty-free specializations of Section 5 and the ruin probability of Section 6 are recovered as projections.

7.1. Definition and Trajectory Decomposition

Definition 7 (Moving-boundary reliability functional). *The moving-boundary reliability functional is the $m \times m$ matrix*

$$[\Phi_v^{\text{ruin}}(\zeta, u, v, \theta, \theta)]_{ij} = \mathbb{E} \left[\zeta^v u^{S_{v-1}^X} v^{S_v^X} e^{-\theta \tau_{v-1}} e^{-\theta \tau_v} \mathbf{1}_{\{v < \infty\}} \mathbf{1}_{\{J(\tau_v)=j\}} \mid J(\tau_0) = i \right], \quad (41)$$

with ruin index $v = \inf\{n \geq 1 : S_n^X > u_0 + cn\}$ of (5) and the conventions $S_{-1}^X = 0$, $\tau_{-1} = 0$ of [2] at $v = 0$. We adopt the following throughout the reset-phase convention of [2]: $J(\tau_n) \sim \boldsymbol{\alpha}$, so that the matrix LST $H(\theta) = \mathbf{v}(\theta) \boldsymbol{\alpha}$ of Definition 1 carries the joint law of one inter-shock duration and the phase at the next epoch.

Lemma 2 (Trajectory decomposition). Writing $\omega := \vartheta + \theta$,

$$\Phi_v^{\text{ruin}} = \Phi^{(0)}(v, \theta) + \Phi^{(+)}(\xi, u, v, \vartheta, \theta), \quad (42)$$

where $\Phi^{(0)}$ collects the immediate-ruin paths $\{v = 0\} = \{X_0 > u_0\}$ and $\Phi^{(+)}$ the delayed paths $\{v \geq 1\}$.

Proof. The events $\{v = 0\}$ and $\{v \geq 1\}$ partition $\{v < \infty\}$, exactly as in ([2], eq. (trajectory-split)). At $v = 0$, the conventions $S_{-1}^X = \tau_{-1} = 0$ make $\xi^0 = u^0 = e^{-\vartheta \cdot 0} = 1$, leaving only the failure damage $v^{S_0} = v^{X_0}$ and failure time $e^{-\theta \tau_0}$. $\square$

7.2. The Immediate-Ruin Term

Proposition 4 (Immediate-ruin term).

$$\Phi^{(0)}(v, \theta) = a_0^{>u_0}(v) H_0(\theta), \quad a_0^{>u_0}(v) := \sum_{x > u_0} \mathbb{P}(X_0 = x) v^x, \quad (43)$$

where $a_0^{>u_0}$ is the upper-tail PGF of the delay claim and $H_0(\theta)$ the delay matrix LST.

Proof. On $\{v = 0\} = \{X_0 > u_0\}$, the integrand of (41) reduces to $v^{X_0} e^{-\theta \tau_0} \mathbf{1}_{\{X_0 > u_0\}} \mathbf{1}_{\{J(\tau_0)=j\}}$. By the independence of damage and time ([2], (A2)) the expectation factors into $\sum_{x > u_0} \mathbb{P}(X_0 = x) v^x$ and the delay matrix LST $[H_0(\theta)]_{ij} = \mathbb{E}[e^{-\theta \tau_0} \mathbf{1}_{\{J(\tau_0)=j\}} \mid J(\tau_0^-) = i]$, giving (43). $\square$

7.3. The Delayed-Ruin Term and the Moving-Boundary Resummation

The delayed term carries the moving boundary. Where the fixed-threshold functional of [2] applies the $\mathfrak{D}$ -operator truncation and resums a homogeneous matrix-geometric series by Sherman–Morrison, the moving boundary forces a first-passage resummation governed by the ladder structure of Section 6. Because the phase resets at each claim epoch (D3), the delayed term is rank-one in the reset generator, and its scalar content is the moving-boundary first-passage transform.

Definition 8 (Post-delay first-passage transform). For a reduced reserve $r \in \mathbb{Z}_+$, the post-delay first-passage transform $\Omega_r(\xi, u, v, \omega, \theta)$ is the unique solution of the finite linear recursion

$$\Omega_r = \xi h(\theta) \sum_{X: X-c > r} \mathbb{P}(X=X) v^X + \xi h(\omega) \sum_{X: X-c \leq r} \mathbb{P}(X=X) (uv)^X \Omega_{r-(X-c)}, \quad (44)$$

with $\Omega_{r'}$ for $r' < 0$ given by the immediate-crossing (fatal) term alone. The first sum is the fatal step (an up-crossing claim $X - c > r$, contributing the failure damage v^X and the failure-interval transform $h(\theta)$); the second is a pre-fatal step (a non-crossing claim, contributing $(uv)^X$ since the claim enters both S_{v-1} and S_v , and the pre-failure interval transform $h(\omega)$).

Theorem 7 (Delayed-ruin term). Under Assumption 1, with $\omega = \vartheta + \theta$,

$$\Phi^{(+)}(\xi, u, v, \vartheta, \theta) = \left[\sum_{x=0}^{u_0} \mathbb{P}(X_0 = x) (uv)^x \Omega_{u_0-x}(\xi, u, v, \omega, \theta) \right] H_0(\omega), \quad (45)$$

rank-one in the delay generator $H_0(\omega)$, with Ω_r the post-delay first-passage transform of Definition 8.

Proof. Condition on the delay claim $X_0 = x \leq u_0$ (so $v \geq 1$). The delay claim enters both the pre-failure damage $S_{v-1} = x + \sum_{i=1}^{v-1} X_i$ and the failure damage $S_v = x + \sum_{i=1}^v X_i$,

contributing the factor $(uv)^x$. The delay duration τ_0 enters both $\tau_{\nu-1}$ and τ_ν (since $\tau_0 \leq \tau_{\nu-1} \leq \tau_\nu$ for $\nu \geq 1$), contributing $e^{-(\vartheta+\theta)\tau_0} = e^{-\omega\tau_0}$, i.e., the delay matrix LST $H_0(\omega)$ carrying the reset-tagged phase. After the delay, the post-delay walk must cross the reduced reserve $r = u_0 - x$. Decomposing the post-delay time as $\tau_{\nu-1} - \tau_0 = T_1 + \dots + T_{\nu-1}$ and $\tau_\nu - \tau_0 = T_1 + \dots + T_\nu$ gives $e^{-\vartheta(\tau_{\nu-1}-\tau_0)-\theta(\tau_\nu-\tau_0)} = e^{-\omega(T_1+\dots+T_{\nu-1})}e^{-\theta T_\nu}$, so each of the $\nu - 1$ pre-failure intervals carries $h(\omega)$ and the failure interval carries $h(\theta)$. Conditioning on the first post-delay claim yields the recursion (44): a crossing claim ($X - c > r$) ends the walk with failure damage v^X and failure-interval $h(\theta)$; a non-crossing claim ($X - c \leq r$) is a pre-failure step contributing $(uv)^X$ and $h(\omega)$ and shifting the level to $r - (X - c)$. The index ξ^ν accrues one ξ per step. Phase resets at every epoch (D3), so the matrix content is the rank-one $H_0(\omega)$, completing (45). $\square$

7.4. Closed Form and Span

Theorem 8 (Moving-boundary central formula). *Under Assumption 1, the moving-boundary reliability functional is*

$$\Phi_v^{\text{ruin}}(\xi, u, v, \vartheta, \theta) = a_0^{>u_0}(v) H_0(\theta) + \left[\sum_{x=0}^{u_0} \mathbb{P}(X_0 = x) (uv)^x \Omega_{u_0-x}(\xi, u, v, \omega, \theta) \right] H_0(\omega), \quad (46)$$

$\omega = \vartheta + \theta$, lying in the two-dimensional matrix subspace $\text{span}\{H_0(\theta), H_0(\omega)\}$. When the delay shares the post-delay phase law ($\text{PH}(\alpha_0, S_0) = \text{PH}(\alpha, S)$, $H_0 = H$), this is $\text{span}\{H(\theta), H(\omega)\}$, the same subspace as the fixed-threshold functional Φ_v of [2]: the moving boundary deforms the scalar coefficients through the first-passage transform Ω_r but leaves the matrix subspace invariant.

Proof. Combine Lemma 2, Proposition 4, and Theorem 7. The first term is a scalar multiple of $H_0(\theta)$ and the second of $H_0(\omega)$, giving the span. The comparison with [2] follows from the identical rank-one generators; only the scalar coefficients differ, the fixed-threshold $\mathfrak{D}$ -operator truncation being replaced by the moving-boundary recursion Ω_r of Definition 8. $\square$

Remark 19 (The first-passage transform replaces Sherman–Morrison). *In [2], the delayed term is a homogeneous matrix-geometric series $\sum_{n \geq 1} \xi^n [a(uv)H(\omega)]^{n-1}$ resummed in closed form by the Sherman–Morrison identity (Proposition 1(v)), the threshold entering only through the $\mathfrak{D}$ -operator truncation $\mathfrak{D}_x^{p_1}$. The moving boundary $u_0 + cn$ makes the per-step contribution inhomogeneous: the n -th step is tested against $u_0 + cn$, inserting a level shift that depends on n . This is precisely the structure that the running maximum/ladder reformulation of Section 6 resolves: the inhomogeneous series is resummed not by Sherman–Morrison but by the first-passage recursion Ω_r , whose closed form is governed by the ladder scalar $g(c, \theta)$ and the ladder-height law χ of Proposition 2. The two devices coincide at $c = 0$, where the boundary is fixed and Ω_r collapses to the Sherman–Morrison geometric; see Remark 21.*

Remark 20 (On the closed-form status of Ω_r). *The term “closed form” admits two readings, and we clarify which applies to Ω_r . The first-passage transform Ω_r does not admit a single elementary expression in r of the form “rational function of model primitives evaluated at r ”; it is defined by the finite linear recursion (44), which has $J+1 = d - c + 1$ terms with d the maximal claim and which involves only model primitives and the variables $(\xi, u, v, \omega, \theta)$. Solving the recursion for any specified r thus reduces to $O(r)$ evaluations of $J+1$ rational terms; alternatively, the recursion is a linear difference equation whose characteristic polynomial is the Lundberg polynomial of Definition 4 at the deformed arguments $(uvz; \xi h(\omega))$, so Ω_r admits the explicit representation*

$$\Omega_r(\xi, u, v, \omega, \theta) = \sum_{j=1}^J A_j(\xi, u, v, \omega, \theta) \rho_j(\xi, u, v, \omega)^r + A_0(\xi, u, v, \omega, \theta), \quad (47)$$

where $\rho_1, \dots, \rho_l$ are the outside roots of the deformed Lundberg polynomial (cf. Proposition 2) and the coefficients A_j are rational in the model primitives, determined by the boundary conditions $\Omega_{r' < 0}$ together with the fixed point at $r = \infty$. In this sense, Ω_r is closed form to the same extent as the maximum-distribution PGF (17): a finite-sum geometric expression whose decay rates are roots of a polynomial built from the model primitives. For computation, we report the recursion directly, since the recursion is finite, numerically stable, and equally inexpensive.

7.5. Recovery Relations

Proposition 5 (Recovery of the ruin functional and ruin probability). *The matrix Gerber–Shiu functional of Definition 6 is recovered term by term (Proposition 3):*

$$\mathbf{M}(\theta, y_-, y_+) = y_-^{u_0} y_+^{-u_0} \Phi^{(0)}(y_+, \theta) + y_-^{u_0-c} y_+^{-u_0} \Phi^{(+)}((y_-/y_+)^c, y_-^{-1}, y_+, 0, \theta), \quad (48)$$

and the phase-resolved ruin probability is the further specialization $\xi = u = v = 1, \vartheta = \theta = 0$: $\alpha \Phi_v^{\text{ruin}}(1, 1, 1, 0, 0) \mathbf{e} = \psi(u_0)$ of Corollary 4.

Proof. The substitution is Proposition 3; the penalty-free projection is the trajectory sum $\Phi^{(0)}(1, 0) + \Phi^{(+)}(1, 1, 1, 0, 0)$ projected by $\alpha(\cdot) \mathbf{e}$, which reduces Ω_r to the compound-geometric tail $\mathbb{P}(M_{\text{walk}} > r)$ of Theorem 3. $\square$

Proposition 6 (Recovery of the phase-tagged first-excess functional). *Suppressing pre-failure tracking ($u = 1, \vartheta = 0$) reduces Φ_v^{ruin} to the moving-boundary analogue of the phase-tagged first-excess functional Ψ of [1]: $\Psi^{\text{ruin}}(\xi, \theta, v) = \Phi_v^{\text{ruin}}(\xi, 1, v, 0, \theta)$.*

Remark 21 (The $c \rightarrow 0$ limit and the fixed-threshold functional). *At $c = 0$, the boundary is the fixed level u_0 , the ruin index becomes the first-excess index over u_0 , and the recursion (44) becomes homogeneous in the level, collapsing to the matrix-geometric series of [1] with threshold $p_1 = u_0$. The first-passage transform Ω_r then admits the Sherman–Morrison closed form, and $\Phi_v^{\text{ruin}}|_{c=0}$ coincides with the fixed-threshold functional Φ_v of [2]. For $c \geq 1$, the boundary genuinely moves and the first-passage resummation is essential; this is the structural content of the moving-boundary extension.*

8. Numerical Example

We illustrate the moving-boundary apparatus on a concrete ruin model and verify the closed-form indices against Monte Carlo simulation. As in [2], the model primitives are chosen rational so that the Lundberg polynomial, the ladder-height law, and every closed-form index reduce to elementary root-finding and finite recursion.

8.1. Model Specification

We instantiate the premium drift model of Section 2.1 as follows.

Inter-shock times.

The post-delay inter-claim times T_n ($n \geq 1$) are two-phase Erlang with rate $\mu = 1/2$ per phase, matching the example of [2]:

$$\alpha = (1, 0), \quad S = \begin{pmatrix} -\mu & \mu \\ 0 & -\mu \end{pmatrix}, \quad h(\theta) = \left(\frac{\mu}{\mu + \theta} \right)^2, \quad \mu_T = \mathbb{E}[T_1] = 4.$$

The delay T_0 is exponential with rate $\mu_0 = 1$, so $h_0(\theta) = \mu_0/(\mu_0 + \theta)$ and $\mu_{T_0} = 1$.

Claim sizes.

The post-delay claims X_n ($n \geq 1$) are integer-valued on $\{0, 1, 2, 3, 4\}$ with

$$\mathbb{P}(X_1 = k) = (0.35, 0.25, 0.20, 0.12, 0.08)_k, \quad \mathbb{E}[X_1] = 1.33.$$

The delay claim X_0 has a distinct law $\mathbb{P}(X_0 = k) = (0.30, 0.25, 0.20, 0.15, 0.10)_k$, $\mathbb{E}[X_0] = 1.50$, exercising the $a_0 \neq a$ structure.

Premium and reserve.

The premium rate is $c = 2$ per period and the initial reserve $u_0 = 5$. The net-profit condition $\mathbb{E}[X_1] = 1.33 < 2 = c$ holds, so ruin is not certain.

The maximal up-jump is $J = \max \text{supp}(a) - c = 4 - 2 = 2$, so the ladder-height law is genuinely non-degenerate (supported on $\{1, 2\}$); this is the generic compound-geometric regime of Theorem 3, in contrast to the skip-free case $J = 1$ of Corollary 3.

8.2. Closed-Form Computation

The Lundberg polynomial $Q(z) = z^2 - a(z)$ has dominant root $z^* = 2.090026$ outside the unit disk, giving the ladder scalar and adjustment coefficient

$$g(c) = 1/z^* = 0.478463, \quad R = \log z^* = 0.737177.$$

The two outside roots of Q yield, via Proposition 2, the continuation probability $p_L = \mathbb{P}(M > 0) = 0.350526$ and the ascending ladder-height law

$$\chi(z) = 0.665161 z + 0.334839 z^2 \quad \text{on } \{1, 2\}.$$

From these, the maximum of the net-claim walk has the rational generating function (17), and the closed-form ruin indices follow. Table 1 reports them alongside Monte Carlo estimates.

Table 1. Closed-form versus simulated values of the principal ruin indices for the model of Section 8.1, over $N = 4 \times 10^5$ Monte Carlo trajectories. Each index is reported in joint $(\cdot; \text{ruin})$ form, i.e., as $\mathbb{E}[\cdot \mathbf{1}_{\{\tau_v < \infty\}}]$, the unconditional quantity weighted by the ruin indicator; the corresponding *conditional* expectations $\mathbb{E}[\cdot \mid \text{ruin}]$, obtained by dividing each joint value by $\psi(u_0)$, are reported separately in Section 8.3 and differ from the joint values by approximately two orders of magnitude. For all four indices, the absolute error between the closed form and the Monte Carlo mean lies below the 95% Monte Carlo standard error. The same model is used for the figure of Section 8.3; the larger sample size $N = 4 \times 10^5$ is used here so that the four indices of the table each attain comparable precision, while the figure (reporting only $\psi(u_0)$ at seven reserves) uses $N = 3 \times 10^5$.

Index	Closed Form	Monte Carlo	MC s.e.
Ruin probability $\psi(u_0)$	0.049897	0.049615	0.000343
$\mathbb{E}[U^+; \text{ruin}]$ (deficit)	0.058865	0.058498	0.000427
$\mathbb{E}[U^-; \text{ruin}]$ (surplus prior)	0.027477	0.027400	0.000258
$\mathbb{E}[\tau_v; \text{ruin}]$ (ruin time)	1.066193	1.060529	0.010200

8.3. Monte Carlo Verification

We ran $N = 4 \times 10^5$ independent trajectories, each generated by sampling X_0 and the exponential delay, then iterating the Erlang-2 inter-claim time and the claim until the net-claim walk first crosses u_0 , recording the time of ruin, the deficit, and the surplus prior to ruin. Table 1 compares the empirical means against the closed-form values, with the 95% Monte Carlo standard error.

Dividing by the ruin probability gives the conditional quantities most meaningful to an actuary: given that ruin occurs, the expected deficit is $\mathbb{E}[U^+ \mid \text{ruin}] = 1.179729$ (simulated 1.179029), the expected surplus just before ruin is $\mathbb{E}[U^- \mid \text{ruin}] = 0.550676$ (simulated 0.552252), and the expected time of ruin is $\mathbb{E}[\tau_v \mid \text{ruin}] = 21.367945$ (simulated 21.375162). All four closed-form indices agree with simulation to within Monte Carlo noise, confirming the moving-boundary central formula and its specializations. Figure 1 plots the ruin probability $\psi(u_0)$ as a function of the initial reserve u_0 , comparing the closed-form compound-geometric value with the pure exponential Lundberg asymptote and Monte Carlo estimates.

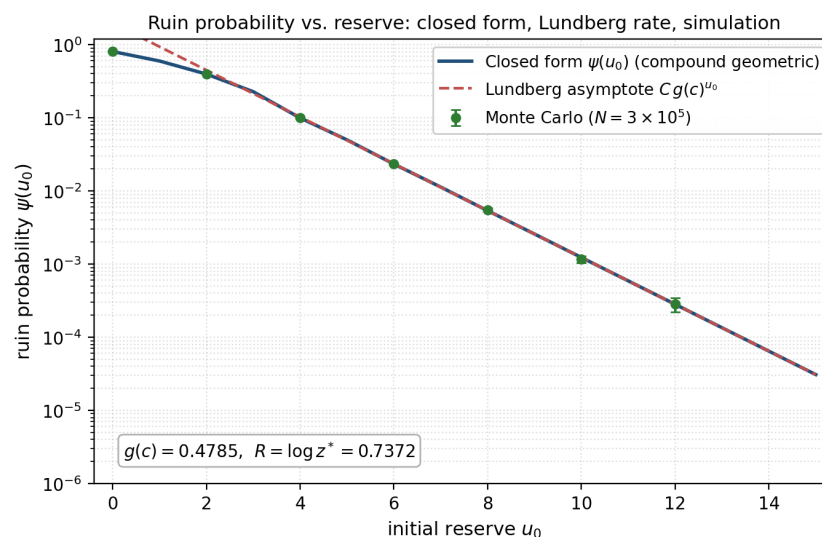


Figure 1. Ruin probability $\psi(u_0)$ as a function of the initial reserve u_0 for the model of Section 8.1. The solid curve is the closed-form compound-geometric value (17); the dashed line is the pure exponential Lundberg asymptote $Cg(c)^{u_0}$ with decay rate $g(c) = 0.478463$ (adjustment coefficient $R = \log z^* = 0.737177$); the points are Monte Carlo estimates over $N = 3 \times 10^5$ trajectories with 95% error bars. The exact curve and the Lundberg asymptote share the same geometric slope on the logarithmic scale, confirming $g(c)$ as the exact decay rate, while the compound-geometric closed form is required for the finite-reserve values near the origin where the pure exponential overestimates.

8.4. Robustness Across Parameter Regimes

To assess the framework outside the specific parameter values of Section 8.1, we evaluate the closed-form indices on three additional model variants chosen to probe distinct features of the moving boundary: a higher premium $c = 3$, a heavier-tailed claim distribution with mass on the upper end of the support, and a regime close to the net-profit boundary $\mathbb{E}[X_1]/c \approx 1$. In every variant, the inter-shock and delay distributions of Section 8.1 are retained; only the claim law and the premium are altered. Table 2 reports the resulting indices alongside Monte Carlo estimates.

The closed-form indices agree with the Monte Carlo estimates in every variant. Three observations are worth noting. First, $\psi(u_0)$ varies over more than *two orders of magnitude* across the four variants, from 1.0×10^{-3} (higher premium) to 6.2×10^{-1} (near boundary), and the geometric decay rate $g(c)$ correspondingly varies from 0.09 to 0.89, yet each value is reproduced by the closed form via the same polynomial root-finding. Second, the skip-free case (variant ii) exhibits exactly the degeneracies anticipated by Corollary 3: the ladder-height law collapses to a unit mass at 1, the deficit law is the ladder-height law, and the surplus prior to ruin is identically zero. The non-skip-free variants (iii) and (iv) by contrast show genuinely two-dimensional ladder-height laws with the moving-boundary structure of the general central formula. Third, comparison of variants (i) and (iv) shows the expected

approach to certain ruin as $\mathbb{E}[X_1]/c \uparrow 1$: the ladder continuation probability p_L rises from 0.35 to 0.86 and the mean ruin time $\mathbb{E}[\tau_v \mid \text{ruin}]$ rises from 21.4 to 170.0, both reflecting the vanishing safety loading.

Table 2. Closed-form ruin indices across four parameter regimes. Variant (ii) shifts the premium to $c = 3$ on the baseline claim law, reducing the maximal up-jump to $J = 1$ and producing the skip-free regime of Corollary 3—note that the deficit U^+ equals the ladder-height law in this case, which is degenerate at 1, and the surplus prior to ruin is identically zero. Variant (iii) uses $p = (0.45, 0.30, 0.05, 0.05, 0.15)$ on $\{0, \dots, 4\}$, shifting mass toward larger claims while preserving net profit; the ladder-height law becomes $\chi = (0.4385, 0.5615)$, putting majority mass on the larger of the two up-jumps. Variant (iv) uses $p = (0.20, 0.20, 0.25, 0.20, 0.15)$, giving $\mathbb{E}[X_1] = 1.90$ very close to $c = 2$; the ruin probability rises by an order of magnitude and the mean ruin time by two orders. Monte Carlo trajectories: $N = 4 \times 10^5$ for (i)–(iii) at horizon 400, and $N = 2 \times 10^5$ at horizon 5000 for (iv) since the mean ruin time of ≈ 105 requires longer trajectories (see Remark 22).

Variant	$\mathbb{E}[X_1]/c$	$g(c)$	J	$\psi(u_0)$	$\mathbb{E}[U^+; \mathbf{r}]$	$\mathbb{E}[\tau_v; \mathbf{r}]$
(i) Baseline (cf. Table 1)	0.665	0.4785	2	0.0499	0.0589	1.066
(ii) Higher premium ($c = 3$)	0.443	0.0931	1	0.00101	0.00101	0.01139
(iii) Heavier-tailed claims	0.575	0.5070	2	0.0616	0.0737	0.957
(iv) Near net-profit boundary	0.950	0.8944	2	0.6158	0.7647	104.7
Monte Carlo $\psi(u_0)$ for comparison:						
(i) 0.0506 ± 0.00035 , (ii) 0.00110 ± 0.00005 ,				(iii) 0.0614 ± 0.00038 ,	(iv) 0.6143 ± 0.00109	

Remark 22 (Monte Carlo horizon near the net-profit boundary). *The simulation horizon required to validate variant (iv) is an order of magnitude longer than that of (i)–(iii): mean ruin times rise from ≈ 1 in the baseline to ≈ 105 near the boundary, and a horizon of 400 truncates a non-negligible fraction of slow ruin paths. The closed-form indices, by contrast, are evaluated by polynomial root-finding and finite recursion regardless of the regime, with no convergence parameter that deteriorates as $\mathbb{E}[X_1]/c \rightarrow 1$. The contrast illustrates a structural advantage of the closed-form framework over direct simulation: as the safety loading shrinks, ruin times grow, simulation cost grows with them, but the closed-form cost remains constant.*

Three features of the example are worth noting. *First*, the entire computation reduces to root-finding on the degree-2 Lundberg polynomial followed by finite recursion; no matrix inversion or iteration is required, and the closed-form indices are evaluated in negligible time compared with the $N = 4 \times 10^5$ simulation. *Second*, the identity $U^- + U^+ = X_v - c$ of Theorem 5 is visible at the mean level: $\mathbb{E}[U^- \mid \text{ruin}] + \mathbb{E}[U^+ \mid \text{ruin}] = 0.5507 + 1.1797 = 1.7304$, the conditional mean of the fatal net increment $X_v - c$. *Third*, the non-degenerate ladder-height law χ (with mass on both 1 and 2) shows that the ruin probability is a genuine compound geometric rather than the pure geometric of the skip-free case; the geometric decay rate is nonetheless exactly $g(c) = 0.478463$, the Lundberg estimate, while the finite-reserve value $\psi(5) = 0.049897$ carries the Cramér–Lundberg constant determined by χ and p_L .

9. Discussion

We have extended the phase-tagged fluctuation framework from a fixed failure threshold to a linearly moving boundary, the premium drift regime of insurance ruin theory, and obtained the full five-variable moving-boundary functional in closed form. We close by recording the structural lessons, the limitations of the present treatment, and directions for further work.

9.1. What the Drift Changes, and What It Does Not

The central structural finding is a clean separation between matrix geometry and scalar content. The rank-one phase-reset matrix $\mathbf{e}\alpha$ is the common generator across all three settings: it produces the first-passage matrices of the fixed-threshold matrix-analytic bridge [1,2], it generates the low-dimensional span of the phase-tagged reliability functional, and, through the discrete Lundberg deformation $g(c) \mathbf{e}\alpha$, it generates the moving-boundary ladder matrix of Theorem 1. In all three, the rank-one structure is a direct consequence of the phase resetting to α at each renewal; the premium drift enters only through the scalar Lundberg deformation $1 \mapsto g(c)$ and leaves the matrix subspace invariant. The moving boundary thus changes the *scalar* resummation, replacing the $\mathfrak{D}$ -operator partial sum and its Sherman–Morrison geometric by a first-passage transform with a genuinely different analytic character, while preserving the *matrix* subspace span $\{H(\theta), H(\omega)\}$.

This separation also delimits the method. The rank-one structure, and with it the closed forms, rests on the phase-reset hypothesis. For Markovian-arrival (MAP) claim streams, where the phase does not reset at each claim, the ladder matrix is full rank, the discrete Lundberg scalar is replaced by a matrix adjustment equation, and the closed forms give way to the iterative matrix-analytic machinery of the standard ruin literature [3]. This is the same boundary already visible in the fixed-threshold theory [1,2], now inherited by the moving-boundary extension.

9.2. Limitations

Three modelling choices bound the present treatment. (i) *Integer premium and lattice boundary*. The bounded-jump hypothesis (D1) keeps the moving boundary on the integer lattice and truncates the ladder-height support, which is what renders the ladder-height law a finite product over polynomial roots. A continuous premium rate replaces the polynomial Lundberg equation by a transcendental one and the finite ladder-height law by a continuous overshoot distribution; the rank-one matrix structure survives, but the algebraic closed forms do not. (ii) *Net-profit condition*. We assume $\mathbb{E}[X_1] < c$, so that ruin is not certain and the ladder process is defective; this is the standard positive-safety-loading regime. (iii) *Renewal claim sizes with phase reset*. As noted above, MAP claims fall outside the rank-one regime.

9.3. Directions

Several extensions appear within reach. The continuous premium case requires replacing the Lundberg polynomial by its transcendental analogue and the ladder-height product by a contour-integral overshoot representation; the matrix span statement should persist. Dividend-barrier and tax-threshold variants introduce a second moving boundary and would test whether the first-passage transform composes. Finally, the MAP-claim case, where the rank-one structure fails, is the natural setting in which to quantify precisely how much of the closed-form tractability is lost and what minimal matrix apparatus replaces the scalar Lundberg deformation, mirroring, for the moving boundary, the analysis of where rank-one structure breaks in the fixed-threshold theory.

The broader point is methodological. The phase-tagged fluctuation functional, the ladder/first-passage apparatus, and the matrix-analytic phase machinery have largely developed as separate traditions. The moving-boundary functional shows they compose: ladder heights supply the scalar first-passage content that the moving boundary demands, phase tagging supplies the operational phase at ruin, and the rank-one reset structure keeps the combination in closed form. The result is a phase-resolved, joint Gerber–Shiu analysis (time of ruin, deficit, and surplus prior to ruin, tagged by phase) of a kind not previously available for phase-type inter-claim times.

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Data Availability Statement: The data presented in this study consist of Monte Carlo simulation results generated by a Python 3.11 (Python Software Foundation, Wilmington, DE, USA; <https://python.org>) implementation of the closed-form formulas described in Section 8. The code that generates the data is available from the corresponding author upon reasonable request.

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Appendix A. Self-Contained Statement of Results Imported from [1]

This appendix collects, in self-contained form, the results from [1] that are imported into the present paper. The statements below are paraphrased from [1] with notation adapted to the present setting; the proofs are not reproduced. The reader who wishes to follow only the moving-boundary analysis of the main text needs no further material than what is collected here.

Appendix A.1. The Phase-Type Machinery

The phase-type distribution $\text{PH}(\boldsymbol{\alpha}, S)$ with m -state initial vector $\boldsymbol{\alpha}$ (row, $\boldsymbol{\alpha} \mathbf{e} = 1$) and sub-generator S ($m \times m$, with strictly negative real-part eigenvalues and absorption vector $\mathbf{s} = -S\mathbf{e}$) is the distribution of the absorption time of a continuous-time Markov chain on m transient states. The scalar Laplace–Stieltjes transform is $h(\theta) = \mathbb{E}[e^{-\theta T}] = \boldsymbol{\alpha}(\theta I - S)^{-1}\mathbf{s}$ for $\text{Re}(\theta) \geq 0$, and the matrix LST is

$$H(\theta) := (\theta I - S)^{-1}\mathbf{s}\boldsymbol{\alpha} = \mathbf{v}(\theta)\boldsymbol{\alpha}, \quad \mathbf{v}(\theta) := (\theta I - S)^{-1}\mathbf{s}.$$

The matrix $H(\theta)$ is rank-one with $h(\theta) = \boldsymbol{\alpha} H(\theta) \mathbf{e}$ and satisfies the four algebraic properties of Proposition 1, all imported verbatim from ([1], Proposition 2.5, Theorem 2.6): (i) $h(\theta) = \boldsymbol{\alpha} H(\theta) \mathbf{e}$; (ii) $H(\theta) = \mathbf{v}(\theta)\boldsymbol{\alpha}$ is rank-one; (iii) $H(\theta)^2 = h(\theta) H(\theta)$ (rank-one idempotency); (iv) $H(0) = \mathbf{e}\boldsymbol{\alpha}$; (v) Sherman–Morrison: for any scalar c with $|c| \cdot |h(\theta)| < 1$,

$$(I - c H(\theta))^{-1} = I + \frac{c}{1 - c h(\theta)} H(\theta).$$

These properties are the algebraic engine of every closed-form result in both [1,2] and the present paper. They reduce every matrix Neumann series in $H(\theta)$ to a scalar geometric series times $H(\theta)$, collapsing the matrix complexity into the two-dimensional span $\text{span}\{I, H(\theta)\}$.

Appendix A.2. The Phase-Tagged First-Excess Functional

The setting of [1] is a phase-type compound renewal process: a sequence of renewal epochs $\tau_0 < \tau_1 < \dots$ with inter-renewal times $T_n \sim \text{PH}(\boldsymbol{\alpha}, S)$ ($n \geq 1$), independent batch arrivals X_n with PGF $a(u)$, and the cumulative process $A_n = X_0 + X_1 + \dots + X_n$. Fix an integer threshold $p_1 \in \mathbb{Z}_+$ and let

$$v_1 := \inf\{n \geq 0 : A_n > p_1\}$$

be the first-excess index. The *phase-tagged first-excess functional* of ([1], Definition 3.1) is the $m \times m$ matrix

$$\boldsymbol{\Psi}(\boldsymbol{\zeta}, \theta, u) := \mathbb{E}[\boldsymbol{\zeta}^{v_1} e^{-\theta \tau_{v_1}} u^{A_{v_1}}; J(\tau_{v_1})], \quad (\text{A1})$$

whose (i, j) entry tracks the joint distribution of the excess index ν_1 , the excess time τ_{ν_1} , the excess level A_{ν_1} , and the phase at the excess epoch, conditional on starting in phase i .

Theorem A1 (Central formula of [1], Theorem 3.5). *Under the standard assumptions of ([1], Assumption 2.9),*

$$\frac{1}{h_0(\theta)} \Psi(\xi, \theta, u) = a_0(u) I - (I - \xi a(u) H(\theta)) \mathfrak{D}_x^{p_1} \left\{ \frac{a_0(ux) I}{I - \xi a(ux) H(\theta)} \right\}, \quad (\text{A2})$$

where $\mathfrak{D}_x^{p_1}$ is the $\mathfrak{D}$ -operator of Proposition 1's preamble. The matrix Neumann series $(I - \xi a(ux) H(\theta))^{-1}$ is collapsed by Sherman–Morrison (Proposition 1(v)) into the form $\phi_1(\xi, \theta, u) I + \phi_2(\xi, \theta, u) H(\theta)$ for scalar ϕ_1, ϕ_2 , and the $\mathfrak{D}$ -operator then acts entrywise via ([1], Lemma 2.13).

The structural consequence is the span-reduction theorem of ([1], Theorem 3.6): $\Psi(\xi, \theta, u) \in \text{span}\{I, H(\theta)\}$ for all admissible (ξ, θ, u) , despite the matrix being $m \times m$. The scalar functional of Dshalalow [7,8] is recovered as $\mathbb{E}[\xi^{\nu_1} e^{-\theta \tau_{\nu_1}} u^{A_{\nu_1}}] = \alpha \Psi(\xi, \theta, u) \mathbf{e}$ ([1], Theorem 3.8).

Appendix A.3. Boundary Evaluations Giving Neuts's $\mathbf{G}$

Theorem A2 ([1], Theorem 5.1). *Set $p_1 = 0$, $\alpha_0 = \alpha$, $S_0 = S$. Then Neuts's first-passage matrix $\mathbf{G}$ is*

$$\mathbf{G} = \Psi(1, 0, 1)|_{p_1=0} = (1 - p_{00}) I + p_{00} \mathbf{e} \alpha, \quad (\text{A3})$$

where $p_{00} = \mathbb{P}(X_0 = 0)$. The matrix $\mathbf{G}$ is stochastic ($\mathbf{G} \mathbf{e} = \mathbf{e}$), has eigenvalues 1 (multiplicity 1) and $1 - p_{00}$ (multiplicity $m - 1$), and satisfies the fixed-point equation $\mathbf{G} = (1 - p_{00}) I + p_{00} \mathbf{e} \alpha$ ([1], Theorem 5.2).

Theorem A3 ([1], Theorem 5.5; discounted Neuts $\mathbf{G}$). *For $\text{Re}(\theta) > 0$, the discounted Neuts matrix is*

$$\mathbf{G}(\theta) = \Psi(1, \theta, 1)|_{p_1=0} = h_0(\theta) \left[(1 - p_{00}) I + \kappa_0(\theta) H(\theta) \right], \quad (\text{A4})$$

where $\kappa_0(\theta) = p_{00} + (h(\theta) - 1) p_{00} p_0 / (1 - p_0 h(\theta))$ and $p_0 = a(0)$. The undiscounted matrix is recovered as $\mathbf{G} = \lim_{\theta \rightarrow 0^+} \mathbf{G}(\theta)$.

The contrast with the present paper is captured in Remark 12: the matrix $\mathbf{G}(\theta)$ of (A4) lives in $\text{span}\{I, H(\theta)\}$ and its matrix content moves with θ ; the moving-boundary discounted ladder matrix $\mathbf{G}^\uparrow(c, \theta) = g(c, \theta) \mathbf{e} \alpha$ of Theorem 2 lives in $\text{span}\{\mathbf{e} \alpha\}$ and is θ -invariant in its matrix factor, with the entire discount content riding on the scalar $g(c, \theta)$.

Appendix A.4. The Factorization Theorem

Theorem A4 ([1], Theorem 7.1; matrix Wiener–Hopf factorization). *The phase-tagged first-excess functional factorizes as*

$$\Psi(\xi, \theta, u) = \Psi^{(0)}(\xi, \theta, u) + \mathbf{T}(\xi, \theta, u) \xi a(u) H(\theta) \mathbf{G}(\theta), \quad (\text{A5})$$

where $\Psi^{(0)} = h_0(\theta)(a_0(u) - d_1(u)) I$ is the immediate-excess contribution and $\mathbf{T}$ is the taboo functional of ([1], Theorem 6.4). This factorization is the matrix analogue of the classical scalar Wiener–Hopf factorization [3,4], with $\mathbf{T} \cdot \xi a(u) H(\theta)$ playing the role of the ascending and $\mathbf{G}(\theta)$ that of the descending ladder transform.

The factorization (A5) is the structural device that lets us, in the present paper, separate the moving-boundary first crossing into an ascending ladder process (Section 3) governed

by the drift ladder matrix $G^\uparrow(c)$, and a passage-time analysis governed by the discounted ladder $G^\uparrow(c, \theta)$.

Appendix A.5. Imported Items Used in This Paper, Indexed

For convenience, we list the specific imported items of [1] and where each is used in the present paper:

- Algebraic properties of $H(\theta)$ ([1], Proposition 2.5, Theorem 2.6) $\rightarrow$ Proposition 1; used throughout.
- Phase-reset hypothesis $J(\tau_n) \sim \alpha$ ([1], Assumption 2.9 and Section 2.1.2) $\rightarrow$ Assumption 1(D3); foundational for the rank-one structure of $G^\uparrow(c)$.
- Central formula for Ψ ([1], Theorem 3.5; Theorem A1 above) $\rightarrow$ referenced in Sections 6 and 7.
- Span reduction $\Psi \in \text{span}\{I, H(\theta)\}$ ([1], Theorem 3.6) $\rightarrow$ contrasted with the present two-dimensional span $\text{span}\{H(\theta), H(\omega)\}$ in Section 6.
- Boundary evaluation: $G = \Psi(1, 0, 1)|_{p_1=0}$ ([1], Theorem 5.1; Theorem A2) $\rightarrow$ analogue role for the present $G^\uparrow(c)$ at the moving boundary, cf. Definition 3.
- Fixed-point equation for G ([1], Theorem 5.2) $\rightarrow$ cited in Remark 9 as the structural source of the naive level-decrement argument that does not transport to the moving-boundary setting.
- Discounted Neuts matrix ([1], Theorem 5.5; Theorem A3) $\rightarrow$ contrasted with the θ -invariant matrix factor of $G^\uparrow(c, \theta)$ in Remark 12.
- Factorization theorem ([1], Theorem 7.1; Theorem A4) $\rightarrow$ structural source of the ascending/descending ladder decomposition we deploy in Section 3.

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