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Hyers–Ulam Stability of a Multi-Variable Additive-Quadratic Functional Equation

Jae-Hyeong Bae ¹ and Won-Gil Park ^{2,*}¹ School of Liberal Studies, Kyung Hee University, Yongin 17104, Republic of Korea; jhbae@khu.ac.kr² Department of Mathematics Education, College of Education, Mokwon University, Daejeon 35349, Republic of Korea

* Correspondence: wgpark@mokwon.ac.kr

Abstract

In this paper, we investigate the generalized Hyers–Ulam stability of a multi-variable additive-quadratic functional equation in Banach spaces. Unlike traditional mixed-type equations that segregate variables, the proposed equation embeds a dense combinatorial cross-variable interaction. To overcome the complex analytical challenges arising from asymmetric scaling weights, we prove a complete structural equivalence theorem between this multivariate framework and the classical additive-quadratic system of equations. We establish comprehensive stability results utilizing two distinct methodological paradigms: the direct constructive aggregation method and the generalized metric fixed-point alternative. Furthermore, we provide a rigorous mathematical justification of an analytical singularity by constructing a counterexample that explicitly proves the inherent instability of the system at the critical exponent threshold $p = 3$.

Keywords: Banach space; functional equation; Hyers–Ulam stability; fixed point**MSC:** 39B52; 39B82

1. Introduction

The question of the stability in functional equations stems from a fundamental query regarding the relationship between mathematical models and their structural perturbations. Specifically, it concerns under what conditions an approximate solution to a functional equation guarantee the existence of a true solution near it? This formulation was first proposed by S. M. Ulam [1] in 1940 before the Mathematical Colloquium at the University of Wisconsin, concerning the stability of group homomorphisms. Specifically, given a group G_1 , a metric group (G_2, d) , and a positive constant ε , does there exist a positive constant δ such that if a mapping $f : G_1 \rightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a true homomorphism $H : G_1 \rightarrow G_2$ exists with $d(f(x), H(x)) < \varepsilon$?

In 1941, D. H. Hyers [2] provided the first celebrated affirmative answer to Ulam’s question for the case of additive mappings on Banach spaces, proving that the Cauchy functional equation $f(x + y) = f(x) + f(y)$ is stable under bounded error bounds. This pioneering result established the concept now known as “classical Hyers–Ulam stability.” In 1950, T. Aoki [3] generalized Hyers’ theorem by allowing the error bound to be unbounded, replacing the constant bound with a sum of powers of norms. Subsequently, Th. M. Rassias [4] independently rediscovered and significantly extended this concept by considering the unbounded control function involving the sum of powers of norms, $\theta(\|x\|^p + \|y\|^p)$,



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for $p < 1$ or $p > 1$. This approach, which allows the error term to scale with the magnitude of the inputs, is known as the generalized Hyers–Ulam–Rassias stability. A further critical expansion was achieved by P. Găvruta [5], who substituted the power-law control bounds with a control function $\phi(x, y)$, satisfying a certain convergence condition.

In parallel with the development of stability theory for purely additive mappings, substantial research has been dedicated to the quadratic functional equation of the form:

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad (1)$$

whose solutions are termed quadratic mappings because they map elements to quadratic forms when restricted to real vector spaces. The stability of Equation (1) was extensively studied by several authors (e.g., Aczél and Dhombres [6], J.-H. Bae and I.-S. Chang [7], and Bae and Jun [8], etc.). More recently, researchers have focused on mixed-type functional equations, such as additive-quadratic or cubic-quadratic systems, which generalize individual algebraic structures into combined multi-variable frameworks.

Definition 1 ([9]). A mapping $f : X \times Y \rightarrow Z$ is called an additive-quadratic mapping if f satisfies the system of two functional equations

$$\begin{aligned} f(x + y, z) &= f(x, z) + f(y, z), \\ f(x, y + z) + f(x, y - z) &= 2f(x, y) + 2f(x, z). \end{aligned} \quad (2)$$

In 2005, W.-G. Park, J.-H. Bae and B.-H. Chung [9] obtained the general solution of the additive-quadratic functional equation

$$f(x + y, u + v) + f(x + y, u - v) = 2f(x, u) + 2f(x, v) + 2f(y, u) + 2f(y, v) \quad (3)$$

and its stability. In ref. [9], they proved that a mapping $f : X \times Y \rightarrow Z$ satisfies (2) if and only if it satisfies (3).

In 2001, J.-H. Bae and I.-S. Chang [7] introduced the following quadratic functional equation:

$$f(x + y + z) + f(x - y) + f(y - z) + f(x - z) = 3f(x) + 3f(y) + 3f(z). \quad (4)$$

In ref. [7], they obtained the solution to the Equation (4), proved the Hyers–Ulam stability of the equation in Banach spaces, and also proved the Hyers–Ulam stability of the equation on bounded domains. In 2001, J.-H. Bae and K.-W. Jun [8] investigated the following multi-variable quadratic functional equation:

$$f(x_1 + x_2 + \cdots + x_n) + \sum_{1 \leq i < j \leq n} f(x_i - x_j) = n \sum_{i=1}^n f(x_i), \quad (5)$$

where n is an integer greater than 1. In ref. [8], they proved the generalized Hyers–Ulam–Rassias stability of the Equation (5) in Banach spaces.

1.1. Structural Novelty and Cross-Variable Interactions

While the literature contains numerous results on mixed-type equations ([10,11], etc.), this paper introduces and analyzes a novel multi-variable additive-quadratic functional equation of the following form:

$$f\left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\right) + \sum_{1 \leq i < j \leq n} f\left(\sum_{k=1}^n x_k, y_i - y_j\right) = n \sum_{i=1}^n \sum_{j=1}^n f(x_i, y_j). \quad (6)$$

where $n > 1$ is a fixed integer, and the unknown mapping $f : X \times Y \rightarrow Z$ is expected to be additive in the first variable and quadratic in the second variable. From refs. [12,13], we obtain interesting results concerning multi-variable additive-quadratic mappings. These include stability results for equations closely related to Equation (6).

Equation (6) differs fundamentally from previously studied mixed or multi-variable functional equations in terms of its ideal structure. In conventional mixed-type multi-variable equations, variables are typically segregated or coupled in linear combinations that do not scale with the system size n . In contrast, Equation (6) embeds a dense cross-variable interaction. The second term on the left-hand side, $\sum_{1 \leq i < j \leq n} f(\sum_{k=1}^n x_k, y_i - y_j)$, couples the collective sum of the additive components ($\sum_{k=1}^n x_k$) with all possible pairwise differences of the quadratic components ($y_i - y_j$).

1.2. Analytical Challenges in Stability Derivations

This structural configuration introduces several severe analytical difficulties that prevent the straightforward application of classical stability techniques:

In the proof of intertwined system equivalence, establishing that any solution to (6) is strictly additive-quadratic requires isolating the individual variables from highly dense summations. Because the variables are deeply coupled inside the combinatorial operators, standard substitutions (e.g., setting elements to zero) yield large remaining coefficients that must be canceled out using exact algebraic representations, which requires the representation theorem by Aczél and Dhombres [6].

To place this study within the broader context of modern stability analysis, readers may wish to explore the following related phenomena:

- **Hyperstability:** Cases where an approximate solution satisfying a specific control condition is forced to be an exact solution. (see refs. [14,15])
- **Superstability:** Situations where the boundedness of the functional evaluator automatically implies that the mapping belongs to a strict algebraic class. (See refs. [16,17])

The primary objective of this paper is to overcome these analytical barriers. We establish the complete equivalence between (6) and the standard additive-quadratic system. Furthermore, we provide a unified presentation of generalized Hyers–Ulam stability under various control conditions, utilizing both a direct constructive methodology and a fixed-point alternative approach, thereby ensuring mathematical rigor and comprehensive completion for multi-variable systems.

2. Main Results

Lemma 1. *A mapping $f : X \rightarrow Z$ satisfies (1) if and only if it satisfies (5).*

Proof. Assume that f satisfies the functional Equation (1). Letting $x = y = 0$ in (1), we gain $f(0) = 0$. Putting $x = 0$ in (1), we get $f(-y) = f(y)$ for all $y \in X$. In Proposition 1, Section 11.1 in ref. [6], Aczél and Dhombres showed that there exists a symmetric bi-additive mapping $B : X \times X \rightarrow Z$ such that $f(x) = B(x, x)$ for all $x \in X$. Proposition 1 in ref. [6] holds true even if both the domain and the codomain are real vector spaces or complex vector spaces. Specifically, B can be defined as

$$B(x, y) = \frac{1}{4}[f(x + y) - f(x - y)]$$

for all $x, y \in X$. Hence, we obtain that

$$\begin{aligned}
& f(x_1 + x_2 + \cdots + x_n) + \sum_{1 \leq i < j \leq n} f(x_i - x_j) \\
&= \left[\sum_{i=1}^n f(x_i) + 2 \sum_{1 \leq i < j \leq n} B(x_i, x_j) \right] + \left[(n-1) \sum_{i=1}^n f(x_i) - 2 \sum_{1 \leq i < j \leq n} B(x_i, x_j) \right] \\
&= n \sum_{i=1}^n f(x_i)
\end{aligned}$$

for all $x_1, \dots, x_n \in X$. Hence, f satisfies the functional Equation (5).

On the contrary, assume that f satisfies the functional Equation (5). By letting $x_1 = \cdots = x_n = 0$ in (5), we have $f(0) = 0$. To show that f satisfies (1), we set $x_1 = x, x_2 = y$ and $x_3 = \cdots = x_n = 0$ in (5). This yields

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. Thus, f satisfies the functional Equation (1). This completes the proof. $\square$

Theorem 1. For a mapping $f : X \times Y \rightarrow Z$, it satisfies (2) if and only if it satisfies (6).

Proof. Assume that f satisfies the system of Equation (2). Since f is quadratic in the second variable, by Lemma 1, we have

$$f\left(x, \sum_{i=1}^n y_i\right) + \sum_{1 \leq i < j \leq n} f(x, y_i - y_j) = n \sum_{i=1}^n f(x, y_i)$$

for all $x \in X$ and all $y_1, \dots, y_n \in Y$. Replacing x with $\sum_{k=1}^n x_k$ in the above equation, we get

$$f\left(\sum_{k=1}^n x_k, \sum_{i=1}^n y_i\right) + \sum_{1 \leq i < j \leq n} f\left(\sum_{k=1}^n x_k, y_i - y_j\right) = n \sum_{i=1}^n f\left(\sum_{k=1}^n x_k, y_i\right)$$

for all $x_1, \dots, x_n \in X$ and all $y_1, \dots, y_n \in Y$. Since f is additive in the first variable, the right-hand side can be expanded as

$$n \sum_{i=1}^n f\left(\sum_{k=1}^n x_k, y_i\right) = n \sum_{i=1}^n \sum_{k=1}^n f(x_k, y_i) = n \sum_{k=1}^n \sum_{i=1}^n f(x_k, y_i)$$

for all $x_1, \dots, x_n \in X$ and all $y_1, \dots, y_n \in Y$. Thus, f satisfies the functional Equation (6).

On the contrary, assume that f satisfies the functional Equation (6). First, letting $x_1 = \cdots = x_n = 0$ and $y_1 = \cdots = y_n = 0$ in (6), we gain $f(0, 0) = 0$. Next, putting $x_1 = \cdots = x_n = 0$ in (6), we have

$$f(0, y_1 + y_2 + \cdots + y_n) + \sum_{1 \leq i < j \leq n} f(0, y_i - y_j) = n^2 \sum_{j=1}^n f(0, y_j)$$

for all $y_1, \dots, y_n \in Y$. Setting $y_1 = y$ and $y_2 = \cdots = y_n = 0$ in the above equation yields

$$f(0, y) = 0$$

for all $y \in Y$. Taking $y_1 = \cdots = y_n = 0$, we have

$$(n^2 - n + 2) f\left(\sum_{i=1}^n x_i, 0\right) = 2n^2 \sum_{i=1}^n f(x_i, 0)$$

for all $x_1, \dots, x_n \in X$. Setting $x_1 = x$ and $x_2 = \dots = x_n = 0$ in the above equation yields

$$f(x, 0) = 0$$

for all $x \in X$. Letting $x_1 = x$ and $x_2 = \dots = x_n = 0$ in (6), we have

$$f(x, y_1 + y_2 + \dots + y_n) + \sum_{1 \leq i < j \leq n} f(x, y_i - y_j) = n \sum_{j=1}^n f(x, y_j)$$

for all $x \in X$ and all $y_1, \dots, y_n \in Y$. This indicates that for any fixed $x \in X$, the mapping $f(x, \cdot)$ satisfies the n -variable quadratic functional Equation (5). By Lemma 1, $f(x, \cdot)$ satisfies the quadratic Equation (1) as follows:

$$f(x, y + z) + f(x, y - z) = 2f(x, y) + 2f(x, z)$$

for all $x \in X$ and all $y, z \in Y$. Setting $z = y$ in the above equation, we obtain

$$f(x, 2y) = 4f(x, y)$$

for all $(x, y) \in X \times Y$. Therefore, f is quadratic in the second variable, which inherently implies $f(x, -y) = f(x, y)$ for all $(x, y) \in X \times Y$. Finally, we show that f is additive in the first variable. Letting $x_1 = x$, $x_2 = w$, $x_3 = \dots = x_n = 0$, $y_1 = y$, $y_2 = -y$, and $y_3 = \dots = y_n = 0$ in (6), we have

$$2nf(x + w, y) = 2n[f(x, y) + f(w, y)]$$

for all $x, w \in X$ and $y \in Y$. Thus, we obtain

$$f(x + w, y) = f(x, y) + f(w, y)$$

for all $x, w \in X$ and $y \in Y$. This proves that f is additive in the first variable. Consequently, the mapping f satisfies the system of Equation (2). $\square$

From now on, let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be normed linear spaces and let $\mathcal{Z}$ be complete. For a mapping $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$, put

$$\begin{aligned} D_f(x_1, \dots, x_n, y_1, \dots, y_n) \\ := f\left(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\right) + \sum_{1 \leq i < j \leq n} f\left(\sum_{k=1}^n x_k, y_i - y_j\right) - n \sum_{i=1}^n \sum_{j=1}^n f(x_i, y_j) \end{aligned}$$

for all $x_1, \dots, x_n \in \mathcal{X}$ and all $y_1, \dots, y_n \in \mathcal{Y}$.

Theorem 2. Suppose that a mapping $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ satisfies the inequality

$$\|D_f(x_1, \dots, x_n, y_1, \dots, y_n)\| \leq \theta \sum_{i=1}^n (\|x_i\|^p + \|y_i\|^p) \quad (7)$$

for all $x_1, \dots, x_n \in \mathcal{X}$ and all $y_1, \dots, y_n \in \mathcal{Y}$, where $\theta \in [0, \infty)$ and $p \in (0, 3)$. Then there is a unique additive-quadratic mapping $H : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ satisfying the Equation (6) and the inequality

$$\|f(x, y) - H(x, y)\| \leq \frac{n\theta}{n^3 - n^p} \left[\left(1 + \frac{n^p}{n+2}\right) \|x\|^p + \|y\|^p \right] \quad (8)$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

Proof. Letting $x_1 = \cdots = x_n = 0$ and $y_1 = \cdots = y_n = 0$ in (7), we gain $f(0, 0) = 0$. Putting $x_1 = x, x_2 = \cdots = x_n = 0$ and $y_1 = \cdots = y_n = 0$ in (7), we get

$$\|f(x, 0)\| \leq \frac{2\theta}{(n-1)(n+2)} \|x\|^p \quad (9)$$

for all $x \in \mathcal{X}$. Setting $x_i = x$ and $y_i = y$ for all $i = 1, \dots, n$ in (7), we have

$$\left\| f(nx, ny) + \frac{n(n-1)}{2} f(nx, 0) - n^3 f(x, y) \right\| \leq n\theta(\|x\|^p + \|y\|^p)$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. By applying the triangle inequality and using (9), we obtain

$$\begin{aligned} \|f(nx, ny) - n^3 f(x, y)\| &\leq n\theta(\|x\|^p + \|y\|^p) + \frac{n(n-1)}{2} \|f(nx, 0)\| \\ &= n\theta \left(1 + \frac{n^p}{n+2} \right) \|x\|^p + n\theta \|y\|^p \end{aligned}$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Letting $a := n\theta \left(1 + \frac{n^p}{n+2} \right)$ and $b := n\theta$, we get

$$\left\| \frac{f(nx, ny)}{n^3} - f(x, y) \right\| \leq \frac{1}{n^3} (a\|x\|^p + b\|y\|^p)$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

For all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we consider the sequence $\{H_k(x, y)\}$ defined by $H_k(x, y) := \frac{f(n^k x, n^k y)}{n^{3k}}$ for all $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. For any integers $l > k \geq 0$, applying the triangle inequality yields

$$\begin{aligned} \|H_l(x, y) - H_k(x, y)\| &\leq \sum_{j=k}^{l-1} \|H_{j+1}(x, y) - H_j(x, y)\| \\ &= \frac{1}{n^3} \sum_{j=k}^{l-1} \left(\frac{n^p}{n^3} \right)^j (a\|x\|^p + b\|y\|^p) \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Thus, for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $\{H_k(x, y)\}$ is a Cauchy sequence in the complete normed linear space $\mathcal{Z}$. We define a mapping $H : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ by

$$H(x, y) := \lim_{k \rightarrow \infty} H_k(x, y)$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. By letting $k = 0$ and passing the limit $l \rightarrow \infty$, we obtain the error bound (8):

$$\begin{aligned} \|f(x, y) - H(x, y)\| &\leq \frac{1}{n^3} \sum_{j=0}^{\infty} \left(\frac{n^p}{n^3} \right)^j (a\|x\|^p + b\|y\|^p) \\ &= \frac{1}{n^3 - n^p} (a\|x\|^p + b\|y\|^p) = \frac{1}{n^3 - n^p} (a\|x\|^p + b\|y\|^p) \\ &= \frac{n\theta}{n^3 - n^p} \left[\left(1 + \frac{n^p}{n+2} \right) \|x\|^p + \|y\|^p \right] \end{aligned}$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Next, we show that H satisfies the functional Equation (6). For all $x_1, \dots, x_n \in \mathcal{X}$ and all $y_1, \dots, y_n \in \mathcal{Y}$, from the definition of H , we have

$$\begin{aligned}
\|D_H(x_1, \dots, x_n, y_1, \dots, y_n)\| &= \lim_{k \rightarrow \infty} \frac{1}{n^{3k}} \|D_f(n^k x_1, \dots, n^k y_n)\| \\
&\leq \lim_{k \rightarrow \infty} \frac{\theta}{n^{3k}} \sum_{i=1}^n (\|n^k x_i\|^p + \|n^k y_i\|^p) \\
&= \theta \sum_{i=1}^n (\|x_i\|^p + \|y_i\|^p) \lim_{k \rightarrow \infty} \left(\frac{n^p}{n^3}\right)^k = 0,
\end{aligned}$$

which means H satisfies (6) exactly.

Now, let $L : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ be another additive-quadratic mapping satisfying the Equation (6) and the inequality (8). Since both H and L are additive-quadratic mappings, by Theorem 3 in [9], we have $H(n^k x, n^k y) = n^{3k} H(x, y)$ and $L(n^k x, n^k y) = n^{3k} L(x, y)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and all $k \in \mathbb{N}$. According to (8), we obtain

$$\begin{aligned}
\|H(x, y) - L(x, y)\| &= \frac{1}{n^{3k}} \|H(n^k x, n^k y) - L(n^k x, n^k y)\| \\
&\leq \frac{1}{n^{3k}} \left(\|H(n^k x, n^k y) - f(n^k x, n^k y)\| + \|f(n^k x, n^k y) - L(n^k x, n^k y)\| \right) \\
&\leq \frac{2n\theta}{n^3 - n^p} \left(\frac{n^p}{n^3}\right)^k \left[\left(1 + \frac{n^p}{n+2}\right) \|x\|^p + \|y\|^p \right] \rightarrow 0
\end{aligned}$$

as $k \rightarrow \infty$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Therefore, we conclude that $H(x, y) = L(x, y)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. $\square$

For our additive-quadratic Equation (6) with $n = 2$, we can construct a counterexample to show that the stability result does not hold when $p = 3$. We first introduce a lemma required to show that this is a counterexample.

Lemma 2. Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(t) := \begin{cases} t \log_2 |t|, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Then we have

$$|g(x_1 + x_2) - g(x_1) - g(x_2)| \leq \frac{2(|x_1| + |x_2|)}{\ln 2}$$

for all $x_1, x_2 \in \mathbb{R}$.

Proof. If $x_1 = 0$ or $x_2 = 0$, then $g(x_1 + x_2) = g(x_1) + g(x_2)$ and so

$$|g(x_1 + x_2) - g(x_1) - g(x_2)| \leq \frac{2(|x_1| + |x_2|)}{\ln 2}.$$

Let $x_1 \neq 0$ and $x_2 \neq 0$, and let $\ell := |x_1| + |x_2|$ and $A := x_1 + x_2$. Then we have

$$g(A) - g(x_1) - g(x_2) = A \log_2 \frac{|A|}{\ell} + x_1 \log_2 \frac{\ell}{|x_1|} + x_2 \log_2 \frac{\ell}{|x_2|}. \quad (10)$$

Since $|A| \leq \ell$, we have $\frac{|A|}{\ell} \leq 1$, so $\log_2 \frac{|A|}{\ell} \leq 0$ and

$$\left| A \log_2 \frac{|A|}{\ell} \right| = |A| \log_2 \frac{\ell}{|A|} \leq |A| \cdot \frac{\ell - |A|}{|A| \ln 2} = \frac{\ell - |A|}{\ln 2}.$$

For $i = 1, 2$, using $\log_2(1+t) \leq \frac{t}{\ln 2}$ for $t \geq 0$, we have,

$$\left| x_i \log_2 \frac{\ell}{|x_i|} \right| = |x_i| \log_2 \left(1 + \frac{|x_{3-i}|}{|x_i|} \right) \leq |x_i| \cdot \frac{|x_{3-i}|}{|x_i| \ln 2} = \frac{|x_{3-i}|}{\ln 2}.$$

In fact, the bound $\ell - |A| \leq \ell$ used above can be sharpened: $\ell - |A| \leq 2 \min(|x_1|, |x_2|) \leq \ell$, and $\ell - |x_i| = |x_{3-i}| \leq \ell$. Hence we obtain that

$$\begin{aligned} |g(x_1 + x_2) - g(x_1) - g(x_2)| &= |g(A) - g(x_1) - g(x_2)| \leq \frac{\ell - |A|}{\ln 2} + \frac{|x_2|}{\ln 2} + \frac{|x_1|}{\ln 2} \\ &= \frac{2\ell - |A|}{\ln 2} \leq \frac{2\ell}{\ln 2} \leq \frac{2(|x_1| + |x_2|)}{\ln 2}. \end{aligned}$$

for all $x_1, x_2 \in \mathbb{R} - \{0\}$. $\square$

Example 1. Define a mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} xy^2 \log_2(|x| + |y|), & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Then there is a constant $\theta > 0$ such that

$$|D_f(x_1, x_2, y_1, y_2)| \leq \theta(|x_1|^3 + |x_2|^3 + |y_1|^3 + |y_2|^3)$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. But there is no (H, C) such that $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ is an additive-quadratic mapping, $C > 0$ and

$$|f(x, y) - H(x, y)| \leq C(|x|^3 + |y|^3)$$

for all $x, y \in \mathbb{R}$.

Proof. Note that

$$\begin{aligned} D_f(x_1, x_2, y_1, y_2) &= f(x_1 + x_2, y_1 + y_2) + f(x_1 + x_2, y_1 - y_2) \\ &\quad - 2[f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2)] \end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(t) := \begin{cases} t \log_2 |t|, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Consider two functions $f_1, f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f_1(x, y) := y^2 g(x), \quad f_2(x, y) := \begin{cases} xy^2 \log_2 \left(1 + \frac{|y|}{|x|} \right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

For $x \neq 0$ and $y \in \mathbb{R}$, we have

$$f_1(x, y) + f_2(x, y) = y^2 \left[x \log_2 |x| + x \log_2 \left(1 + \frac{|y|}{|x|} \right) \right] = xy^2 \log_2(|x| + |y|) = f(x, y),$$

and at $x = 0$ both sides vanish. Thus $f_1(x, y) + f_2(x, y) = f(x, y)$ everywhere. Hence $D_f(x_1, x_2, y_1, y_2) = D_{f_1}(x_1, x_2, y_1, y_2) + D_{f_2}(x_1, x_2, y_1, y_2)$ for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

For $x \neq 0$ and $y \in \mathbb{R}$, we have

$$|f_2(x, y)| = |x| y^2 \log_2 \left(1 + \frac{|y|}{|x|} \right) \leq |x| y^2 \cdot \frac{|y|}{|x| \ln 2} = \frac{|y|^3}{\ln 2}.$$

Setting $A = x_1 + x_2$, $C = y_1 + y_2$ and $D = y_1 - y_2$, and applying the triangle inequality, we have

$$|D_{f_2}(x_1, x_2, y_1, y_2)| \leq \frac{1}{\ln 2} (|C|^3 + |D|^3 + 4|y_1|^3 + 4|y_2|^3).$$

Since $(a + b)^3 \leq 4(a^3 + b^3)$ for $a, b \geq 0$, we get $|C|^3 + |D|^3 \leq 2(|y_1| + |y_2|)^3 \leq 8(|y_1|^3 + |y_2|^3)$. Thus we have

$$|D_{f_2}(x_1, x_2, y_1, y_2)| \leq \frac{12(|y_1|^3 + |y_2|^3)}{\ln 2}. \quad (11)$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Since $f_1(x, y) = y^2 g(x)$ and $h(y) := y^2$ satisfies $h(y_1 + y_2) + h(y_1 - y_2) = 2h(y_1) + 2h(y_2)$, a direct calculation gives

$$\begin{aligned} D_{f_1}(x_1, x_2, y_1, y_2) &= g(A)[h(C) + h(D)] - 2[g(x_1) + g(x_2)][h(y_1) + h(y_2)] \\ &= [h(C) + h(D)][g(A) - g(x_1) - g(x_2)] \\ &\quad + [g(x_1) + g(x_2)] \underbrace{[h(C) + h(D) - 2h(y_1) - 2h(y_2)]}_{=0}. \end{aligned}$$

Thus we have

$$D_{f_1}(x_1, x_2, y_1, y_2) = 2(y_1^2 + y_2^2)[g(x_1 + x_2) - g(x_1) - g(x_2)].$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. It remains to bound $|g(x_1 + x_2) - g(x_1) - g(x_2)|$.

Applying Lemma 2, we have

$$|D_{f_1}(x_1, x_2, y_1, y_2)| = 2(y_1^2 + y_2^2)|g(x_1 + x_2) - g(x_1) - g(x_2)| \leq \frac{4(y_1^2 + y_2^2)(|x_1| + |x_2|)}{\ln 2}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Since $a^2 b \leq \frac{2}{3}a^3 + \frac{1}{3}b^3$ for $a, b \geq 0$, each term $y_i^2 |x_j|$ satisfies $y_i^2 |x_j| \leq \frac{2|y_i|^3 + |x_j|^3}{3}$. Summing over $i, j \in \{1, 2\}$, we have

$$\begin{aligned} (y_1^2 + y_2^2)(|x_1| + |x_2|) &= \sum_{i,j=1}^2 y_i^2 |x_j| \leq \frac{1}{3} \sum_{i,j=1}^2 (2|y_i|^3 + |x_j|^3) \\ &= \frac{4|y_1|^3 + 4|y_2|^3 + 2|x_1|^3 + 2|x_2|^3}{3}. \end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Hence, we have

$$\begin{aligned} |D_{f_1}(x_1, x_2, y_1, y_2)| &\leq \frac{4}{\ln 2} \cdot \frac{4|y_1|^3 + 4|y_2|^3 + 2|x_1|^3 + 2|x_2|^3}{3} \\ &= \frac{16}{\ln 2} \cdot \frac{2(|y_1|^3 + |y_2|^3) + (|x_1|^3 + |x_2|^3)}{3} \leq \frac{16}{3 \ln 2} \sum_{i=1}^2 (|x_i|^3 + |y_i|^3). \end{aligned} \quad (12)$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Combining (11) and (12), we find that

$$|D_f(x_1, x_2, y_1, y_2)| \leq |D_{f_1}(x_1, x_2, y_1, y_2)| + |D_{f_2}(x_1, x_2, y_1, y_2)| \\ \leq \left(\frac{16}{3\ln 2} + \frac{12}{\ln 2}\right) \sum_{i=1}^2 (|x_i|^3 + |y_i|^3) = \frac{52}{3\ln 2} \sum_{i=1}^2 (|x_i|^3 + |y_i|^3).$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Hence the bound holds with $\theta = \frac{52}{3\ln 2}$.

Suppose for contradiction that stability holds for $p = 3$: there exists an additive-quadratic mapping $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $C > 0$ such that

$$|f(x, y) - H(x, y)| \leq C(|x|^3 + |y|^3) \quad \text{for all } x, y \in \mathbb{R}. \quad (13)$$

Since H is additive-quadratic, it satisfies $H(2x, 2y) = 8H(x, y)$ for all $x, y \in \mathbb{R}$. Replacing (x, y) by $(2^k x, 2^k y)$ in (13) and dividing by 8^k , we have

$$\left| \frac{f(2^k x, 2^k y)}{8^k} - H(x, y) \right| \leq C(|x|^3 + |y|^3). \quad (14)$$

for all $x, y \in \mathbb{R}$ and all $k \in \mathbb{N}$. From the definition of f , we have

$$f(2^k x, 2^k y) = (2^k x)(2^k y)^2 \log_2(2^k |x| + 2^k |y|) = 8^k xy^2 [\log_2(|x| + |y|) + k] \\ = 8^k f(x, y) + k \cdot 8^k xy^2$$

for all $x, y \in \mathbb{R}$ and all $k \in \mathbb{N}$. Dividing by 8^k and substituting into (14), we obtain

$$|f(x, y) + kxy^2 - H(x, y)| \leq C(|x|^3 + |y|^3) \quad \text{for all } x, y \in \mathbb{R} \text{ and all } k \in \mathbb{N}.$$

For any fixed x, y with $xy^2 \neq 0$, the left-hand side tends to ∞ as $k \rightarrow \infty$, while the right-hand side is a fixed constant. This is a contradiction. Therefore, Hyers–Ulam stability does not hold for the critical exponent $p = 3$. $\square$

Theorem 3. Suppose that a mapping $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ satisfies $f(x, 0) = 0$ and the inequality

$$\|D_f(x_1, \dots, x_n, y_1, \dots, y_n)\| \leq \theta \prod_{i=1}^n \|x_i\|^{q_i} \prod_{j=1}^n \|y_j\|^{r_j} \quad (15)$$

for all $x_1, \dots, x_n \in \mathcal{X}$ and $y_1, \dots, y_n \in \mathcal{Y}$, where $\theta \geq 0$ and $q_i, r_j \geq 0$ are constants such that $p := \sum_{i=1}^n q_i + \sum_{j=1}^n r_j > 3$. Then there is a unique additive-quadratic mapping $H : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ satisfying the functional Equation (6) and the inequality

$$\|f(x, y) - H(x, y)\| \leq \frac{\theta}{n^p - n^3} \left(\frac{n^{p+1}}{n+2} \|x\|^p + \|x\|^q \|y\|^r \right), \quad (16)$$

where $q = \sum_{i=1}^n q_i$ and $r = \sum_{j=1}^n r_j$, for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

Proof. Letting $x_1 = \dots = x_n = 0$ and $y_1 = \dots = y_n = 0$ in (15), we gain $f(0, 0) = 0$. Putting $x_1 = x, x_2 = \dots = x_n = 0$, and $y_1 = \dots = y_n = 0$ in (15), we get

$$\|f(x, 0)\| \leq \frac{2\theta}{(n-1)(n+2)} \|x\|^p \quad (17)$$

for all $x \in \mathcal{X}$. Note that in cases where $r > 0$ or $q_i > 0$ for $i \geq 2$, we have $f(x, 0) = 0$ for all $x \in \mathcal{X}$. Setting $x_1 = \dots = x_n = x$ and $y_1 = \dots = y_n = y$ in (15), we have

$$\left\| f(nx, ny) + \frac{n(n-1)}{2} f(nx, 0) - n^3 f(x, y) \right\| \leq \theta \|x\|^q \|y\|^r$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Applying the triangle inequality and utilizing (17), we have

$$\|f(nx, ny) - n^3 f(x, y)\| \leq \theta \|x\|^q \|y\|^r + \frac{n^{p+1}\theta}{n+2} \|x\|^p$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Replacing x by $\frac{x}{n}$ and y by $\frac{y}{n}$ in the above inequality, we obtain

$$\left\| f(x, y) - n^3 f\left(\frac{x}{n}, \frac{y}{n}\right) \right\| \leq \frac{\theta}{n^p} \|x\|^q \|y\|^r + \frac{n\theta}{n+2} \|x\|^p$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

For all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we consider the sequence $\{H_k(x, y)\}$ defined by $H_k(x, y) := n^{3k} f\left(\frac{x}{n^k}, \frac{y}{n^k}\right)$ for all $k \in \mathbb{N}$. For any positive integers $l > k \geq 0$, we see that

$$\begin{aligned} \|H_l(x, y) - H_k(x, y)\| &\leq \sum_{m=k}^{l-1} \left\| n^{3(m+1)} f\left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}}\right) - n^{3m} f\left(\frac{x}{n^m}, \frac{y}{n^m}\right) \right\| \\ &= \sum_{m=k}^{l-1} n^{3m} \left\| n^3 f\left(\frac{x}{n^{m+1}}, \frac{y}{n^{m+1}}\right) - f\left(\frac{x}{n^m}, \frac{y}{n^m}\right) \right\| \\ &\leq \sum_{m=k}^{l-1} n^{3m} \left(\frac{\theta}{n^p} \left\| \frac{x}{n^m} \right\|^q \left\| \frac{y}{n^m} \right\|^r + \frac{n\theta}{n+2} \left\| \frac{x}{n^m} \right\|^p \right) \\ &= \sum_{m=k}^{l-1} n^{3m} \left(\frac{\theta}{n^p n^{m(q+r)}} \|x\|^q \|y\|^r + \frac{n\theta}{(n+2)n^{mp}} \|x\|^p \right) \\ &= \sum_{m=k}^{l-1} n^{(3-p)m} \left(\frac{\theta}{n^p} \|x\|^q \|y\|^r + \frac{n\theta}{n+2} \|x\|^p \right) \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Thus, for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $\{H_k(x, y)\}$ is a Cauchy sequence in the complete normed linear space $\mathcal{Z}$. We define a mapping $H : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ by

$$H(x, y) := \lim_{k \rightarrow \infty} H_k(x, y)$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

By letting $k = 0$ and passing the limit $l \rightarrow \infty$, we obtain the inequality (16). Since

$$\begin{aligned} \|D_H(x_1, \dots, x_n, y_1, \dots, y_n)\| &= \lim_{k \rightarrow \infty} n^{3k} \left\| D_f\left(\frac{x_1}{n^k}, \dots, \frac{x_n}{n^k}, \frac{y_1}{n^k}, \dots, \frac{y_n}{n^k}\right) \right\| \\ &\leq \lim_{k \rightarrow \infty} n^{3k} \theta \prod_{i=1}^n \left\| \frac{x_i}{n^k} \right\|^{q_i} \prod_{j=1}^n \left\| \frac{y_j}{n^k} \right\|^{r_j} \\ &= \lim_{k \rightarrow \infty} n^{(3-p)k} \theta \prod_{i=1}^n \|x_i\|^{q_i} \prod_{j=1}^n \|y_j\|^{r_j} = 0 \end{aligned}$$

for all $x_1, \dots, x_n \in \mathcal{X}$ and all $y_1, \dots, y_n \in \mathcal{Y}$, H satisfies the functional Equation (6).

Now, let $L : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ be another additive-quadratic mapping satisfying the Equation (6) and the inequality (16). Since both H and L are additive-quadratic mappings, by Theorem 3 in [9], we have $H\left(\frac{x}{n^k}, \frac{y}{n^k}\right) = \frac{1}{n^{3k}} H(x, y)$ and $L\left(\frac{x}{n^k}, \frac{y}{n^k}\right) = \frac{1}{n^{3k}} L(x, y)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ and all $k \in \mathbb{N}$. It follows from (16) that

$$\begin{aligned}
\|H(x, y) - L(x, y)\| &= n^{3k} \left\| H\left(\frac{x}{n^k}, \frac{y}{n^k}\right) - L\left(\frac{x}{n^k}, \frac{y}{n^k}\right) \right\| \\
&\leq n^{3k} \left(\left\| H\left(\frac{x}{n^k}, \frac{y}{n^k}\right) - f\left(\frac{x}{n^k}, \frac{y}{n^k}\right) \right\| + \left\| f\left(\frac{x}{n^k}, \frac{y}{n^k}\right) - L\left(\frac{x}{n^k}, \frac{y}{n^k}\right) \right\| \right) \\
&\leq 2n^{3k} \frac{\theta}{n^p - n^3} \left(\frac{n^{p+1}}{n+2} \left\| \frac{x}{n^k} \right\|^p + \left\| \frac{x}{n^k} \right\|^q \left\| \frac{y}{n^k} \right\|^r \right) \\
&= \frac{2\theta}{n^p - n^3} n^{(3-p)k} \left(\frac{n^{p+1}}{n+2} \|x\|^p + \|x\|^q \|y\|^r \right) \rightarrow 0
\end{aligned}$$

as $k \rightarrow \infty$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Therefore, we conclude that $H(x, y) = L(x, y)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, which proves the uniqueness of H . $\square$

As a direct application of our fixed-point approach, we can establish a generalized stability for the n -dimensional quadratic functional equation introduced by Bae and Jun [8] in 2001.

Theorem 4. Suppose that a mapping $f : \mathcal{X} \rightarrow \mathcal{Z}$ satisfies $f(0) = 0$ and the inequality

$$\left\| f(x_1 + x_2 + \cdots + x_n) + \sum_{1 \leq i < j \leq n} f(x_i - x_j) - n \sum_{i=1}^n f(x_i) \right\| \leq \varphi(x_1, \dots, x_n) \quad (18)$$

for all $x_1, \dots, x_n \in \mathcal{X}$, where $\varphi : \mathcal{X}^n \rightarrow [0, \infty)$ is a given control function. If there exists a constant $L \in (0, 1)$ such that

$$\varphi(nx_1, \dots, nx_n) \leq n^2 L \varphi(x_1, \dots, x_n) \quad (19)$$

for all $x_1, \dots, x_n \in \mathcal{X}$, then there exists a unique strictly quadratic mapping $Q : \mathcal{X} \rightarrow \mathcal{Z}$ satisfying the functional equation (5) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{1}{n^2(1-L)} \varphi(x, \dots, x) \quad (20)$$

for all $x \in \mathcal{X}$.

Proof. Let S be the set of all mappings $g : \mathcal{X} \rightarrow \mathcal{Z}$ satisfying $g(0) = 0$. We introduce a generalized metric d on S given by

$$d(g, h) := \inf\{\mu \in [0, \infty] \mid \|g(x) - h(x)\| \leq \mu \varphi(x, \dots, x) \text{ for all } x \in \mathcal{X}\}$$

for all $g, h \in S$. It is a well-known fact that (S, d) is a complete generalized metric space.

We define a linear mapping $J : S \rightarrow S$ by

$$Jg(x) := \frac{1}{n^2} g(nx)$$

for all $x \in \mathcal{X}$. Let $g, h \in S$ with $d(g, h) = \mu$. By the inequality (19), we have

$$\|Jg(x) - Jh(x)\| = \frac{1}{n^2} \|g(nx) - h(nx)\| \leq \frac{\mu}{n^2} \varphi(nx, \dots, nx) \leq \mu L \varphi(x, \dots, x)$$

for all $x \in \mathcal{X}$. This implies $d(Jg, Jh) \leq Ld(g, h)$, meaning J is a strictly contractive mapping with the Lipschitz constant L .

Letting $x_1 = \cdots = x_n = x$ in (18), we have

$$\left\| \frac{f(nx)}{n^2} - f(x) \right\| \leq \frac{1}{n^2} \varphi(x, \dots, x)$$

for all $x \in \mathcal{X}$. This means that $d(Jf, f) \leq \frac{1}{n^2}$.

By the fixed-point alternative theorem of Diaz and Margolis [18], there exists a unique fixed point Q of J in the set $S^* = \{g \in S \mid d(f, g) < \infty\}$. This unique fixed point Q satisfies

$$Q(x) = \lim_{k \rightarrow \infty} (J^k f)(x) = \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{2k}}$$

for all $x \in \mathcal{X}$. Furthermore, the fixed-point alternative theorem guarantees that

$$d(f, Q) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{n^2(1-L)},$$

which exactly corresponds to the desired error bound (20):

$$\|f(x) - Q(x)\| \leq \frac{1}{n^2(1-L)} \varphi(x, \dots, x)$$

for all $x \in \mathcal{X}$.

By the inequalities (18) and (19), we obtain

$$\begin{aligned} & \left\| Q(x_1 + x_2 + \dots + x_n) + \sum_{1 \leq i < j \leq n} Q(x_i - x_j) - n \sum_{i=1}^n Q(x_i) \right\| \\ &= \lim_{k \rightarrow \infty} \frac{1}{n^{2k}} \left\| f(n^k(x_1 + x_2 + \dots + x_n)) + \sum_{1 \leq i < j \leq n} f(n^k x_i - n^k x_j) - n \sum_{i=1}^n f(n^k x_i) \right\| \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{n^{2k}} \varphi(n^k x_1, \dots, n^k x_n) \leq \lim_{k \rightarrow \infty} L^k \varphi(x_1, \dots, x_n) = 0 \end{aligned}$$

for all $x_1, \dots, x_n \in \mathcal{X}$. Thus, Q exactly satisfies the functional Equation (5). $\square$

Theorem 5. Suppose that a mapping $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ satisfies $f(x, 0) = 0$ and the inequality

$$\|D_f(x_1, \dots, x_n, y_1, \dots, y_n)\| \leq \Phi(x_1, \dots, x_n, y_1, \dots, y_n) \quad (21)$$

for all $x, x_1, \dots, x_n \in \mathcal{X}$ and $y_1, \dots, y_n \in \mathcal{Y}$, where $\Phi : \mathcal{X}^n \times \mathcal{Y}^n \rightarrow [0, \infty)$ is a given control function. If there exists a constant $L \in (0, 1)$ such that

$$\Phi(nx_1, \dots, nx_n, ny_1, \dots, ny_n) \leq n^3 L \Phi(x_1, \dots, x_n, y_1, \dots, y_n) \quad (22)$$

for all $x_1, \dots, x_n \in \mathcal{X}$ and $y_1, \dots, y_n \in \mathcal{Y}$, then there exists a unique additive-quadratic mapping $H : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ satisfying the functional Equation (6) and the inequality

$$\|f(x, y) - H(x, y)\| \leq \frac{1}{n^3(1-L)} \Phi(x, \dots, x, y, \dots, y) \quad (23)$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

Proof. Let S be the set of all mappings $g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Z}$ such that $g(x, 0) = 0$ for all $x \in \mathcal{X}$. We introduce a generalized metric d on S given by

$$\begin{aligned} & d(g, h) \\ &:= \inf\{\mu \in [0, \infty] \mid \|g(x, y) - h(x, y)\| \leq \mu \Phi(x, \dots, x, y, \dots, y) \text{ for all } (x, y) \in \mathcal{X} \times \mathcal{Y}\} \end{aligned}$$

for all $g, h \in S$. It is a well-known fact that (S, d) is a complete generalized metric space.

We define a linear mapping $J : S \rightarrow S$ by

$$Jg(x, y) := \frac{1}{n^3} g(nx, ny)$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Let $g, h \in S$ with $d(g, h) = \mu$. By the inequality (22), we obtain

$$\begin{aligned} \|Jg(x, y) - Jh(x, y)\| &= \frac{1}{n^3} \|g(nx, ny) - h(nx, ny)\| \\ &\leq \frac{\mu}{n^3} \Phi(nx, \dots, nx, ny, \dots, ny) \leq \mu L \Phi(x, \dots, x, y, \dots, y) \end{aligned}$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. This implies $d(Jg, Jh) \leq Ld(g, h)$, meaning J is a strictly contractive mapping with the Lipschitz constant L .

Letting $x_1 = \dots = x_n = x$ and $y_1 = \dots = y_n = y$ in (21), we have

$$\left\| \frac{f(nx, ny)}{n^3} - f(x, y) \right\| \leq \frac{1}{n^3} \Phi(x, \dots, x, y, \dots, y)$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. This means that $d(Jf, f) \leq \frac{1}{n^3}$.

By the fixed-point alternative theorem of Diaz and Margolis [18], there exists a unique fixed point H of J in the set $S^* = \{g \in S \mid d(f, g) < \infty\}$. This unique fixed point H satisfies

$$H(x, y) = \lim_{k \rightarrow \infty} (J^k f)(x, y) = \lim_{k \rightarrow \infty} \frac{f(n^k x, n^k y)}{n^{3k}}$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Furthermore, the fixed-point alternative theorem guarantees that

$$d(f, H) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{n^3(1-L)},$$

which implies to the desired inequality (23).

Finally, we must show that H satisfies the functional Equation (6). From the definition of H and the condition (22), we have

$$\begin{aligned} \|D_H(x_1, \dots, x_n, y_1, \dots, y_n)\| &= \lim_{k \rightarrow \infty} \frac{1}{n^{3k}} \|D_f(n^k x_1, \dots, n^k x_n, n^k y_1, \dots, n^k y_n)\| \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{n^{3k}} \Phi(n^k x_1, \dots, n^k x_n, n^k y_1, \dots, n^k y_n) \\ &\leq \lim_{k \rightarrow \infty} L^k \Phi(x_1, \dots, x_n, y_1, \dots, y_n) = 0 \end{aligned}$$

for all $x_1, \dots, x_n \in \mathcal{X}$ and $y_1, \dots, y_n \in \mathcal{Y}$. Thus the mapping H satisfies the functional Equation (6). $\square$

3. Conclusions

This study provides a comprehensive analysis of the Hyers–Ulam stability of a multi-variable additive-quadratic functional equation in Banach spaces, utilizing two complementary approaches. First, through the direct constructive method, we proved stability results where the approximation error is controlled by the sum of powers of the norms (Theorem 2). This approach clearly proves the existence and uniqueness of an additive-quadratic mapping that approximates the given function. Second, we extended the study by applying the fixed-point method. Using the fixed-point alternative theorem, we proved stability for more general control functions (Theorem 4). This methodology facilitates a

generalized metric space setting and confirms that stability holds as long as the control function satisfies certain contraction property.

Therefore, this study demonstrates the robustness of the multi-variable additive-quadratic functional equation using both classical and modern techniques. These results provide a rigorous mathematical foundation for approximating such mappings. Furthermore, as stability criteria can vary significantly depending on the underlying geometric structure, future research should focus on extending these results to more generalized frameworks. Potential areas of exploration include quasi-Banach spaces, non-Archimedean spaces, and 2-Banach spaces, where unique norm properties and stability conditions may require new analytical insights. The stability bounds of the multi-variable additive-quadratic functional equation can be utilized in estimating errors for multi-dimensional real-world phenomena, such as complex network modeling or signal processing.

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References

1. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. <https://doi.org/10.1073/pnas.27.4.222>.
3. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66.
4. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. <https://doi.org/10.1090/S0002-9939-1978-0507327-1>.
5. Găvruta, P. A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **1994**, *184*, 431–436. <https://doi.org/10.1006/jmaa.1994.1211>.
6. Aczél, J.; Dhombres, J. *Functional Equations in Several Variables*; Cambridge University Press: Cambridge, UK, 1989.
7. Bae, J.-H.; Chang, I.-S. On the Ulam stability problem of a quadratic functional equation. *J. Anal. Math. Inform.* **2001**, *8*, 561–567.
8. Bae, J.-H.; Jun, K.-W. On the generalized Hyers–Ulam–Rassias stability of an n -dimensional quadratic functional equation. *J. Math. Anal. Appl.* **2001**, *258*, 183–193. <https://doi.org/10.1006/jmaa.2000.7372>.
9. Park, W.-G.; Bae, J.-H.; Chung, B.-H. On an additive-quadratic functional equation and its stability. *J. Appl. Math. Comput.* **2005**, *18*, 563–572.
10. Donganont, S.; Bodaghi, A. A general system of functional equations deriving from additive, quadratic, cubic, and quartic mappings. *J. Math.* **2026**, *2026*, 1–15. <https://doi.org/10.1155/jom/6669054>.
11. Tongsonporn, J.; Phonrakkhet, S. A generalized bi-quadratic–Drygas functional system in non-Archimedean normed spaces over p -adic numbers. *Symmetry* **2026**, *18*, 514. <https://doi.org/10.3390/sym18030514>.
12. Bahyrycz, A.; Ciepliński, K.; Olko, J. On an equation characterizing multi-additive-quadratic mappings and its Hyers–Ulam stability. *Appl. Math. Comput.* **265**, 2015, 448–455. <http://dx.doi.org/10.1016/j.amc.2015.05.037>.
13. Bodaghi, A.; Mahzoon, H.; Mikaeilvand, N. The system of mixed type additive-quadratic equations and approximations. *J. Inequal. Appl.* **2024**, *2024*, 1–17. <https://doi.org/10.1186/s13660-024-03180-8>.
14. Baza, A.; Rossafi, M.; Mouniane, M. Hyperstability of some functional equations in modular spaces. *Methods Func. Anal. Top.* **2025**, *31*, 310–324. https://doi.org/10.31392/MFAT-npu26_4.2025.02.
15. Karthikeyan, S.; Donganont, S.; Park, C.; Tamilvanan, K.; Wang, Y. Hyperstability of functional equation deriving from quintic mapping in Banach spaces by fixed point method. *Euro. J. Pure Appl. Math.* **18**, 2025, 1–18. <https://doi.org/10.29020/nybg.ejppam.v18i1.5757>.
16. Brzdęk, J. Some comments on the superstability of a general functional equation. *Symmetry* **2024**, *16*, 1654. <https://doi.org/10.3390/sym16121654>.

17. Najati, A.; Tareeghee, M.; Rassias, T. Stability and superstability of a linear functional equation on restricted domains. *J. Func. Spaces* **2026**, 2026, 1–8. <https://doi.org/10.1155/jofs/7336932>.
18. Diaz, J.B.; Margolis, B. A fixed point theorem of the alternative, for contractions on a generalized complete metric space. *Bull. Am. Math. Soc.* **1968**, 74, 305–309. <https://doi.org/10.1090/S0002-9904-1968-11933-0>.

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