

Article

Design of Trivariate Random Preventive Maintenance for Systems Under Random Discrete Warranty with Cost Charging

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Abstract

In product lifecycle management, warranty and post-warranty maintenance are critical for system reliability but suffer from notable limitations: Traditional warranty models lack profit-generating mechanisms and ignore user usage intensity data, while post-warranty strategies fail to account for system heterogeneity (e.g., diverse usage habits and reliability degradation patterns). To fill these gaps, this paper proposes two random warranty models: random discrete repair warranty considering cost charging first (RDRW-CCF) and random discrete repair warranty considering cost charging last (RDRW-CCL). Both employ failure thresholds and time spans to classify user usage rates, enabling manufacturers to control costs by providing free repairs for low-usage users and charging high-usage users to generate profits. Additionally, this study develops a trivariate random preventive maintenance (TRPM) strategy for post-warranty systems. Drawing on three metrics (failure threshold, limited mission cycles, and time span threshold) from RDRW-CCF, TRPM categorizes systems into usage-rate and reliability scenarios and customizes maintenance actions accordingly. Numerical experiments show that RDRW-CCF is more profitable than RDRW-CCL, and TRPM outperforms the bivariate random preventive maintenance (BRPM) strategy in both cost efficiency and reliability.

Keywords: random warranty; post-warranty system; trivariate random preventive maintenance; usage rate; reliability heterogeneity

MSC: 90B25

1. Introduction

In modern product lifecycle management, ensuring the reliability of systems (including physical products and industrial equipment) is a cornerstone for improving societal value and market competitiveness. A foundational step is the implementation of burn-in tests, which, as extensively studied by Safaei et al. [1] and Ye et al. [2], provide rigorous screening for early defects and verify that systems meet predefined operational standards prior to their market launch. Beyond this pre-launch phase, reliability management throughout the entire service lifecycle of a system is a dynamic and multi-faceted task primarily divided into two interconnected core domains: warranty-stage reliability management and post-warranty reliability management. These two domains are, respectively, supported by specialized tools, namely warranty models for governing reliability during the warranty period and post-warranty maintenance strategies for sustaining performance after the warranty expires. Given their pivotal roles in optimizing system lifecycle

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performance, reducing operational risks, and balancing cost efficiency with user satisfaction, these tools have long been intensively studied by scholars, engineers, and practitioners worldwide.

In recent years, the global industry has undergone a profound digital transformation (DT) driven by the rapid advancement and widespread adoption of advanced digital technologies (ADTs), such as the Internet of Things (IoT), sensor technology, industrial internet platforms, blockchain, and 5G communication. This transformation has reshaped the strategic priorities of various sectors, shifting the focus toward objectives like cost reduction, operational efficiency improvement, and the exploration of innovative business models that leverage data as a core asset. Within the specific context of warranty systems, which form an integral part of product-service ecosystems, this digital shift has led to a clear dichotomy in existing warranty models: ADT-driven warranty models and classic warranty models. The key distinguishing factor between these two categories lies in their utilization of real-time operational data. ADT-driven warranty models harness real-time performance degradation data and usage data to dynamically adjust warranty terms and service provisions, while classic models operate without such data integration, relying instead on pre-set rules and historical experience.

ADT-driven warranty models can be further subclassified into two types based on the nature of the data they utilize and the failure modes they address. The first type is random warranty models, which are specifically designed for products with self-announcing failures that are immediately detectable and do not require complex monitoring of degradation processes, often with limited mission cycles serving as warranty coverage constraints. Notable examples of such models include the 2D and 3D random repair warranties proposed by Shang et al. [3, 4], as well as their subsequent extensions and refinements (Shang et al. [5]). The second type is condition-based warranty models, which rely on the modeling of degradation processes to predict and address failures. This category encompasses a wide range of research, such as the work by Ye et al. [6] on degradation-based warranty design; Wang et al. [7, 8] on integrating degradation data with warranty service optimization; Zhang et al. [9] on addressing external impacts on degradation processes; and Zhou et al. [10], Xu et al. [11], Yang et al. [12], Qiu and Cui [13], and Qiu et al. [14] on various specialized degradation modeling techniques. Specific applications within this subclass include a Gamma-process-based non-renewable replacement warranty model in Zhang et al. [15] and an inverse Gaussian-process-based renewable replacement model in Shang et al. [16], with recent advancements further expanding this body of knowledge (Wang et al. [17]; Liang et al. [18]; Wang et al. [19]; Cha et al. [20]).

In contrast, classic warranty models, developed prior to the widespread adoption of ADTs, lack the integration of real-time data and instead depend on traditional maintenance actions such as minimal repairs, preventive maintenance (PM), or full replacements to manage warranty obligations. For instance, Yeh et al. [21] and Yun et al. [22] constructed models for free repair warranties using minimal repairs, which focus on restoring failed systems to a functional state without addressing underlying degradation. Chukova et al. [23] and Blischke et al. [24], on the other hand, centered their research on replacement-based warranty frameworks, where failed components or systems are entirely replaced rather than repaired. More recently, Wang et al. [25] and Huang et al. [26] incorporated PM into warranty models, using scheduled maintenance activities to proactively manage reliability during the warranty period and introducing the concept of PM warranties. Additional contributions to classic warranty modeling include the work of Iskandar and Pasaribu [27] on warranty cost analysis under consumer protection laws and Banerjee and Bhattacharjee [28] on two-dimensional warranty servicing strategies with imperfect repair options.

Despite the significant progress made in both ADT-driven and classic warranty models, a critical limitation persists across most existing frameworks: their failure to recognize that usage data, quantified by metrics such as mission cycle frequency (how often a system is operated) and mission cycle duration (how long each operation lasts), contain valuable insights into user-specific usage-rate scenarios. These scenarios, which reflect the unique ways in which different users interact with a system, are pivotal for two key objectives: effective cost management (by aligning warranty costs with actual usage intensity) and the design of profit-oriented warranty terms (by identifying opportunities to monetize high-usage scenarios). Compounding this limitation is the fact that existing warranty models are predominantly designed from the manufacturer's perspective, with objectives centered on cost reduction, boosting sales, or enhancing customer loyalty rather than treating the warranty itself as a potential profit center. This narrow focus prevents manufacturers from generating direct revenue through warranty services, limiting the strategic value of warranties in the broader product-service ecosystem.

While warranty models play a crucial role in governing reliability during the warranty stage, they are inherently inadequate for managing reliability in the post-warranty phase: a period where systems are no longer covered by the manufacturer's standard warranty and require tailored maintenance strategies to extend their service life and minimize operational costs. To address this gap, researchers have developed a diverse array of post-warranty maintenance strategies, which, similarly to warranty models, can be broadly categorized into ADT-driven and classic approaches. ADT-driven post-warranty strategies leverage real-time degradation data and usage data to enable proactive maintenance decisions, such as the random maintenance strategy proposed by Shang et al. [29] and the condition-based replacement strategy developed by Shang et al. [16]. Classic post-warranty strategies, by contrast, rely on heuristic methods that do not integrate real-time data, including age replacement (where systems are replaced after a predefined operational age) and fixed-interval PM. Examples of such classic strategies include the work of Liu & Wang [30] on extended warranty and post-warranty maintenance integration; Park et al. [31, 32] on optimizing periodic PM after warranty expiration; Qiao et al. [33] on three-dimensional warranty-linked post-warranty strategies; and Karar et al. [34] on maintenance strategy selection using statistical process modeling.

A key shortcoming of existing post-warranty maintenance strategies is their neglect of the heterogeneous characteristics of post-warranty systems. These heterogeneous traits cover user-specific usage habits, reliability consumption patterns, and historical service time utilization, all of which shape distinct system scenarios and determine optimal maintenance decisions. Two major research gaps can be identified from this limitation: Existing strategies rarely adjust maintenance plans to match unique operational profiles, and they fail to fully exploit reliability information embedded in historical failure data. Post-warranty equipment carries complete historical mission data representing its operational profiles, yet most maintenance policies adopt one-size-fits-all schedules without customized adjustments. Meanwhile, historical failure data that reflect distinct reliability profiles are largely underutilized, which leads to either costly over-maintenance or risky under-maintenance. Failing to tailor maintenance to usage patterns wastes the chance to extend system service life, while ignoring reliability degradation trends results in higher repair frequencies. Although classifying systems by usage rate and reliability can resolve this long-standing flaw in post-warranty reliability management, such data-driven, reliability-oriented maintenance solutions have not been sufficiently investigated in academic and industrial research.

To address the identified limitations of existing warranty models and post-warranty maintenance strategies summarized in the previous section, this paper makes two primary contributions targeting these research gaps. First, it introduces two innovative

random warranty models: the random discrete repair warranty considering cost charging first (RDRW-CCF) and the random discrete repair warranty considering cost charging last (RDRW-CCL). These models leverage a sophisticated screening mechanism that integrates a failure threshold (a predefined number of allowable failures) and a time span parameter (a predefined duration) to systematically classify user usage-rate scenarios, enabling manufacturers to optimize their profit margins by charging repair costs on high-usage users while offering free repairs to low-usage users. Second, this paper proposes a novel post-warranty maintenance strategy: the trivariate random PM (TRPM) strategy. This strategy utilizes three key metrics from the RDRW-CCF model, failure threshold, limited mission cycles, and time span threshold to classify post-warranty systems into distinct usage rate and reliability scenarios and then tailors maintenance actions (e.g., PM intervals, preventive replacement timing) to each scenario. By doing so, the TRPM strategy addresses the heterogeneity of post-warranty systems and enhances both cost efficiency and reliability performance.

The core innovations of this paper are summarized in three aspects: (a) two novel warranty models, RDRW-CCF and RDRW-CCL, are developed under dual threshold criteria to establish segmented charging rules and generate independent warranty profits; (b) the TRPM strategy is proposed to categorize equipment into four heterogeneous groups and deploy customized maintenance schedules, thus avoiding over- or under-maintenance; (c) an integrated full-lifecycle optimization framework is constructed, which transfers warranty classification data to support post-warranty maintenance decision-making.

The remainder of this paper is structured as follows: Section 2 outlines the key research assumptions, defines and models the two proposed random warranty systems (RDRW-CCF and RDRW-CCL) and derives additional variant warranty types through parameter analysis. Section 3 introduces the framework for post-warranty maintenance strategies, starting with the definition and modeling of the core TRPM strategy and then developing and formalizing variant strategies (e.g., TRPM considering only usage-rate heterogeneity) via parameter adjustments to demonstrate adaptability across diverse scenarios. Section 4 conducts numerical experiments on the proposed warranty models and maintenance strategies, using smart air conditioners as a practical case study to uncover underlying mechanisms and derive actionable managerial insights. Finally, Section 5 summarizes the key findings of the research, highlights its limitations, and outlines directions for future studies.

2. Random Warranties

The assumptions adopted in this study are specified as follows: First, the arrival time X of the first failure follows a distribution function $F(x) = 1 - \exp\left(-\int_0^x \lambda(u) du\right)$ with a corresponding failure-rate function $\lambda(u)$. Second, the system operates in a self-announcing failure mode and executes tasks through random mission cycles Y_i ($i = 1, 2, \dots, \infty$). These mission cycles adhere to an exponential distribution function expressed as $G(y)$; in other words, all random mission cycles are independent and identically distributed (i.i.d.) exponential random variables. This modeling choice is justified by the fact that consecutive random mission epochs do not exhibit a consistent monotonic trend, making the exponential distribution an appropriate fit. Third, the time consumed for repair, replacement, and PM is considered negligible and thus excluded from the analytical framework.

2.1. Random Warranty A

In conventional warranty systems, manufacturers, frequently the primary decision-makers, exercise subjective control over the scope and terms of warranties, with goals including cost reduction, customer loyalty enhancement, and sales growth. This aligns with the observation in the Introduction that conventional warranties are “unilaterally formulated by manufacturers from a subjective perspective” to cater to strategic objectives such as cost-cutting and sales promotion. Nevertheless, two fundamental limitations of these models, emphasized in the Introduction, have persistently been overlooked. Primarily, the absence of a profit-generating mechanism precludes manufacturers from generating direct revenue from warranty services. Secondly, there is an absence of the utilization of usage data, which is quantified by metrics such as mission cycle frequency (how often a system is operated) and mission cycle duration (how long each operation lasts). These metrics accumulate through the system’s mission cycles and are embedded in operational time.

In order to address these gaps, a novel random warranty model is proposed below.

2.1.1. The Definition of Random Warranty A

Assume that m and N are represented by two non-negative integers. Let w denote a time span. With all failures corrected by minimal repairs, a random warranty is defined as follows:

- The warranty service will expire upon the end of the N^{th} mission cycle.
- If the warranty service does not expire by the m^{th} failure or time span w (whichever occurs first), the manufacturer charges fees for all subsequent repair work.

In this warranty model, all failures are rectified through minimal repairs. The warranty coverage is determined by the end of the N^{th} mission cycle (i.e., discrete data). The initial part of the coverage is the free-of-charge period, which is bounded by either the m^{th} failure occurrence or the expiration of the time span w , whichever comes first. During this period, the manufacturer bears all costs associated with rectifying any failures that occur. In contrast, the latter part of the coverage is the chargeable period. When failure occurs within this part, the manufacturer charges a cost for the repair of each subsequent failure. Consequently, it is designated as a random discrete repair warranty considering cost charging first (RDRW-CCF).

Evidently, the period defined by the first occurrence of either the m^{th} failure or the expiration of the time span w serves as an effective means to categorize usage patterns. When the N^{th} mission cycle concludes within this defined period, it indicates a lower-intensity usage type by the user; conversely, if the mission cycle extends beyond this timeframe, it signifies a higher-intensity usage type. Under this warranty framework, profit can be generated when the user exhibits a higher-intensity usage type. This mechanism demonstrates how manufacturers can leverage usage types to manage costs and generate profits. From the users’ standpoint, the distinction between free-of-charge and chargeable service terms (i.e., repair terms) is determined by their own usage behaviors, offering a more adaptable and user-centric warranty experience. Consequently, this warranty approach represents an effective solution for surmounting the limitations inherent in traditional warranty systems, offering a mutually beneficial arrangement for both manufacturers and users.

2.1.2. The Measurement Modeling of the RDRW-CCF

Let S_N be a random sum produced by N mission cycles; let T_m be the arrival time of the m^{th} failure. Then, in the context of the RDRW-CCF, the scenarios in which cost charging occurs can be categorized into two distinct cases: The m^{th} failure and the

end of the N^{th} mission cycle occur after the time span w , which is named case A and can be modeled as a set of inequalities $w < \text{Min}\{T_m, S_N\}$; the m^{th} failure occurs before the time span w or the end of the N^{th} mission cycle, whichever comes first. The latter case is named case B and can be modeled as a set of inequalities $T_m < \text{Min}\{w, S_N\}$. Using these two types of inequality sets, the occurrence probabilities p_i ($i = A, B$) of the respective cases can be obtained as

$$p_i = \begin{cases} \Pr\{w < \text{Min}\{T_m, S_N\}\} & \text{for case A} \\ \Pr\{T_m < \text{Min}\{w, S_N\}\} & \text{for case B} \end{cases} = \begin{cases} \bar{F}_m(w)\bar{G}^{(N)}(w) & \text{for case A} \\ \int_0^w \bar{G}^{(N)}(t)dF_m(t) & \text{for case B} \end{cases} \quad (1)$$

where $G^{(N)}(\cdot)$ and $\bar{G}^{(N)}(\cdot)$ are distribution and reliability functions of the random sum S_N , respectively; $\bar{F}_m(\cdot)$ and $F_m(\cdot)$ are reliability and reliability functions of the arrival time T_m , respectively.

In the RDRW-CCF framework, the situations where no cost is levied can be classified into two distinct cases: Both the N^{th} mission cycle ends before the time span w and the m^{th} failure occurs after the time span w , which is named case a and can be modeled as a set of inequalities $S_N \leq w < T_m$; the N^{th} mission cycle ends before the m^{th} failure that occurs prior to the time span w , which is named case b and can be modeled as a set of inequalities $S_N \leq T_m \leq w$. Using these two types of inequality sets, the occurrence probabilities, $\bar{p}_i$ ($i = a, b$), of the respective cases can be obtained as

$$\bar{p}_i = \begin{cases} \Pr\{S_N \leq w < T_m\} & \text{for case a} \\ \Pr\{S_N \leq T_m \leq w\} & \text{for case b} \end{cases} = \begin{cases} G^{(N)}(w)\bar{F}_m(w) & \text{for case a} \\ \int_0^w G^{(N)}(t)dF_m(t) & \text{for case b} \end{cases} \quad (2)$$

Clearly, the sum of these occurrence probabilities is equal to one, i.e.,

$$p_A + p_B + \bar{p}_a + \bar{p}_b = \bar{F}_m(w)\bar{G}^{(N)}(w) + \int_0^w \bar{G}^{(N)}(t)dF_m(t) + G^{(N)}(w)\bar{F}_m(w) + \int_0^w G^{(N)}(t)dF_m(t) = \bar{F}_m(w) + F_m(w) = 1 \quad (3)$$

which satisfies the probability characteristic of mutually exclusive events.

When case A occurs, failures occurring between time span w and the end of the N^{th} mission cycle will incur repair charges. Under this case, the mean number of failures

that incur chargeable repair costs can be expressed as $\int_w^{S_N} \lambda(t)dt$. When case B transpires, the failures that occur within the interval from the arrival time T_m to the end of the N^{th} mission cycle are removed in the form of charging costs. Under this case, the mean number of failures that incur chargeable repair costs can be expressed as $\int_{T_m}^{S_N} \lambda(t)dt$. Moreover,

by making use of the probabilities p_i , the mean value, denoted as MN_{cc}^f , of the mean failures that incur chargeable costs can be derived as

$$\begin{aligned} MN_{cc}^f &= \bar{F}_m(w) \int_w^\infty \left(\int_w^s \lambda(t)dt \right) dG^{(N)}(s) + \int_0^w \left(\int_t^\infty \left(\int_t^s \lambda(u)du \right) dG^{(N)}(s) \right) dF_m(t) \\ &= \bar{F}_m(w) \int_w^\infty \bar{G}^{(N)}(s) \lambda(s) ds + \int_0^w \left(\int_t^\infty \bar{G}^{(N)}(s) \lambda(s) ds \right) dF_m(t) \\ &= \int_w^\infty \bar{G}^{(N)}(s) \lambda(s) ds + \int_0^w F_m(s) \bar{G}^{(N)}(s) \lambda(s) ds \end{aligned} \quad (4)$$

When the RDRW-CCF expires at the end of the N^{th} mission cycle, the total mean value TMV of failures over the warranty coverage can be given by

$$TMV = \int_0^\infty \left(\int_0^s \lambda(u)du \right) dG^{(N)}(s) = \int_0^\infty \bar{G}^{(N)}(s) \lambda(s) ds \quad (5)$$

Let c_m denote the unit cost for fixing each failure. Additionally, let c_r represent the cost levied for fixing each failure. Then, through algebraic operations, the profit-generating PM^f of the RDRW-CCF can be obtained as

$$PM^f = c_r \cdot MN_{cc}^f - c_m \cdot TMV = c_r \left(\int_w^\infty \bar{G}^{(N)}(s) \lambda(s) ds + \int_0^w F_m(s) \bar{G}^{(N)}(s) \lambda(s) ds \right) - c_m \int_0^\infty \bar{G}^{(N)}(s) \lambda(s) ds \quad (6)$$

where $c_r > c_m$.

Significantly, the implementation of the inequality $c_r > c_m$ is aimed at securing the profit-generating operations of manufacturers.

Letting WP^f be the warranty period of the RDRW-CCF, the warranty period WP^f of the RDRW-CCF can be obtained as

$$WP^f = \int_0^\infty s dG^{(N)}(s) = \int_0^\infty \bar{G}^{(N)}(s) ds \quad (7)$$

which implies that the warranty period WP^f is not influenced by either the failure number m or the time span w .

2.2. Random Warranty B

In RDRW-CCF, the m^{th} failure and the time span w are used as two scales to control costs in the form of ‘whichever comes first’. Here, revising ‘whichever comes first’ in the RDRW-CCF as ‘whichever comes last’, the second random warranty is defined and modeled below.

2.2.1. The Definition of Random Warranty B

- The warranty service will expire upon the end of the N^{th} mission cycle.
- If warranty service remains active beyond the m^{th} failure or time span w (whichever occurs last), then the manufacturer will charge a cost for each subsequent failure.

Given the application of the ‘whichever comes last’ principle, this warranty model is classified as a random discrete repair warranty considering cost charging last (RDRW-CCL). One of the key differences from the RDRW-CCF lies in the principle governing the period for classifying usage patterns: the RDRW-CCL adopts a ‘whichever comes last’ rule instead of the ‘whichever comes first’ principle used in the RDRW-CCF. The common ground between the two lies in classifying the warranty period into a free-of-charge period and a chargeable period. Therefore, the RDRW-CCL also represents an effective solution for surmounting the limitations inherent in traditional warranty systems, offering a mutually beneficial arrangement for both manufacturers and users.

2.2.2. The Measurement Modeling of the RDRW-CCL

In the RDRW-CCL framework, the circumstances under which cost charging takes place can be categorized into two distinct situations: The m^{th} failure occurs within the interval from the time span w to the end of the N^{th} mission cycle, which is named situation A and can be modeled as a set of inequalities $w < T_m \leq S_N$; the m^{th} failure occurs before the time span w and the N^{th} mission cycle ends after the time span w . The latter situation is named situation B and can be modeled as a set of inequalities $T_m \leq w < S_N$. Using these two types of inequality sets, the occurrence probabilities Q_i ($i = A, B$) of the respective cases can be obtained as

$$Q_i = \begin{cases} \Pr\{w < T_m \leq S_N\} & \text{for situation A} \\ \Pr\{T_m \leq w < S_N\} & \text{for situation B} \end{cases} = \begin{cases} \int_w^\infty \bar{G}^{(N)}(t) dF_m(t) & \text{for situation A} \\ F_m(w) \bar{G}^{(N)}(w) & \text{for situation B} \end{cases} \quad (8)$$

In the RDRW-CCL model, the scenarios where no cost is levied can be decomposed into two distinct situations: The N^{th} mission cycle ends before the time span w and the m^{th} failure occurs after the time span w , which is named situation a and can be modeled as a set of inequalities $S_N \leq w < T_m$; the m^{th} failure occurs after the end of the N^{th} mission cycle and before the time span w , which is named situation b and can be modeled as a set of inequalities $S_N \leq T_m \leq w$. Using these two types of inequality sets, the occurrence probabilities $\bar{Q}_i$ ($i = a, b$) of the respective cases can be obtained as

$$\bar{Q}_i = \begin{cases} \Pr\{S_N \leq w < T_m\} & \text{for situation a} \\ \Pr\{S_N \leq T_m \leq w\} & \text{for situation b} \end{cases} = \begin{cases} G^{(N)}(w)\bar{F}_m(w) & \text{for situation a} \\ \int_0^w G^{(N)}(t)dF_m(t) & \text{for situation b} \end{cases} \quad (9)$$

Clearly, the sum of these four probabilities equals 1, i.e., $Q_A + Q_B + \bar{Q}_a + \bar{Q}_b = 1$, which satisfies the probability characteristic of mutually exclusive events.

When situation A occurs, the failures that occur within the interval from the arrival time T_m to the end of the N^{th} mission cycle are removed in the form of charging costs. Under this situation, the mean number of failures that incur chargeable repair costs can

be expressed as $\int_{T_m}^{S_N} \lambda(t)dt$. When situation B transpires, the failures that occur within the interval from the arrival time T_m to the end of the N^{th} mission cycle are removed in the form of charging costs. Under this situation, the mean number of failures that incur chargeable repair costs can be expressed as $\int_w^{S_N} \lambda(t)dt$. Moreover, by making use of the probabilities Q_i , the mean value, denoted as MN_{cc}^I , of the mean failures that incur chargeable costs can be derived as

$$\begin{aligned} MN_{cc}^I &= \int_w^\infty \left(\int_t^\infty \left(\int_t^s \lambda(t)dt \right) dG^{(N)}(s) \right) dF_m(t) + F_m(w) \int_w^\infty \left(\int_w^s \lambda(u)du \right) dG^{(N)}(s) \\ &= \int_w^\infty \left(\int_t^\infty \bar{G}^{(N)}(s) \lambda(s)ds \right) dF_m(t) + F_m(w) \int_w^\infty \bar{G}^{(N)}(s) \lambda(s)ds \\ &= \int_w^\infty F_m(s) \bar{G}^{(N)}(s) \lambda(s)ds \end{aligned} \quad (10)$$

When the RDRW-CCL expires at the end of the N^{th} mission cycle, the total mean value of failures over warranty coverage is equal to TMV . Then, through algebraic operations, the profit-generating PM^I of the RDRW-CCL can be obtained as

$$PM^I = c_r \cdot MN_{cc}^I - c_m \cdot TMV = c_r \int_w^\infty F_m(s) \bar{G}^{(N)}(s) \lambda(s)ds - c_m \int_0^\infty \bar{G}^{(N)}(s) \lambda(s)ds \quad (11)$$

Based on the definition of the RDRW-CCL, its coverage concludes at the termination of the N^{th} mission cycle. Consequently, the warranty period WP^I of the RDRW-CCL is equivalent to that of the RDRW-CCF, namely $WP^I = WP^f$.

2.3. Discussion and Analysis

2.3.1. The Limit Warranties of the RDRW-CCF

According to reliability theory, the limit $\lim_{m \rightarrow \infty} F_m(s) = 0$ holds true. Therefore, using such a limit, the limit value of the profit-generating PM^f of the RDRW-CCF can be expressed as

$$\lim_{m \rightarrow \infty} PM^f = c_r \int_w^\infty \bar{G}^{(N)}(s) \lambda(s)ds - c_m \int_0^\infty \bar{G}^{(N)}(s) \lambda(s)ds \quad (12)$$

The relationship $m \rightarrow \infty$ implies that the m^{th} failure always occurs after time span w . This shows that, in the case of using ‘whichever comes first’, the scale—that is, the failure number m —is removed from the RDRW-CCF. Consequently, the relationship

$m \rightarrow \infty$ can reduce the RDRW-CCF to a random discrete repair warranty considering cost charging (RDRW-CC), for which its profit generation can be expressed by Equation (12). As stated above, the failure number m has no impact on the warranty period. Therefore, the warranty period of the RDRW-CC is the same as that of the RDRW-CCF.

When $m \rightarrow \infty$ and $w \rightarrow \infty$, the limit value of the profit-generating PM^f of the RDRW-CCF can be listed as

$$\lim_{\substack{m \rightarrow \infty \\ w \rightarrow \infty}} PM^f = -c_m \int_0^\infty \bar{G}^{(N)}(s) \lambda(s) ds \quad (13)$$

The relationships $m \rightarrow \infty$ and $w \rightarrow \infty$ imply that the time span w necessarily expires after the termination of the N^{th} mission cycle, and the m^{th} failure also occurs after the termination of the N^{th} mission cycle. This shows that, in the case of using ‘whichever comes first’, two scales, which are the failure number m and the time span w , are removed from the RDRW-CCF. Consequently, the relationships $m \rightarrow \infty$ and $w \rightarrow \infty$ can reduce the RDRW-CCF to a random discrete repair warranty (RDRW), for which its profit generation is expressed by Equation (13). As stated above, the time span w and the m^{th} failure have no impact on the warranty period. Therefore, the warranty period of the RDRW is the same as that of the RDRW-CCF.

According to reliability theory, the limit $\lim_{m \rightarrow 0} F_m(s) = 1$ holds true. Therefore, using such a limit, the limit value of the profit-generating PM^f for the RDRW-CCF can be obtained as

$$\lim_{\substack{m \rightarrow 0 \\ w \rightarrow 0}} PM^f = (c_r - c_m) \int_0^\infty \bar{G}^{(N)}(s) \lambda(s) ds \quad (14)$$

The relationships $m \rightarrow 0$ and $w \rightarrow 0$ indicate that cost charging will take place for every failure that occurs throughout the entire warranty coverage. Consequently, the relationships $m \rightarrow 0$ and $w \rightarrow 0$ can reduce the RDRW-CCF to a random discrete repair warranty considering full cost charging (RDRW-FCC), for which its profit can be expressed by Equation (14). Similarly, the warranty period of the RDRW-CCF is the same as that of the RDRW-CCF.

2.3.2. The Limit Warranties of the RDRW-CCL

When $m \rightarrow 0$, the limit value of the profit-generating PM^l for the RDRW-CCL can be obtained as

$$\lim_{m \rightarrow 0} PM^l = c_r \int_w^\infty \bar{G}^{(N)}(s) \lambda(s) ds - c_m \int_0^\infty \bar{G}^{(N)}(s) \lambda(s) ds \quad (15)$$

The relationship $m \rightarrow 0$ means that, in the case of using ‘whichever comes last’, the scale, i.e., the failure number m , is removed from the RDRW-CCL. As a result, the relationship $m \rightarrow 0$ can reduce the RDRW-CCL to an RDRW-CC for which its profit generation is equal to the result presented by Equation (15) and is identical to the result in Equation (12).

$$\lim_{\substack{m \rightarrow 0 \\ w \rightarrow 0}} PM^l = (c_r - c_m) \int_0^\infty \bar{G}^{(N)}(s) \lambda(s) ds \quad (16)$$

The relationships $m \rightarrow 0$ and $w \rightarrow 0$ imply that the failure number m and the time span w are removed from the RDRW-CCL. Consequently, the relationships $m \rightarrow 0$ and $w \rightarrow 0$ can reduce the RDRW-CCL to an RDRW-CCF, for which its profit generation is expressed by Equation (16) and is identical to the result presented by Equation (14).

When $m \rightarrow \infty$ and $w \rightarrow \infty$, the limit value of the profit-generating PM^l for the RDRW-CCL can be obtained as

$$\lim_{\substack{m \rightarrow \infty \\ w \rightarrow \infty}} PM^I = -c_m \int_0^\infty \bar{G}^{(N)}(s) \lambda(s) ds \quad (17)$$

The relationships $m \rightarrow \infty$ and $w \rightarrow \infty$ imply that the N^{th} mission cycle terminates before the failure number m or the time span w , whichever comes last. These conditions ensure that all failures within the warranty coverage are rectified free of charge. As a result, the relationships $m \rightarrow \infty$ and $w \rightarrow \infty$ can reduce the RDRW-CCL to an

RDRW, for which its profit is expressed by $\lim_{\substack{m \rightarrow \infty \\ w \rightarrow \infty}} PM^I$. Evidently, this outcome aligns precisely with the result derived from Equation (13).

We investigate the inherent transformation relationships of all proposed warranty models through limit analyses of threshold parameters m and w . As illustrated in Figure 1, RDRW serves as the fundamental base model, from which two derivative warranty schemes, RDRW-CCF and RDRW-CCL, are generated under distinct charge-triggering rules. Specifically, WCF adopts the “whichever occurs first” principle, while WCL follows the “whichever occurs last” principle. A set of simplified models including RDRW-CC and RDRW-FCC can be derived by assigning extreme values to the two thresholds. It is worth noting that the TRPM post-warranty maintenance strategy developed in Section 3 relies entirely on the classification metrics extracted from the RDRW-CCF framework.

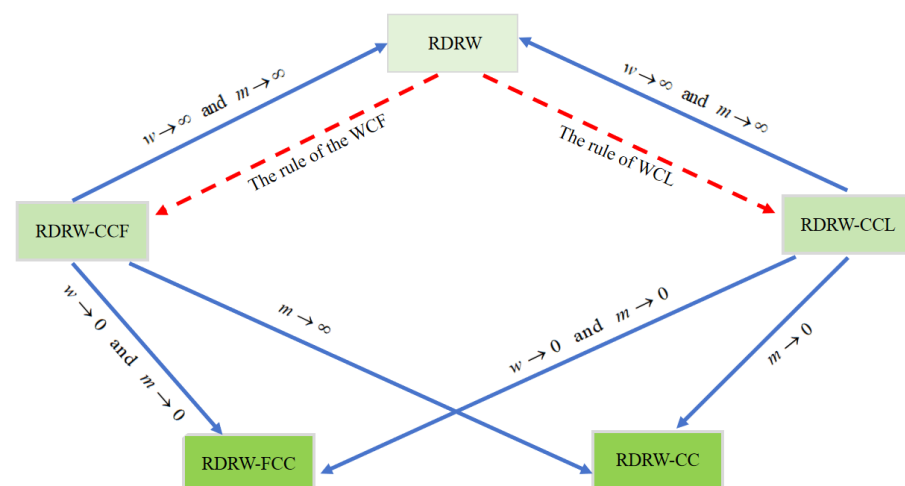


Figure 1. Transformation relationships of RDRW-series models.

3. Post-Warranty Maintenance Strategy

To address the gap of neglecting heterogeneous characteristics of post-warranty systems highlighted in the Introduction and to complement the RDRW-CCF, a profit-oriented warranty model proposed in Section 2 that leverages usage data for user classification, this section customizes and models a post-warranty maintenance strategy.

Under the RDRW-CCF, the warranty service concludes at the end of the N^{th} mission cycle. With the time span w serving as the classification boundary, we further delineate usage rate scenarios and reliability scenarios. In this section, a light usage type is defined as a situation where the N^{th} mission cycle concludes prior to the time span w , whereas a heavy usage type is characterized by the N^{th} mission cycle ending after the time span w . Moreover, a lower reliability type is identified when the m^{th} failure occurs before the time span w , and a higher reliability type is defined as the m^{th} failure taking place after the time span w . Leveraging these definitions, a post-warranty maintenance

strategy is formulated based on these scenario types, as elaborated in the following sections.

3.1. The Definition of Strategy

In this paper, we use R_i to denote reliability heterogeneities and U_j to represent usage rate heterogeneities. An ordered pair $H_{ij} = (R_i, U_j)$ is introduced to model these heterogeneities, where $i = l$ indicates lower reliability and $i = h$ implies higher reliability; $j = l$ represents a light usage rate, while $j = h$ signifies a heavy usage rate. As a result, within the RDRW-CCF framework, the heterogeneities can be categorized into four mutually exclusive scenarios: H_{lh} , H_{hh} , H_{ll} , and H_{hl} .

From a stochastic standpoint, the accumulated operating time (past age) of the system in light-usage scenarios is less than that in heavy-usage scenarios. This leads to the following observations: For systems in light-usage scenarios, the remaining service life is longer, and the subsequent failure frequency is lower due to the relatively small accumulated age. Conversely, for systems in heavy-usage scenarios, the remaining service life is shorter, and the subsequent failure frequency is higher because of the greater accumulated age.

Consequently, maximizing the utilization of the remaining service life is a prudent objective for systems in light-usage scenarios. For systems in heavy-usage scenarios, given that the higher failure frequency results in increased costs and more significant downtime losses, adopting measures (such as improving reliability) to reduce both costs and losses becomes a more rational goal. To achieve these objectives, we propose a post-warranty maintenance strategy that accounts for usage rates and reliability heterogeneities.

Let K , L , and M be three non-negative integers with $L \geq K$. All failures are corrected by minimal repairs. The post-warranty maintenance strategy is defined as follows:

- ◆ **The post-warranty system subject to heterogeneity scenario H_{lh} will undergo the following maintenance terms:**
 - ① It will undergo PM first upon the expiry of the warranty and then at intervals of consecutive K mission cycles in sequential order;
 - ② While repeating PMs at intervals of consecutive K mission cycles, if the $(M-1)^{\text{th}}$ PM is completed, the system undergoes preventive replacement (PR) at the end of the M^{th} interval, and no further PMs will be performed.
- ◆ **The post-warranty system subject to heterogeneity scenario H_{hh} will undergo the following maintenance terms:**
 - ① It will undergo PM at intervals of consecutive K mission cycles;
 - ② During the repetition of the term ①, upon completion of the $(M-1)^{\text{th}}$ PM, the system undergoes the PR at the end of the M^{th} interval, and no further PM will be performed.
- ◆ **The post-warranty system, which is characterized by heterogeneity scenario H_{ll} , will undergo the following maintenance terms:**
 - ① The system will undergo PMs, commencing upon the warranty expiration and then at intervals of consecutive L mission cycles in sequence;
 - ② During the repetition of PMs at intervals of consecutive L mission cycles, once the $(M-1)^{\text{th}}$ PM is completed, the system undergoes the PR at the end of the M^{th} interval, and no further PMs will be performed.
- ◆ **The post-warranty system, which is characterized by heterogeneity scenario H_{hl} , will adhere to the following maintenance protocols:**
 - ① It will undergo PM at intervals of consecutive L mission cycles;

- ② In the process of repeating the term ①, if the $(M-1)^{\text{th}}$ PM is completed, the system undergoes the PR at the end of the M^{th} interval while not performing any PM.

For post-warranty systems with lower reliability, specifically those of heterogeneity scenarios H_{lh} and H_{ll} , conducting PM immediately upon warranty expiration serves to boost reliability. This, in turn, decreases costs and losses by reducing the frequency of failures. On the other hand, for post-warranty systems with light usage, namely those of heterogeneity scenarios H_{hl} and H_{ll} , modeling the PM interval (PMI) as the intervals of consecutive L rather than consecutive K mission cycles is aimed at maximizing the utilization of their remaining service life. Given these considerations, this strategy can effectively achieve multiple objectives: cost reduction, minimization of losses, and extension of the remaining service life. With three key metrics K , L , and M , this strategy is called the trivariate random PM (TRPM) strategy.

To intuitively illustrate the hierarchical differentiated decision rules of the TRPM strategy, we establish a full schematic framework of this strategy, as presented in Figure 2.

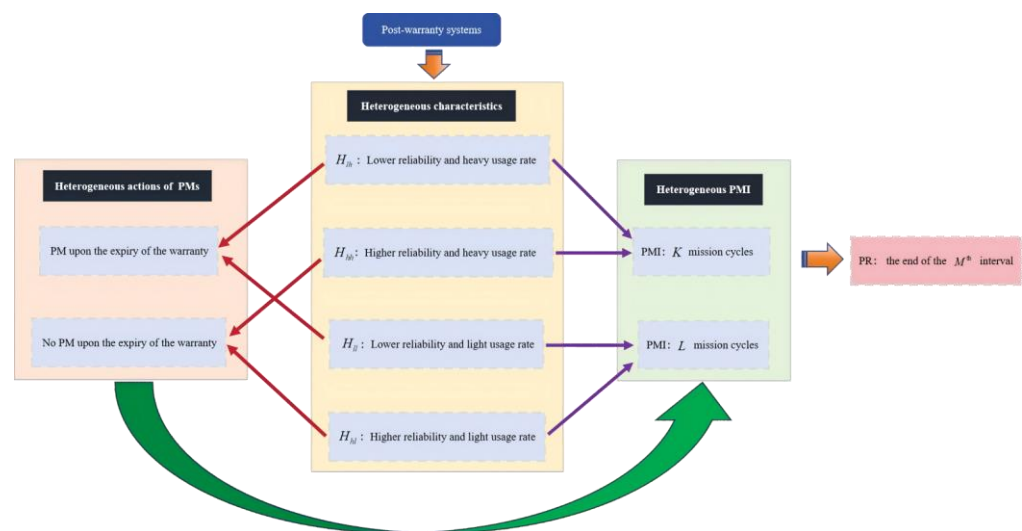


Figure 2. Schematic framework of the TRPM.

3.2. The Objective Function Modeling of the TRPM

The scenario H_{lh} of the heterogeneity type can be modeled by the inequality $T_m \leq w < S_N$. The scenario H_{hh} of the heterogeneity type can be modeled by $w < \text{Min}\{T_m, S_N\}$. The scenario H_{ll} of heterogeneity type is represented by $w \geq \text{Min}\{T_m, S_N\}$, and the scenario H_{hl} of heterogeneity type can be modeled as $S_N \leq w < T_m$. Leveraging these inequalities, the occurrence probabilities q_{ij} ($i = l, h$ and $j = l, h$) of the four aforementioned scenarios can be expressed as

$$q_{ij} = \begin{cases} \Pr\{T_m \leq w < S_N\} & \text{for scenario } H_{lh} \\ \Pr\{w < \text{Min}\{T_m, S_N\}\} & \text{for scenario } H_{hh} \\ \Pr\{w \geq \text{Min}\{T_m, S_N\}\} & \text{for scenario } H_{ll} \\ \Pr\{S_N \leq w < T_m\} & \text{for scenario } H_{hl} \end{cases} = \begin{cases} F_m(w)\bar{G}^{(N)}(w) & \text{for scenario } H_{lh} \\ \bar{F}_m(w)\bar{G}^{(N)}(w) & \text{for scenario } H_{hh} \\ F_m(w)G^{(N)}(w) & \text{for scenario } H_{ll} \\ G^{(N)}(w)\bar{F}_m(w) & \text{for scenario } H_{hl} \end{cases} \quad (18)$$

Clearly, the sum of these probabilities equals 1, i.e., $q_{lh} + q_{hh} + q_{ll} + q_{hl} = 1$, which satisfies the probability characteristic of mutually exclusive events.

When performing PM at the warranty expiration, it is possible to reduce the corresponding PMI, denoted as S_N . Let σS_N ($0 \leq \sigma \leq 1$) represent the virtual interval after such a PM. Then, the reduction in such a PMI can be expressed as $S_N - \sigma S_N = (1 - \sigma)S_N = \bar{\sigma}S_N$, where $\bar{\sigma} = 1 - \sigma$. This reduction $\bar{\sigma}S_N$ serves as an indicator of reliability improvement. Similarly, for PMs carried out at intervals of consecutive K mission cycles, let $S_K = \sum_{i=1}^K Y_i$ be the random sum formed by these K mission cycles, which defines a PMI. When σS_K represents the virtual interval after each PM, the reduction in each PMI can be expressed as $S_K - \sigma S_K = (1 - \sigma)S_K = \bar{\sigma}S_K$, which also quantifies the enhancement in reliability. Other approaches for modeling the enhancement in reliability have been presented in Ruan et al. [35] and Azevedo et al. [36]. Engineering experience shows that the cost of enhancing reliability is a monotonically increasing function of the degree of reliability improvement. Therefore, the cost $C_{PM}(S_i)$ associated with performing PM at the warranty expiration, as well as the cost for each PM carried out at intervals of consecutive K and L mission cycles, can be formulated as

$$C_{PM}(S_i) = \begin{cases} c_s + c_l (\bar{\sigma}S_N)^a & \text{for PM upon the expiry of the warranty} \\ c_s + c_l (\bar{\sigma}S_K)^a & \text{for each PM at intervals of consecutive } K \text{ mission cycles} \\ c_s + c_l (\bar{\sigma}S_L)^a & \text{for each PM at intervals of consecutive } L \text{ mission cycles} \end{cases} \quad (19)$$

where $i = N, K, L$, c_s is the setup cost of unit PM, $c_l > 0$, and $a > 0$.

Under the framework of the RDRW-CCF, the total PM cost related to four mutually exclusive scenarios can be listed as

$$\begin{cases} c_s + c_l (\bar{\sigma}S_N)^a + (M-1)(c_s + c_l (\bar{\sigma}S_K)^a) & \text{for scenario } H_{lh} \\ (M-1)(c_s + c_l (\bar{\sigma}S_K)^a) & \text{for scenario } H_{hh} \\ c_s + c_l (\bar{\sigma}S_N)^a + (M-1)(c_s + c_l (\bar{\sigma}S_L)^a) & \text{for scenario } H_{ll} \\ (M-1)(c_s + c_l (\bar{\sigma}S_L)^a) & \text{for scenario } H_{hl} \end{cases} \quad (20)$$

By means of the occurrence probabilities q_{ij} of four scenarios, the total PM cost $TC_{PM}(K, L, M | S_K, S_L)$ of the TRPM can be calculated as

$$TC_{PM}(K, L, M | S_K, S_L) = \begin{pmatrix} F_m(w) \left(\int_w^\infty (c_s + c_l (\bar{\sigma}s)^a) dG^{(N)}(s) + \bar{G}^{(N)}(w) (M-1)(c_s + c_l (\bar{\sigma}S_K)^a) \right) + \\ \bar{F}_m(w) \bar{G}^{(N)}(w) (M-1)(c_s + c_l (\bar{\sigma}S_K)^a) + \\ F_m(w) \left(\int_0^w (c_s + c_l (\bar{\sigma}s)^a) dG^{(N)}(s) + G^{(N)}(w) (M-1)(c_s + c_l (\bar{\sigma}S_L)^a) \right) + \\ G^{(N)}(w) \bar{F}_m(w) (M-1)(c_s + c_l (\bar{\sigma}S_L)^a) \end{pmatrix} \quad (21)$$

In the case of letting $G^{(K)}(S_K)$ be the distribution function of the random sum S_K and $G^{(L)}(S_L)$ be the distribution function of the random sum S_L , the expected value $TC_{PM}(K, L, M)$ of the total PM cost $TC_{PM}(K, L, M | S_K, S_L)$ can be represented as

$$TC_{PM}(K, L, M) = \begin{pmatrix} F_m(w) \int_w^\infty (c_s + c_l (\bar{\sigma}s)^a) dG^{(N)}(s) + \bar{G}^{(N)}(w) (M-1) \left(c_s + c_l \int_0^\infty (\bar{\sigma}s)^a dG^{(K)}(s_K) \right) + \\ F_m(w) \int_0^w (c_s + c_l (\bar{\sigma}s)^a) dG^{(N)}(s) + G^{(N)}(w) (M-1) \left(c_s + c_l \int_0^\infty (\bar{\sigma}s)^a dG^{(L)}(s_L) \right) \end{pmatrix} \quad (22)$$

When the scenario H_{lh} of the heterogeneity type occurs, the corresponding system will undergo PM immediately upon warranty expiration. Based on the preceding discussion, the system's age following this PM can be expressed as σS_N . Similarly, when the

scenario H_{ll} of heterogeneity type arises, the associated system will also receive PM at the end of the warranty period, and the post-PM age of the system is denoted as σS_N . In contrast, when either scenario H_{hh} or H_{hl} of the heterogeneity type occurs, the system's age can be represented as S_N . According to reliability theory, the failure-rate functions corresponding to these four mutually exclusive scenarios can be enumerated as

$$\begin{cases} \lambda(\sigma S_N + u) & \text{for scenario } H_{lh} \\ \lambda(S_N + u) & \text{for scenario } H_{hh} \\ \lambda(\sigma S_N + u) & \text{for scenario } H_{ll} \\ \lambda(S_N + u) & \text{for scenario } H_{hl} \end{cases} \quad (23)$$

In the interval from the end of the i^{th} ($0 \leq i \leq M-1$) PM to the occurrence of the $(i+1)^{\text{th}}$ PM, all failures are removed through minimal repairs. According to reliability theory, the total repair cost related to four mutually exclusive scenarios can be listed as

$$\begin{cases} c_u \sum_{i=0}^{M-1} \int_{i\sigma S_K}^{i\sigma S_K + S_K} \lambda(\sigma S_N + u) du & \text{for scenario } H_{lh} \\ c_u \sum_{i=0}^{M-1} \int_{i\sigma S_K}^{i\sigma S_K + S_K} \lambda(S_N + u) du & \text{for scenario } H_{hh} \\ c_u \sum_{i=0}^{M-1} \int_{i\sigma S_L}^{i\sigma S_L + S_L} \lambda(\sigma S_N + u) du & \text{for scenario } H_{ll} \\ c_u \sum_{i=0}^{M-1} \int_{i\sigma S_L}^{i\sigma S_L + S_L} \lambda(S_N + u) du & \text{for scenario } H_{hl} \end{cases} \quad (24)$$

where $c_u = c_m + c_f$.

By means of the occurrence probabilities q_{ij} of four mutually exclusive scenarios, the total repair cost $TC_m(K, L, M | S_K, S_L)$ of the TRPM can be calculated as

$$TC_m(K, L, M | S_K, S_L) = c_u \begin{pmatrix} F_m(w) \int_w^\infty \left(\sum_{i=0}^{M-1} \int_{i\sigma S_K}^{i\sigma S_K + S_K} \lambda(\sigma s + u) du \right) dG^{(N)}(s) + \bar{F}_m(w) \int_w^\infty \left(\sum_{i=0}^{M-1} \int_{i\sigma S_K}^{i\sigma S_K + S_K} \lambda(s + u) du \right) dG^{(N)}(s) + \\ F_m(w) \int_0^w \left(\sum_{i=0}^{M-1} \int_{i\sigma S_L}^{i\sigma S_L + S_L} \lambda(\sigma s + u) du \right) dG^{(N)}(s) + \bar{F}_m(w) \int_0^w \left(\sum_{i=0}^{M-1} \int_{i\sigma S_L}^{i\sigma S_L + S_L} \lambda(s + u) du \right) dG^{(N)}(s) \end{pmatrix} \quad (25)$$

Using the distribution function $G^{(K)}(s_K)$ of the random sum S_K and the distribution function $G^{(L)}(s_L)$ of the random sum S_L , the expected value $TC_m(K, L, M)$ of the total repair cost $TC_m(K, L, M | S_K, S_L)$ can be represented as

$$TC_m(K, L, M) = c_u \begin{pmatrix} \int_0^\infty \left(F_m(w) \int_w^\infty \left(\sum_{i=0}^{M-1} \int_{i\sigma s_K}^{i\sigma s_K + s_K} \lambda(\sigma s + u) du \right) dG^{(N)}(s) + \bar{F}_m(w) \int_w^\infty \left(\sum_{i=0}^{M-1} \int_{i\sigma s_K}^{i\sigma s_K + s_K} \lambda(s + u) du \right) dG^{(N)}(s) \right) dG^{(K)}(s_K) + \\ \int_0^\infty \left(F_m(w) \int_0^w \left(\sum_{i=0}^{M-1} \int_{i\sigma s_L}^{i\sigma s_L + s_L} \lambda(\sigma s + u) du \right) dG^{(N)}(s) + \bar{F}_m(w) \int_0^w \left(\sum_{i=0}^{M-1} \int_{i\sigma s_L}^{i\sigma s_L + s_L} \lambda(s + u) du \right) dG^{(N)}(s) \right) dG^{(L)}(s_L) \end{pmatrix} \quad (26)$$

Through algebraic operations, the expected total cost $TC(K, L, M)$ resulting from the TRPM can be obtained as

$$TC(K, L, M) = TC_{PM}(K, L, M) + TC_m(K, L, M) + c_R \quad (27)$$

where c_R represents the unit cost of unit replacement.

According to the definition of the TRPM, the replacement times related to four mutually exclusive scenarios can be listed as

$$\begin{cases} MS_K & \text{for scenario } H_{lh} \\ MS_K & \text{for scenario } H_{hh} \\ MS_L & \text{for scenario } H_{ll} \\ MS_L & \text{for scenario } H_{hl} \end{cases} \quad (28)$$

By means of the distribution function $G^{(K)}(s_K)$ and $G^{(L)}(s_L)$, the expected replacement times related to four mutually exclusive scenarios can be given by

$$\begin{cases} M \int_0^\infty s_K dG^{(K)}(s_K) & \text{for scenario } H_{lh} \\ M \int_0^\infty s_K dG^{(K)}(s_K) & \text{for scenario } H_{hh} \\ M \int_0^\infty s_L dG^{(L)}(s_L) & \text{for scenario } H_{ll} \\ M \int_0^\infty s_K dG^{(L)}(s_L) & \text{for scenario } H_{hl} \end{cases} = \begin{cases} M \int_0^\infty \bar{G}^{(K)}(s_K) ds_K & \text{for scenario } H_{lh} \\ M \int_0^\infty \bar{G}^{(K)}(s_K) ds_K & \text{for scenario } H_{hh} \\ M \int_0^\infty \bar{G}^{(L)}(s_L) ds_L & \text{for scenario } H_{ll} \\ M \int_0^\infty \bar{G}^{(L)}(s_L) ds_L & \text{for scenario } H_{hl} \end{cases} \quad (29)$$

By means of the occurrence probabilities q_{ij} of four mutually exclusive scenarios, the expected replacement time $ERT(K, L, M)$ of the TRPM can be given by

$$ERT(K, L, M) = \begin{pmatrix} F_m(w) \bar{G}^{(N)}(w) \cdot M \int_0^\infty \bar{G}^{(K)}(s_K) ds_K + \\ \bar{F}_m(w) \bar{G}^{(N)}(w) \cdot M \int_0^\infty \bar{G}^{(K)}(s_K) ds_K + \\ F_m(w) G^{(N)}(w) \cdot M \int_0^\infty \bar{G}^{(L)}(s_L) ds_L + \\ G^{(N)}(w) \bar{F}_m(w) \cdot M \int_0^\infty \bar{G}^{(L)}(s_L) ds_L \end{pmatrix} = M \left(\bar{G}^{(N)}(w) \int_0^\infty \bar{G}^{(K)}(s_K) ds_K + G^{(N)}(w) \int_0^\infty \bar{G}^{(L)}(s_L) ds_L \right) \quad (30)$$

The availability objective function is commonly used (see Qiu et al. [37]), while the total cost is a frequently employed objective function (refer to Peng et al. [38]; Yang et al. [39]). Besides these two types of objective functions, based on the renewal reward theorem in Barlow and Proschan [40], the cost rate function becomes the commonly used objective function due to the combined use of time and cost measurements. Here, the cost rate function is used as the objective function of the TRPM, and thus, the cost rate function of the TRPM can be obtained as

$$CR(K, L, M) = \frac{FC + TC(K, L, M)}{WP^f + ERT(K, L, M)} = \frac{FC + TC_{PM}(K, L, M) + TC_m(K, L, M) + c_R}{WP^f + ERT(K, L, M)} \quad (31)$$

where FC is the total cost produced by the RDRW-CCF and is obtained by replacing $-c_m$ in the profit-generating PM^f with $+c_f$.

3.3. Special Strategies

When PM is terminated upon the expiration of the warranty, the heterogeneities related to both the usage rate and reliability are simplified, leaving only the heterogeneities associated with the usage rate. By setting the PM costs to zero at the end of the warranty period, all customized PM activities within the TRPM framework, specifically those scheduled for execution upon warranty expiration, are completely eliminated. Under these circumstances, TRPM becomes a strategy that focuses solely on usage-rate heterogeneity, and the corresponding cost rate can be expressed as

$$CR(K, L, M) = \frac{FC + \left(\bar{G}^{(N)}(w)(M-1) \left(c_s + c_i \int_0^\infty (\bar{\sigma} s_K)^a dG^{(K)}(s_K) \right) + G^{(N)}(w)(M-1) \left(c_s + c_i \int_0^\infty (\bar{\sigma} s_L)^a dG^{(L)}(s_L) \right) \right) + c_u \left(\int_0^\infty \left(\int_w^\infty \left(\sum_{i=0}^{M-1} \int_{i\sigma s_K}^{i\sigma s_K + s_K} \lambda(s+u) du \right) dG^{(N)}(s) \right) dG^{(K)}(s_K) + \int_0^\infty \left(\int_0^w \left(\sum_{i=0}^{M-1} \int_{i\sigma s_L}^{i\sigma s_L + s_L} \lambda(s+u) du \right) dG^{(N)}(s) \right) dG^{(L)}(s_L) \right) + c_R}{WP^f + ERT(K, L, M)} \quad (32)$$

For the sake of convenience, the above-mentioned strategy is named the TRPM considering only usage-rate heterogeneity (TRPM-URH).

When $L = K$ holds, all PMIs are identical. Under such circumstances, the heterogeneities considered are limited solely to reliability differences. As a result, this relationship causes the TRPM to transform into a bivariate random PM strategy that takes into account reliability heterogeneity (BRPM-RH). The cost rate of the BRPM-RH can be formulated as

$$CR(K, M) = \frac{FC + F_m(w) \int_0^\infty (c_s + c_l (\bar{\sigma}s)^a) dG^{(N)}(s) + (M-1) \left(c_s + c_l \int_0^\infty (\bar{\sigma}s_K)^a dG^{(K)}(s_K) \right) + c_u \left(F_m(w) \int_0^\infty \left(\int_0^\infty \left(\sum_{i=0}^{M-1} \int_{i\sigma s_K}^{i\sigma s_K + s_K} \lambda(\sigma s + u) du \right) dG^{(N)}(s) + \bar{F}_m(w) \int_0^\infty \left(\sum_{i=0}^{M-1} \int_{i\sigma s_K}^{i\sigma s_K + s_K} \lambda(s + u) du \right) dG^{(N)}(s) \right) dG^{(K)}(s_K) \right) + c_R}{WP^f + M \int_0^\infty \bar{G}^{(K)}(s_K) ds_K} \quad (33)$$

When all PM activities are terminated upon the expiration of the warranty and the equation $L = K$ is met, the heterogeneities associated with both the usage rate and reliability are eliminated. Under these circumstances, the TRPM is simplified to a bivariate random PM (BRPM) strategy. The cost rate of this BRPM can be expressed as

$$CR(K, M) = \frac{FC + (M-1) \left(c_s + c_l \int_0^\infty (\bar{\sigma}s_K)^a dG^{(K)}(s_K) \right) + c_u \int_0^\infty \left(\int_0^\infty \left(\sum_{i=0}^{M-1} \int_{i\sigma s_K}^{i\sigma s_K + s_K} \lambda(s + u) du \right) dG^{(N)}(s) \right) dG^{(K)}(s_K) + c_R}{WP^f + M \int_0^\infty \bar{G}^{(K)}(s_K) ds_K} \quad (34)$$

Up until this point, four strategies have been proposed to ensure the post-warranty reliability of the system under RDRW-CCF: TRPM, TRPM-URH, BRPM-RH, and BRPM. The first three strategies are customized approaches, as they take into account various heterogeneities. In contrast, the last strategy, BRPM, does not consider any heterogeneity, making it a standardization strategy. Additionally, other strategies can be derived by exploring different parameter settings within the RDRW-CCF framework. However, due to space limitations, these additional strategies will not be presented here.

4. Numerical Investigations

Numerous random warranty and post-warranty maintenance strategies have been proposed and modeled to effectively manage the service-life-cycle reliability of systems that operate under mission cycles. To gain deeper insights, numerical analysis will be employed. Specifically, a system equipped with ADTs has been integrated into smart air conditioners (SACs). This advanced management system enables real-time monitoring, recording, and transmission of on-site data, including the number of mission cycles, the duration of each cycle, and failure information. The SAC offers an ideal practical case study for numerically evaluating the proposed methods, with RDRW-CCF, RDRW-CCL, TRPM, and BRPM serving as key examples. The analytical approaches employed in this study are applicable to other methods like RDRW-CC, RDRW, TRPM-URH, and the like. Detailed descriptions of these methodologies will not be repeated herein, as they can be easily extended to other contexts.

In practical engineering applications, the failure frequency of a SAC tends to increase with its service time, a trend that can be accurately captured through continuous monitoring of mission-critical information. Therefore, the failure-rate function can be set as an increasing function: $\lambda(u) = \alpha(u)^\beta$, where $\alpha > 0$ and $\beta > 0$. The exponential distribution function $G(y)$ of mission cycles Y_i is assumed as $G(y) = 1 - \exp(-y\tau)$, where $\tau > 0$. Some of the parameters are assigned as $\alpha = 0.8$, $\beta = 1$, $a = 1$, $c_s = 0.05$, $c_m = 0.1$, $c_r = 0.3$, $c_l = 0.5$, $c_u = 0.2$, and $c_R = 15$. Other parameters are assigned wherever used.

4.1. Numerical Investigations of the Proposed Warranties

4.1.1. Numerical Investigations of the RDRW-CCF

Figure 3 presents the effects of the time span w and failure count m on the profitability of the RDRW-CCF, using $N = 15$ and $\tau = 3$. As depicted in Figure 3, the curves associated with the time span w clearly show that profitability diminishes as w increases. Similarly, the graphs related to the failure number m indicate that the

profitability also declines with an increase in m . The reason is that an increase in either w or m shortens the chargeable period while expanding the free-of-charge period, thereby decreasing the overall profit.

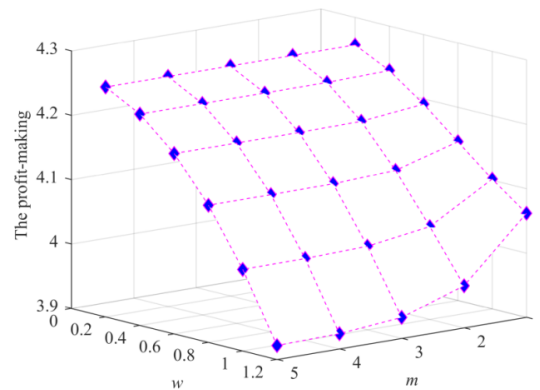


Figure 3. The impacts of key parameters on the profit-generating of the RDRW-CCF.

To elucidate the influence of the mission number N and the exponential distribution parameter τ on the profitability of the RDRW-CCF, Figure 4 was constructed using $w=2$ and $m=5$. As shown in Figure 4, the curves corresponding to the mission number N demonstrate that the profitability of the RDRW-CCF improves as N increases. Conversely, the graphs associated with the exponential distribution parameter τ reveal that profitability diminishes as τ grows. The rationale behind the former trend is that an increment in N expands the chargeable period, consequently boosting the overall profit. For the latter trend, the explanation lies in the fact that an increase in τ reduces the chargeable period, thereby leading to a decrease in the overall profit.

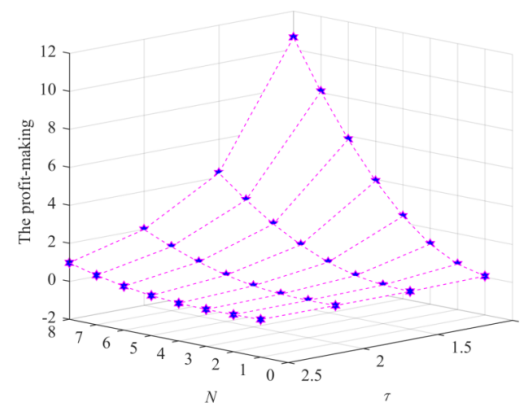


Figure 4. The trends in the profit-generating of the RDRW-CCF.

4.1.2. Numerical Investigations of the RDRW-CCL

To illustrate the impact of the time span w and the failure number m on the profitability of the RDRW-CCL, Figure 5 was generated using $N=15$ and $\tau=3$.

As depicted in Figure 5, the curves associated with the time span w clearly show that the profitability of the RDRW-CCL diminishes as w increases. Similarly, the graphs related to the failure number m indicate that the profitability also declines with an increase in m . Obviously, these trends are identical to those in Figure 3. Therefore, the related reason is that an increase in either w or m reduces the chargeable period while expanding the free-of-charge period, thereby decreasing the overall profit.

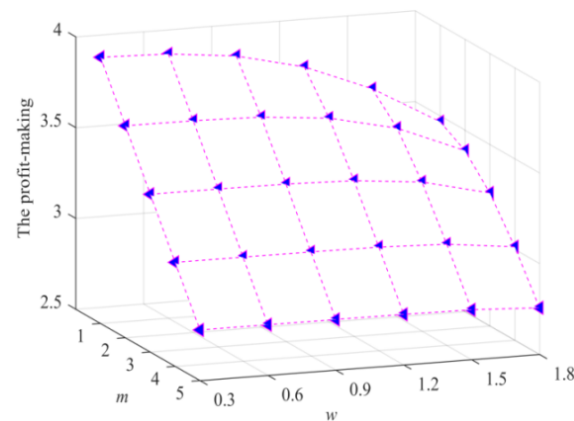


Figure 5. The affect of key parameters on the profit generation of the RDRW-CCL.

From the single-parameter curve results, we can observe obvious monotonic variation characteristics of the objective profit. We further quantify the maximum and minimum variation range of profit in the given parameter interval, and we identify the critical inflection point where the optimal maintenance threshold changes abruptly. From a theoretical perspective, when this parameter varies together with the contract duration parameter, there exists an obvious substitution interaction effect, which jointly determines the upper limit of the total leasing revenue.

To elucidate the influence of the mission number N and the exponential distribution parameter τ on the profitability of the RDRW-CCL, Figure 6 was constructed using $w = 2$ and $m = 5$.

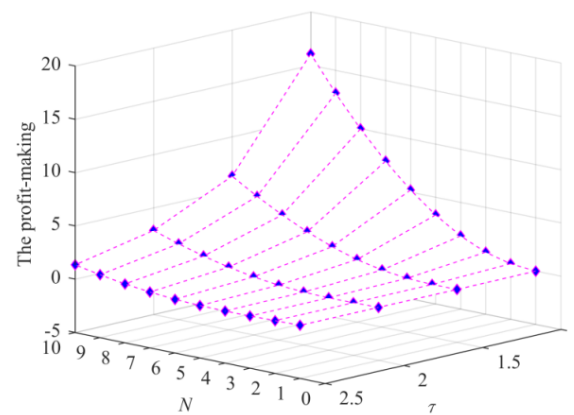


Figure 6. The impacts of key parameters on the profit generation of the RDRW-CCL.

As shown in Figure 6, the curves corresponding to the mission number N demonstrate that the profitability of the RDRW-CCL improves as N increases. Conversely, the graphs associated with the exponential distribution parameter τ reveal that profitability diminishes as τ grows. The rationale behind the former trend is that an increment in N expands the chargeable period, consequently boosting the overall profit. For the latter trend, the explanation lies in the fact that an increase in τ reduces the chargeable period, thereby leading to a decrease in the overall profit.

These curves illustrate the performance gap between our TRPM strategy and the classic benchmark maintenance policies. We calculate the specific profit improvement rate of our proposed model at several key parameter nodes. For the two core decision parameters in this three-dimensional optimization framework, their joint constraint relationship will compress the feasible solution space of the optimal policy, which is the inherent reason for the gradual convergence of curves in the later stage.

4.1.3. The Performance Comparison

To compare the performance disparities between the RDRW-CCF and the RDRW-CCL, Figure 7 was created by utilizing $w = 2$, $m = 5$, $N = 15$, and $\tau = 3$. As illustrated in Figure 7, the profitability curve of the RDRW-CCF consistently lies above that of the RDRW-CCL. This visual representation clearly indicates that the RDRW-CCF outperforms the RDRW-CCL in terms of profit-generating ability.

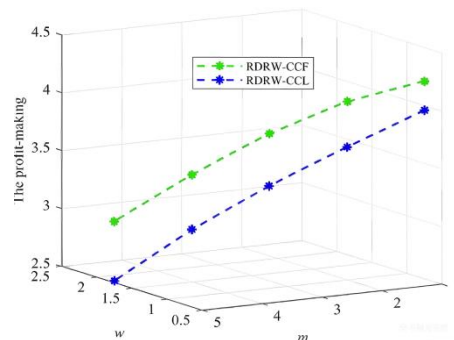


Figure 7. The performance comparison.

4.2. Numerical Investigations of the TRPM and BRPM

4.2.1. Numerical Investigations of the TRPM

To explore the uniqueness of the optimal TRPM, Table 1 was created by utilizing specific values of relevant parameters, including $w = 2$, $m = 5$, $N = 10$, $\tau = 2$, and $\delta = 0.3$. As illustrated in Table 1, when $K = 4$, $L = 6$, and $M = 6$, the minimum cost rate exists. This indicates that the TRPM strategy is effective. Note that values in bold denote minimum values.

Table 1. The effectiveness of the optimal TRPM.

L	K	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 6$	$M = 7$	$M = 8$	$M = 9$
		$CR(K, L, M)$ WP	$CR(K, L, M)$ WP	$CR(K, L, M)$ WP	$CR(K, L, M)$ WP	$CR(K, L, M)$ WP	$CR(K, L, M)$ WP	$CR(K, L, M)$ WP	$CR(K, L, M)$ WP	$CR(K, L, M)$ WP
1	1	4.7720	4.5460	4.3622	4.2115	4.0873	3.9846	3.8996	3.8295	3.7717
2	2	4.5139	4.1745	3.9380	3.7700	3.6500	3.5650	3.5061	3.4673	3.4444
3	3	4.3077	3.9218	3.6881	3.5443	3.4581	3.4109	3.3914	3.3923	3.4086
4	4	4.1425	3.7485	3.5413	3.4349	3.3888	3.3818	3.4015	3.4404	3.4933
5	5	4.0100	3.6305	3.4605	3.3951	3.3896	3.4214	3.4782	3.5524	3.6393
6	6	3.9040	3.5530	3.4244	3.4005	3.4342	3.5029	3.5945	3.7019	3.8207
2	1	4.7691	4.5413	4.3563	4.2049	4.0802	3.9772	3.8922	3.8220	3.7644
3	2	4.5115	4.1711	3.9342	3.7662	3.6463	3.5614	3.5028	3.4643	3.4417
4	3	4.3057	3.9194	3.6856	3.5420	3.4560	3.4091	3.3899	3.3911	3.4078
5	4	4.1408	3.7466	3.5396	3.4334	3.3876	3.3809	3.4009	3.4401	3.4933
6	5	4.0086	3.6291	3.4593	3.3942	3.3890	3.4211	3.4782	3.5526	3.6398
7	6	3.9028	3.5519	3.4236	3.4000	3.4339	3.5029	3.5948	3.7025	3.8216
3	1	4.7663	4.5369	4.3509	4.1989	4.0741	3.9711	3.8863	3.8165	3.7593
4	2	4.5092	4.1680	3.9309	3.7629	3.6433	3.5588	3.5006	3.4626	3.4405
5	3	4.3039	3.9171	3.6834	3.5401	3.4545	3.4080	3.3892	3.3909	3.4079
6	4	4.1393	3.7449	3.5382	3.4323	3.3869	3.3806	3.4010	3.4406	3.4942
7	5	4.0072	3.6279	3.4584	3.3936	3.3888	3.4213	3.4787	3.5536	3.6411
8	6	3.9017	3.5510	3.4230	3.3998	3.4341	3.5034	3.5957	3.7037	3.8231
4	1	4.7636	4.5328	4.3460	4.1938	4.0689	3.9663	3.8819	3.8127	3.7562

5	2	4.5070	4.1651	3.9279	3.7602	3.6410	3.5570	3.4994	3.4621	3.4406
6	3	4.3021	3.9151	3.6816	3.5386	3.4535	3.4075	3.3893	3.3916	3.4092
7	4	4.1378	3.7434	3.5370	3.4316	3.3867	3.3809	3.4018	3.4418	3.4959
8	5	4.0060	3.6268	3.4577	3.3934	3.3890	3.4219	3.4798	3.5551	3.6431
9	6	3.9007	3.5502	3.4226	3.3998	3.4346	3.5043	3.5970	3.7054	3.8252
5	1	4.7610	4.5289	4.3416	4.1893	4.0647	3.9627	3.8790	3.8107	3.7553
6	2	4.5049	4.1625	3.9254	3.7580	3.6394	3.5562	3.4993	3.4628	3.4422
7	3	4.3004	3.9132	3.6801	3.5376	3.4531	3.4078	3.3903	3.3932	3.4115
8	4	4.1364	3.7422	3.5362	3.4313	3.3869	3.3817	3.4032	3.4438	3.4985
9	5	4.0049	3.6259	3.4572	3.3935	3.3896	3.4231	3.4815	3.5573	3.6458
10	6	3.8998	3.5496	3.4225	3.4002	3.4354	3.5057	3.5988	3.7077	3.8279

To elucidate the influence of the time span w and the failure number m on the optimal TRPM, Table 2 was constructed by applying $N=10$, $\tau=2$, and $\delta=0.3$. As clearly shown in Table 2, the optimal mission number L^* exhibits a non-increasing trend with an increase in the time span w . Conversely, the optimal PM number M^* demonstrates a non-decreasing pattern as the time span w grows. Additionally, the minimum cost rate $CR(K^*, L^*, M^*)$ shows a non-increasing behavior with the extension of the time span w . These trends indicate that, as the time span w increases, the PMI corresponding to the heterogeneity scenario H_{ll} and H_{hl} shortens, the number of PM actions increases, and the minimum cost rate decreases.

Table 2. The impact of w and w on the optimal TRPM.

m	$w=2$				$w=2.5$				$w=3$			
	K^*	L^*	M^*	$CR(K^*, L^*, M^*)$	K^*	L^*	M^*	$CR(K^*, L^*, M^*)$	K^*	L^*	M^*	$CR(K^*, L^*, M^*)$
3	4	6	6	3.1174	4	5	7	2.9248	4	5	7	2.8502
4	4	6	6	3.2626	4	5	7	3.0157	4	5	7	2.8698
5	4	6	6	3.3806	4	6	6	3.1274	4	5	7	2.9179

Furthermore, as also shown in Table 2, the optimal mission number L^* exhibits a non-decreasing trend with an increase in the failure number m , while the optimal PM number M^* demonstrates a non-decreasing pattern as the failure number m increases. Additionally, the minimum cost rate $CR(K^*, L^*, M^*)$ shows a non-increasing behavior with the extension of the failure number m . These trends indicate that as the failure number m increases, the PMI corresponding to the heterogeneity scenario H_{ll} and H_{hl} enlarges, the number of PM actions decreases, and the minimum cost rate increases.

To explore the impact of the PM parameter δ and the mission number N on the optimal TRPM, Table 3 was generated by incorporating $w=2$, $\tau=2$, and $m=3$.

Table 3. The impact of δ and N on the optimal TRPM.

N	$\delta=0.4$				$\delta=0.5$				$\delta=0.6$			
	K^*	L^*	M^*	$CR(K^*, L^*, M^*)$	K^*	L^*	M^*	$CR(K^*, L^*, M^*)$	K^*	L^*	M^*	$CR(K^*, L^*, M^*)$
10	5	7	4	3.2822	6	8	3	3.4063	8	9	2	3.5089
13	5	6	4	3.6205	6	7	3	3.7491	8	9	2	3.8547
16	5	6	5	3.9947	5	6	5	4.1354	8	9	2	4.2479

As Table 3 clearly illustrates, both the optimal mission numbers exhibit a non-increasing trend as the mission number N rises. In contrast, the optimal PM number

shows a non-decreasing pattern with the growth of the mission number N . Moreover, the minimum cost rate $CR(K^*, L^*, M^*)$ demonstrates a non-decreasing behavior as the mission number N extends. These trends suggest that as the mission number N increases, the PMI shortens, the frequency of PM actions rises, and the minimum cost rate climbs.

Furthermore, Table 3 also reveals that the optimal mission numbers display a non-decreasing trend as the PM parameter δ increases. Similarly, the optimal PM number follows a non-increasing pattern with the growth of the PM parameter δ . Additionally, the minimum cost rate $CR(K^*, L^*, M^*)$ shows a non-decreasing behavior as the PM parameter δ increases. These findings indicate that when the PM parameter δ increases, the PMI lengthens, the frequency of PM actions decreases, and the minimum cost rate increases.

4.2.2. Numerical Investigations of the BRPM

To explore the uniqueness of the optimal BRPM, Figure 8 was created by utilizing $w=3$, $m=5$, $N=10$, $\tau=2$, and $\delta=0.3$. As illustrated in Figure 8, the minimum value exists. This signals that the BRPM is effective.

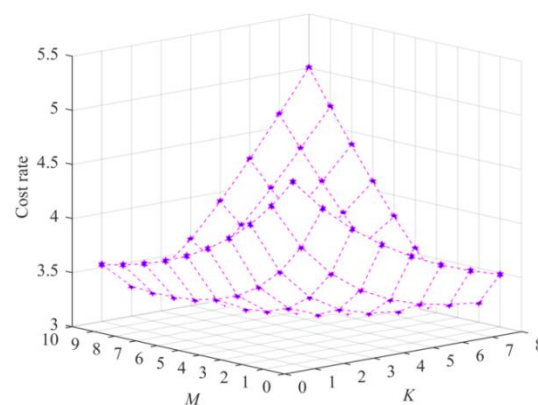


Figure 8. The effectiveness of the optimal BRPM.

To elucidate the influence of the time span w and the failure number m on the optimal BRPM, Table 4 was constructed by applying $N=10$, $\tau=2$, and $\delta=0.3$.

Table 4. The impact of w and m on the optimal BRPM.

m	$w=2$			$w=2.5$			$w=3$		
	K^*	M^*	$CR(K^*, M^*)$	K^*	M^*	$CR(K^*, M^*)$	K^*	M^*	$CR(K^*, M^*)$
3	5	4	3.5386	5	4	3.5303	4	5	3.5276
4	5	4	3.5306	4	5	3.5157	4	5	3.5094
5	5	4	3.5263	4	5	3.5047	4	5	3.4930

As clearly shown in Table 4, the optimal mission number K^* exhibits a non-decreasing trend with an increase in the time span w , and the optimal PM number M^* also demonstrates a non-decreasing pattern as the time span w grows. Additionally, the minimum cost rate $CR(K^*, M^*)$ shows a non-decreasing behavior with the extension of the time span w . These trends indicate that as the time span w increases, the PMI shortens, the number of PM actions increases, and the minimum cost rate decreases.

To explore the impact of the PM parameter δ and the mission number N on the optimal BRPM, Table 5 was generated by incorporating $w = 2$, $m = 5$, and $\tau = 2$. As Table 5 clearly illustrates, the optimal mission number K^* exhibits a non-increasing trend as the mission number N rises. Moreover, the minimum cost rate $CR(K^*, M^*)$ demonstrates a non-decreasing behavior as the mission number N increases. These trends suggest that as the mission number N increases, the PMI shortens, and the minimum cost rate increases. Furthermore, Table 5 also reveals that the optimal mission number K^* displays a non-decreasing trend as the PM parameter δ increases. Similarly, the optimal PM number follows a non-increasing pattern with the growth of the PM parameter δ . Additionally, the minimum cost rate $CR(K^*, M^*)$ shows a non-decreasing behavior as the PM parameter δ increases. These findings indicate that when the PM parameter δ increases, the PMI lengthens, the frequency of PM actions decreases, and the minimum cost rate increases.

Table 5. The impact of δ and N on the optimal BRPM.

N	$\delta = 0.4$			$\delta = 0.5$			$\delta = 0.6$		
	K^*	M^*	$CR(K^*, M^*)$	K^*	M^*	$CR(K^*, M^*)$	K^*	M^*	$CR(K^*, M^*)$
10	5	4	3.6036	5	3	3.6515	7	2	3.6843
13	4	4	3.9947	5	3	4.0316	7	2	4.0571
16	4	4	4.4227	5	3	4.4544	7	2	4.4754

4.2.3. Numerical Investigations of the Performances

Multiple solutions and models have been proposed for the managing post-warranty reliability of systems that operate under mission cycles. Comparing their performances is an intriguing topic. Taking a comparison between TRPM and BRPM as an example, Table 6 (where $w = 2$, $\tau = 2$, $N = 10$ and $\delta = 0.3$) is presented, wherein $RC(K^*, L^*, M^*)$ and $RT(K^*, L^*, M^*)$ represent the numerators of both Equations (31) and (34), and both $CR^b(K^*, L^*, M^*)$ and $CR^b(K^*, M^*)$ represent the denominators of Equations (31) and (34) for TRPM and BRPM, respectively.

Table 6. The performance comparison.

m	TRPM		BRPM		Total Cost Measurements		Relationship
	$RC(K^*, L^*, M^*)$	$RT(K^*, L^*, M^*)$ $CR^b(M^*, T^*)$	$RC(K^*, M^*)$	$RT(K^*, M^*)$	$CM(K^*, L^*, M^*)$	$CM(K^*, M^*)$ $CR^b(M^*, T^*)$	
3	53.1476	17.0488	53.0785	15	797.2140	904.9247	$CM(K^*, L^*, M^*) < CM(K^*, M^*)$
4	55.6242	17.0488	53.0546	15	834.3630	904.5173	$CM(K^*, L^*, M^*) < CM(K^*, M^*)$
5	57.6350	17.0488	52.9890	15	864.5250	903.3989	$CM(K^*, L^*, M^*) < CM(K^*, M^*)$

Other proposed solutions to manage reliability can be compared similarly. The related details are no longer provided to control the length of this paper. The methods offered in Shang et al. [5] have been used to obtain the above comparison results.

5. Conclusions

This study addresses critical limitations in existing warranty models and post-warranty maintenance strategies by leveraging advanced digital technologies (ADTs) to optimize system lifecycle reliability management for mission-cycled systems. The key findings and contributions are summarized as follows.

First, this paper proposes two innovative random discrete repair warranty models, RDRW-CCF (cost charging first) and RDRW-CCL (cost charging last), to fill the gaps of traditional warranties, which lack profit mechanisms and ignore user usage-rate data. Both models integrate a predefined failure threshold and time span to classify user usage patterns. Low-usage users receive free repairs within the initial free-of-charge period, while high-usage users incur charges for failures in the subsequent chargeable period.

Second, for post-warranty maintenance, this paper develops a trivariate random PM (TRPM) strategy to address the neglect of system heterogeneity in existing approaches. TRPM utilizes three metrics from RDRW-CCF, including failure threshold, limited mission cycles, and time span threshold, to categorize post-warranty systems into four mutually exclusive scenarios combining usage rates (light or heavy) and reliability levels (lower or higher). It then tailors maintenance actions. Systems with lower reliability undergo immediate PM at warranty expiration to improve reliability, while light-usage systems use longer PM intervals to maximize the remaining service life.

Third, this study derives and validates variant models and strategies to enhance adaptability. For warranties, adjusting parameters of RDRW-CCF and RDRW-CCL yields limit models such as RDRW-CC, RDRW, and RDRW-FCC that cater to different application scenarios. For post-warranty maintenance, parameter adjustments generate TRPM-URH, which is focused on usage-rate heterogeneity; BRPM-RH, which is focused on reliability heterogeneity; and the bivariate random PM (BRPM). These options provide manufacturers with flexible choices based on data availability and business needs.

Finally, comparative numerical analyses conducted with smart air conditioners (SACs) as a practical case study show that TRPM outperforms the BRPM strategy in both cost efficiency and reliability. For instance, TRPM achieves a lower minimum cost rate by adjusting PM intervals and replacement timing according to scenario-specific characteristics. This approach avoids over-maintenance or under-maintenance caused by the one-size-fits-all standardized BRPM strategy.

In this paper, the time spent on repair, replacement and preventive maintenance is assumed negligible and excluded from the analytical framework. This simplification is adopted to reduce the complexity of the three-dimensional joint optimization problem so that we can concentrate on balancing leasing profit and multi-type warranty maintenance costs. However, this setting limits the generalizability of the derived optimal policies. This model is only suitable for equipment with short maintenance operations, in which maintenance downtime occupies a trivial fraction of the total leasing contract period. For equipment requiring long-duration disassembly, overhaul, or off-site repair, non-zero maintenance downtime will reduce the total available working time, cut down the number of completed tasks, and further change the optimal failure threshold and preventive maintenance cycle solved by the TRPM strategy.

If non-negligible maintenance time is introduced into the optimization framework, the original three-dimensional optimization problem will expand to four dimensions, introducing an extremely heavy computational burden. We theoretically infer the evolution law of optimal policies under positive maintenance time: Longer maintenance downtime leads to higher loss of production capacity during maintenance activities. To avoid long-duration corrective repair triggered by severe equipment failure, the optimal strategy will select a lower failure threshold and carry out preventive maintenance more frequently. We leave the quantitative sensitivity analysis of maintenance time as an important direction for future extended research.

In this paper, equipment states are divided into only two categories, normal operating state and failure state, and we here justify the rationality of this binary classification setting. Our primary research goal is to optimize the joint decision of the failure threshold, contract term, and task cycle for leased equipment, and the binary state division is enough

to fully characterize the core balance between leasing income and maintenance costs. Meanwhile, from the practical application perspective of the smart air conditioner case, once the equipment breaks down, it cannot continue to complete leasing tasks, and there is no stable semi-degraded intermediate state that supports normal task execution. Introducing extra intermediate state categories would bring more state transition parameters and decision variables, further aggravating the heavy computational burden of the three-dimensional optimization model.

Nevertheless, multi-stage degradation states can be introduced as a valuable direction for follow-up research. We can subdivide equipment performance into mild, moderate, and severe degradation states between normal operations and complete failure. Such multi-state classification can more finely depict the gradual deterioration process of equipment performance, yet it will significantly expand the dimension of the optimization model and multiply the calculation workload. We regard extending binary classification to multi-stage degradation classification as an important direction for future research.

Notably, this study focuses on how usage rates affect cost control and profit generation under the proposed warranties. However, it does not consider user participation in warranty design to improve user benefits. Additionally, the complex and diverse needs of different user groups across various scenarios are not fully explored. These areas will be prioritized as key directions in future research.

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