

Article

Generalized Wright Noise Stochastic Calculus and Applications

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Abstract

This paper primarily aims to derive combinatorial properties of generalized Wright falling operators and establish their connections with a generalization of Appell polynomials associated with infinite-dimensional generalized Wright measures. Building on these results, we investigate the stochastic counterpart of generalized Wright analysis. We introduce a novel class of non-Gaussian processes, termed the two-parameter generalized Wright process (tpgWp) and we analyze their joint probability density and covariance. As applications of our results, we first prove the existence of local times for the generalized Wright process. Next, we characterize the two-parameter generalized Wright Ornstein–Uhlenbeck process as the unique solution to a Langevin equation driven by the (tpgWp), as well as a linear Wick-type stochastic differential equation driven by multi-two-parameter generalized Wright processes (mtpgWp). We establish that these solutions lie within a suitable distribution space.

Keywords: appell polynomials; generalized Wright noise; two-parameter generalized Wright process; Wright falling operators

MSC: 11C08; 26C05; 42C05; 33E12; 60J45

1. Introduction

In recent decades, the field of non-Gaussian analysis has undergone a transformative evolution, emerging as an infinite-dimensional distribution theory marked by rapid structural advancements. Its extensions have played an intrinsic role in numerous applications spanning stochastic partial differential equations, quantum physics, and various arithmetical and combinatorial frameworks. Furthermore, it offers powerful tools for resolving integral and differential equations, exploring quantum probability and statistics, and analyzing problems that involve special functions of mathematical physics along with their extensions and generalizations. Initial steps in this direction were taken in [1] utilizing the Poisson measure. This approach was subsequently generalized by employing biorthogonal systems known as Appell systems (see, e.g., refs. [2–4]) which are applicable to a broad class of measures, including both the Gaussian and Poisson measures [5]). Subsequently, Bender, Pipiras, and Taqqu [6,7] constructed fractional Brownian motions and fractional white noises for Hurst parameters $0 < H < 1$ and show its connection with fractional calculus. Building upon these developments, Laskin, Mainardi et al. [8–10] investigated a fractional non-Markovian Poisson stochastic process. These foundational results were later extended to infinite dimensions via biorthogonal generalized Appell systems, a direction that has since been addressed by several authors (see, e.g., refs. [11–14]). In a related direction, Mittag-Leffler analysis was established in [15] to generalize the machinery of white



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noise calculus to settings where the exponential function in the Gaussian characteristic functional is replaced by a Mittag-Leffler function. The grey noise measure arises as a special case within this broader class of Mittag-Leffler measures, enabling the application of Mittag-Leffler analysis to fractional differential equations, such as those describing fractional diffusion, relaxation-type phenomena, and viscoelasticity. A prominent stochastic process embedded in this framework is grey Brownian motion (gBm). Originally introduced in [16,17] as an extension of standard Brownian motion (Bm) to model anomalous diffusion with non-Gaussian marginals, gBm has attracted intensive study [18,19] due to its unique properties and its natural applications in fields ranging from mathematical finance to telecommunications engineering.

Within this framework, we replace for the first time the key role traditionally played by the Mittag-Leffler function in infinite-dimensional analysis with the generalized Wright functions, which include the Mittag-Leffler function as a special case. The Wright function [20] appear in solutions of various types of equations with fractional operators. Indeed, its applications cover the fractional partial differential equations generalizing the classical diffusion and wave equations. Some results can be found in [21] including the connection of the generalized Wright function with the Mittag-Leffler function via the Laplace transform. This motivates the present study to focus on measures whose characteristic functions are expressed via generalized Wright functions, thereby enabling their application to specific classes of stochastic differential equations.

This article pursues a threefold objective: First, it demonstrates the importance of degenerate falling operators in establishing key properties of Appell polynomials associated with infinite-dimensional generalized Wright measures. In this context, we present a general mathematical construction that allows us to define a parametric class of non-Gaussian processes, namely the generalized Wright process (gWp) and the two-parameter generalized Wright process (tpgWp). Next, our second main goal is to prove the existence of local times for the generalized Wright process. The formulation of long-term memory in time series dates back to Hurst's (1951) study on the long-term fluctuations of the Nile River reservoir (see [22]). Physically, these fluctuations are driven by a complex aggregation of upstream factors: tributary confluence, glacial melt, riverbed geometry, groundwater discharge, and localized rainfall. This raises a fundamental question: how can we realistically account for the massive number of underlying noise sources feeding into a single reservoir? To resolve this, we model each individual source k as a generalized Wright process, defined as follows:

$$\mathbb{X}_t^{H_k}(\varepsilon, \gamma) = \varepsilon \mathcal{W}_t^{H_k} + \gamma \mathcal{W}_{-t}^{H_k},$$

with an associated impact weight σ_k . The total collective noise of the river system is thus expressed as the linear combination

$$\mathcal{M}_t = \sum_{k=1}^{\infty} \sigma_k \mathbb{X}_t^{H_k}, \quad (1)$$

which we introduce as the multi-two-parameter generalized Wright process (mtpgWp). Motivated by these physical considerations, our third objective is to analyze the stochastic processes in (1) and investigate their properties. Specifically, we construct solutions to linear SDEs and a generalized Wright version of the Ornstein–Uhlenbeck equation involving Wick multiplicative generalized Wright noise, interpreted as distributions in a suitable Wright noise space.

The paper is structured as follows. In Section 2, we introduce the infinite-dimensional version of the generalized Wright measure by deriving its characteristic functionals and applying Minlos' theorem. We show that the generalized Wright measure has finite mixed moments of all orders and is a mixture of Gaussian white noise measures with the Fox- H

density. In Section 3, we discuss basic results on falling operators, which serve to establish certain properties of a new class of generalized Wright Appell polynomials. Section 4 is devoted to introducing the two-parameter generalized Wright processes (tpgWps) and analyzing their joint probability densities and covariances. As an application, in the final section, we establish the existence of local times for the generalized Wright process. Furthermore, we derive the two-parameter generalized Wright Ornstein–Uhlenbeck process as the unique solution to a Langevin equation driven by the two-parameter generalized Wright process (tpgWp), and study a linear Wick-type stochastic differential equation associated with the multi-two-parameter generalized Wright processes.

2. Infinite Dimensional Generalized Wright Measure

Let $m, n, p, q \in \mathbb{N}$ such that $0 \leq m \leq q$ and $0 \leq n \leq p$. Furthermore, let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j > 0$ for $i = 1, \dots, p$ and $j = 1, \dots, q$. The Fox H -function is defined via the Mellin–Barnes integral as follows:

$$H_{p,q}^{m,n} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| z \right] := \frac{1}{2i\pi} \int_{\mathcal{C}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds, \quad z \in \mathbb{C},$$

where

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^m \Gamma(b_j + s\beta_j) \prod_{i=1}^n \Gamma(1 - a_i - s\alpha_i)}{\prod_{i=n+1}^p \Gamma(a_i + s\alpha_i) \prod_{j=m+1}^q \Gamma(1 - b_j - s\beta_j)}$$

and $\mathcal{C}$ is a contour that separates the poles of $\mathcal{H}_{p,q}^{m,n}(s)$ given by $\prod_{j=1}^m \Gamma(b_j + s\beta_j)$ from those of $\prod_{i=1}^n \Gamma(1 - a_i - s\alpha_i)$. For further details regarding the existence of these functions, we refer the reader to [23]. The generalized Wright function are formally defined by the power series:

$${}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(B_i + n\beta_i)}{\prod_{j=1}^p \Gamma(A_j + n\alpha_j)} \frac{z^n}{n!}.$$

In the following, we stipulate the conditions on the parameters m, p, a_i, b_j, α_i , and β_j required to characterize the class of Fox H -densities possessing finite moments of all orders. Specifically, we make the following assumptions:

- (i) $a_i \in \mathbb{R}, \alpha_i > 0$, for $i = 1, \dots, p$ and $a_i + \alpha_i > 0$, for $i = 1, \dots, p$;
- (ii) $b_j \in \mathbb{R}, \beta_j > 0$, for $j = 1, \dots, m$ and $b_j + \beta_j > 0$, for $j = 1, \dots, m$;
- (iii) $a^* := \sum_{j=1}^m \beta_j - \sum_{i=1}^p \alpha_i > 0$ or $a^* = 0$ and

$$\tau := \sum_{j=1}^m b_j - \sum_{i=1}^p a_i - (p - m)/2 < -1;$$

- (iv) Let $H_{p,m}^{m,0}(\cdot)$ be non-negative on $(0, +\infty)$.

The corresponding Fox- H density ρ is defined by

$$\rho(x) = \frac{1}{\delta} H_{p,m}^{m,0}(x), \quad x > 0,$$

where

$$\delta := \frac{\prod_{i=1}^m \Gamma(b_i + \beta_i)}{\prod_{i=1}^p \Gamma(a_i + \alpha_i)}, \quad (2)$$

and its Laplace transform is given by

$$\mathcal{L}_\rho(s) := \int_{\mathbb{R}} e^{sx} \rho(x) dx = \frac{1}{\delta} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -s \right], \quad s \geq 0. \quad (3)$$

Furthermore, ρ has finite moments of all orders and we have

$$\int_0^\infty x^n \rho(x) dx = \frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(n+1))}{\delta \prod_{i=1}^p \Gamma(a_i + \alpha_i(n+1))}, \quad n = 0, 1, \dots$$

Remark 1. Throughout the paper, we recover the cases of white noise analysis and Mittag-Leffler analysis by assigning specific values to the parameters as follows:

- (i) The Mittag-Leffler function, $E_\nu(\cdot)$ for $\nu \in (0, 1]$, is represented via the generalized Wright function (gWf) by choosing $m = 1, p = 1, B_1 = 1, \beta_1 = 1, A_1 = 1$, and $\alpha_1 = \nu$ (see Section 6 in [24] and Example 5.5 in [25]).
- (ii) The white noise case is obtained by setting $m = p = 0$ (see Corollary 4.6 in [25]).

In what follows, white noise analysis is naturally viewed as a special case of Mittag-Leffler analysis, which in turn embeds into generalized Wright analysis, given that $E_1(\cdot) = \exp(\cdot)$ when $\nu = 1$.

Now our first objective is give the construction of the infinite-dimensional version of the generalized Wright measure via the Minlos' theorem. Let $K = L^2(\mathbb{R}, dt)$ be the real Hilbert space equipped with the norm $\|\cdot\|_0$ induced by the inner product $\langle \cdot, \cdot \rangle$. Consider the harmonic oscillator operator $A = 1 + t^2 - \frac{d^2}{dt^2}$, which is a densely defined and self-adjoint operator on H , and assume that A^{-r} is of Hilbert–Schmidt type for some $r > 0$. For $p > 0$, let $\text{Dom}(A^p)$ denote the domain of A^p , which forms a dense subspace of K , denoted by $\mathcal{E}_p$. It is obvious that $\mathcal{E}_p$ is a Hilbert space when endowed with the norm $\|\cdot\|_p$, defined by

$$\|\phi\|_p := \|A^p \phi\|_0 < \infty, \quad \phi \in \text{Dom}(A^p).$$

Let $\mathcal{E}_{-p}$ be the completion of K with respect to the norm

$$\|\phi\|_{-p} := \|A^{-p} \phi\|_0, \quad \phi \in K.$$

Under the identification of K with its dual space, we establish the real standard nuclear triple

$$\mathcal{E} := \bigcap_{p \geq 0} \mathcal{E}_p \subset K \subset \bigcup_{p \geq 0} \mathcal{E}_{-p} := \mathcal{E}'.$$

In accordance with the inner product structure of K , the canonical bilinear form on $\mathcal{E}' \times \mathcal{E}$ is also denoted by $\langle \cdot, \cdot \rangle$. Through complexification, we obtain the corresponding complex Gel'fand triple:

$$\mathcal{E}_{\mathbb{C}} \subset \mathcal{K} \subset \mathcal{E}'_{\mathbb{C}}$$

where $\mathcal{E}_{\mathbb{C}}, \mathcal{K}$, and $\mathcal{E}'_{\mathbb{C}}$ represent the complexified versions of $\mathcal{E}, K$, and $\mathcal{E}'$, respectively. For notational convenience, the $\mathbb{C}$ -bilinear form on $\mathcal{E}'_{\mathbb{C}} \times \mathcal{E}_{\mathbb{C}}$ is denoted by the same symbol so that $\|\phi\|_0^2 = \langle \bar{\phi}, \phi \rangle$ holds for $\phi \in \mathcal{K} := K_{\mathbb{C}}$. Let $\tilde{\phi}$ the Fourier transform of the function ϕ defined by

$$\tilde{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} \phi(y) dy.$$

Note that $\mathcal{E}$ is the Schwartz space $\mathcal{S}(\mathbb{R})$, which may also be endowed with the following scalar product

$$\langle \phi_1, \phi_2 \rangle_H = \kappa(H) \int_{\mathbb{R}} \overline{\tilde{\phi}_1(x)} \tilde{\phi}_2(x) |x|^{1-2H} dx, \quad \phi_1, \phi_2 \in \mathcal{E},$$

where $H \in (0, 1)$ is the Hurst parameter and $\kappa(H)$ is given by

$$\kappa(H) = \Gamma(2H + 1) \sin(\pi H).$$

Furthermore, for any $p \in \mathbb{R}$, we denote the norm on $\mathcal{E}_p$ by $|\cdot|_{p,H}$. For any $t \geq 0$, the function $\mathbf{1}_{[0,t)}$ belongs to $\mathcal{E}$. In fact for $0 < H < 1$, we have

$$\|\mathbf{1}_{[a,t)}\|_0^2 = \frac{\kappa(H)}{2\pi} \int_{\mathbb{R}} \frac{2}{|x|^{1+2H}} (1 - \cos tx) = t^{2H}.$$

For more details see [7].

Lemma 1. Consider the mapping $\mathcal{C}_\Psi : \mathcal{E} \rightarrow \mathbb{C}$ defined by

$$\mathcal{C}_\Psi(\phi) = \frac{1}{\delta} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -\frac{\langle \phi, \phi \rangle}{2} \right], \quad \phi \in \mathcal{E}, \quad (4)$$

with normalization constant δ given in (2). Then $\mathcal{C}_\Psi$ is a characteristic function on $\mathcal{E}$.

Proof. It is clear that $\mathcal{C}_\Psi$ is continuous on $\mathcal{E}$ and $\mathcal{C}_\Psi(0) = 1$. For any $n \in \mathbb{N}$, $z_1, z_2, \dots, z_n \in \mathbb{C}$, by using (3) we obtain

$$\begin{aligned} \sum_{j,k=1}^n z_j \bar{z}_k \mathcal{C}_\Psi(\phi_i - \phi_j) &= \frac{1}{\delta} \sum_{j,k=1}^n z_{jm} \Psi_p \left(-\frac{\langle \phi_i - \phi_j, \phi_i - \phi_j \rangle}{2} \right) \bar{z}_k \\ &= \frac{1}{\delta} \int_0^\infty \sum_{j,k=1}^n z_j e^{-s \langle \phi_i - \phi_j, \phi_i - \phi_j \rangle / 2} \bar{z}_k H_{p,m}^{m,0} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,m} \end{matrix} \middle| s \right] ds \\ &= \frac{1}{\delta} \int_0^\infty \sum_{j,k=1}^n z_j \mathcal{C}_G(\phi) \bar{z}_k H_{p,m}^{m,0} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,m} \end{matrix} \middle| s \right] ds \geq 0, \end{aligned}$$

where $\mathcal{C}_G(\cdot)$ is the characteristic function of the Gaussian white noise measure. This gives the statement. $\square$

By applying the Bochner–Minlos theorem, we define the following probability measure on $\mathcal{E}'$, equipped with its cylindrical σ -algebra:

Definition 1. The generalized Wright noise measure is defined to be the unique probability measure denoted by Λ_Ψ on $\mathcal{E}'$ such that

$$\int_{\mathcal{E}'} e^{i\langle v, \phi \rangle} \Lambda_\Psi(dv) = \frac{1}{\delta} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -\frac{\langle \phi, \phi \rangle}{2} \right], \quad \phi \in \mathcal{E}.$$

Lemma 2. The measure Λ_Ψ has moments of any order. In fact, for $\phi \in \mathcal{E}$ and $n \in \mathbb{N}$ we get

$$\begin{aligned} 1. \quad & \int_{\mathcal{E}'} \langle v, \phi \rangle^{2n+1} \Lambda_\Psi(dv) = 0; \\ 2. \quad & \int_{\mathcal{E}'} \langle v, \phi \rangle^{2n} \Lambda_\Psi(dv) = \frac{1}{\delta} \frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(n+1))}{\prod_{i=1}^p \Gamma(a_i + \alpha_i(n+1))} \frac{(2n)!}{2^n n!} \|\phi\|_0^{2n}. \end{aligned}$$

In particular

$$\|\langle \cdot, \phi \rangle\|_{\Lambda_\Psi}^2 = \frac{1}{\delta} \frac{\prod_{i=1}^m \Gamma(b_i + 2\beta_i)}{\prod_{i=1}^p \Gamma(a_i + 2\alpha_i)} \|\phi\|_0^2.$$

Moreover by applying the polarization identity, we get

$$\int_{\mathcal{E}'} \langle w, \phi_1 \rangle \langle w, \phi_2 \rangle \Lambda_\Psi(dv) = \frac{1}{\delta} \frac{\prod_{i=1}^m \Gamma(b_i + 2\beta_i)}{\prod_{i=1}^p \Gamma(a_i + 2\alpha_i)} \langle \phi_1, \phi_2 \rangle, \quad \phi_1, \phi_2 \in \mathcal{E}.$$

Proof. Under the assumptions ${}_m\Psi_p$ possess the following series representation:

$${}_m\Psi_p \left[\begin{matrix} (b_i + \beta_i, \beta_i)_{1,m} \\ (a_j + \alpha_j, \alpha_j)_{1,p} \end{matrix} \middle| -\frac{z}{2} \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(n+1))}{\prod_{j=1}^p \Gamma(a_j + \alpha_j(n+1))} \frac{(-z)^n}{n! 2^n}. \quad (5)$$

Thus by using (4) and (5) we have

$$\int_{\mathcal{E}'} \sum_{n=0}^{\infty} i^n \frac{\langle v, \phi \rangle^n}{n!} \Lambda_{\Psi}(dv) = \frac{1}{\delta} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(n+1))}{\prod_{i=1}^p \Gamma(a_i + \alpha_i(n+1))} \frac{(-1)^n \|\phi\|_0^{2n}}{n! 2^n}.$$

Hence we deduce all moments of the measure Λ_{Ψ} as follows:

$$\int_{\mathcal{E}'} \langle v, \phi \rangle^{2n+1} \Lambda_{\Psi}^{(\beta)}(dv) = 0$$

and

$$\int_{\mathcal{E}'} \langle v, \phi \rangle^{2n} \Lambda_{\Psi}(dv) = \frac{1}{\delta} \frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(n+1))}{\prod_{i=1}^p \Gamma(a_i + \alpha_i(n+1))} \frac{(2n)!}{2^n n!} \|\phi\|_0^{2n}.$$

□

Proposition 1. The generalized Wright noise measure Λ_{Ψ} is an integral of centered Gaussian measures $\mu_{s,G}$ with respect to the Fox-H density ρ , i.e.,

$$\Lambda_{\Psi} = \int_0^{\infty} \mu_{s,G} \rho(s) ds, \quad (6)$$

where $\mu_{s,G}$ denote the Gaussian white noise measures on $\mathcal{E}'$, i.e.,

$$\int_{\mathcal{E}'} e^{i\langle v, \phi \rangle} \mu_{s,G}(dv) = \exp\left(-\frac{s}{2} \langle \phi, \phi \rangle\right), \quad \phi \in \mathcal{E}. \quad (7)$$

Proof. Denote the right hand side of (6) by

$$\nu := \int_0^{\infty} \mu_{s,G} \rho(s) ds.$$

Hence, by using (3) and (7), for all $\phi \in \mathcal{E}$ we obtain

$$\begin{aligned} \int_{\mathcal{E}'} e^{i\langle v, \phi \rangle} \nu(dw) &= \int_0^{\infty} \int_{\mathcal{E}'} \exp(i\langle v, \phi \rangle) \mu_{s,G}(dw) \rho(s) ds \\ &= \int_0^{\infty} \exp\left(-\frac{s}{2} \langle \phi, \phi \rangle\right) \rho(s) ds \\ &= \frac{1}{\delta} {}^m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -\frac{1}{2} \langle \phi, \phi \rangle \right] \\ &= C_{\Psi}(\phi). \end{aligned}$$

This implies $\Lambda_{\Psi} = \nu$. □

3. Generalized Wright Noise Analysis

Our goal is to study the generalized Appell polynomials with respect to the infinite-dimensional generalized Wright measure Λ_{Ψ} . So to solve the problem of existence of Appell system we need to introduce Generalized Wright falling operators $Y_{\Psi}(n, k)$ and $\Xi_{\Psi}(n, k)$.

Let ${}_m\Psi_p : \mathcal{E}_{\mathbb{C}} \rightarrow \mathcal{E}_{\mathbb{C}}$ defined by

$${}_m\Psi_p(\varphi) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(n+1))}{\prod_{j=1}^p \Gamma(a_j + \alpha_j(n+1))} \frac{\varphi^n}{n!}, \quad (8)$$

and denote by ${}_m\Theta_p$ the inverse function of ${}_m\Psi_p$ defined via its Taylor expansion by

$${}_m\Theta_p(\varphi) = \sum_{n=0}^{\infty} \frac{{}_m\Theta_p^{(n)}(0)}{n!} \varphi^n. \quad (9)$$

By using the kernel theorem, obviously one can see that the function

$$\mathcal{E}_{\mathbb{C}} \ni \phi \mapsto \exp(\langle v, {}_m\Psi_p(\phi) \rangle) \in \mathbb{C}$$

has the decomposition in terms of the generalized Wright falling factorials on $\mathcal{E}'_{\mathbb{C}}$ given by

$$\exp(\langle v, {}_m\Psi_p(\phi) \rangle) = \sum_{k=0}^{\infty} \frac{1}{k!} \langle (v)_{k,\Psi}, \phi^{\otimes k} \rangle. \quad (10)$$

Definition 2. The generalized Wright falling operators of the first kind is defined as a linear operators $Y_{\Psi}(n, k) : \mathcal{E}_{\mathbb{C}}^{\otimes n} \longrightarrow \mathcal{E}_{\mathbb{C}}^{\otimes k}$ satisfying

$$\langle (v)_{n,\Psi}, \phi^{(n)} \rangle = \sum_{k=0}^n \langle v^{\otimes k}, Y_{\Psi}(n, k) \phi^{(n)} \rangle, \quad \phi^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\otimes n}, v \in \mathcal{E}'_{\mathbb{C}}, \quad (11)$$

and the generalized Wright falling operators of the second kind as the linear operators $\Xi_{\Psi}(n, k) : \mathcal{E}_{\mathbb{C}}^{\otimes n} \longrightarrow \mathcal{E}_{\mathbb{C}}^{\otimes k}$ satisfying

$$\langle v^{\otimes n}, \phi^{(n)} \rangle = \sum_{k=0}^n \langle (v)_{k,\Psi}, \Xi_{\Psi}(n, k) \phi^{(n)} \rangle, \quad \phi^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\otimes n}, v \in \mathcal{E}'_{\mathbb{C}}. \quad (12)$$

Let $n, k \in \mathbb{N}, k \leq n$ and $i_1, \dots, i_k \in \mathbb{N}$ such that $i_1 + \dots + i_k = n$ and define the linear continuous operator $\mathbb{D}_{i_1, \dots, i_k}^{(n)} \in \mathcal{L}(\mathcal{E}_{\mathbb{C}}^{\otimes n}, \mathcal{E}_{\mathbb{C}}^{\otimes k})$ by

$$(\mathbb{D}_{i_1, \dots, i_k}^{(n)} \phi^{(n)})(x_1, \dots, x_k) := \frac{1}{k!} \sum_{\tau \in S_k} \phi^{(n)}(\underbrace{x_{\tau(1)}, \dots, x_{\tau(1)}}_{i_1 \text{ times}}, \dots, \underbrace{x_{\tau(k)}, \dots, x_{\tau(k)}}_{i_k \text{ times}}),$$

where S_k is symmetric group of degree n . In particular, we have

$$\mathbb{D}_{i_1, \dots, i_k}^{(n)} \phi^{\otimes n} = \phi^{i_1} \hat{\otimes} \dots \hat{\otimes} \phi^{i_k}, \quad \phi \in \mathcal{E}_{\mathbb{C}}. \quad (13)$$

Theorem 1. The operators $Y_{\Psi}(n, k)$ and $\Xi_{\Psi}(n, k)$ possess the following properties:

1. For any $n, k \in \mathbb{N}$ such that $k \leq n$, $Y_{\Psi}(n, k)$ and $\Xi_{\Psi}(n, k)$ admits an explicit representation in terms of the operator $\mathbb{D}_{i_1, \dots, i_k}^{(n)}$ as follows:

$$Y_{\Psi}(n, k) = \frac{n!}{k!} \sum_{\substack{(i_1, \dots, i_k) \in \mathbb{N}^k \\ i_1 + \dots + i_k = n}} \prod_{j=1}^k \left(\frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(i_j + 1))}{\prod_{j=1}^p \Gamma(a_j + \alpha_j(i_j + 1)) i_j!} \right) \mathbb{D}_{i_1, \dots, i_k}^{(n)}, \quad (14)$$

and

$$\Xi_{\Psi}(n, k) = \frac{n!}{k!} \sum_{\substack{(i_1, \dots, i_k) \in \mathbb{N}^k \\ i_1 + \dots + i_k = n}} \prod_{j=1}^k \frac{{}_m\Theta_p^{(i_j)}(0)}{i_j!} \mathbb{D}_{i_1, \dots, i_k}^{(n)}. \quad (15)$$

2. For each $k \in \mathbb{N}$ and $\phi \in \mathcal{E}_{\mathbb{C}}$, we have

$$\sum_{n=k}^{\infty} \frac{1}{n!} Y_{\Psi}(n, k) \phi^{\otimes k} = \frac{{}_m\Psi_p(\phi)^{\otimes k}}{k!}, \quad (16)$$

$$\sum_{n=k}^{\infty} \frac{1}{n!} \Xi_{\Psi}(n, k) \phi^{\otimes k} = \frac{{}_m\Theta_p(\phi)^{\otimes k}}{k!}. \quad (17)$$

3. For $v \in \mathcal{E}'_{\mathbb{C}}$ and $\phi \in \mathcal{E}_{\mathbb{C}}$, we have

$$\langle v^{\otimes k}, Y_{\Psi}(n, k) \phi^{\otimes n} \rangle = \frac{1}{k!} \frac{d^n}{dz^n} \Big|_{z=0} \langle v, {}_m\Psi_p(z\phi) \rangle, \quad (18)$$

$$\langle v^{\otimes k}, \Xi_{\Psi}(n, k) \phi^{\otimes n} \rangle = \frac{1}{k!} \frac{d^n}{dz^n} \Big|_{z=0} \langle w, {}_m\Theta_p(z\phi) \rangle. \quad (19)$$

Proof. 1. It follows directly from (10) that

$$\begin{aligned}
& \langle (v)_{n,\Psi}, \phi^{\otimes n} \rangle \\
&= \sum_{k=1}^n \frac{n!}{k!} \sum_{\substack{(i_1, \dots, i_k) \in \mathbb{N}^k \\ i_1 + \dots + i_k = n}} \prod_{j=1}^k \left(\frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(i_j + 1))}{\prod_{j=1}^p \Gamma(a_j + \alpha_j(i_j + 1)) i_j!} \right) \langle v, \phi^{i_1} \rangle \dots \langle v, \phi^{i_k} \rangle \\
&= \sum_{k=1}^n \frac{n!}{k!} \sum_{\substack{(i_1, \dots, i_k) \in \mathbb{N}^k \\ i_1 + \dots + i_k = n}} \prod_{j=1}^k \left(\frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(i_j + 1))}{\prod_{j=1}^p \Gamma(a_j + \alpha_j(i_j + 1)) i_j!} \right) \langle v^{\otimes k}, \phi^{i_1} \hat{\otimes} \dots \hat{\otimes} \phi^{i_k} \rangle.
\end{aligned}$$

Hence by using (8)–(13) we obtain the representation of $Y_\Psi(n, k)$ as (14). By a similar calculus and with help of (9) we find the result of (15).

2. We have

$$\begin{aligned}
\frac{{}_m\Theta_p(\phi)^{\otimes k}}{k!} &= \frac{1}{k!} \left(\sum_{i=1}^{\infty} \frac{{}_m\Theta_p^{(i)}(0) \phi^i}{i!} \right)^{\otimes k} \\
&= \frac{1}{k!} \sum_{n=k}^{\infty} \sum_{\substack{(i_1, \dots, i_k) \in \mathbb{N}^k \\ i_1 + \dots + i_k = n}} \frac{{}_m\Theta_p^{(i_1)}(0) \dots {}_m\Theta_p^{(i_k)}(0)}{i_1! \dots i_k!} \phi^{i_1} \hat{\otimes} \dots \hat{\otimes} \phi^{i_k}.
\end{aligned}$$

Hence, (17) follows from (15), and the proof of (16) proceeds similarly.

3. By using (17), for $v \in \mathcal{E}'_{\mathbb{C}}, z \in \mathbb{C}$ and $\phi \in \mathcal{E}_{\mathbb{C}}$, we get

$$\begin{aligned}
\sum_{n=k}^{\infty} \frac{z^n}{n!} \langle v^{\otimes k}, \Xi_\Psi(n, k) \phi^{\otimes n} \rangle &= \langle v^{\otimes k}, \frac{1}{k!} ({}_m\Theta_p(z\phi))^{\otimes k} \rangle \\
&= \frac{1}{k!} \langle v, ({}_m\Theta_p(z\phi))^k \rangle.
\end{aligned}$$

Viewing this as an equality of formal power series in z , we directly obtain (19). The derivation of (18) proceeds in an identical manner.

□

Remark 2. The Wright falling operators are continuous and so their adjoints $Y_\Psi(n, k)^*$ and $\Xi_\Psi(n, k)^*$ are well defined, that is,

$$Y_\Psi(n, k)^* : \mathcal{E}'_{\mathbb{C}}^{\hat{\otimes} k} \longrightarrow \mathcal{E}'_{\mathbb{C}}^{\hat{\otimes} n}, \quad \Xi_\Psi(n, k)^* : \mathcal{E}'_{\mathbb{C}}^{\hat{\otimes} k} \longrightarrow \mathcal{E}'_{\mathbb{C}}^{\hat{\otimes} n}$$

and satisfy

$$\begin{aligned}
\langle v^{(k)}, Y_\Psi(n, k) \phi^{(n)} \rangle &= \langle Y_\Psi(n, k)^* v^{(k)}, \phi^{(n)} \rangle, \\
\langle v^{(k)}, \Xi_\Psi(n, k) \phi^{(n)} \rangle &= \langle \Xi_\Psi(n, k)^* w^{(k)}, \phi^{(n)} \rangle,
\end{aligned}$$

for all $v^{(k)} \in \mathcal{E}'_{\mathbb{C}}^{\hat{\otimes} k}$ and $\phi^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\hat{\otimes} n}$. Consequently, Equations (11) and (12) imply that

$$(v)_{n,\Psi} = \sum_{k=1}^n Y_\Psi(n, k)^* v^{\otimes k} \quad \text{and} \quad w^{\otimes n} = \sum_{k=1}^n \Xi_\Psi(n, k)^* (v)_{k,\Psi}.$$

The Laplace transform $\mathcal{L}_{\Lambda_\Psi}(\cdot)$ admits a decomposition in terms of the moment kernels $\mathcal{M}_n^{(\Psi)}$, given by

$$\mathcal{L}_{\Lambda_\Psi}(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \mathcal{M}_n^{(\Psi)}, \phi^{\otimes n} \rangle, \quad \phi \in \mathcal{E}_{\mathbb{C}}, \quad \mathcal{M}_n^{(\Psi)} \in \mathcal{E}'_{\mathbb{C}}^{\hat{\otimes} n}. \quad (20)$$

Definition 3. The normalized Laplace-exponential function is defined by

$$\text{Exp}_{\Lambda_\Psi}(\phi, v) = \frac{\exp(\langle v, \phi \rangle)}{\mathcal{L}_{\Lambda_\Psi}(\phi)}, \quad v \in \mathcal{E}'_{\mathbb{C}}, \quad \phi \in \mathcal{E}_{\mathbb{C}}. \quad (21)$$

Since $\mathcal{L}_{\Lambda_{\Psi}}(0) = 1$ and $\mathcal{L}_{\Lambda_{\Psi}}$ is holomorphic, then $\text{Exp}_{\Lambda_{\Psi}}(\phi, v)$ can be expanded in the following form

$$\text{Exp}_{\Lambda_{\Psi}}(\phi, v) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle A_{n,\Psi}(v), \phi^{\otimes n} \rangle, \quad v \in \mathcal{E}'_{\mathbb{C}}, \quad (22)$$

for suitable $A_{n,\Psi}(v) \in \mathcal{E}'_{\mathbb{C}}^{\otimes n}$. Define the modified normalized Laplace-exponential function

$$\begin{aligned} \text{Exp}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi), v) &= \frac{\exp(\langle v, {}_m\Psi_p(\phi) \rangle)}{\mathcal{L}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi))} \\ &= \frac{\delta e^{\langle w, {}_m\Psi_p(\phi) \rangle}}{{}_m\Psi_p\left(\frac{1}{2}\langle {}_m\Psi_p(\phi), {}_m\Psi_p(\phi) \rangle\right)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_{n,\Psi}(v), \phi^{\otimes n} \rangle, \end{aligned} \quad (23)$$

where the family

$$\mathbb{P}_{\Psi} = \left\{ \langle P_{n,\Psi}(v), \phi^{(n)} \rangle, \phi^{(n)} \in \mathcal{E}'_{\mathbb{C}}^{\otimes n}, n \in \mathbb{N} \right\}$$

is called the generalized Appell system associated to the generalized Wright noise measure Λ_{Ψ} .

Theorem 2. For any $v_1, v_2 \in \mathcal{E}'_{\mathbb{C}}, n \in \mathbb{N}$, the following properties hold:

(P1) $P_{n,\Psi}(v_1 + v_2) = \sum_{k=0}^n \binom{n}{k} P_{k,\Psi}(v_1) \hat{\otimes} (v_2)_{n-k,\Psi}$, where $(v)_{m,\Psi}$ are the falling factorials on $\mathcal{E}'_{\mathbb{C}}$ given by Equation (10).

(P2) $P_{n,\Psi}(v_1) = \sum_{m=0}^n Y_{\Psi}(n, m)^* A_{m,\Psi}(v_1)$, where the $A_{m,\Psi}(v_1)$ are those in (22).

(P3) $v_1^{\otimes n} = \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \Xi_{\Psi}(k, m)^* P_{m,\Psi}(v_1) \hat{\otimes} \mathcal{M}_{n-k}^{(\Psi)}$, where $\mathcal{M}_{n-k}^{(\Psi)} \in \mathcal{E}'_{\mathbb{C}}^{\otimes n}$ are the moment kernels of Λ_{Ψ} determined by

$$\mathcal{L}_{\Lambda_{\Psi}}(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \mathcal{M}_n^{(\Psi)}, \phi^{\otimes n} \rangle, \quad \phi \in \mathcal{E}_{\mathbb{C}}, \quad \mathcal{M}_n^{(\Psi)} \in \mathcal{E}'_{\mathbb{C}}^{\otimes n}.$$

(P4) $P_{n,\Psi}(v_1 + v_2) = \sum_{k+l+m=n} \frac{n!}{k!l!m!} P_{k,\Psi}(v_1) \hat{\otimes} P_{l,\Psi}(v_2) \hat{\otimes} \mathcal{M}_m^{(\Psi)}({}_m\Psi_p)$, where $\mathcal{M}_n^{(\Psi)}({}_m\Psi_p) \in \mathcal{E}'_{\mathbb{C}}^{\otimes n}$ is defined via the following Laplace transform

$$\mathcal{L}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi)) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \mathcal{M}_n^{(\Psi)}({}_m\Psi_p), \phi^{\otimes n} \rangle, \quad \phi \in \mathcal{E}_{\mathbb{C}}.$$

(P5) $\mathbb{E}_{\Psi}(\langle P_{n,\Psi}(\cdot), \phi^{(n)} \rangle) = \delta_{n,0}$, where $\phi^{(n)} \in \mathcal{E}'_{\mathbb{C}}^{\otimes n}$ and $\mathbb{E}_{\Psi}$ is the expectation with respect to the measure Λ_{Ψ} .

Proof.

(P1) It can be noted that

$$\text{Exp}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi), v_1 + v_2) = \text{Exp}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi), v_1) \exp(\langle v_2, {}_m\Psi_p(\phi) \rangle).$$

Thus by using Equation (10) we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_{n,\Psi}(v_1 + v_2), \phi^{\otimes n} \rangle &= \sum_{k=0}^{\infty} \frac{1}{k!} \langle P_{k,\Psi}(v_1), \phi^{\otimes k} \rangle \sum_{m=0}^{\infty} \frac{1}{m!} \langle (v_2)_{m,\Psi}, \phi^{\otimes m} \rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{k=0}^n \binom{n}{k} P_{k,\Psi}(v_1) \hat{\otimes} (v_2)_{n-k,\Psi}, \phi^{\otimes n} \right\rangle. \end{aligned}$$

Hence we obtain

$$P_{n,\Psi}(v_1 + v_2) = \sum_{k=0}^n \binom{n}{k} P_{k,\Psi}(v_1) \widehat{\otimes} (v_2)_{n-k,\Psi}.$$

(P2) By using Equations (16) and (22), we have

$$\begin{aligned} \text{Exp}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi), v) &= \frac{\exp(\langle v, {}_m\Psi_p(\phi) \rangle)}{\mathcal{L}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi))} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \langle A_{m,\Psi}(v), {}_m\Psi_p(\phi)^{\otimes m} \rangle \\ &= \sum_{m=0}^{\infty} \left\langle A_{m,\Psi}(v), \sum_{n=m}^{\infty} \frac{1}{n!} Y_{\Psi}(n, m) \phi^{\otimes n} \right\rangle \\ &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{1}{n!} \langle Y_{\Psi}(n, m)^* A_{m,\Psi}(v), \phi^{\otimes n} \rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{m=0}^n Y_{\Psi}(n, m)^* A_{m,\Psi}(v), \phi^{\otimes n} \right\rangle. \end{aligned}$$

Moreover, in view of Equation (23), a comparison of the both series for $\text{Exp}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi), w)$ gives

$$P_{n,\Psi}(w) = \sum_{m=0}^n Y_{\Psi}(n, m)^* A_{m,\Psi}(v).$$

(P3) By using Equations (17) and (22), we have

$$\begin{aligned} \text{Exp}_{\Lambda_{\Psi}}(\phi, w) &= \sum_{m=0}^{\infty} \frac{1}{m!} \langle P_{m,\Psi}(v), {}_m\Theta_p(\phi)^{\otimes m} \rangle \\ &= \sum_{m=0}^{\infty} \left\langle P_{m,\Psi}(v), \sum_{n=m}^{\infty} \frac{1}{n!} \Xi(n, m) \phi^{\otimes n} \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{m=0}^n \Xi(n, m)^* P_{m,\Psi}(v), \phi^{\otimes n} \right\rangle. \end{aligned}$$

Then by comparing the above series $\text{Exp}_{\Lambda_{\Psi}}(\phi, v)$ and Equation (22), we get

$$A_{n,\Psi}(v) = \sum_{m=0}^n \Xi_{\Psi}(n, m)^* P_{m,\Psi}(v). \quad (24)$$

On the other since

$$\exp(\langle v, \phi \rangle) = \text{Exp}_{\Lambda_{\Psi}}(\phi, v) \mathcal{L}_{\Lambda_{\Psi}}(\phi),$$

we conclude that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \langle v^{\otimes n}, \phi^{\otimes n} \rangle &= \sum_{m=0}^{\infty} \frac{1}{m!} \langle A_{m,\Psi}(v), \phi^{\otimes m} \rangle \sum_{k=0}^{\infty} \frac{1}{k!} \langle \mathcal{M}_k^{(\Psi)}, \phi^{\otimes k} \rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{k=0}^n \binom{n}{k} A_{k,\Psi}(v) \widehat{\otimes} \mathcal{M}_{n-k}^{(\Psi)}, \phi^{\otimes n} \right\rangle. \end{aligned}$$

Hence with the help of (24), we obtain

$$\begin{aligned} v^{\otimes n} &= \sum_{k=0}^n \binom{n}{k} A_{k,\Psi}(w) \widehat{\otimes} \mathcal{M}_{n-k}^{(\Psi)} \\ &= \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} \Xi_{\Psi}(k, m)^* P_{m,\Psi}(v) \widehat{\otimes} \mathcal{M}_{n-k}^{(\Psi)}. \end{aligned}$$

(P4) Firstly one can see that

$$\text{Exp}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi), v_1 + v_2) = \text{Exp}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi), v_1) \text{Exp}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi), v_2) \mathcal{L}_{\Lambda_{\Psi}}({}_m\Psi_p(\phi)).$$

Thus

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_{n,\Psi}(v_1 + v_2), \phi^{\otimes n} \rangle \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \langle P_{k,\Psi}(v_2), \phi^{\otimes k} \rangle \sum_{l=0}^{\infty} \frac{1}{l!} \langle P_{l,\Psi}(v_1), \phi^{\otimes l} \rangle \sum_{r=0}^{\infty} \frac{1}{r!} \langle \mathcal{M}_r^{(\Psi)}(m\Psi_p), \phi^{\otimes r} \rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \sum_{k+l+r=n} \frac{n!}{k!l!r!} P_{k,\Psi}(v_1) \hat{\otimes} P_{l,\Psi}(v_2) \hat{\otimes} \mathcal{M}_r^{(\Psi)}(m\Psi_p), \phi^{\otimes n} \right\rangle.
\end{aligned}$$

Thus, comparing the coefficients of like powers on both sides of the preceding equation yields the desired result.

(P5) Note that for $\phi \in \mathcal{E}_{\mathbb{C}}$, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{E}_{\Psi}(\langle P_{k,\Psi}(v), \phi^{\otimes k} \rangle) &= \mathbb{E}_{\Psi}(\exp_{\Lambda_{\Psi}}(m\Psi_p(\phi), v)) \\
&= \frac{\mathbb{E}_{\Psi}(\exp(\langle \cdot, m\Psi_p(\phi) \rangle))}{\mathcal{L}_{\Lambda_{\Psi}}(m\Psi_p(\phi))} = 1.
\end{aligned}$$

The result follows by polarization identity and comparison of coefficients.

□

4. Two-Parameter Generalized Wright Process

Define the operator $M_{\pm}^H$ on the Schwartz test function space $\mathcal{E}$ by

$$M_{\pm}^H \varphi := \begin{cases} \sqrt{\kappa_H} D_{\pm}^{-(2H-1)/2} \varphi, & H \in (0, \frac{1}{2}) \\ \varphi, & H = \frac{1}{2} \\ \sqrt{\kappa_H} I_{\pm}^{-(2H-1)/2} \varphi, & H \in (\frac{1}{2}, 1), \end{cases}$$

where $H \in (0, 1)$ is the Hurst parameter and $D_{\pm}^r, I_{\pm}^r$ denote the right-sided and left-sided fractional derivative and fractional integral of order r in the sense of Riemann–Liouville, respectively, defined as follows:

$$\begin{aligned}
(D_-^r f)(x) &= \frac{-1}{\Gamma(1-r)} \frac{d}{dx} \int_x^{+\infty} f(t)(t-x)^{-r} dt, \\
(I_-^r f)(x) &= \frac{1}{\Gamma(r)} \int_x^{+\infty} f(t)(t-x)^{r-1} dt, \\
(D_+^r f)(x) &= \frac{1}{\Gamma(1-r)} \frac{d}{dx} \int_{-\infty}^x f(t)(x-t)^{-r} dt, \\
(I_+^r f)(x) &= \frac{1}{\Gamma(r)} \int_{-\infty}^x f(t)(x-t)^{r-1} dt, \quad x \in \mathbb{R}.
\end{aligned}$$

We refer to [26] for the details on these operators. We point out the following facts in order to consistently define the stochastic processes and study their properties. Note that $M_{\pm}^H \mathbf{1}_{[a,b]} \in L^2(\mathbb{R}, dx)$ and for $t, s \geq 0$, we have

$$\langle M_-^H \mathbf{1}_{[0,t]}, M_-^H \mathbf{1}_{[0,s]} \rangle = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

We refer to [27] for the proof and further details. Thus the following definition makes sense:

Definition 4. The stochastic process

$$\mathcal{E}' \ni v \mapsto \mathcal{W}_t^H(v) := \begin{cases} \langle v, M_-^H \mathbf{1}_{[0,t]} \rangle, & t > 0, \\ 0, & t = 0, \\ \langle v, M_-^H \mathbf{1}_{[0,-t]} \rangle, & t < 0, \end{cases} \quad (25)$$

is called generalized Wright process and by using Lemma 2 we obtain the following properties:

1. $\mathbb{E}(\mathcal{W}_t^H) = 0$ and $\mathbb{E}((\mathcal{W}_t^H)^2) = \frac{1}{\delta} \frac{\prod_{i=1}^m \Gamma(b_i + \beta_i(n+1))}{\prod_{i=1}^p \Gamma(a_i + \alpha_i(n+1))} t^{2H}$.
2. The auto-covariance function is

$$\mathbb{E}(\mathcal{W}_t^H \mathcal{W}_s^H) = \mathcal{R}_H(t, s) := \frac{1}{2\delta} \frac{\prod_{i=1}^m \Gamma(b_i + 2\beta_i)}{\prod_{i=1}^p \Gamma(a_i + 2\alpha_i)} (t^{2H} + s^{2H} - |t - s|^{2H}). \quad (26)$$

3. For any $t, s \in \mathbb{R}$, the characteristic function of the increment is

$$\mathbb{E}[e^{i\lambda(\mathcal{W}_t^H - \mathcal{W}_s^H)}] = \frac{1}{\delta} {}_m\Psi_p\left(-\frac{\lambda^2}{2}|t - s|^{2H}\right), \quad \lambda \in \mathbb{R}. \quad (27)$$

Now our objective is to generalize the preceding construction to define a broad class of processes, termed two-parameter generalized Wright processes (tpgWp). This class encompasses Brownian motion, fractional Brownian motion, and other more general stochastic processes.

Definition 5. For any two real constants (ε, γ) such that $(\varepsilon, \gamma) \neq (0, 0)$, a two-parameter generalized Wright process (tpgWp) is a process

$$\mathbb{X}^H = \{\mathbb{X}_t^H, t \geq 0\}$$

defined on the probability space $(\mathcal{E}', \mathcal{B}(\mathcal{E}'), \Lambda_\Psi)$ by

$$\mathbb{X}_t^H := \mathbb{X}_t^H(\varepsilon, \gamma) = \varepsilon \mathcal{W}_t^H + \gamma \mathcal{W}_{-t}^H. \quad (28)$$

Lemma 3. The tpgWp $\{\mathbb{X}_t^H(\varepsilon, \gamma)\}_{t \in \mathbb{R}_+}$ satisfies the following properties:

- (i) $\mathbb{X}^H$ is a centered process, i.e., $\mathbb{E}(\mathbb{X}_t^H(\varepsilon, \gamma)) = 0$.
- (ii) For all $s, t \in \mathbb{R}_+$, we have

$$\begin{aligned} \mathcal{R}_{H, \varepsilon, \gamma}(t, s) &:= \text{Cov}(\mathbb{X}_t^H(\varepsilon, \gamma), \mathbb{X}_s^H(\varepsilon, \gamma)) \\ &= K(\delta)(\varepsilon + \gamma)^2(t^{2H} + s^{2H}) - 2K(\delta)\varepsilon\gamma|t + s|^{2H} \\ &\quad - K(\delta)(\varepsilon^2 + \gamma^2)|t - s|^{2H}, \end{aligned}$$

where

$$K(\delta) := \frac{1}{2\delta} \frac{\prod_{i=1}^m \Gamma(b_i + 2\beta_i)}{\prod_{i=1}^p \Gamma(a_i + 2\alpha_i)}.$$

- (iii) For all $t \in \mathbb{R}_+$, we have

$$\mathbb{E}(\mathbb{X}_t^H(\varepsilon, \gamma)^2) = 2K(\delta)(\varepsilon^2 + \gamma^2 - (2^{2H} - 2)\varepsilon\gamma)t^{2H}.$$

Proof. By using Equation (26), we have

$$\begin{aligned} \text{Cov}(\mathbb{X}_t^H(\varepsilon, \gamma), \mathbb{X}_s^H(\varepsilon, \gamma)) &= \mathbb{E}(\mathbb{X}_t^H(\varepsilon, \gamma) \mathbb{X}_s^H(\varepsilon, \gamma)) \\ &= \varepsilon^2 \mathbb{E}(\mathcal{W}_t^H \mathcal{W}_s^H) + \varepsilon\gamma \mathbb{E}(\mathcal{W}_t^H \mathcal{W}_{-s}^H) + \varepsilon\gamma \mathbb{E}(\mathcal{W}_{-t}^H \mathcal{W}_s^H) + \gamma^2 \mathbb{E}(\mathcal{W}_{-t}^H \mathcal{W}_{-s}^H) \\ &= K(\delta)\varepsilon^2(t^{2H} + s^{2H} - |t - s|^{2H}) \\ &\quad + K(\delta)\varepsilon\gamma(t^{2H} + (-s)^{2H} - |t + s|^{2H}) \\ &\quad + K(\delta)\varepsilon\gamma((-t)^{2H} + s^{2H} - |t + s|^{2H}) \\ &\quad + K(\delta)\gamma^2((-t)^{2H} + (-s)^{2H} - |t - s|^{2H}) \\ &= K(\delta)(\varepsilon + \gamma)^2(t^{2H} + s^{2H}) - 2K(\delta)\varepsilon\gamma|t + s|^{2H} \\ &\quad - K(\delta)(\varepsilon^2 + \gamma^2)|t - s|^{2H}. \end{aligned}$$

Moreover we get

$$\begin{aligned} \mathbb{E}(\mathbb{X}_t^H(\varepsilon, \gamma)^2) &= 2K(\delta)(\varepsilon + \gamma)^2 t^{2H} - 2K(\delta)\varepsilon\gamma 2^{2H} t^{2H} \\ &= 2K(\delta)(\varepsilon^2 + \gamma^2 - (2^{2H} - 2)\varepsilon\gamma) t^{2H}. \end{aligned}$$

□

Now our goal to characterize the two-parameter generalized Wright process through its finite dimensional structure. In the multidimensional case, given a sequence of real numbers $\{\vartheta_1, \vartheta_2, \dots, \vartheta_n\}$, for any collection

$$\{\mathbb{X}_{t_1}^H(\varepsilon, \gamma), \dots, \mathbb{X}_{t_n}^H(\varepsilon, \gamma)\}$$

with $0 < t_1 < \dots < t_n$, using linearly again, one can see that

$$\begin{aligned} \mathbb{E} \left[\exp \left(i \vartheta_j \sum_{j=1}^n \mathbb{X}_{t_j}^H(\varepsilon, \gamma) \right) \right] \\ = {}_m\Psi_p \left(- \frac{1}{2K(\delta)(\varepsilon^2 + \gamma^2 - (2^{2H} - 2)\varepsilon\gamma)} \sum_{i,j=1}^n \vartheta_i \vartheta_j \mathcal{R}_{H,\varepsilon,\gamma}(t_i, t_j) \right), \end{aligned} \quad (29)$$

where $\mathcal{R}_{H,\varepsilon,\gamma}$ is the auto-covering matrix of $\mathbb{X}_t^H(\varepsilon, \gamma)$ given by Equation (26).

Theorem 3. The joint probability density function of the collection $\{\mathbb{X}_{t_1}^H(\varepsilon, \gamma), \dots, \mathbb{X}_{t_n}^H(\varepsilon, \gamma)\}$ is given by

$$\Omega_{H,\varepsilon,\gamma}(x_1, \dots, x_n, \mathcal{R}_{H,\varepsilon,\gamma}) = \frac{(2\pi)^{-\frac{n+1}{2}} C_H^{n/2}(\delta, \varepsilon, \gamma)}{\delta \sqrt{2 \det \mathcal{R}_{H,\varepsilon,\gamma}}} \int_0^\infty \frac{1}{\tau^{\frac{n}{2}}} G\left(\frac{\zeta}{\tau^{1/2}}\right) H_{p,m}^{m,0}(\tau) d\tau, \quad (30)$$

where

$$\begin{aligned} C_H(\delta, \varepsilon, \gamma) &:= 2K(\delta)(\varepsilon^2 + \gamma^2 - (2^{2H} - 2)\varepsilon\gamma), \\ \zeta &= \left(C_H(\delta, \varepsilon, \gamma) \sum_{i,j=1}^n x_i \mathcal{R}_{H,\varepsilon,\gamma}^{-1}(t_i, t_j) x_j \right)^{1/2}, \end{aligned}$$

and G is the Gaussian probability density function given by

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Proof. We need to calculate the n -dimensional Fourier transform of Equation (30) and we find that is equal to Equation (29). We have

$$\begin{aligned} \int_{\mathbb{R}^n} \exp \left(i \sum_{j=1}^n \vartheta_j x_j \right) \Omega_{H,\varepsilon,\gamma}(x_1, \dots, x_n, \mathcal{R}_{H,\varepsilon,\gamma}) d^n x \\ = \frac{(2\pi)^{-\frac{n+1}{2}} C_H^{n/2}(\delta, \varepsilon, \gamma)}{\delta \sqrt{\det \mathcal{R}_{H,\varepsilon,\gamma}}} \\ \times \int_0^\infty \frac{1}{\tau^{\frac{n}{2}}} H_{p,m}^{m,0}(\tau) \int_{\mathbb{R}^n} \exp \left(i \sum_{j=1}^n \vartheta_j x_j \right) G\left(\frac{\zeta}{\tau^{1/2}}\right) d^n x d\tau \\ = \int_0^\infty \frac{1}{\tau^{\frac{n}{2}}} H_{p,m}^{m,0}(\tau) \int_{\mathbb{R}^n} \exp \left(i \sum_{j=1}^n \vartheta_j x_j \right) \frac{(2\pi)^{-n/2} C_H^{n/2}(\delta, \varepsilon, \gamma)}{\delta \sqrt{\det \mathcal{R}_{H,\varepsilon,\gamma}}} \\ \exp \left(- \sum_{i,j=1}^n \frac{C_H(\delta, \varepsilon, \gamma) x_i \mathcal{R}_{H,\varepsilon,\gamma}^{-1}(t_i, t_j) x_j}{2\tau} \right) d^n x d\tau. \end{aligned}$$

If we make the change of variables $x = \frac{\tau^{1/2} y}{C_H^{1/2}(\delta, \varepsilon, \gamma)}$, with $x, y \in \mathbb{R}^n$, and by using Equation (3) we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n} \exp \left(i \sum_{j=1}^n \vartheta_j x_j \right) \Omega_{H,\varepsilon,\gamma}(x_1, \dots, x_n, \mathcal{R}_{H,\varepsilon,\gamma}) d^n x \\
&= \int_0^\infty H_{p,m}^{m,0}(\tau) \int_{\mathbb{R}^n} \exp \left(i \frac{\tau^{1/2}}{C_H^{1/2}(\delta, \varepsilon, \gamma)} \sum_{j=1}^n \vartheta_j y_j \right) \frac{(2\pi)^{-n/2}}{\delta \sqrt{\det \mathcal{R}_{H,\varepsilon,\gamma}}} \\
&\quad \exp \left(- \sum_{i,j=1}^n \frac{y_i \mathcal{R}_{H,\varepsilon,\gamma}^{-1}(t_i, t_j) y_j}{2} \right) d^n y d\tau \\
&= \int_0^\infty \frac{1}{\delta} H_{p,m}^{m,0}(\tau) \exp \left[- \tau \sum_{i,j=1}^n \frac{\vartheta_i \mathcal{R}_{H,\varepsilon,\gamma}(t_i, t_j) \vartheta_j}{C_H(\delta, \varepsilon, \gamma)} \right] d\tau \\
&= \int_0^\infty \frac{1}{\delta} H_{p,m}^{m,0}(\tau) e^{-\tau s} d\tau \\
&= {}_m\Psi_p(-s),
\end{aligned}$$

where

$$s = \sum_{i,j=1}^n \frac{\vartheta_i \vartheta_j \mathcal{R}_{H,\varepsilon,\gamma}(t_i, t_j)}{C_H(\delta, \varepsilon, \gamma)}.$$

This gives the statement. $\square$

5. Applications

Our objective now is to investigate our Wright calculus in order to:

- (1) Prove the existence of local times for the generalized Wright process;
- (2) Characterize the two-parameter generalized Wright Ornstein–Uhlenbeck process as the unique solution to a Langevin equation driven by the tpgWp;
- (3) Analyze a linear Wick-type stochastic differential equation driven by multi-two-parameter generalized Wright processes (mtpgWps).

5.1. Local Times for Generalized Wright Process

The existence of local times for the generalized Wright processes. The theory of local times for random processes was pioneered by Lévy (1940). Recently, there has been growing interest in the local times of multi-parameter random processes. While this theory is well-established for standard Brownian motion and generalized grey Brownian motion, it remains an open problem for processes that further generalize these classes. This gap in the literature provides the primary motivation for investigating this problem within the framework of generalized Wright processes. In this section, we solve this problem by applying Berman's criterion (see [28], Lemma 3.1). For the reader's convenience, we begin by recalling the concepts of occupation measure and occupation density.

Let $f : [0, T] \rightarrow \mathbb{R}$ be a measurable function. For any Borel set $B \in \mathcal{B}(\mathbb{R})$, the occupation measure μ_f of f on $[0, T]$ is defined by

$$\mu_f(B) := \int_0^T \mathbf{1}_B(f(s)) ds = \lambda(\{t \in [0, T] : f(t) \in B\}),$$

where λ denotes the Lebesgue measure on $[0, T]$. Thus, $\mu_f(B)$ measures the amount of time f spends in B throughout the time period $[0, T]$. Analogously, for a stochastic process $X = (X_t)_{t \in [0, T]}$, the occupation measure of the trajectory

$$t \mapsto X_t(\omega), \quad t \in [0, T]$$

is defined in the same manner. However, $\mu_X(\omega)$ is now a random measure that depends explicitly on the sample point ω from the probability space. The function f is said to have an occupation density over $[0, T]$ if its occupation measure μ_f is absolutely continuous

with respect to the Lebesgue measure λ . Denoting this Radon-Nikodym derivative by $L^f(\cdot, [0, T])$, we write for any $x \in \mathbb{R}$:

$$L^f(x, [0, T]) = \frac{d\mu_f}{d\lambda}(x).$$

It follows that for every $F \in \mathcal{B}(\mathbb{R})$,

$$\int_0^T \mathbf{1}_F(f(s)) ds = \int_F L^f(x, [0, T]) dx.$$

Definition 6. (see [28]) A continuous stochastic process X is said to possess an occupation density on $[0, T]$ if, for almost all $\omega \in \Omega$, the sample path $X(\omega)$ has an occupation density $L^X(\cdot, [0, T])$. This density is also referred to as the local time of X .

The criteria governing the existence of local times for stochastic processes were established by Berman in [28], Section 3. More precisely, a stochastic process X is shown to admit a local time if and only if

$$\int_{\mathbb{R}} \left| \int_0^1 \int_0^1 \mathbb{E}(e^{i\lambda(X(t)-X(s))}) ds dt \right| d\lambda < \infty. \quad (31)$$

In the following we show that (31) is fulfilled if the stochastic process X is the Generalized Wright Process $\mathcal{W}_t^H$. In fact, according to (27), the characteristic function of the increments of the generalized Wright process (gwp) $\mathcal{W}_t^H - \mathcal{W}_s^H$ is given by

$$\mathbb{E}[e^{i\lambda(\mathcal{W}_t^H - \mathcal{W}_s^H)}] = \frac{1}{\delta} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -\frac{\lambda^2}{2} |t-s|^{2H} \right], \quad \lambda \in \mathbb{R}.$$

By using the change of variables $r = (2)^{-1/2} \lambda |t-s|^H$, we have

$$\int_{\mathbb{R}} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -\frac{\lambda^2}{2} |t-s|^{2H} \right] d\lambda = \frac{\sqrt{2}}{|t-s|^H} \int_{\mathbb{R}} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -r^2 \right] dr.$$

Now we need to calculate the following integral

$$I = \int_{\mathbb{R}} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -r^2 \right] dr = 2 \int_0^\infty {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -r^2 \right] dr.$$

If we consider the Mellin transform

$$M(s) = \int_0^\infty r^{s-1} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -r^2 \right] dr,$$

then by making the change of variable $t = r^2$, we obtain

$$M(s) = \frac{1}{2} \int_0^\infty t^{\frac{s}{2}-1} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -t \right] dt.$$

Alternatively, using the Mellin transform representation of the generalized Wright function, it follows that

$$\int_0^\infty t^{\tau-1} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -t \right] dt = \Gamma(\tau) \frac{\prod_{i=1}^m \Gamma(B_i - \beta_i \tau)}{\prod_{j=1}^p \Gamma(A_j - \alpha_j \tau)},$$

we deduce that

$$M(s) = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \frac{\prod_{i=1}^m \Gamma\left(B_i - \frac{\beta_i s}{2}\right)}{\prod_{j=1}^p \Gamma\left(A_j - \frac{\alpha_j s}{2}\right)}.$$

Evaluating this expression at $s = 1$ gives

$$I = 2M(1) = \sqrt{\pi} \frac{\prod_{i=1}^m \Gamma\left(B_i - \frac{\beta_i}{2}\right)}{\prod_{j=1}^p \Gamma\left(A_j - \frac{\alpha_j}{2}\right)}.$$

Finally, the integration with respect to t and s is performed as follows. Since the integrand is symmetric with respect to the diagonal $s = t$, we can rewrite the integral as

$$\int_0^1 \int_0^1 \frac{1}{|t-s|^H} ds dt = 2 \int_0^1 \int_0^t (t-s)^{-H} ds dt.$$

For a fixed t , make the change of variable $u = t - s$, we get

$$\int_0^t (t-s)^{-H} ds = \int_0^t u^{-H} du = \frac{t^{1-H}}{1-H},$$

because $0 < H < 1$. Substituting into the outer integral gives

$$\int_0^1 \int_0^1 \frac{1}{|t-s|^H} ds dt = \frac{2}{1-H} \int_0^1 t^{1-H} dt = \frac{2}{(1-H)(2-H)}.$$

Therefore, putting all together, we obtain

$$\int_{\mathbb{R}} \left| \int_0^1 \int_0^1 \mathbb{E}(e^{i\lambda(X(t)-X(s))}) ds dt \right| d\lambda = \frac{2\sqrt{2\pi}}{(1-H)(2-H)} \frac{\prod_{i=1}^m \Gamma\left(B_i - \frac{\beta_i}{2}\right)}{\prod_{j=1}^p \Gamma\left(A_j - \frac{\alpha_j}{2}\right)} < \infty.$$

Thus, we have shown the main result of this subsection which we state in the following theorem.

Theorem 4. *The generalized Wright process $\mathcal{W}^H$ admits a λ -square integrable local time $L^{\mathcal{W}^H}(\cdot, [0, T])$ almost surely.*

Corollary 1. *The existence of the local time $L^{\mathcal{W}^H}(\cdot, [0, T])$ immediately implies the following occupation formula:*

$$\int_0^T f(\mathcal{W}_t^H) dt = \int_{\mathbb{R}} f(x) L^{\mathcal{W}^H}(x, [0, T]) dx, \quad a.s.$$

5.2. Two-Parameter Generalized Wright Ornstein–Uhlenbeck Process

Now our goal to study the two-parameters Generalized Wright Ornstein–Uhlenbeck process. Define the space of smooth polynomials $\mathcal{P}(\mathcal{E}')$ on $\mathcal{E}'$ by

$$\mathcal{P}(\mathcal{E}') = \left\{ \phi(v) = \sum_{n=0}^{N_\phi} \langle P_{n,\Psi}(v), \phi^{(n)} \rangle, \phi^{(n)} \in \mathcal{E}_{\mathbb{C}}^{\otimes n}, v \in \mathcal{E}', N_\phi \in \mathbb{N} \right\}. \quad (32)$$

For $\phi \in \mathcal{P}(\mathcal{E}')$ as in Equation (32) we define for any $q_1, q_2 \in \mathbb{N}$ the norm

$$\|\phi\|_{q_1, q_2, \Lambda_\Psi}^2 := \sum_{n=0}^{N_\phi} (n!)^2 2^{nq_2} |\phi^{(n)}|_{q_1, H}^2,$$

and we denote by $(\mathcal{E}_{q_1})_{q_2, H}$ the completion of $\mathcal{P}(\mathcal{E}')$ with respect to $\|\cdot\|_{q_1, q_2, \Lambda_\Psi}^2$. So the space of Wright noise test functions $(\mathcal{E})_H$ is defined as a projective system of Hilbert space $\{(\mathcal{E}_{q_1})_{q_2, H}; q_1, q_2 \in \mathbb{N}\}$. The dual space $(\mathcal{E}_{-q_1})_{-q_2, H}$ is a subset of $\mathcal{P}'(\mathcal{E}')$ such that if $\Phi \in (\mathcal{E}_{-q_1})_{-q_2}$, then

$$\|\Phi\|_{-q_1, -q_2, \Lambda_\Psi}^2 := \sum_{n=0}^{\infty} 2^{-nq_2} |\Phi^{(n)}|_{-q_1, H}^2 < \infty$$

It holds that $(\mathcal{E}_{-q_1})_{-q_2, H}$ is the dual of $(\mathcal{E}_{q_1})_{q_2, H}$ and the space of Wright noise generalized functions is defined as the inductive limit of $(\mathcal{E}_{-q_1})_{-q_2}$ i.e.,

$$(\mathcal{E})_H^* \equiv \bigcup_{p, q \in \mathbb{N}} (\mathcal{E}_{-q_1})_{-q_2, H}.$$

Then we obtain the so-called Wright noise Gel'fand triple of Hilbert spaces

$$(\mathcal{E})_H \subset L^2(\mathcal{E}', \Lambda_\Psi) \subset (\mathcal{E})_H^*,$$

with the $\mathbb{C}$ -bilinear form on $(\mathcal{E})_H^* \times (\mathcal{E})_H$ given by

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the canonical $\mathbb{C}$ -bilinear form on $\mathcal{E}'^{\otimes n} \times \mathcal{E}^{\otimes n}$ which is compatible with the inner product of $\mathcal{K}^{\otimes n}$. The set of normalized Laplace-exponential function with respect to the measure Λ_Ψ

$$\left\{ \text{Exp}_{\Lambda_\Psi}(\cdot, \phi) := \frac{e^{\langle \cdot, \phi \rangle}}{\mathcal{L}_{\Lambda_\Psi}(\phi)}; \phi \in \mathcal{U}_{q_1, q_2} \right\},$$

where

$$\mathcal{U}_{q_1, q_2} := \left\{ \phi \in \mathcal{E}_{\mathbb{C}}; |\phi|_{q_1, H} < 2^{-q_2} \right\}$$

which forms a total set in $(\mathcal{E}_{q_1})_{q_2, H}$. So we can introduce the S_{Λ_Ψ} -transform, which is used to characterize the space $(\mathcal{E})_H^*$.

Definition 7. For any $\Phi \in (\mathcal{E})_H^*$ and $\phi \in \mathcal{U}_{q_1, q_2}$, we define

$$S_{\Lambda_\Psi} \Phi(\phi) := \frac{\langle\langle \Phi, e^{\langle \cdot, \phi \rangle} \rangle\rangle_H}{\mathcal{L}_{\Lambda_\Psi}(\phi)} = \frac{1}{m \Psi_p(-\frac{1}{2} \|\phi\|_0^2)} \langle\langle \Phi, e^{\langle \cdot, \phi \rangle} \rangle\rangle. \quad (33)$$

We denote by $Hol_0(\mathcal{E}_{\mathbb{C}})$ the space of holomorphic functions at zero where we identify two functions which coincide in a neighborhood of zero. $Hol_0(\mathcal{E}_{\mathbb{C}})$ is the union of the spaces

$$\left\{ F \in Hol_0(\mathcal{E}_{\mathbb{C}}); \sup_{|\phi|_{q_1, H} < 2^{-q_2}} |F(\phi)| < \infty \right\}, \quad q_1, q_2 \in \mathbb{N}.$$

The following proposition is proven in [4]:

Proposition 2. The S_{Λ_Ψ} -transform is a topological isomorphism from $(\mathcal{E})_H^*$ to $Hol_0(\mathcal{E}_{\mathbb{C}})$.

Theorem 5. The solution of the Langevin equation driven by (tpgWp)

$$\begin{cases} dY_H(t) = -\lambda Y_H(t) + \mu d\mathbb{X}_t^H, \\ Y_H(0) = Y_0, \mu \in \mathbb{R}, \lambda > 0, \end{cases} \quad (34)$$

is given explicitly by

$$Y_H(t) = Y_0 e^{-\lambda t} + \mu \langle \cdot, \psi_H \rangle, \quad (35)$$

where

$$F_H = \varepsilon M_-^H \mathbf{1}_{[0,t)} + \gamma M_-^H \mathbf{1}_{(-t,0]} - \lambda \int_0^t e^{\lambda(s-t)} (\varepsilon M_-^H \mathbf{1}_{[0,s)} + \gamma M_-^H \mathbf{1}_{(-s,0]}) ds.$$

Proof. Note that the stochastic differential Equation (34) is interpreted in the integral form by

$$Y_H(t) = Y_0 - \lambda \int_0^t Y_H(s) ds + \mu \mathbb{X}_t^H.$$

For any $f \in K$, the $S_{\Lambda\Psi}$ -transform of the random variable $\langle \cdot, f \rangle$ is given by

$$S_{\Lambda\Psi} \langle \cdot, f \rangle (\phi) = \int_{\mathcal{E}'} \frac{d}{ds} e^{\langle w, \phi + sf \rangle} \Big|_{s=0} \Lambda\Psi(dw). \quad (36)$$

On the other hand, it easy to prove that

$$\left| \frac{d}{ds} e^{\langle v, \phi + sf \rangle} \right| = |\langle v, f \rangle| e^{\langle v, \phi + sf \rangle} \in L^1(\Lambda\Psi),$$

therefore, interchanging derivative and integral in Equation (36), we have

$$S_{\Lambda\Psi} \langle \cdot, f \rangle (\phi) = \frac{d}{ds} \left({}_m\Psi_p \left(\frac{1}{2} \langle \phi + sf, \phi + sf \rangle_H \right) \right) \Big|_{s=0}.$$

Using the fact that

$$\begin{aligned} \frac{d}{dz} {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| z \right] &= \frac{d}{dz} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(B_i + n\beta_i)}{\prod_{j=1}^p \Gamma(A_j + n\alpha_j)} \frac{z^n}{n!} \\ &= \sum_{n=1}^{\infty} \frac{\prod_{i=1}^m \Gamma(B_i + n\beta_i)}{\prod_{j=1}^p \Gamma(A_j + n\alpha_j)} \frac{z^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^m \Gamma(B_i + n\beta_i + \beta_i)}{\prod_{j=1}^p \Gamma(A_j + n\alpha_j + \alpha_j)} \frac{z^n}{n!} \\ &= {}_m\Psi_p \left[\begin{matrix} (B_i + \beta_i, \beta_i)_{1,m} \\ (A_j + \alpha_j, \alpha_j)_{1,p} \end{matrix} \middle| z \right]. \end{aligned}$$

Then we obtain

$$S_{\Lambda\Psi} \langle \cdot, f \rangle (\phi) = \langle f, \phi \rangle {}_m\Psi_p \left[\begin{matrix} (B_i + \beta_i, \beta_i)_{1,m} \\ (A_j + \alpha_j, \alpha_j)_{1,p} \end{matrix} \middle| \frac{1}{2} \|\phi\|_H^2 \right].$$

Hence by computing the $S_{\Lambda\Psi}$ -transform of $Y_H(t)$ we get

$$\begin{aligned} S_{\Lambda\Psi} Y_H(t)(\phi) &= Y_0 - \lambda \int_0^t S_{\Lambda\Psi} X_H(s)(\phi) ds \\ &\quad + \frac{\mu {}_m\Psi_p \left[\begin{matrix} (B_i + \beta_i, \beta_i)_{1,m} \\ (A_j + \alpha_j, \alpha_j)_{1,p} \end{matrix} \middle| \frac{1}{2} \|\phi\|_H^2 \right]}{{}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| \frac{1}{2} \|\phi\|_H^2 \right]} \left(\langle \phi, \varepsilon M_-^H \mathbf{1}_{[0,t)} + \gamma M_-^H \mathbf{1}_{(-t,0]} \rangle \right). \end{aligned}$$

Let us denote $Y(s) := S_{\Lambda\Psi} Y_H(t)(\phi)$, then the previous equation is equivalent to

$$Y'(t) = -\lambda Y(t) + \mu R_H(\phi) \frac{d}{dt} \langle \phi, \varepsilon M_-^H \mathbf{1}_{[0,t)} + \gamma M_-^H \mathbf{1}_{(-t,0]} \rangle,$$

where

$$R_H(\phi) := \frac{{}_m\Psi_p \left[\begin{matrix} (B_i + \beta_i, \beta_i)_{1,m} \\ (A_j + \alpha_j, \alpha_j)_{1,p} \end{matrix} \middle| \frac{1}{2} \|\phi\|_H^2 \right]}{{}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| \frac{1}{2} \|\phi\|_H^2 \right]}.$$

Then it is easy to compute the solution of this ordinary differential equation and we get

$$\begin{aligned}
Y(t) &= Y_0 e^{-\lambda t} + \mu R_H(\varphi) \left(\langle \varphi, \varepsilon M_-^H \mathbf{1}_{[0,t]} + \gamma M_-^H \mathbf{1}_{(-t,0]} \rangle \right. \\
&\quad \left. - \lambda \int_0^t e^{\lambda(s-t)} \langle \varphi, \varepsilon M_-^H \mathbf{1}_{[0,s]} + \gamma M_-^H \mathbf{1}_{(-s,0]} \rangle ds \right).
\end{aligned}$$

Thus, the S_{Λ_Ψ} -transform of $Y_H(t)$ is

$$\begin{aligned}
S_{\Lambda_\Psi} Y_H(t) &= Y_0 e^{-\lambda t} + \mu R_H(\varphi) \left(\langle \varphi, \varepsilon M_-^H \mathbf{1}_{[0,t]} + \gamma M_-^H \mathbf{1}_{(-t,0]} \rangle \right. \\
&\quad \left. - \lambda \int_0^t e^{\lambda(s-t)} \langle \varphi, \varepsilon M_-^H \mathbf{1}_{[0,s]} + \gamma M_-^H \mathbf{1}_{(-s,0]} \rangle ds \right) \\
&= Y_0 e^{-\lambda t} + \mu R_H(\varphi) \left(\langle \varphi, \varepsilon M_-^H \mathbf{1}_{[0,t]} + \gamma M_-^H \mathbf{1}_{(-t,0]} \rangle \right. \\
&\quad \left. - \lambda \int_0^t e^{\lambda(s-t)} \langle \varphi, \varepsilon M_-^H \mathbf{1}_{[0,s]} + \gamma M_-^H \mathbf{1}_{(-s,0]} \rangle ds \right)
\end{aligned}$$

Denoting

$$F_H = \varepsilon M_-^H \mathbf{1}_{[0,t]} + \gamma M_-^H \mathbf{1}_{(-t,0]} - \lambda \int_0^t e^{\lambda(s-t)} (\varepsilon M_-^H \mathbf{1}_{[0,s]} + \gamma M_-^H \mathbf{1}_{(-s,0]}) ds,$$

then the S_{Λ_Ψ} -transform of $X_H(t)$ is represented as

$$S_{\Lambda_\Psi} Y_H(t) = Y_0 e^{-\lambda t} + \mu R_H(\varphi) \langle \varphi, F_H \rangle.$$

Finally, by inverting the S_{Λ_Ψ} -transform we conclude that

$$Y_H(t) = Y_0 e^{-\lambda t} + \mu \langle \varphi, F_H \rangle.$$

□

Definition 8. The solution $Y(t), t \geq 0$ given by Equation (35), of the Langevin Equation (34) is called a two-parameter generalized Wright Ornstein–Uhlenbeck process (tpgWOU). Moreover we have

$$\begin{aligned}
\mathbb{E}(Y_H(t)) &= Y_0 e^{-\lambda t}, \\
\mathbb{E}(e^{ikY_H(t)}) &= \exp(ikY_0 e^{-\lambda t}) {}_m\Psi_p \left[\begin{matrix} (B_i, \beta_i)_{1,m} \\ (A_j, \alpha_j)_{1,p} \end{matrix} \middle| -\frac{k^2}{2} \|F_H\|_0^2 \right].
\end{aligned}$$

The numerical simulation presented in Figure 1 plots the Euler–Maruyama approximation of Equation (34) and the discretised explicit solution (35) on the same figure for the parameter choice $H = 0.75$, $\lambda = 1$, $\mu = 1$, $\varepsilon = 1$, $\gamma = 0.5$, and $Y_0 = 0.5$ on $[0, 1]$ with $N = 1000$ time steps. The two curves are visually indistinguishable at this resolution, confirming the consistency of the explicit formula with the Euler–Maruyama scheme and validating Theorem 5 numerically. Setting $H = 0.5$, $\varepsilon = 1$, $\gamma = 0$ recovers the standard Gaussian Ornstein–Uhlenbeck process as a special case, providing a direct benchmark.

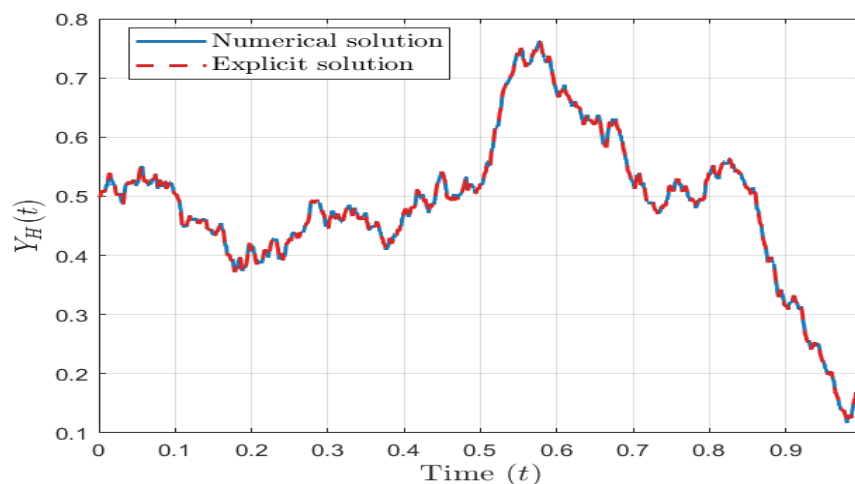


Figure 1. Trajectory simulation of the two-parameter generalized Wright Ornstein–Uhlenbeck process.

6. Linear Wick-Type Stochastic Differential Equation Associated with the Multi-Two-Parameter Generalized Wright Processes

Definition 9. Let $(\sigma_k)_{k \in \mathbb{N}}$ be a sequence satisfying $\sum_{k=1}^{\infty} \sigma_k < \infty$. Given a sequence $(H_k)_{k \in \mathbb{N}}$, the multi-two-parameter generalized Wright processes (mgwp) is defined by

$$\mathcal{M}_t = \sum_{k=1}^{\infty} \sigma_k \mathbb{X}_t^{H_k}, \quad (37)$$

where $\mathbb{X}^{H_k}, k \in \mathbb{N}$ are independent gwp's.

It is well established that the Wick product is a well-defined operation within both Gaussian and non-Gaussian frameworks (see, e.g., [2,4]). For any $\Phi, \Psi \in (\mathcal{E})^*$, the Wick product $\Phi \diamond \Psi$ is defined by

$$S_{\Lambda_{\Psi}}(\Phi \diamond \Psi) = S_{\Lambda_{\Psi}}(\Phi) \cdot S_{\Lambda_{\Psi}}(\Psi). \quad (38)$$

As $Hol_0(\mathcal{E}_{\mathbb{C}})$ is an algebra, it follows from Proposition 2 that there exists an element $\Phi \diamond \Psi \in (\mathcal{E})^*$ such that (38) holds. Since the exponential function is entire, the Wick exponential $\exp^{\diamond}(\Phi)$ is well-defined and given by

$$\exp^{\diamond}(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{\diamond n}.$$

is well defined for all $\Phi \in (\mathcal{E})_H^*$. Our goal now is to investigate Wick-type stochastic differential equations (SDEs), namely equations of the form

$$\begin{cases} dZ(t) = -\lambda Z(t) + Z(t) \diamond d\mathcal{M}_t, \\ Z(0) = \Phi_0 \in (\mathcal{E})_H^*. \end{cases} \quad (39)$$

Lemma 4. Each multi-two-parameter generalized Wright processes

$$(0, +\infty[\ni t \mapsto \mathcal{M}_t \in (\mathcal{E})_H^*,$$

is differentiable, i.e.,

$$\lim_{h \rightarrow 0} \frac{\mathcal{M}_{t+h} - \mathcal{M}_t}{h}$$

exists and converges to some element in $(\mathcal{E})_H^*$ denoted by $\mathcal{N}_t$ fulfilling

$$(S_{\Lambda_{\Psi}} \mathcal{N}_t)(\varphi) = \sum_{k=1}^{\infty} \sigma_k \left(\varepsilon(M_+^{H_k} \varphi)(t) + \gamma(M_+^{H_k} \varphi)(-t) \right) R_{H_k}(\varphi). \quad (40)$$

Proof. Given a sequence $(h_n)_{n \in \mathbb{N}}$ converging to 0, we consider for each $n \in \mathbb{N}$ and $t > 0$ the element $\Theta_n \in (\mathcal{E})_H^*$ defined by

$$\Theta_n := \frac{\mathcal{M}_{t+h_n} - \mathcal{M}_t}{h_n}.$$

By using the fact that

$$(S_{\Lambda_{\Psi}} \mathcal{W}_t^{H_k})(\varphi) = R_{H_k}(\varphi) \langle \varphi, M_-^{H_k} \mathbf{1}_{[0,t)} \rangle = R_{H_k}(\varphi) \int_0^t (M_+^{H_k} \varphi)(x) dx,$$

we obtain

$$\begin{aligned} S_{\Lambda_{\Psi}} \Theta_n(\varphi) &= \frac{1}{h_n} \sum_{k=1}^{\infty} \sigma_k S_{\Lambda_{\Psi}} (\mathbb{X}_{t+h_n}^{H_k} - \mathbb{X}_t^{H_k})(\varphi) \\ &= \frac{1}{h_n} \sum_{k=1}^{\infty} \sigma_k \left[\varepsilon \int_t^{t+h_n} (M_+^{H_k} \varphi)(x) dx + \gamma \int_{-t}^{-t-h_n} (M_+^{H_k} \varphi)(x) dx \right] R_{H_k}(\varphi). \end{aligned}$$

Since M_+^H is continuous, see [6], we have

$$\lim_{n \rightarrow \infty} \frac{1}{h_n} \left[\varepsilon \int_t^{t+h_n} (M_+^{H_k} \varphi)(x) dx + \gamma \int_{-t}^{-t-h_n} (M_+^{H_k} \varphi)(x) dx \right] = \varepsilon(M_+^{H_k} \varphi)(t) + \gamma(M_+^{H_k} \varphi)(-t).$$

Consequently, $(S_{\Lambda_{\Psi}} \Theta_n(\varphi))_{n \in \mathbb{N}}$ forms a Cauchy sequence. The holomorphy of the generalized Wright functions implies that there exist $p, q \in \mathbb{N}$ and $C_1 > 0$ such that

$$|R_{H_k}(\varphi)| \leq C_1, \quad \text{for all } \varphi \in \mathcal{U}_{p,q}.$$

Moreover, Theorem 2.3 in [6] ensures the existence of $p' \in \mathbb{N}$ and $C_2 > 0$ such that for all $x \in \mathbb{R}$,

$$|\varepsilon(M_+^{\alpha_k} \varphi)(x) + \gamma(M_+^{\alpha_k} \varphi)(-x)| \leq (|\varepsilon| + |\gamma|) C_2 |\varphi|_{p'}.$$

Choose $p^* > \max(p, p')$, then for all $n \in \mathbb{N}$, we have

$$\left| \frac{S_{\Lambda_{\Psi}} (\mathcal{M}_{t+h_n} - \mathcal{M}_t)(\varphi)}{h_n} \right| \leq \sum_{k=1}^{\infty} \sigma_k (|\varepsilon| + |\gamma|) C_1 C_2 |\varphi|_{p^*} < \infty.$$

Finally, applying Proposition 2, we see that the sequence $(\Theta_n)_{n \in \mathbb{N}}$ converges in $(\mathcal{E})_H^*$ to a distribution $\mathcal{N}_t$ satisfying

$$(S_{\Lambda_{\Psi}} \mathcal{N}_t)(\varphi) = \lim_{n \rightarrow \infty} (S_{\Lambda_{\Psi}} \Theta_n)(\varphi), \quad \varphi \in \mathcal{U}_{p^*,q}.$$

□

Theorem 6. The solution $Z(t), t \geq 0$ of the linear stochastic differential equation with Wick-multiplicative generalized Wright noise (39) is given by

$$Z(t) = e^{-\lambda t} \Phi_0 \diamond \exp^{\diamond}(\mathcal{N}_t),$$

and it's an element in $(\mathcal{E})_H^*$. Its generalized expectation and $S_{\Lambda_{\Psi}}$ -transform for a suitable element $\varphi \in \mathcal{U}_{p^*,q}$ are given by

$$\begin{aligned} S_{\Lambda_{\Psi}} Z(t)(0) &= e^{-\lambda t} \mathbb{E}(\Phi_0) \\ S_{\Lambda_{\Psi}} Z(t)(\varphi) &= e^{-\lambda t} S_{\Lambda_{\Psi}} \Phi_0(\varphi) \exp(S_{\Lambda_{\Psi}} \mathcal{N}_t(\varphi)). \end{aligned}$$

Proof. First, Equation (39) is interpreted in its integral form by

$$Z(t) = \Phi_0 - \lambda \int_0^t Z(s) ds + \int_0^t Z(t) \diamond \mathcal{N}_s ds. \quad (41)$$

Applying the S_{Λ_Ψ} -transform to Equation (41) and using (40), we obtain

$$\begin{aligned} S_{\Lambda_\Psi} Z(t)(\varphi) &= S_{\Lambda_\Psi} \Phi_0(\varphi) - \lambda \int_0^t S_{\Lambda_\Psi} Z(s)(\varphi) ds \\ &+ \int_0^t S_{\Lambda_\Psi} Z(s)(\varphi) \sum_{k=1}^{\infty} \sigma_k (\varepsilon M_+^{H_k} \varphi(s) + \gamma M_+^{H_k} \varphi(-s)) R_{H_k}(\varphi) ds. \end{aligned}$$

Denote $z(t) := S_{\Lambda_\Psi} Z(t)(\varphi)$, then

$$z'(t) = -\lambda z(t) + z(t) \sum_{k=1}^{\infty} \sigma_k (\varepsilon M_+^{H_k} \varphi(t) + \gamma M_+^{H_k} \varphi(-t)) R_{H_k}(\varphi)$$

and the solution of this ordinary differential equation is given by

$$z(t) = y(0) e^{-\lambda t} \exp \left(\sum_{k=1}^{\infty} \sigma_k (\varepsilon M_+^{H_k} \varphi(t) + \gamma M_+^{H_k} \varphi(-t)) R_{H_k}(\varphi) \right).$$

By virtue of Equation (40), the S_{Λ_Ψ} -transform of $Z(t)$ takes the form

$$S_{\Lambda_\Psi} Z(t)(\varphi) = e^{-\lambda t} S_{\Lambda_\Psi} \Phi_0(\varphi) \exp(S_{\Lambda_\Psi} \mathcal{N}_t(\varphi)).$$

Finally, taking the inverse S_{Λ_Ψ} -transform, we obtain

$$Z(t) = e^{-\lambda t} \Phi_0 \diamond \exp^\diamond(\mathcal{N}_t).$$

□

7. Conclusions

Until now, the study of Appell polynomials within the framework of infinite-dimensional generalized Wright calculus has remained less developed than in the case of Mittag-Leffler analysis. In this study, we introduce a new class of generalized Wright lowering operators, which play an intrinsic role in establishing several combinatorial properties of Appell polynomials associated with infinite-dimensional generalized Wright measures. In terms of applications, our work establishes the existence of local times for the generalized Wright process and develops the stochastic counterpart of generalized Wright analysis. This allows us to characterize the two-parameter generalized Wright Ornstein–Uhlenbeck process as the solution to a Langevin equation driven by the tpgWp. This combinatorial study will serve us in our future research to develop some stochastic applications, with the aim of obtaining practical applications. As example the study of the Green measure for a class of non-Gaussian processes in $\mathbb{R}^d$ associated with the family of two-parameter generalized Wright process. Another application by choosing a specific nuclear triple, we may define a generalized stochastic process that can be realized in different forms using known processes for similar examples. Specifically, it is expected that these processes behave like anomalous diffusions and their fractional Fokker–Planck equations. Due to their analytical properties and continued fraction representations, the latter have inspired major advances in several mathematical fields, particularly regarding integral equations and the Feynman–Kac formula. A detailed study of the Feynman–Kac formula in this setting is reserved for future research.

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