

Article

Bi-Univalent Functions Associated with Class Bazilevič Functions Defined by Gregory Polynomials

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Abstract

In this paper, we introduce and investigate a new subclass of bi-univalent Bazilevič-type functions associated with the Gregory generating function. The proposed class is defined by means of subordination and depends on the parameters ν and α , which provides a broader framework containing several previously studied subclasses as special cases. By using the coefficient comparison method together with standard estimates for functions with positive real part, we obtain upper bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Furthermore, the Fekete–Szegő functional $|a_3 - \xi a_2^2|$ is estimated for functions belonging to this class. Several special cases are also derived to demonstrate the connection between the present results and earlier known results related to Gregory-type bi-univalent functions.

Keywords: analytic (holomorphic) function; bi-univalent function; Bazilevič functions; Fekete–Szegő inequality; Gregory polynomial; starlike function

MSC: 30C45; 30C50

1. Introduction

Let Σ denote the class of bi-univalent functions. It can be formulated as follows:

$$f(\zeta) = \zeta + \sum_{n=1}^{\infty} a_n \zeta^n. \quad (1)$$

These functions are defined analytically on the open unit disc.

$$\Omega = \{\zeta : \zeta \in \mathbb{C} : |\zeta| < 1\}.$$

Consider the following subclass:

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is an analytic and injective function in } \Omega\}.$$

This class includes functions of set $\mathcal{A}$ univalent within the disc Ω . According to The Koebe One-Quarter theorem [1], every function within the class $f \in \mathcal{S}$ represents the open unit disc Ω on a domain containing a disc of radius $\frac{1}{4}$. Consequently, every function $f \in \mathcal{S}$ has an inverse function f^{-1} that satisfies the following conditions:

$$f^{-1}(f(\zeta)) = \zeta \quad \text{and} \quad f(f^{-1}(\omega)) = \omega \quad \text{for } |\omega| < R_0(f),$$



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where $R_0(f) \geq \frac{1}{4}$. The inverse function

$$g(\omega) = f^{-1}(\omega),$$

can be expressed as follows when developed into a sequence:

$$g(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots$$

The symbol Σ denotes the class of all bi-univalent functions defined on the open unit disk Ω . This class includes a number of distinctive functions that have been extensively studied within the framework of bi-univalent function theory, most notably the following:

$$f_1(\zeta) = \frac{\zeta}{1-\zeta}, \quad f_2(\zeta) = \log\left(\frac{1}{1-\zeta}\right), \quad f_3(\zeta) = \log\sqrt{\frac{1+\zeta}{1-\zeta}}.$$

The inverse representations associated with these functions can be written as follows:

$$f_1^{-1}(\omega) = \frac{\omega}{1+\omega}, \quad f_2^{-1}(\omega) = \frac{e^\omega - 1}{e^\omega}, \quad f_3^{-1}(\omega) = \frac{e^{2\omega} - 1}{e^{2\omega} + 1}.$$

The study of the limits of the coefficients of these functions began with Lewin in (1967) [2], who arrived at an estimate for the second coefficient of $|a_2| \leq 1.51$. Later, Brannan and Taha [3] improved upon this estimate, proposing a better limit of $|a_2| \leq \sqrt{2}$.

Over time, interest in studying prime coefficients has grown significantly resulting in a large body of research in this area. However, the search for general limits of the coefficients, particularly $|a_n|$ for $n \geq 4$, remains incomplete and requires further investigation, see Table 1.

Table 1. A brief review of the literature relating to the a_n operators of functions in the class Σ .

Research Findings	Estimates
Lewin (1967) [2]	$ a_2 \leq 1.51$
Brannan et al. (1980) [3]	$ a_2 \leq \sqrt{2}$
Netanyahu (1969) [4]	$\max_{\Sigma} a_2 = \frac{4}{3}$
Tan (1984) [5]	$ a_2 \leq 1.485$
Different researchers [6,7]	$ a_2 $ and $ a_3 $
Open Problem [8,9]	$ a_n $ unresolved

Let $\mathcal{P}$ be the class of analytic functions defined on unit unit disc Ω , satisfying the condition $p(0) = 1$, and possessing the property of being positive

$$\Re\{p(\zeta)\} > 0, \quad \zeta \in \Omega.$$

Such functions admit the series expansion

$$p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + \dots, \quad p_i \in \mathcal{P}. \quad (2)$$

This class is widely known as the Carathéodory class of functions [10]. The concept of dependency is fundamental to the study of the geometric and analytic properties of several subclasses of univalent functions. Lindelöf [11] established the initial framework for this concept, and the contributions of Rogosinski [12,13] and Littlewood [14] further supported and developed this field.

A function ϖ is defined as a Schwarz function if it is analytic within the open disc Ω , satisfies the condition $c(0) = 0$, and remains bounded such that its absolute value does not exceed 1 at all points of Ω .

$$c(0) = 0 \quad \text{and} \quad |c(\zeta)| < 1 \quad \text{for all } c \in \Omega.$$

In light of this concept, if f and g are analytic functions in Ω , then we describe f as a function of g in Ω , and this is written as follows:

$$f \prec g \quad \text{or} \quad f(\zeta) \prec g(\zeta),$$

whenever there exists a Schwarz function such that

$$f(\zeta) = g(c(\zeta)), \quad \zeta \in \Omega.$$

It should be noted that whenever g is a univalent function in Ω , the following can be deduced:

$$f \prec g \iff f(0) = g(0) \text{ and } f(\Omega) \subset g(\Omega).$$

The concept of subordination is attributed to Robertson [15], who introduced it within the framework of analytic functions. Let f and g be (holomorphic functions) defined on the open unit disk Ω in the complex plane. Then, f is described as subordinate to g , and this relationship is considered one of the fundamental relationships and is referred to in mathematical literature as:

$$f(\zeta) \prec g(\zeta), \quad \zeta \in \Omega,$$

if an analytic function exists ϕ so that the following conditions are met $|\phi(\zeta)| \leq 1$ satisfying $f(\zeta)/\phi(\zeta)$ is holomorphic in Ω and

$$f(\zeta) = \phi(\zeta)g(\varpi(\zeta)).$$

For some Schwartz functions ϖ , when $\varpi(\zeta) = 1$, the subordination is reduced to the classical subordination. If $\varpi(\zeta) = \zeta$, this concept is synonymous with partitioning. The concept of starlike functions is fundamental in geometric function theory. A function $f \in \mathcal{A}$ is described as a starlike function of order β where $(0 \leq \beta < 1)$ if

$$\Re\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > \beta, \quad \zeta \in \Omega.$$

Nasr and Aouf [16] derived a class of starlike functions

$$\Re\left(1 + \frac{1}{\nu}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right)\right) > 0, \quad (\nu \in \mathbb{C} \setminus \{0\}; \zeta \in \Omega).$$

This category is called the symbol $S^*(\beta)$, while in the special case where $\beta = 0$ we use the notation $S^* = S^*(0)$, which embodies the category of starlike functions according to the traditional well-known concept. It is established in the literature that $S^* \subset \mathcal{S}$, and that all functions of this class are univalent within the open unit disc Ω (see [17–20]). Furthermore, the class S^* can be described equivalently by employing the concept of dependency by reformulating its properties based on the dependency relationships that connect its elements:

$$S^* = \left\{ \left(1 + \frac{1}{\nu} \left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 1 \right) \right) \prec \varphi(\zeta), \zeta \in \Omega \right\}.$$

Geometrically speaking, a function f is said to be starlike with respect to the point $\omega_0 = f(0)$ if f is univalent on the open unit disc Ω , and its image $f(\Omega)$ represents a starlike with respect to the point ω_0 . More precisely, this means that every straight line segment reaches the point w_0 , and any point on $f(\Omega)$ remains entirely within this image

$$\{\eta f(0) + (1 - \eta)f(\zeta) : \eta \in [0, 1], \zeta \in \Omega\} \subset f(\Omega).$$

In generalizing the concept of star functions, Ma and Minda substituted the traditional function with a sufficiently general holomorphic function satisfying the following conditions: it maps univalently onto a region that is starlike with respect to the point and exhibits symmetric with respect to the real axis. This approach allowed for the study of a broad and comprehensive class of functions encompassing many known special cases, thus enabling the construction of a unified theoretical framework that integrates these diverse cases.

In mathematical literature, functions belonging to this category are usually referred to as Ma–Minda starlike functions, in recognition of the authors' outstanding contributions to the field. This generalization in turn led to the development of a rich array of subfamilies, which have garnered considerable research attention, as documented in numerous studies [21–25].

Definition 1. The function f is defined as a Bazilevič function of order $\alpha \geq 0$ introduced by Bazilevič in [26], if there exist:

a starlike function $g(\zeta)$ in Ω , and a function $p(\zeta)$ with positive real part satisfying

$$\Re\{p(\zeta)\} > 0 \quad \text{in } \Omega,$$

such that

$$\frac{\zeta f'(\zeta)}{f(\zeta)^\alpha g(\zeta)^{1-\alpha}} = p(\zeta), \quad \zeta \in \Omega.$$

Equivalently,

$$\Re\left\{\frac{\zeta f'(\zeta)}{f(\zeta)^\alpha g(\zeta)^{1-\alpha}}\right\} > 0, \quad \zeta \in \Omega.$$

Remark 1.

1. When $\alpha = 1$, this class is reduced to include the category of starlike functions.
2. When $\alpha = 0$, therefore, this class is reduced to include the category of convex functions.
3. The class of Bazilevič functions generalises several important geometric subclasses of univalent functions.

Definition 2. An analytic function f in Ω is called a Non-Bazilevič function by [27] if it does not satisfy the Bazilevič condition; that is, there does not exist $\alpha \geq 0$, a starlike function $g(\zeta)$, and a function $p(\zeta)$ with $\Re\{p(\zeta)\} > 0$ such that

$$\Re\left\{\frac{\zeta f'(\zeta)}{f(\zeta)^\alpha g(\zeta)^{1-\alpha}}\right\} > 0, \quad \text{for all } \zeta \in \Omega.$$

Definition 3. The class $\mathcal{B}_\alpha^v(\zeta)$ was studied by Daniel Breaz et al. by [28] and others as follows:

$$\mathcal{B}_\alpha^v(\zeta) = \left\{1 + \frac{1}{v} \left(\left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right) \left(\frac{f(\zeta)}{\zeta} \right)^\alpha - 1 \right) \prec \varphi(\zeta), v \in \mathbb{C} \setminus \{0\}; \zeta \in \Omega \right\}, \quad (3)$$

where $\varphi(\zeta)$ is Carathéodory functions.

Campbell and Sherigan [29,30] developed new classes of bi-univalent functions, focusing primarily on defining the limits of the differential coefficients $|a_2|$ and $|a_3|$. In a related context, Yusuf et al. [31] studied the Fekete–Szegő inequality within some special classes of functions. The Fekete–Szegő inequality, introduced in [32], is a fundamental problem in coefficient theory. The Fekete–Szegő inequality can be expressed as follows:

$$|a_3 - \zeta a_2^2| \leq 1 + 2e^{-2\zeta/(1-\zeta)}, \quad \zeta \in \mathbb{R}.$$

Based on the data and results from previous studies, we examine the function g that transforms the open unit disc Ω into a star field centered at point 1. This function is characterized by the fact that the coefficients of its series are given by what are known as Gregory coefficients, which are rational numbers that decrease in value similar to what is known in the case of Bernoulli numbers

$$\frac{1}{2}, -\frac{1}{12}, \frac{1}{24}, -\frac{19}{720}, \frac{3}{160}, -\frac{863}{60480}, \dots$$

Gregory coefficients occupy a prominent place in the fields of numerical analysis and number theory, and they have enjoyed continuous interest since their introduction by Gregory in 1671. They are considered mathematical entities that have been rediscovered repeatedly in various contexts reflecting their importance and rich structure.

These transactions have appeared in scientific literature under various names, including inverse logarithmic numbers, Cauchy numbers, and the second type of Bernoulli numbers, in addition to other names that reflect the diversity of their uses and applications. In this study, our focus is on the generative function G of the Gregory coefficients as presented in [33–37] as follows:

$$G(\zeta) = \frac{\zeta}{\log(1+\zeta)} = \sum_{n=0}^{\infty} \chi_n \zeta^n, \quad 1 + \frac{1}{2}\zeta - \frac{1}{12}\zeta^2 + \frac{1}{24}\zeta^3 + \dots, \quad (|\zeta| < 1). \quad (4)$$

Clearly, for $n = 0, \dots, 5$ the coefficients are given by

$$\chi_0 = 1, \quad \chi_1 = -\frac{1}{2}, \quad \chi_2 = \frac{1}{12}, \quad \chi_3 = -\frac{1}{24}, \quad \chi_4 = \frac{19}{720}, \quad \chi_5 = -\frac{3}{160}.$$

Figure 1 shows the analytical behavior of the Gregory coefficient image within the open unit disc.

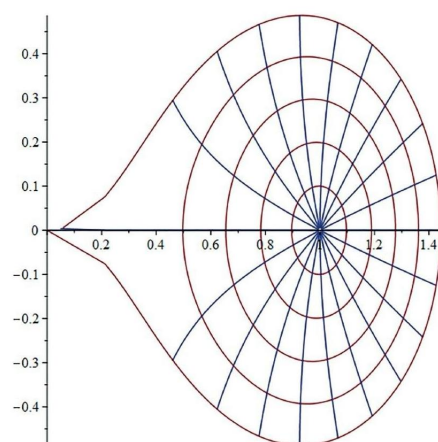


Figure 1. The image of $G(\zeta)$ is given, confined within the open unit disc Ω .

Lemma 1 ([1,38]). We assume that the function c is included in the class $\mathcal{P}$, and it is defined as (2). Then

$$|c_{n+k} - \gamma c_n c_k| \leq 2, \quad \text{if } 0 \leq \gamma \leq 1,$$

$$|c_m c_n - c_k c_l| \leq 4, \quad \text{if } m + n = k + l,$$

$$|c_{n+2k} - \gamma c_n c_k^2| \leq 2(1 + 2\gamma), \quad \text{for } \gamma \in \mathbb{R},$$

$$\left| c_2 - \frac{|c_1|^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

Motivated by the recent growing interest in Gregory generating functions, we introduce a new subclass of Bazilevič-type bi-univalent functions associated with Gregory polynomials. This class generalizes several previously studied subclasses and provides new coefficient estimates for $|a_2|$, $|a_3|$, and the Fekete–Szegő functional. To clarify the novelty and contribution of the present study, we compare our proposed class with the most closely related subclasses previously investigated in the literature. The comparison demonstrates that the current framework extends several existing Gregory-related bi-univalent classes and introduces a Bazilevič-type structure that has not been considered before. The main differences between the proposed class and the related Gregory-type bi-univalent subclasses are summarized in Table 2.

Table 2. Comparison of the proposed class with related Gregory-type bi-univalent subclasses.

Property	[35]	[36]	[37]	Present
Gregory generating function	✓	✓	✓	✓
Bi-univalent framework	✓	✓	✓	✓
Bazilevič structure	×	×	×	✓
Parameter α	×	×	×	✓
Parameter ν	×	Partial	Partial	✓
New class $S_G^*(\nu, \alpha)$	×	×	×	✓
Bound for $ a_2 $	✓	✓	✓	✓
Bound for $ a_3 $	✓	✓	✓	✓
Fekete–Szegő inequality	✓	✓	✓	✓
Contains previous classes	×	Partial	Partial	✓

Remark 2. The classes investigated in Murugusundaramoorthy et al. [35], Al-Hawary et al. [36], and Hadi et al. [37] may be recovered from the present framework under suitable restrictions on the parameters.

However, the present study introduces a new Bazilevič-type bi-univalent class associated with the Gregory generating function, namely

$$S_G^*(\nu, \alpha),$$

which depends simultaneously on the parameters ν and α .

Consequently, the obtained coefficient estimates and the corresponding Fekete–Szegő inequalities constitute genuine extensions of the previously known results rather than simple repetitions thereof.

Definition 4. The class $\mathcal{S}^*(G)$ contains the function $f(\zeta)$ in (1) if

$$\mathcal{S}^*(G) = \left\{ 1 + \frac{1}{\nu} \left(\left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right) \left(\frac{f(\zeta)}{\zeta} \right)^\alpha - 1 \right) \prec G(\zeta), \nu \in \mathbb{C} \setminus \{0\}; \zeta \in \Omega \right\}, \quad (5)$$

where $\nu \in \mathbb{C} \setminus \{0\}$ and $G(\zeta)$ is the generating of Gregory function in (4).

Remark 3.

1. When $\nu = 1$, it is concluded to be the class of Bazilevič functions.
2. When $\nu = 1$ and $\alpha = 0$, it is concluded that class of starlike functions is realized in this context.

2. The Class Coefficient Bound

In the present section, we study a subclass based on Gregory coefficients where these coefficients are generators of a class of bi-univalent analytic functions. Initial estimates of coefficients are also derived, $|a_2|$ and $|a_3|$, along with proof of the Fekete–Szegő inequality. This study relies on the following two principles.

Lemma 2 ([1]). If $\rho \in \mathcal{P}$, then $|s_\kappa| \leq 2$, $\kappa \geq 1$ for each κ , where ρ is an analytic function defined on Ω , and it is written in the following form:

$$\rho(\zeta) = 1 + s_1\zeta + s_2\zeta^2 + \cdots,$$

and the inequality is sharp for all $\kappa \in \mathbb{N}$.

Lemma 3 ([39]). Let $s, e \in \mathbb{R}$ and $\zeta_1, \zeta_2 \in \mathbb{C}$. If $|\zeta_1| < H$ and $|\zeta_2| < H$; then

$$|(s+e)\zeta_1 + (s-e)\zeta_2| \leq \begin{cases} 2|s|H, & \text{for } |s| \geq |e|, \\ 2|e|H, & \text{for } |s| \leq |e|. \end{cases}$$

Definition 5. Let $\alpha \geq 0$ and $\nu \in \mathbb{C} \setminus \{0\}$. Consider a function $f \in \Omega$ as given in (1) belongs to this category $\mathcal{S}^*(G)(\nu, \alpha)$ if the required conditions are subordinations,

$$1 + \frac{1}{\nu} \left(\left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right) \left(\frac{f(\zeta)}{\zeta} \right)^\alpha - 1 \right) \prec G(\zeta), \quad (6)$$

and

$$1 + \frac{1}{\nu} \left(\left(\frac{\omega g'(\omega)}{g(\omega)} \right) \left(\frac{g(\omega)}{\omega} \right)^\alpha - 1 \right) \prec G(\omega), \quad (7)$$

where $\nu \in \mathbb{C} \setminus \{0\}$ and $G(\zeta)$ is given by Equation (4), while $g(\omega) = f^{-1}(\omega)$.

Theorem 1. If $f \in \mathcal{S}^*(G)(\nu, \alpha)$, then

$$|a_2| \leq \min \{ W_1, W_2 \},$$

$$|a_3| \leq \min \{ M_1, M_2 \}.$$

Proof. Let $f \in \mathcal{S}^*(G)(\nu, \alpha)$. Based on the subordinations in (6) and (7), there exist two analytic functions c and b that satisfy the condition $c(0) = 0 = b(0)$, where $|c(\zeta)|, |b(\omega)| < 1$, while

$$1 + \frac{1}{\nu} \left(\left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right) \left(\frac{f(\zeta)}{\zeta} \right)^\alpha - 1 \right) = G(c(\zeta)), \quad \zeta \in \Omega \quad (8)$$

and

$$1 + \frac{1}{\nu} \left(\left(\frac{\omega g'(\omega)}{g(\omega)} \right) \left(\frac{g(\omega)}{\omega} \right)^\alpha - 1 \right) = G(b(\omega)), \quad \omega \in \Omega. \quad (9)$$

Therefore, the function

$$t(\zeta) = \frac{1 + c(\zeta)}{1 - c(\zeta)} = 1 + s_1\zeta + s_2\zeta^2 + \dots \in \mathcal{P},$$

hence

$$c(\zeta) = \frac{s_1}{2}\zeta + \frac{1}{2} \left(s_2 - \frac{s_1^2}{2} \right) \zeta^2 + \frac{1}{2} \left(s_3 - s_1s_2 + \frac{s_1^3}{4} \right) \zeta^3 + \dots,$$

and

$$G(c(\zeta)) = 1 + \frac{s_1}{4}\zeta + \frac{1}{48}(12s_2 - 7s_1^2)\zeta^2 + \frac{1}{192}(17s_1^3 - 56s_1s_2 + 48s_3)\zeta^3 + \dots, \quad \zeta \in \Omega. \quad (10)$$

Also, the function

$$l(\omega) = \frac{1 + b(\omega)}{1 - b(\omega)} = 1 + e_1\omega + e_2\omega^2 + \dots \in \mathcal{P},$$

$$b(\omega) = \frac{e_1}{2}\omega + \frac{1}{2} \left(e_2 - \frac{e_1^2}{2} \right) \omega^2 + \frac{1}{2} \left(e_3 - e_1e_2 + \frac{e_1^3}{4} \right) \omega^3 + \dots,$$

and

$$G(b(\omega)) = 1 + \frac{e_1}{4}\omega + \frac{1}{48}(12e_2 - 7e_1^2)\omega^2 + \frac{1}{192}(17e_1^3 - 56e_1e_2 + 48e_3)\omega^3 + \dots, \quad \omega \in \Omega. \quad (11)$$

Consequently, we get

$$1 + \frac{1}{\nu} \left(\left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right) \left(\frac{f(\zeta)}{\zeta} \right)^\alpha - 1 \right) = 1 + \frac{(\alpha + 1)a_2\zeta}{\nu} + \frac{((\alpha^2 + \alpha - 2)a_2^2 + 2(\alpha + 2)a_3)\zeta^2}{2\nu} + \dots, \quad (12)$$

and

$$1 + \frac{1}{\nu} \left(\left(\frac{\omega g'(\omega)}{g(\omega)} \right) \left(\frac{g(\omega)}{\omega} \right)^\alpha - 1 \right) = 1 - \frac{(\alpha + 1)a_2\omega}{\nu} + \frac{((\alpha^2 + \alpha - 2)a_2^2 + 2(\alpha + 2)[2a_2^2 - a_3])\omega^2}{2\nu} + \dots. \quad (13)$$

By comparing the coefficients of the two series given in Equations (12) and (13), we obtain the following results:

$$\frac{(\alpha + 1)a_2}{\nu} = \frac{s_1}{4}, \quad (14)$$

$$\frac{((\alpha^2 + \alpha - 2)a_2^2 + 2(\alpha + 2)a_3)}{2\nu} = \frac{1}{48}(12s_2 - 7s_1^2), \quad (15)$$

$$-\frac{(\alpha + 1)a_2}{\nu} = \frac{e_1}{4}, \quad (16)$$

and

$$\frac{((\alpha^2 + \alpha - 2)a_2^2 + 2(\alpha + 2)[2a_2^2 - a_3])}{2\nu} = \frac{1}{48}(12e_2 - 7e_1^2). \quad (17)$$

By comparing the coefficients in (14) and (16), we obtain the following relations:

$$s_1 = -e_1. \quad (18)$$

$$s_1^2 + e_1^2 = 32 \left[\frac{(\alpha + 1)^2}{\nu^2} \right] a_2^2. \quad (19)$$

If we add (15) to (17), it follows that

$$\left[\frac{(\alpha^2 + 3\alpha + 2)}{\nu} \right] a_2^2 = \frac{1}{4}(s_2 + e_2) - \frac{7}{48}(s_1^2 + e_1^2). \quad (20)$$

Substituting the value of $s_1^2 + e_1^2$ by comparing the coefficients in (19) and (20), we obtain the following relations:

$$\left\{ \frac{(\alpha + 1)(3\nu(\alpha + 2) + 14(\alpha + 1))}{3\nu^2} \right\} a_2^2 = \frac{1}{4}(s_2 + e_2). \quad (21)$$

Consequently,

$$a_2 = \sqrt{\left| \frac{3\nu^2(s_2 + e_2)}{4[(\alpha + 1)(3\nu(\alpha + 2) + 14(\alpha + 1))]} \right|}.$$

We conclude from Lemma 2 that

$$|a_2| \leq \sqrt{\left| \frac{3\nu^2}{(\alpha + 1)(3\nu(\alpha + 2) + 14(\alpha + 1))} \right|} =: W_1.$$

The other constraint for the coefficient a_2 is directly calculated by

$$|a_2| \leq \left| \frac{\nu}{2(\alpha + 1)} \right| =: W_2.$$

Thus,

$$|a_2| \leq \min\{W_1, W_2\}.$$

Subtracting (17) from (15) with $s_1^2 = e_1^2$, we find that

$$\frac{2(\alpha + 2)a_3 - 2(\alpha + 2)a_2^2}{\nu} = \frac{1}{4}(s_2 - e_2). \quad (22)$$

Solving the equation to find a_3 and using the previous quadratic terms, we obtain

$$a_3 = a_2^2 + \frac{\nu(s_2 - e_2)}{8(\alpha + 2)}. \quad (23)$$

The above equation with (19) becomes

$$a_3 = \frac{\nu^2(s_1^2 + e_1^2)}{32(\alpha + 1)^2} + \frac{\nu(s_2 - e_2)}{8(\alpha + 2)}. \quad (24)$$

Based on the last relationship, and with applying Lemma 2 combination along with using the triangle inequality, we deduce the following result:

$$|a_3| \leq \left| \frac{\nu^2}{8(\alpha + 1)^2} \right| + \left| \frac{\nu}{4(\alpha + 2)} \right| := M_1, \quad (25)$$

and using our first assertion with (21), it follows that

$$a_3 = \frac{3\nu^2(s_2 + e_2)}{4(\alpha + 1)(3\nu(\alpha + 2) + 14(\alpha + 1))} + \frac{\nu(s_2 - e_2)}{8(\alpha + 2)}. \quad (26)$$

Based on the triangle inequality and a straightforward application of the Lemma 2 formula, we achieve the following results relating to the last two equations:

$$|a_3| \leq \left| \frac{\nu^2}{(\alpha+1)(3\nu(\alpha+2)+14(\alpha+1))} + \frac{\nu}{4(\alpha+2)} \right| := M_2. \quad (27)$$

This completes the proof. $\square$

The following corollaries are obtained by assigning particular values to the involved parameters in Theorem 1.

For $\alpha = 0$, the below result can be obtained.

Corollary 1. For $f(\zeta) \in \mathcal{B}_\alpha^\nu(\zeta)$, we have

$$|a_2| \leq \min \left\{ \sqrt{\left| \frac{3\nu^2}{2(3\nu+7)} \right|}, \left| \frac{\nu}{2} \right| \right\} = \sqrt{\left| \frac{3\nu^2}{2(3\nu+7)} \right|},$$

and

$$|a_3| \leq \min \left\{ \frac{|\nu(\nu+1)|}{8}, \frac{|3\nu(\nu+1)|}{|12\nu+28|} \right\} = \frac{|3\nu(\nu+1)|}{|12\nu+28|}.$$

For $\nu = 1$, the below result can be obtained.

Corollary 2. For $f(\zeta) \in \mathcal{B}_\alpha^\nu(\zeta)$, we have

$$|a_2| \leq \min \left\{ \sqrt{\frac{3}{(\alpha+1)(17\alpha+20)}}, \frac{1}{2(\alpha+1)} \right\} = \sqrt{\frac{3}{(\alpha+1)((17\alpha+20))}},$$

and

$$\begin{aligned} |a_3| &\leq \min \left\{ \frac{1}{(\alpha+1)(17\alpha+20)} + \frac{1}{4(\alpha+2)}, \frac{1}{8(\alpha+1)^2} + \frac{1}{4(\alpha+2)} \right\} \\ &= \frac{1}{8(\alpha+1)^2} + \frac{1}{4(\alpha+2)}. \end{aligned}$$

For $\nu = 1, \alpha = 0$, the below result can be obtained.

Corollary 3. For $f(\zeta) \in \mathcal{B}_\alpha^\nu(\zeta)$, we have

$$|a_2| \leq \min \left\{ \sqrt{\frac{3}{20}}, \frac{1}{2} \right\} = \sqrt{\frac{3}{20}},$$

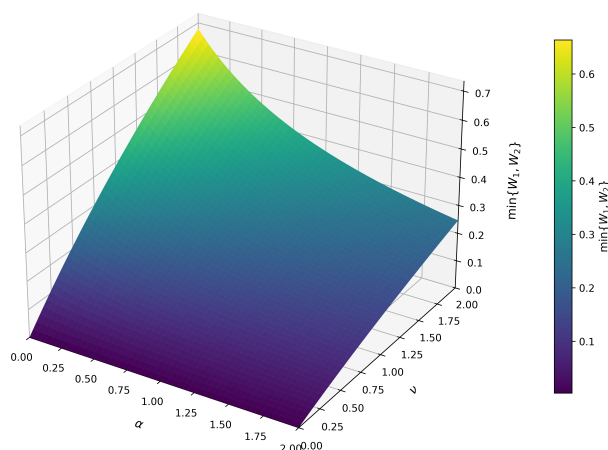
and

$$|a_3| \leq \min \left\{ \frac{1}{4}, \frac{28}{160} \right\} = \frac{28}{160}.$$

To provide a clearer interpretation of the obtained coefficient estimates, we present selected numerical values of the upper bound of $|a_2|$ for different choices of the parameters α and ν . These values illustrate how the bound varies as the parameters change, see Table 3 and Figure 2 as follows:

Table 3. Numerical values of the coefficient bounds for selected parameters.

α	ν	W_1	W_2	$\min\{W_1, W_2\}$
0	0.5	0.2722	0.2500	0.2500
0	1.0	0.3873	0.5000	0.3873
0.5	1.0	0.2955	0.3333	0.2955
1.0	1.0	0.2434	0.2500	0.2434
1.0	1.5	0.3371	0.3750	0.3371
2.0	1.0	0.1873	0.1667	0.1667

Surface behavior of the upper bound for $|a_2|$ **Figure 2.** Surface plot illustrating the behavior of the upper bound of $|a_2|$ with respect to the parameters α and ν .

3. Fekete–Szegő Inequality

Based on the previous results the Fekete–Szegő inequality associated with the Gregory function and the class of bi-univalent functions can be deduced.

Theorem 2. Let $f(\zeta) \in \mathcal{B}_\alpha^\nu(\zeta)$. If $\xi \in \mathbb{R}$, then

$$|a_3 - \xi a_2^2| \leq \frac{\nu}{4} \begin{cases} \frac{2}{(\alpha+2)}, & |1 - \xi| \leq \frac{3\nu((\alpha+1)(3\nu(\alpha+2)+14(\alpha+1))}{6\nu(\alpha+2)} \\ \frac{12|1-\xi|\nu}{3\nu((\alpha+1)(3\nu(\alpha+2)+14(\alpha+1))}, & |1 - \xi| \geq \frac{3\nu((\alpha+1)(3\nu(\alpha+2)+14(\alpha+1))}{6\nu(\alpha+2)} \end{cases}.$$

Proof. We infer from (21) and (23) that

$$\begin{aligned} a_3 - \xi a_2^2 &= \frac{\nu(s_2 - e_2)}{8(\alpha+2)} + (1 - \xi)a_2^2 \\ &= \frac{\nu(s_2 - e_2)}{8(\alpha+2)} + \frac{3\nu^2(1 - \xi)(s_2 + e_2)}{4[(\alpha+1)(3\nu(\alpha+2) + 14(\alpha+1))]} \\ &= \frac{\nu}{4} \left[\frac{3\nu(1 - \xi)(s_2 + e_2)}{(\alpha+1)(3\nu(\alpha+2) + 14(\alpha+1))} + \frac{(s_2 - e_2)}{2(\alpha+2)} \right] \\ &= \frac{\nu}{4} \left[\left(W_\nu^{\alpha;\xi} + \frac{1}{2(\alpha+2)} \right) s_2 + \left(W_\nu^{\alpha;\xi} - \frac{1}{2(\alpha+2)} \right) e_2 \right], \end{aligned}$$

where

$$W_\nu^{\alpha;\xi} := \frac{3\nu(1 - \xi)}{(\alpha+1)(3\nu(\alpha+2) + 14(\alpha+1))}.$$

Taking the module value, we get

$$|a_3 - \zeta a_2^2| \leq \frac{\nu}{4} \left| \left(W_{\nu}^{\alpha; \zeta} + \frac{1}{2(\alpha+2)} \right) + \left(W_{\nu}^{\alpha; \zeta} - \frac{1}{2(\alpha+2)} \right) \right|.$$

Employing Lemma 3 yields

$$|a_3 - \zeta a_2^2| \leq \frac{\nu}{4} \begin{cases} \frac{2}{(\alpha+2)}, & 0 \leq |W_{\nu}^{\alpha; \zeta}| \leq \frac{1}{2(\alpha+2)} \\ 4|W_{\nu}^{\alpha; \zeta}|, & |W_{\nu}^{\alpha; \zeta}| \geq \frac{1}{2(\alpha+2)} \end{cases},$$

Consequently,

$$|a_3 - \zeta a_2^2| \leq \frac{\nu}{4} \begin{cases} \frac{2}{(\alpha+2)}, & |1 - \zeta| \leq \frac{(\alpha+1)(3\nu(\alpha+2)+14(\alpha+1))}{2(\alpha+2)} \\ \frac{4|1-\zeta|}{(\alpha+1)(3\nu(\alpha+2)+14(\alpha+1))}, & |1 - \zeta| \geq \frac{(\alpha+1)(3\nu(\alpha+2)+14(\alpha+1))}{2(\alpha+2)} \end{cases},$$

which is analogous to the accurate outcome. $\square$

Remark 4. The obtained Fekete–Szegő inequality reduces to several known inequalities under suitable choices of α and ν . In particular, for $\alpha = 0$, the results reduce to the Gregory-type starlike bi-univalent case.

Corollary 4. Case 1: Taking $\nu = 1$, we have

$$|a_3 - \zeta a_2^2| \leq \frac{1}{4} \begin{cases} \frac{2}{(\alpha+2)}, & |1 - \zeta| \leq \frac{(\alpha+1)(3(\alpha+2)+14(\alpha+1))}{2(\alpha+2)} \\ \frac{4|1-\zeta|}{(\alpha+1)(3(\alpha+2)+14(\alpha+1))}, & |1 - \zeta| \geq \frac{(\alpha+1)(3(\alpha+2)+14(\alpha+1))}{2(\alpha+2)} \end{cases}.$$

Case 2: Taking $\alpha = 0$, we get

$$|a_3 - \zeta a_2^2| \leq \frac{\nu}{4} \begin{cases} 1, & |1 - \zeta| \leq \frac{(3\nu+7)}{2} \\ \frac{2|1-\zeta|}{(3\nu+7)}, & |1 - \zeta| \geq \frac{(3\nu+7)}{2} \end{cases},$$

Case 3: Taking $\nu = 1$ and $\alpha = 0$, we have

$$|a_3 - \zeta a_2^2| \leq \frac{1}{4} \begin{cases} 1, & |1 - \zeta| \leq 5 \\ \frac{|1-\zeta|}{5}, & |1 - \zeta| \geq 5 \end{cases}.$$

4. Concluding and Valuable Remark

In this paper, we introduced a new subclass of bi-univalent Bazilevič functions associated with Gregory polynomials. By employing subordination techniques and coefficient comparison methods, we derived estimates for the initial Taylor coefficients and established the corresponding Fekete–Szegő inequality. The obtained results generalize several previously known results in the literature. Future work may include the investigation of higher-order Hankel determinants, Toeplitz determinants, and extensions to harmonic and q -analogue settings.

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