

Article

Intrinsic Ultracontractivity for Markov Semigroups and Diffusion Models in Population Dynamics

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Abstract

This paper investigates the long-term behavior of diffusion models in population dynamics before absorption. By establishing the intrinsic ultracontractivity of the associated Markov semigroups, we obtain sharp quantitative estimates for three fundamental limiting objects: the quasi-stationary distribution, the quasi-ergodic distribution, and the mean quasi-ergodic distribution. Specifically, we determine their domains of attraction and prove that convergence to each occurs at an exponential rate. Moreover, we show that the conditioned Q -process is uniformly exponentially ergodic, and we characterize the mixing speed of the original process in terms of the spectral gap. These results provide a unified framework for understanding pre-absorption dynamics in population processes.

Keywords: diffusion models in population dynamics; intrinsic ultracontractivity; quasi-stationary distribution; quasi-ergodic distribution; Q -process; mixing speed

MSC: 47D07; 60J25; 92D25; 37A30

1. Introduction

1.1. Background and Motivation

Understanding how demographic stochasticity affects the long-term behavior of a density-regulated population is a central question in population dynamics. The stochastic logistic Feller diffusion model serves as a canonical framework for this inquiry. Its most profound feature is not the deterministic sigmoidal growth curve, but the way stochasticity fundamentally reshapes long-term behavior—when population size becomes sufficiently small, fluctuations in birth and death processes can no longer be neglected, and extinction is almost certain. In this regime, the population's transient dynamics are precisely characterized by the quasi-stationary distribution and the quasi-ergodic distribution [1,2], which are conditioned on non-extinction.

Mathematically, the model is a nonlinear diffusion process with an absorbing boundary at zero, where the infinitesimal generator is degenerate. This degeneracy prevents direct application of classical ergodic theory [3]. While recent advances in large deviation theory, spectral theory, and functional inequalities have significantly improved our understanding of the existence, uniqueness, and convergence rates of quasi-stationary distributions [4–7], the explicit analytic expressions for the quasi-stationary distribution, the quasi-ergodic distribution, and the stationary distribution obtained via the Doob h -transform remain scattered across the literature and lack systematic synthesis.



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Such explicit expressions are not only of theoretical interest—they provide tractable tools for applied problems, including extinction risk assessment in population ecology, the maintenance of rare alleles in evolutionary biology, and minimal residual disease dynamics in tumor growth models [8,9].

1.2. The Model

We consider the stochastic differential equation

$$dZ_t = \sqrt{Z_t}dB_t + (rZ_t - cZ_t^2)dt, \quad Z_0 > 0,$$

where $(B_t; t \geq 0)$ is a standard one-dimensional Brownian motion, the initial state Z_0 is given, and r and c are assumed to be positive. The minimal solution Z is a logistic Feller diffusion process on $E = (0, \infty)$ with a lifetime of $\delta = \inf\{t \geq 0 : Z_t = 0 \text{ or } \infty\}$.

To better investigate the properties of $\{Z_t, t \geq 0\}$, we introduce the process $\{X_t, t \geq 0\}$ defined by $X_t = 2\sqrt{Z_t}$. Then we have

$$dX_t = dB_t - q(X_t)dt, \quad q(x) = \frac{1}{2x} - \frac{rx}{2} + \frac{cx^3}{8}.$$

Process X is still absorbed at 0, and limiting properties of $\{Z_t, t \geq 0\}$ can be readily deduced from $\{X_t, t \geq 0\}$. Throughout the paper, we focus on the diffusion process X .

1.3. Literature Review

Quasi-stationary distributions and quasi-ergodic distributions describe the behavior of Markov processes conditioned on long-term survival, which is a key problem in the study of absorbed Markov processes. In recent years, many authors have made contributions to this area. On the one hand, using purely probabilistic methods, several important results on quasi-stationary distributions and quasi-ergodic distributions of Markov processes have been obtained; see [10–12]. The latest results in [11] provide necessary and sufficient conditions for exponential convergence to a unique quasi-stationary distribution in the total variation norm. These conditions also guarantee the existence and exponential ergodicity of the Q -process.

On the other hand, spectral theory of self-adjoint operators has also yielded profound results; see [13–17]. In [14], the authors prove that if P_t is an irreducible, trace class, and ultracontractive semigroup, then the unique quasi-stationary distribution attracts all initial distributions supported in the allowed states at an exponential rate. Moreover, the process admits a quasi-ergodic distribution, and its L^p -spectral bounds are independent of p for $1 \leq p \leq \infty$. In [17], the authors give the relationship between uniform convergence to a unique quasi-stationary distribution, the existence of the Q -process for a symmetric Markov process, and the intrinsic ultracontractivity of its associated semigroup.

For the diffusion process X , ref. [18] proved that a unique quasi-stationary distribution exists and that the conditional distribution of X converges uniformly exponentially to the quasi-stationary distribution. Our approach differs from that of [18], and we conduct a more in-depth study of X . In fact, our approach builds upon and extends the results of [19].

1.4. Novelty and Contribution

Compared to existing works on quasi-stationary distributions for diffusion processes, the present paper provides the following new contributions:

1. **First explicit spectral gap estimates** for the stochastic logistic Feller diffusion with absorption, providing sharp exponential convergence rates to the quasi-stationary distribution, the quasi-ergodic distribution, and the mean quasi-ergodic distribution.

2. **Unified treatment** of quasi-stationary distribution, quasi-ergodic distribution, mean quasi-ergodic distribution, and mixing speed within a single spectral framework based on intrinsic ultracontractivity.
3. **Explicit expressions** for the convergence rates in terms of the first two eigenvalues λ_1, λ_2 and eigenfunctions φ_1, φ_2 of the generator.
4. **Uniform exponential ergodicity** of the Q -process, with explicit characterization of its transition density.

These results extend and sharpen previous findings by [7,17,18] by providing quantitative convergence rates and explicit limiting objects for this specific but important class of population models.

1.5. Organization of the Paper

The remainder of this paper is organized as follows. Section 2 presents preliminary material, including notation, key definitions, and known results from spectral theory. Section 3 states all major results, including exponential convergence to the quasi-stationary distribution, the Q -process and its ergodicity, the quasi-ergodic distribution and mean quasi-ergodic distribution, and mixing speed with limit theorems. Section 4 provides a discussion of the results, including biological interpretation, comparison with the literature, limitations, and future directions. All proofs are collected in the Appendix A.

2. Preliminaries

2.1. Notation and Basic Setup

We define the speed measure m of the diffusion process X on E :

$$m(x) = \int_1^x e^{-Q(y)} dy,$$

where

$$Q(y) = 2 \int_1^y q(x) dx = \log y + \frac{r(1-y^2)}{2} + \frac{c(y^4-1)}{16}.$$

The process X is an m -symmetric diffusion process with state space $E = (0, \infty)$ and lifetime $\zeta = \inf\{t \geq 0 : X_t = 0 \text{ or } \infty\}$.

We denote by $\mathcal{P}(E)$ the set of probability measures on $(E, \mathcal{B})$. The norm in the usual Banach space $L^p(E; m)$ is written as $\|\cdot\|_{p,m}$, $1 \leq p \leq \infty$. The inner product and norm in $L^2(E; m)$ are written as $(\cdot, \cdot)_m$ and $\|\cdot\|_m$, respectively.

2.2. Markov Semigroup and Generator

Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $L^2(E; m)$ generated by $X = (X_t, \mathbb{P}_x, \zeta)$, $(-\mathcal{A}, \mathcal{D}(\mathcal{A}))$ be its generator, and $\{P_t = e^{-\mathcal{A}t}\}$ be its associated irreducible semigroup. The semigroup P_t acts on $f \in L^2(E; m)$ as

$$P_t f(x) = \mathbb{E}_x(f(X_t) \mathbb{1}_{\{t < \zeta\}}) = \int_E p(t, x, y) f(y) m(dy),$$

where $p(t, x, y)$ is the jointly continuous, strictly positive integral kernel of P_t , and $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$.

Denote by λ_1 the infimum of the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$:

$$\lambda_1 = \inf \sigma(\mathcal{A}) = \inf\{(\mathcal{A}u, u) : u \in D(\mathcal{A}), \|u\| = 1\}.$$

2.3. Spectral Properties

The following properties of the semigroup P_t are known from [19–22]:

Proposition 1. P_t is trace class for all $t > 0$.

Proposition 2. P_t is intrinsically ultracontractive for all $t > 0$.

As a consequence of these properties, the generator $\mathcal{A}$ has purely discrete spectrum with eigenvalues $\{\lambda_i\}_{i=1}^\infty$ satisfying $0 < \lambda_1 < \lambda_2 < \dots \rightarrow \infty$, and there exists a complete orthonormal basis $\{\varphi_i\}$ of $L^2(E; m)$. Each $\varphi_i \in \mathcal{D}(\mathcal{A})$ is a continuous function on E with $\mathcal{A}\varphi_i = \lambda_i\varphi_i$, $i \geq 1$, and φ_1 can be chosen to be strictly positive on E .

We recall the equivalent condition for intrinsic ultracontractivity (see [23]):

Proposition 3. If $p(t, x, y) > 0$ and P_t is trace class, then P_t is intrinsically ultracontractive if and only if for any $t > 0$ there exists $c_t \in (0, \infty)$ such that

$$p(t, x, y) \geq c_t b_t(x) b_t(y), \quad \forall x, y \in E,$$

where $b_t \in L^1(E; m) \cap L^\infty(E; m)$ and $b_t(x) = \sqrt{p(t, x, x)}$.

The following lemma summarizes key properties of the heat kernel and eigenfunctions (see [17]):

Lemma 1. The heat kernel $p(t, x, y)$ on $(0, \infty) \times E \times E$ satisfies:

1. *Eigenfunction expansion:*

$$p(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y),$$

where the series converges absolutely and uniformly for $(t, x, y) \in [\varepsilon, \infty) \times E \times E$ for any $\varepsilon > 0$.

2. *Integral representation:* For any $\varepsilon > 0$, $p \in [1, \infty]$, $t > 0$, $f \in L^p(E; m)$, and $x \in E$,

$$P_t f(x) = \sum_{i=1}^{\infty} (\varphi_i, f) \varphi_i(x) = \int_E p(t, x, y) f(y) m(dy),$$

where the series converges absolutely and uniformly for $(t, x) \in [\varepsilon, \infty) \times E$ for any $\varepsilon > 0$.

3. b_t is a continuous function in $L^2(E; m)$ and $|\varphi_i(x)| \leq e^{\lambda_i t/2} b_t(x)$ for all $i \geq 1$, $x \in E$, and $t > 0$.

4. For each $x \in E$, $b_t(x)$ and $e^{\lambda_1 t/2} b_t(x)$ are analytic, logarithmically convex, monotonically decreasing functions of t .

Lemma 2. The eigenfunctions satisfy:

1. $\varphi_1 \in L^1(E; m) \cap L^\infty(E; m)$ and there exists a decreasing function $\tilde{c}_t$ such that

$$|\varphi_i(x)| \leq \tilde{c}_t e^{(\lambda_i - \lambda_1)t} \varphi_1(x), \quad \forall i \geq 1, x \in E, t > 0.$$

2. For any $\varepsilon > 0$ there exists $t(\varepsilon)$ such that for all $t > t(\varepsilon)$,

$$1 - \varepsilon \leq \frac{e^{\lambda_1 t} p(t, x, y)}{\varphi_1(x) \varphi_1(y)} \leq 1 + \varepsilon, \quad x, y \in E,$$

and consequently,

$$(1 - \varepsilon) m(\varphi_1) \varphi_1 \leq e^{\lambda_1 t} P_t \mathbb{1} \leq (1 + \varepsilon) m(\varphi_1) \varphi_1.$$

2.4. Key Definitions

We now recall the definitions of quasi-stationary distribution and related concepts.

Definition 1. A probability measure ν on E is called a quasi-stationary distribution of the diffusion process X if

$$\mathbb{P}_\nu(X_t \in \cdot \mid \zeta > t) = \nu(\cdot), \quad t \geq 0,$$

where $\mathbb{P}_\nu(\cdot) = \int_E \mathbb{P}_x(\cdot) \nu(dx)$.

Definition 2. A probability measure ρ on E is called a quasi-limiting distribution of the diffusion process X if

$$\lim_{t \rightarrow \infty} \mathbb{P}_\rho(X_t \in F \mid \zeta > t) = \nu(F), \quad F \in \mathcal{B}(E),$$

for some probability measure ν . We then say that ρ is attracted to ν , or is in the domain of attraction of ν , under conditional evolution.

Definition 3. The Q -process is defined as the diffusion process X conditioned to never exit E . More precisely, if the following limit is well defined,

$$Q_x(A) = \lim_{t \rightarrow \infty} \mathbb{P}_x(A \mid \zeta > t), \quad A \in \mathcal{F}_s, \quad s \geq 0,$$

then $\{Q_x\}_{x \in E}$ is the law of a continuous process with transition semigroup given by Doob's h -transform:

$$P_t^{\varphi_1} f = e^{\lambda_1 t} \frac{P_t(\varphi_1 f)}{\varphi_1}, \quad t > 0, \quad f \in L^2(E; \varphi_1^2 \cdot m).$$

3. Major Results

In this section, we state the major results of the paper. All proofs are deferred to the Appendix A.

3.1. Asymptotic Behavior of the Semigroup

As established in Proposition 2, the semigroup P_t is intrinsically ultracontractive. This property, together with the trace class property, implies the following asymptotic behavior, which is crucial for studying long-term dynamics.

Lemma 3. For all $f \in L^p(E; m)$, $1 \leq p, p^* \leq \infty$ with $\frac{1}{p} + \frac{1}{p^*} = 1$, and for any $t > s > 0$, we have

$$|e^{\lambda_1 t} p(t, x, y) - \varphi_1(x) \varphi_1(y)| \leq e^{\lambda_2 s} e^{-(\lambda_2 - \lambda_1)t} b_s(x) b_s(y), \quad x, y \in E.$$

Consequently,

$$\lim_{t \rightarrow \infty} \sup_{\rho \in \mathcal{P}(E)} |\rho(e^{\lambda_1 t} P_t f(x)) - \|\varphi_1(x)\|_{1, \rho}(\varphi_1, f)| = 0.$$

3.2. Exponential Convergence to the Quasi-Stationary Distribution

Our first major result establishes the existence, uniqueness, and exponential convergence to the quasi-stationary distribution.

Theorem 1. The diffusion process X has a unique quasi-stationary distribution given by

$$\nu = \frac{\varphi_1 \cdot m}{\|\varphi_1\|_1}.$$

This quasi-stationary distribution attracts all probability measures $\rho \in \mathcal{P}(E)$ uniformly exponentially. Specifically,

$$\begin{aligned} & \lim_{t \rightarrow \infty} e^{(\lambda_2 - \lambda_1)t} \|\mathbb{P}_\rho(X_t \in \cdot \mid \zeta > t) - \nu(\cdot)\|_{TV} \\ &= \sup_{F \in \mathcal{B}(E)} \left| \frac{[(\varphi_2, \mathbb{1}_F) - (\varphi_2, \mathbb{1})\nu(F)]\rho(\varphi_2)}{\|\varphi_1\|_1 \rho(\varphi_1)} \right|. \end{aligned}$$

3.3. The Q-Process and Its Ergodicity

The next result concerns the Q-process, which represents the original process conditioned on never being absorbed.

Theorem 2. Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form and X be its associated diffusion process. Then the Q-process of X exists and coincides with X^{φ_1} . That is, for all $s \geq 0$ and $x \in E$,

$$\lim_{t \rightarrow \infty} \sup_{F \in \mathcal{F}_s} |\mathbb{P}_x(F \mid \zeta > t) - \mathbb{P}_x^{\varphi_1}(F)| = 0.$$

Moreover, X^{φ_1} admits a $\varphi_1^2 \cdot m$ -symmetric transition function $P_t^{\varphi_1}$ and a transition density function

$$q(t, x, y) = \frac{e^{\lambda_1 t} p(t, x, y)}{\varphi_1(x) \varphi_1(y)}.$$

Theorem 3. For all $s > 0$,

$$\lim_{t \rightarrow \infty} \sup_{\rho \in \mathcal{P}(E)} \|\mathbb{P}_\rho^{\varphi_1}(X_s \in \cdot) - \mathbb{P}_\rho(X_s \in \cdot \mid \zeta > t)\|_{TV} = 0,$$

and

$$\lim_{t \rightarrow \infty} \sup_{\rho_1, \rho_2 \in \mathcal{P}(E)} \|\mathbb{P}_{\rho_1}^{\varphi_1}(X_t \in \cdot) - \mathbb{P}_{\rho_2}^{\varphi_1}(X_t \in \cdot)\|_{TV} = 0.$$

3.4. Quasi-Ergodic Distribution and Mean Quasi-Ergodic Distribution

The following results describe the quasi-ergodic behavior of the process.

Theorem 4. The diffusion process X has a unique quasi-ergodic distribution given by

$$\mu = \varphi_1^2 \cdot m.$$

This quasi-ergodic distribution attracts all probability measures $\rho \in \mathcal{P}(E)$ uniformly exponentially, as follows:

$$\begin{aligned} & \lim_{t \rightarrow \infty} e^{(\lambda_2 - \lambda_1)t} \|\mathbb{P}_\rho^{\varphi_1}(X_t \in \cdot) - \mu(\cdot)\|_{TV} \\ &= \sup_{F \in \mathcal{B}(E)} |(\varphi_1 \mathbb{1}_F, \varphi_2) \rho(\varphi_2 \varphi_1^{-1})|. \end{aligned}$$

Theorem 5. Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form and X be its associated diffusion process. For all $\rho \in \mathcal{P}(E)$ and $0 < \alpha \leq 1$, the mean quasi-ergodic distribution satisfies

$$\lim_{t \rightarrow \infty} \left\| \mathbb{E}_\rho \left(\frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{1}_A(X_s) ds \mid \zeta > t \right) - \mu(A) \right\|_{TV} = 0, \quad \forall A \in \mathcal{B}(E).$$

Moreover, the convergence rate is proportional to time t : for any $t^* > 0$,

$$\begin{aligned} & \lim_{t \rightarrow \infty} \sup_{A \in \mathcal{B}(E)} \alpha t \left\| \mathbb{E}_\rho \left(\frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{1}_A(X_s) ds \mid \zeta > t \right) - \mu(A) \right\|_{TV} \\ & \leq 4t^* + \frac{[2e^{\lambda_1 t^*} \|b_{t^*}\| \|b_{t^*}\|_1 + m(\varphi_1)] e^{\lambda_1 t^*} \rho(b_{t^*}) \|b_{t^*}\|}{\rho(\varphi_1) m(\varphi_1) (\lambda_2 - \lambda_1)}. \end{aligned}$$

3.5. Mixing Speed and Limit Theorems

Finally, we characterize the mixing speed of the diffusion process and establish related limit theorems.

Theorem 6. Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form and X be its associated diffusion process. For all $\rho \in \mathcal{P}(E)$, $\nu = \varphi_1 \cdot m / \|\varphi_1\|_1$, and $\mu = \varphi_1^2 \cdot m$:

1. For any $\alpha = a \wedge (1 - a)$ with $0 < a < 1$, there exists a finite function Φ_1 such that

$$\begin{aligned} & \lim_{t \rightarrow \infty} e^{(\lambda_2 - \lambda_1)\alpha t} \sup_{A, B \in \mathcal{B}(E)} |\mathbb{E}_\rho(\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t) \mid \zeta > t) - \mu(A)\nu(B)| \\ &= \sup_{A, B \in \mathcal{B}(E)} |\Phi_1(\rho, \alpha, A, B)|. \end{aligned}$$

2. For any $\beta = a \wedge (b - a) \wedge (1 - b)$ with $0 < a < b < 1$, there exists a finite function Φ_2 such that

$$\begin{aligned} & \lim_{t \rightarrow \infty} e^{(\lambda_2 - \lambda_1)\beta t} \sup_{A, B \in \mathcal{B}(E)} |\mathbb{E}_\rho(\mathbb{1}_A(X_{at})\mathbb{1}_B(X_{bt}) \mid \zeta > t) - \mu(A)\mu(B)| \\ &= \sup_{A, B \in \mathcal{B}(E)} |\Phi_2(\rho, \beta, A, B)|. \end{aligned}$$

The explicit forms of Φ_1 and Φ_2 are given in the Appendix A.

Theorem 7. For all $\rho \in \mathcal{P}(E)$ and $0 < \alpha, \beta \leq 1$,

$$\lim_{t \rightarrow \infty} \sup_{A, B \in \mathcal{B}(E)} \left| \mathbb{E}_\rho \left(\frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{1}_A(X_s) ds \cdot \frac{1}{\beta t} \int_0^{\beta t} \mathbb{1}_B(X_r) dr \mid \zeta > t \right) - \mu(A)\mu(B) \right| = 0.$$

Theorem 8. For all $\rho \in \mathcal{P}(E)$, $0 < \alpha \leq 1$, and $\epsilon > 0$,

$$\lim_{t \rightarrow \infty} \sup_{A \in \mathcal{B}(E)} \mathbb{P}_\rho \left(\left| \frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{1}_A(X_s) ds - \mu(A) \right| > \epsilon \mid \zeta > t \right) = 0.$$

Moreover, for any $p > 0$,

$$\lim_{t \rightarrow \infty} \sup_{A \in \mathcal{B}(E)} \mathbb{E}_\rho \left(\left| \frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{1}_A(X_s) ds - \mu(A) \right|^p \mid \zeta > t \right) = 0.$$

4. Discussion

In this section, we interpret our results biologically, compare them with the existing literature, discuss limitations, and outline future directions.

4.1. Recapitulation of Main Findings

In this paper, we have developed a complete spectral theory of quasi-stationarity for the stochastic logistic Feller diffusion. Our main results are:

- Explicit exponential convergence rates to the quasi-stationary distribution, the quasi-ergodic distribution, and the mean quasi-ergodic distribution, all expressed in terms of $\lambda_1, \lambda_2, \varphi_1, \varphi_2$.
- Uniform exponential ergodicity of the Q -process, with explicit transition density.
- Sharp estimates of mixing speed, mixed limit theorems, and a weak law of large numbers for the conditioned process.

4.2. Biological Interpretation

Our results carry concrete biological interpretations. The quasi-stationary distribution ν answers the following question: *Given that the population has survived up to now, what is its current size distribution?* The exponential convergence rate $\lambda_2 - \lambda_1$ quantifies the memory time of the population; a larger gap means faster forgetting of initial conditions. The quasi-ergodic distribution μ describes the time-averaged behavior along a single long surviving trajectory, while the mean quasi-ergodic distribution corresponds to the ensemble average over all surviving paths. The distinction matters in practice; the quasi-ergodic distribution is relevant to predicting the long-term behavior of a *specific* population, whereas the mean quasi-ergodic distribution is relevant to predicting the average behavior across *multiple independent* populations.

The spectral gap $\lambda_2 - \lambda_1$ also controls the mixing speed. A small gap indicates long memory — extinction risk and population size remain correlated for extended periods. Conversely, a large gap implies rapid relaxation to the quasi-stationary regime.

4.3. Position Within the Literature

Compared to previous works, our contribution provides several new elements:

- While [7] established the existence of quasi-stationary distributions for logistic Feller diffusions, we provide explicit exponential convergence rates via spectral gap estimates.
- Unlike [18] which used probabilistic methods, our spectral approach yields explicit expressions for the convergence rates and the limiting objects in terms of φ_1, φ_2 .
- Compared to [17] which studied general symmetric Markov processes, we apply the framework to a concrete biological model and obtain sharp, interpretable results.
- To the best of our knowledge, this is the first work that simultaneously characterizes quasi-stationary distributions, quasi-ergodic distributions, mean quasi-ergodic distributions, and mixing speed for this model within a unified spectral framework.

4.4. Limitations and Priorities for Future Work

We acknowledge several limitations.

4.4.1. Robustness of Intrinsic Ultracontractivity

Our analysis relies on the intrinsic ultracontractivity of P_t , which holds for the present generator with symmetric drift. For non-symmetric drifts, singular coefficients, or non-local operators (e.g., Lévy-type generators), the spectral decomposition may fail.

4.4.2. Vanishing Spectral Gap

In metastable or highly heterogeneous environments, $\lambda_2 - \lambda_1$ may become arbitrarily small. Exponential convergence then gives way to polynomial or sub-exponential rates.

4.4.3. Non-Autonomous and Perturbed Dynamics

Time-dependent environments break the semigroup property. Whether our estimates persist under small perturbations remains open.

Among these limitations, the most pressing is numerical validation, as it would provide immediate empirical testing of our theoretical rates.

4.5. Outlook

Several promising directions remain for future work. We intend to:

- **Numerical validation:** Implement an Euler–Maruyama scheme with an absorbing boundary at zero, estimate the quasi-stationary distribution via histograms of surviving trajectories, and compare the empirical convergence rate with $\lambda_2 - \lambda_1$.

- **Concrete applications:** Apply our model to real-world systems—single-species population dynamics in ecology, continuous stirred-tank reactors in chemical engineering, and disease extinction in epidemiology.
- **Theoretical extensions:** Generalize to multi-dimensional or interacting particle systems, models with Allee effects or other nonlinearities, and quasi-stationary behavior in the presence of catastrophes or resetting events.

4.6. Concluding Remark

In summary, this paper provides a mathematically rigorous and biologically interpretable characterization of quasi-stationary and quasi-ergodic behaviors for the stochastic logistic Feller diffusion. The unified spectral framework developed here can serve as a foundation for future theoretical extensions and applied investigations.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proofs

All technical proofs are collected here for completeness.

Appendix A.1. Proof of Proposition 1

From Tomisaki ([22] Corollary 2.7), we can infer that

$$\int_E p(t, x, x) m(dx) = \sum_{i=1}^{\infty} e^{-\lambda_i t} < \infty,$$

where λ_i are the eigenvalues of $\mathcal{A}$. Therefore, $\{P_t\}$ is trace class for all $t > 0$.

Appendix A.2. Proof of Proposition 2

$\{P_t\}_{t>0}$ is proved to be intrinsically ultracontractive by Miura ([19] Theorem 3.4).

Appendix A.3. Proof of Lemma 3

Since P_t is intrinsically ultracontractive and of trace class, for any $\tau, s > 0$, we have

$$\begin{aligned} & |e^{\lambda_1(\tau+s)} p(\tau+s, x, y) - \varphi_1(x) \varphi_1(y)| \\ &= \left| \sum_{i=1}^{\infty} e^{-(\lambda_i - \lambda_1)(\tau+s)} \varphi_i(x) \varphi_i(y) \right| \\ &\leq \left(\sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)(\tau+s)} \varphi_i^2(x) \right)^{\frac{1}{2}} \left(\sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)(\tau+s)} \varphi_i^2(y) \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)s} \varphi_i^2(x) e^{-(\lambda_2 - \lambda_1)\tau} \right)^{\frac{1}{2}} \left(\sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)s} \varphi_i^2(y) e^{-(\lambda_2 - \lambda_1)\tau} \right)^{\frac{1}{2}} \\ &= e^{\lambda_1 s} e^{-(\lambda_2 - \lambda_1)\tau} b_s(x) b_s(y). \end{aligned}$$

Let $\tau = t - s$. Then for any $t > s > 0$, we obtain

$$|e^{\lambda_1 t} p(t, x, y) - \varphi_1(x) \varphi_1(y)| \leq e^{\lambda_1 s} e^{-(\lambda_2 - \lambda_1)(t-s)} b_s(x) b_s(y) = e^{\lambda_2 s} e^{-(\lambda_2 - \lambda_1)t} b_s(x) b_s(y).$$

Then, from Lemma 1,

$$|\rho(e^{\lambda_1 t} P_t f(x)) - \|\varphi_1(x)\|_{1,\rho}(\varphi_1, f)| \leq e^{\lambda_2 s} e^{-(\lambda_2 - \lambda_1)t} \|b_s\|_{p^*} \|b_s\|_{\infty} \|f\|_p.$$

For all $t > s > 0$ and $f \in L^p(E; m)$, $p \in [1, \infty]$, we have

$$\begin{aligned} & |\rho(e^{\lambda_1 t} P_t f(x)) - \|\varphi_1(x)\|_{1,\rho}(\varphi_1, f)| \\ &= |\rho(\sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)t} (\varphi_i, f) \varphi_i)| \\ &\leq \sum_{i=2}^{\infty} e^{\lambda_i s} e^{-(\lambda_i - \lambda_1)t} \|b_s\|_{p^*} \|b_s\|_{\infty} \|f\|_p \rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Appendix A.4. Proof of Theorem 1

For all $F \in \mathcal{B}(E)$ and $\rho \in \mathcal{P}(E)$, we have

$$\begin{aligned} \mathbb{P}_{\rho}(X_t \in F \mid X_t \in E) - \nu(F) &= \frac{\rho(P_t \mathbb{1}_F) - \nu(F) \rho(P_t \mathbb{1})}{\rho(P_t \mathbb{1})} \\ &= \frac{\rho(\sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)t} [(\varphi_i, \mathbb{1}_F) - (\varphi_i, \mathbb{1}) \nu(F)] \varphi_i)}{\rho(e^{\lambda_1 t} P_t \mathbb{1})}. \end{aligned}$$

Multiplying both sides by $e^{(\lambda_2 - \lambda_1)t}$, we deduce that

$$\begin{aligned} & e^{(\lambda_2 - \lambda_1)t} (\mathbb{P}_{\rho}(X_t \in F \mid X_t \in E) - \nu(F)) \\ &= \frac{\rho([(\varphi_2, \mathbb{1}_F) - (\varphi_2, \mathbb{1}) \nu(F)] \varphi_2)}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} \\ &= \frac{\rho(e^{-(\lambda_3 - \lambda_2)t} [(\varphi_3, \mathbb{1}_F) - (\varphi_3, \mathbb{1}) \nu(F)] |\varphi_3|)}{\rho(e^{\lambda_1 t} P_t \mathbb{1})}. \end{aligned}$$

Therefore, for any $0 < \varepsilon < 1$ and $0 < \epsilon < \frac{(\lambda_3 - \lambda_2)}{2(\lambda_3 - \lambda_1)}$, Lemma 2 implies that there exists $t(\varepsilon)$ such that for all $t > t(\varepsilon)$,

$$\begin{aligned} & \frac{\rho(e^{-(\lambda_3 - \lambda_2)t} [(\varphi_3, \mathbb{1}_F) - (\varphi_3, \mathbb{1}) \nu(F)] |\varphi_3|)}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} \\ &\leq \frac{2e^{-(\lambda_3 - \lambda_2)t} m(|\varphi_3|) \rho(|\varphi_3|)}{(1 - \varepsilon) m(\varphi_1) \rho(\varphi_1)} \\ &\leq \frac{2(c_{\varepsilon t}^2) e^{-(\lambda_3 - \lambda_2 - 2\epsilon(\lambda_3 - \lambda_1))t}}{1 - \varepsilon} \rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

This completes the proof.

Appendix A.5. Proof of Theorem 2

The proof follows directly from Champagnat et al. [11] and Li et al. [17], using the intrinsic ultracontractivity of P_t . The explicit form of the transition density is obtained from the definition of Doob's h -transform.

Appendix A.6. Proof of Theorem 3

This follows from Lemma 1, Lemma 2, and Fubini's theorem. For any $F \in \mathcal{B}(E)$ and $\epsilon \in (0, \frac{1}{2})$, there exists a constant $\varepsilon > 0$ such that

$$\begin{aligned} & \sup_{x \in E} |\mathbb{P}_x(X_s \in F \mid \zeta > t) - \mathbb{P}_x^{\varphi_1}(X_s \in F)| \\ & \leq \frac{e^{\lambda_1 s} \mathbb{E}_x[e^{\lambda_1(t-s)} P_{t-s} \mathbb{1}(X_s) - m(\varphi_1) \varphi_1(X_s)]}{e^{\lambda_1 t} P_t \mathbb{1}(x)} \\ & \quad + \frac{|m(\varphi_1) \varphi_1(x) - e^{\lambda_1 t} P_t \mathbb{1}_x| e^{\lambda_1 s} \mathbb{E}_x\{\varphi_1(X_s)\}}{e^{\lambda_1 t} P_t \mathbb{1}(x) \varphi_1(x)} \\ & \leq \frac{e^{\lambda_1 s} \sum_{i=2}^{\infty} e^{-(1-\epsilon)(\lambda_i - \lambda_1)(t-s)} m(|\varphi_i| \tilde{c}_{\epsilon(t-s)} \mathbb{E}_x \varphi_1(X_s))}{(1-\epsilon) m(\varphi_1) \varphi_1(x)} \\ & \quad + \frac{\sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)t} m(|\varphi_i|) |\varphi_i(x)|}{(1-\epsilon) m(\varphi_1) \varphi_1(x)} \\ & \leq \sum_{i=2}^{\infty} (\tilde{c}_{\epsilon(t-s)})^2 e^{-(1-2\epsilon)(\lambda_i - \lambda_1)(t-s)} / (1-\epsilon) \\ & \quad + \sum_{i=2}^{\infty} (\tilde{c}_{\epsilon t})^2 e^{-(1-2\epsilon)(\lambda_i - \lambda_1)t} / (1-\epsilon). \end{aligned}$$

Taking $t \rightarrow \infty$ yields the desired result.

Appendix A.7. Proof of Theorem 4

By Lemma 1, for all $\rho \in \mathcal{P}(E)$ and $F \in \mathcal{B}(E)$, we obtain

$$\begin{aligned} & \mathbb{P}_\rho^{\varphi_1}(X_t \in F) - \mu(F) = \rho(e^{\lambda_1 t} \varphi_1^{-1} P_t(\varphi_1 \mathbb{1}_F)) - \mu(F) \\ & = \rho\left(\sum_{i=2}^{\infty} e^{-(\lambda_i - \lambda_1)t} (\mathbb{1}_F \varphi_1, \varphi_i) \varphi_i \varphi_1^{-1}\right). \end{aligned}$$

Multiplying by $e^{(\lambda_2 - \lambda_1)t}$ and using Lemma 2, for any $0 < \epsilon < \frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}$, we have

$$\begin{aligned} & |e^{(\lambda_2 - \lambda_1)t} (\mathbb{P}_\rho^{\varphi_1}(X_t \in F) - \mu(F)) - (\varphi_1 \mathbb{1}_F, \varphi_2) \rho(\varphi_2 \varphi_1^{-1})| \\ & = |\rho(e^{-(\lambda_3 - \lambda_2)t} (\mathbb{1}_F \varphi_1, \varphi_3) \varphi_3 / \varphi_1)| \\ & \leq e^{-(\lambda_3 - \lambda_2)t} \|\varphi_1\| \|\varphi_3\| \|\varphi_3 \varphi_1^{-1}\|_{1,\rho} \\ & \leq e^{-(1-\epsilon)\lambda_3 - \lambda_2 + \lambda_1 \epsilon} \tilde{c}_{\epsilon t} \rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Appendix A.8. Proof of Theorem 5

For all $\rho \in \mathcal{P}(E)$ and $0 < \alpha \leq 1$, using Lemma 1, Lemma 3, and Fubini's theorem, for any $A \in \mathcal{B}(E)$, $t^* > 0$, and $t > 2t^*/\alpha$, we have

$$\begin{aligned} & \mathbb{E}_\rho\left(\frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{1}_A(X_s) ds \mid \zeta > t\right) - \mu(A) \\ & = \frac{\frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{E}_\rho\{e^{\lambda_1 s} P_s[(\mathbb{1}_A - \mu(A)) e^{\lambda_1(t-s)} P_{t-s} \mathbb{1}]\} ds}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} \\ & = \frac{\frac{1}{\alpha t} \int_0^{t^*} \rho\{e^{\lambda_1 s} P_s[(\mathbb{1}_A - \mu(A)) e^{\lambda_1(t-s)} P_{t-s} \mathbb{1}]\} ds}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} \\ & \quad + \frac{\frac{1}{\alpha t} \int_{t^*}^{\alpha t - t^*} \rho\{e^{\lambda_1 s} P_s[(\mathbb{1}_A - \mu(A)) e^{\lambda_1(t-s)} P_{t-s} \mathbb{1}]\} ds}{\rho(e^{\lambda_1 t} P_t \mathbb{1})} \\ & \quad + \frac{\frac{1}{\alpha t} \int_{\alpha t - t^*}^{\alpha t} \rho\{e^{\lambda_1 s} P_s[(\mathbb{1}_A - \mu(A)) e^{\lambda_1(t-s)} P_{t-s} \mathbb{1}]\} ds}{\rho(e^{\lambda_1 t} P_t \mathbb{1})}. \end{aligned}$$

The first and third terms (over $[0, t^*]$ and $[\alpha t - t^*, \alpha t]$) vanish as $t \rightarrow \infty$. For the middle term, using the Cauchy–Schwarz inequality and Lemma 3, we obtain the bound stated in the theorem. The detailed calculations follow the same lines as in the proof of Lemma 3.

Appendix A.9. Proof of Theorem 6

We begin by establishing a preliminary identity. For any $A, B \in \mathcal{B}(E)$ and $0 < a < b \leq 1$, due to the Markov property of X , we have

$$\begin{aligned} & \mathbb{E}_x[\mathbb{1}_A(X_{at})\mathbb{1}_B(X_{bt})\mathbb{1}_{t < \zeta}] \\ &= \mathbb{E}_x[\mathbb{E}_x(\mathbb{1}_A(X_{at})\mathbb{1}_B(X_{bt})\mathbb{1}_{t < \zeta} | \mathcal{F}_{at})] \\ &= \mathbb{E}_x\{\mathbb{1}_A(X_{at})\mathbb{E}_{X_{at}}[\mathbb{1}_B(X_{(b-a)t})(P_{t-bt}\mathbb{1})(X_{(b-a)t})]\} \\ &= P_{at}\{\mathbb{1}_A P_{(b-a)t}[\mathbb{1}_B(P_{t-bt}\mathbb{1})]\}(x), \quad t \geq 0, x \in E. \end{aligned}$$

Proof of Part (1). For all $\rho \in \mathcal{P}(E)$, by the semigroup property of P_t , (A1) with $b = 1$, and Lemma 1, for all $0 < a < 1$ and $t > 0$, we have

$$\begin{aligned} & \mathbb{E}_\rho[\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t)|\zeta > t] - \mu(A)\nu(B) \\ &= \frac{\rho(P_{at}(\mathbb{1}_A P_{t-at}\mathbb{1}_B)) - \rho(\mu(A)\nu(B)P_t\mathbb{1})}{\rho(P_t\mathbb{1})} \\ &= \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i - \lambda_1)at}([\mathbb{1}_A e^{\lambda_1(t-at)} P_{t-at}\mathbb{1}_B - \mu(A)\nu(B)e^{\lambda_1(t-at)} P_{t-at}\mathbb{1}], \varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t} P_t\mathbb{1})} \quad (\text{A1}) \\ &+ \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i - \lambda_1)(t-at)}(\mathbb{1}_B, \varphi_i)(\varphi_i\mathbb{1}_A, \varphi_1)\varphi_1\}}{\rho(e^{\lambda_1 t} P_t\mathbb{1})}. \end{aligned}$$

Using the same procedure as in the proof of Theorem 1, we conclude that

$$\begin{aligned} & \lim_{t \rightarrow \infty} e^{(\lambda_2 - \lambda_1)at} \sup_{A, B \in \mathcal{B}(E)} |\mathbb{E}_\rho(\mathbb{1}_A(X_{at})\mathbb{1}_B(X_t)|\zeta > t) - \mu(A)\nu(B)| \\ &= \begin{cases} \sup_{A, B \in \mathcal{B}(E)} \left| \frac{(\varphi_1\mathbb{1}_A, \varphi_2)\nu(B)\rho(\varphi_2)}{\rho(\varphi_1)} \right|, & a < \frac{1}{2}, \\ \sup_{A, B \in \mathcal{B}(E)} \left| \frac{(\varphi_2\mathbb{1}_A, \varphi_1)(\mathbb{1}_B, \varphi_2)}{\|\varphi_1\|_1} \right|, & a > \frac{1}{2}, \\ \sup_{A, B \in \mathcal{B}(E)} \left| \frac{(\varphi_1\mathbb{1}_A, \varphi_2)[\nu(B)\rho(\varphi_2)m(\varphi_1) + (\mathbb{1}_B, \varphi_2)\rho(\varphi_1)]}{\rho(\varphi_1)m(\varphi_1)} \right|, & a = \frac{1}{2}. \end{cases} \end{aligned}$$

□

Proof of Part (2). The proof is similar to that of part (1). For all $0 < a < b < 1$ and $t > 0$, we have

$$\begin{aligned} & \mathbb{E}_\rho[\mathbb{1}_A(X_{at})\mathbb{1}_B(X_{bt})|\zeta > t] - \mu(A)\mu(B) \\ &= \frac{\rho\{e^{\lambda_1 at} P_{at}[\mathbb{1}_A e^{(b-a)\lambda_1 t} P_{bt-at}(\mathbb{1}_B e^{(1-b)\lambda_1 t} P_{t-bt}\mathbb{1})]\}}{\rho(e^{\lambda_1 t} P_t\mathbb{1})} \\ &- \frac{\rho\{e^{\lambda_1 at} P_{at}[e^{(b-a)\lambda_1 t} P_{bt-at}(e^{(1-b)\lambda_1 t} P_{t-bt}\mathbb{1})]\}\mu(A)\mu(B)}{\rho(e^{\lambda_1 t} P_t\mathbb{1})} \\ &= \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i - \lambda_1)at}(e^{(b-a)\lambda_1 t} P_{bt-at}(\mathbb{1}_B e^{-(1-b)\lambda_1 t} P_{t-bt}\mathbb{1}), \varphi_i)\varphi_i\}}{\rho(e^{\lambda_1 t} P_t\mathbb{1})} \\ &- \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i - \lambda_1)at}(e^{(b-a)\lambda_1 t} P_{bt-at}(e^{(1-b)\lambda_1 t} P_{t-bt}\mathbb{1}), \varphi_i)\varphi_i\}\mu(A)\mu(B)}{\rho(e^{\lambda_1 t} P_t\mathbb{1})} \\ &+ \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i - \lambda_1)(b-a)t}((e^{(1-b)\lambda_1 t} P_{t-bt}\mathbb{1}, \varphi_i\mathbb{1}_B)\varphi_i, \varphi_1\mathbb{1}_A)\varphi_1\}}{\rho(e^{\lambda_1 t} P_t\mathbb{1})} \\ &+ \frac{\rho\{\sum_{i=2}^\infty e^{-(\lambda_i - \lambda_1)(1-b)t}((\mathbb{1}_B, \varphi_i)\varphi_i, \varphi_1\mathbb{1}_B)\varphi_1, \varphi_1\mathbb{1}_A)\varphi_1\}}{\rho(e^{\lambda_1 t} P_t\mathbb{1})}. \end{aligned}$$

Following the same procedure as in the proof of Theorem 1, we conclude that

$$\lim_{t \rightarrow \infty} e^{(\lambda_2 - \lambda_1)\beta t} \sup_{A, B \in \mathcal{B}(E)} |\mathbb{E}_\rho[\mathbb{1}_A(X_{at})\mathbb{1}_B(X_{bt}) | \zeta > t] - \mu(A)\mu(B)| =$$

$$\left\{ \begin{array}{ll} \sup_{A, B \in \mathcal{B}(E)} \left| \frac{(\varphi_1 \mathbb{1}_A, \varphi_2) \mu(B) \rho(\varphi_2)}{\rho(\varphi_1)} \right|, & a < (b-a) \wedge (1-b), \\ \sup_{A, B \in \mathcal{B}(E)} |(\varphi_2 \mathbb{1}_A, \varphi_1)(\varphi_1, \varphi_2 \mathbb{1}_B)|, & b-a < a \wedge (1-b), \\ \sup_{A, B \in \mathcal{B}(E)} \left| \frac{(\varphi_1, \varphi_2 \mathbb{1}_B) \mu(A) m(\varphi_2)}{m(\varphi_1)} \right|, & 1-b < a \wedge (b-a), \\ \sup_{A, B \in \mathcal{B}(E)} \left| \frac{(\varphi_1 \mathbb{1}_A, \varphi_2) \mu(B) \rho(\varphi_2)}{\rho(\varphi_1)} + (\varphi_2 \mathbb{1}_A, \varphi_1)(\varphi_1, \varphi_2 \mathbb{1}_B) \right|, & (b-a) = a < (1-b), \\ \sup_{A, B \in \mathcal{B}(E)} \left| \frac{(\varphi_1 \mathbb{1}_A, \varphi_2) \mu(B) \rho(\varphi_2)}{\rho(\varphi_1)} + \frac{(\varphi_1, \varphi_2 \mathbb{1}_B) \mu(A) m(\varphi_2)}{m(\varphi_1)} \right|, & (b-a) = (1-b) < a, \\ \sup_{A, B \in \mathcal{B}(E)} \left| (\varphi_2 \mathbb{1}_A, \varphi_1)(\varphi_1, \varphi_2 \mathbb{1}_B) + \frac{(\varphi_1, \varphi_2 \mathbb{1}_B) \mu(A) m(\varphi_2)}{m(\varphi_1)} \right|, & (1-b) = a < (b-a), \\ \sup_{A, B \in \mathcal{B}(E)} \left| \frac{(\varphi_1 \mathbb{1}_A, \varphi_2) \mu(B) \rho(\varphi_2)}{\rho(\varphi_1)} + (\varphi_2 \mathbb{1}_A, \varphi_1)(\varphi_1, \varphi_2 \mathbb{1}_B) + \frac{(\varphi_1, \varphi_2 \mathbb{1}_B) \mu(A) m(\varphi_2)}{m(\varphi_1)} \right|, & \text{otherwise.} \end{array} \right.$$

In particular, the result for two mixed parameters in Theorem 6 can be generalized to a finite number of parameters. $\square$

Appendix A.10. Proof of Theorem 7

For any $\rho \in \mathcal{P}(E)$, $A, B \in \mathcal{B}(E)$, and $0 < \alpha, \beta \leq 1$, using Fubini's theorem and a change of variables, we have

$$\begin{aligned} & \left| \mathbb{E}_\rho \left(\frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{1}_A(X_s) ds \cdot \frac{1}{\beta t} \int_0^{\beta t} \mathbb{1}_B(X_r) dr \mid \zeta > t \right) - \mu(A)\mu(B) \right| \\ & \leq \int_0^1 db \int_0^b \left| \mathbb{E}_\rho(\mathbb{1}_A(X_{a\alpha t})\mathbb{1}_B(X_{b\beta t}) \mid \zeta > t) - \mu(A)\mu(B) \right| da \\ & \quad + \int_0^1 da \int_0^a \left| \mathbb{E}_\rho(\mathbb{1}_A(X_{a\alpha t})\mathbb{1}_B(X_{b\beta t}) \mid \zeta > t) - \mu(A)\mu(B) \right| db. \end{aligned}$$

From Theorem 6 and the dominated convergence theorem, the right-hand side tends to 0 as $t \rightarrow \infty$.

Appendix A.11. Proof of Theorem 8

For all $\rho \in \mathcal{P}(E)$, $0 < \alpha \leq 1$, and $\epsilon > 0$, using Chebyshev's inequality, we obtain

$$\begin{aligned} & \lim_{t \rightarrow \infty} \sup_{A \in \mathcal{B}(E)} \mathbb{P}_\rho \left(\left| \frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{1}_A(X_s) ds - \mu(A) \right| > \epsilon \mid \zeta > t \right) \\ & \leq \lim_{t \rightarrow \infty} \sup_{A \in \mathcal{B}(E)} \mathbb{E}_\rho \left(\left| \frac{1}{\alpha t} \int_0^{\alpha t} \mathbb{1}_A(X_s) ds - \mu(A) \right|^2 \mid \zeta > t \right) / \epsilon^2. \end{aligned}$$

Combining Theorem 5 and Theorem 7 yields the first result. The L^p convergence follows from a standard truncation argument.

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