

Article

Infinitely Many Specific Fuzzy Neural Network Operators Treated as Fuzzy Positive Linear Operators

George A. Anastassiou 

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA;
ganastss@memphis.edu

Abstract

Basic fuzzy multicomposite neural network operators are interpreted as fuzzy positive linear operators, and the related general fuzzy theory is applied. These fuzzy operators are defined by the composition of seven well-known activation functions. We are acting in the univariate case on a compact interval and in the multivariate case on a box, treating continuous functions from these to the fuzzy real numbers. We study quantitatively the rate of convergence of these fuzzy neural network operators to the fuzzy unit operator. Our inequalities involve the related fuzzy moduli of continuity. We derive pointwise, uniform and L_p ($p \geq 1$) fuzzy convergences. The amazing fact here is that these fuzzy convergences are enforced by the properties of the deterministic corresponding real-valued operators. We derive a plethora of infinitely many such fuzzy positive linear operators with the above behaviors. These are based on the infinitely many multicomposite combinations of our seven fundamental defining activation functions.

Keywords: fuzzy positive linear operator; fuzzy modulus of continuity; multicomposite activation functions; overactivation in neural networks; fuzzy differentiation

MSC: 26E50; 28E10; 41A17; 41A25; 41A36; 47S40

1. Introduction

1.1. Activation in Fuzzy Neural Networks

Fuzziness in neural networks refers to integrating fuzzy logic with artificial neural networks (ANNs) to handle uncertainty, imprecision, and vague data, enhancing both interpretability and learning capabilities. These hybrid models (neuro-fuzzy systems) use membership functions instead of crisp, binary inputs to represent human-like reasoning, allowing networks to process intermediate truth states rather than strict “true or false” values.

Key Features of Fuzzy Neural Networks Fuzzy neural networks have the following attributes:

- **Fuzzy Components:** Fuzziness is implemented by defining neural weights, biases, or inputs as fuzzy sets (e.g., Low, Medium, and High) rather than precise numbers.
- **Structure:** They typically include a fuzzification layer (input processing), a rule-based hidden layer (fuzzy inference), and a defuzzification layer (turning fuzzy results into a crisp output).
- **Hybrid Strengths:** The system combines the learning, adaptation, and pattern recognition capabilities of neural networks with the human-like interpretability of fuzzy systems.



Academic Editors: Vladimir Balan
and Francisco Chiclana

Received: 30 April 2026

Revised: 6 July 2026

Accepted: 6 July 2026

Published: 13 July 2026

Copyright: © 2026 by the author.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

- **Reduced Complexity:** Some studies indicate that fuzzy neural networks can achieve high-quality results with smaller network architectures, improving efficiency.

Applications and Benefits

Fuzzy NNs are ideal for systems where precise mathematical models are difficult to define. Examples include the following:

- **Expert Systems and Control:** Used in modeling human decision-making and control systems.
- **Classification and Pattern Recognition:** Capable of handling vague data, thus enhancing accuracy.
- **Interpretable Modeling:** Unlike “black-box” neural networks, these provide insights into how decisions are made.

While some consider traditional fuzzy logic less favored than pure deep learning by the priorities of modern industry, fuzzy neural networks remain valuable for tasks requiring both high adaptability and explainable AI.

1.2. Multiactivation in Fuzzy Neural Networks

Fuzziness in multicomposite activation neural networks refers to the integration of fuzzy logic principles—which handle ambiguity and continuous, uncertain, or linguistic data—with traditional deep neural networks that use multiple, combined, or non-traditional activation functions to enhance learning, interpretability, and robustness. These networks are frequently referred to as hybrid fuzzy-neural networks or neuro-fuzzy systems.

The core concepts of fuzziness in networks are as follows:

- **Fuzzy Membership Functions:** Instead of relying on crisp (1 or 0) inputs, these networks use membership functions—such as Gaussian, S-shaped, or Z-shaped—to convert numerical data to a range, representing degrees of truth (e.g., “slow,” “medium,” and “fast”).
- **Fuzzy Rule Integration:** The hidden layers of these networks are often designed to implement fuzzy if–then rules, allowing the system to learn interpretable, human-readable logic while maintaining the adaptive power of neural networks.
- **Handling Uncertainty:** By utilizing fuzzy set membership, these models handle indistinct boundaries, noise, and overlapping classes better than conventional neural networks do, making them effective for image processing and complex control systems with ambiguous data.

The following multicomposite activation features are useful in neuro-fuzzy systems:

- **Beyond ReLU:** While standard networks use ReLU, multicomposite activation networks often combine different functions (e.g., Sigmoid, Tanh, Leaky ReLU) in different layers or within the same neuron ensemble to capture diverse nonlinear relationships.
- **Overcoming “Dead Neurons”:** The use of LeakyReLU as part of the composition helps alleviate the “dying ReLU” problem by allowing a small non-zero gradient slope for negative inputs, keeping neurons active during training.
- **Fuzzy-Guided Learning:** Fuzziness is used as a guidance mechanism, where a “fuzzy branch” provides supplementary information to multi-granular feature extraction modules, allowing the network to focus on essential features while ignoring noise.

Some of the applications and performance benefits of neuro-fuzzy networks are as follows:

- **Image Segmentation and Classification:** Fuzzy CNNs can better classify images with blurry boundaries by assigning pixels a degree of membership, reducing noise interference.
- **IoT and Fault Detection:** Fuzzy CNN models have demonstrated superior accuracy in predicting mechanical faults, even in the presence of noise, by consolidating data through fuzzification layers.
- **Faster Convergence:** Fuzzy neural networks with Ordered Fuzzy Numbers (OFN) or Type 2 fuzzy sets can lead to faster training and fewer required neurons to achieve the same accuracy as traditional networks, making them ideal for IoT devices.
- **Robust Control Systems:** These networks are used in intelligent controllers, such as those in vehicle braking systems, to manage nonlinear, uncertain dynamics.

The key components of a fuzzy system are as follows:

1. **Fuzzification Layer:** This layer converts input data to fuzzy sets (using membership functions).
2. **Fuzzy Rule Computation Layer:** This layer processes rules (AND/OR operations) using fuzzy logic.
3. **Defuzzification Layer:** This layer translates the fuzzy output back to a precise numerical (crisp) value.

These networks are generally more interpretable than deep networks, as the internal rules can be translated into linguistic “if-then” descriptions.

In Section 2, we provide the necessary basic fuzzy background; in Section 3, we describe the essentials of deterministic multicomposite neural networks. Our main convergence results, univariate and multivariate, are presented in Section 4. Our results are well described above in the abstract. Our monograph on Fuzziness [1] was a major source of guidance for the conduct of this work. Furthermore, this work is supported greatly by our other monographs [2–4]. Interesting related research can be found in [5–17].

We achieve the following:

Core Function: Our fuzzy neural network operators function as fuzzy positive linear operators, created by assembling seven foundational activation functions.

Applicability: They process continuous functions that map real intervals (univariate) or boxes (multivariate) to fuzzy real numbers.

Convergence Rates: We quantified their convergence to the “fuzzy unit” operator using inequalities derived from fuzzy moduli of continuity.

Convergence Limits: The study establishes exact boundaries for pointwise, uniform, and L_p convergence.

Key Finding: Complex fuzzy convergence behaviors are fundamentally guaranteed and controlled by the properties of their deterministic, real-valued equivalents.

Versatility: Because the seven base activation functions can be combined in nearly limitless ways, this framework yields an infinite variety of operators.

2. Fuzzy Background

We start with the following:

Definition 1 (see [18]). Let $\mu : \mathbb{R} \rightarrow [0, 1]$ with the following properties:

- (i) It is normal, i.e., $\exists x_0 \in \mathbb{R}; \mu(x_0) = 1$.
- (ii) $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in \mathbb{R}, \forall \lambda \in [0, 1]$ (μ is called a convex fuzzy subset).
- (iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall x_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon, \forall x \in V(x_0)$.

(iv) The set $\overline{\text{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\text{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

For example, $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0 .

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known [19] that for each $r \in [0, 1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we uniquely define the sum $u \oplus v$ and the product $\lambda \odot u$ as

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda[u]^r, \quad \forall r \in [0, 1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [18]).

Notice that $1 \odot u = u$; it holds true that

$$u \oplus v = v \oplus u, \quad \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$, then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually, $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall r \in [0, 1]$.

We denote $u \lesssim v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, all $r \in [0, 1]$. Define

$$D: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_+$$

as

$$D(u, v) := \sup_{r \in [0, 1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\},$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We know that D is a metric on $\mathbb{R}_{\mathcal{F}}$. It follows that $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space (see [18,20]) with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u, v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) &= |k|D(u, v), \quad \forall u, v \in \mathbb{R}_{\mathcal{F}}, \forall k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u, w) + D(v, e), \quad \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $f, g: [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$, $[a, b] \subseteq \mathbb{R}$ be fuzzy real-number-valued functions. The distance between f and g is defined by

$$D^*(f, g) := \sup_{x \in [a, b]} D(f(x), g(x)).$$

Additionally, by Σ^* , we denote the fuzzy summation.

We use the following

Definition 2. Let $f: [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy real-number-valued function. We define the (first) fuzzy modulus of continuity of f as

$$\omega_1^{(\mathcal{F})}(f, \delta) := \sup_{\substack{x, y \in [a, b] \\ |x - y| \leq \delta}} D(f(x), f(y)),$$

for any $0 < \delta \leq b - a$.

Definition 3. Let $f: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$. We say that f is fuzzy continuous at $x_0 \in [a, b]$ iff whenever $x_n \rightarrow x_0$, then $D(f(x_n), f(x_0)) \rightarrow 0$, as $n \rightarrow \infty$, $n \in \mathbb{N}$. We call f fuzzy continuous iff it is fuzzy continuous $\forall x \in [a, b]$, and we denote the space of fuzzy continuous functions by $C_{\mathcal{F}}([a, b])$.

We denote $[f]^r = [f_-^{(r)}, f_+^{(r)}]$, and we mean

$$[f(x)]^r = [f_-^{(r)}(x), f_+^{(r)}(x)], \quad \forall x \in [a, b], \text{ all } r \in [0, 1].$$

Let $f, g \in C_{\mathcal{F}}([a, b])$. Then, we say that f is fuzzy larger than g pointwise, and we denote it as $f \succsim g$ iff $f(x) \succsim g(x)$ iff $f_-^{(r)}(x) \geq g_-^{(r)}(x)$ and $f_+^{(r)}(x) \geq g_+^{(r)}(x)$, $\forall x \in [a, b]$, $\forall r \in [0, 1]$, iff $f_-^{(r)} \geq g_-^{(r)}$, $f_+^{(r)} \geq g_+^{(r)}$, $\forall r \in [0, 1]$.

Let L be a map from $C_{\mathcal{F}}([a, b])$ to itself; we call it a fuzzy linear operator iff

$$L(c_1 \odot f_1 \oplus c_2 \odot f_2) = c_1 \odot L(f_1) \oplus c_2 \odot L(f_2),$$

for any $c_1, c_2 \in \mathbb{R}$, $f_1, f_2 \in C_{\mathcal{F}}([a, b])$. We say that L is a fuzzy positive linear operator iff for $f, g \in C_{\mathcal{F}}([a, b])$ with $f \succsim g$ we obtain $L(f) \succsim L(g)$ iff $(L(f))_-^{(r)} \geq (L(g))_-^{(r)}$ and $(L(f))_+^{(r)} \geq (L(g))_+^{(r)}$ on $[a, b]$ for all $r \in [0, 1]$.

From [1], p. 101, we have the following:

Lemma 1. Let $f \in C_{\mathcal{F}}([a, b])$, $[a, b] \subseteq \mathbb{R}$. Then it holds true that

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup_{r \in [0, 1]} \max\{\omega_1(f_-^{(r)}, \delta), \omega_1(f_+^{(r)}, \delta)\},$$

for any $0 < \delta \leq b - a$, where ω_1 is the basic usual first modulus of continuity.

Note 1. For $f \in C_{\mathcal{F}}([a, b])$ we can show that f is fuzzy bounded and $\omega_1^{(\mathcal{F})}(f, \delta)$ is finite for all $0 < \delta \leq b - a$. Furthermore, $f_{\pm}^{(r)}$ are continuous on $[a, b]$, and $\omega_1(f_{\pm}^{(r)}, \delta)$ are finite as well, all $r \in [0, 1]$.

We need the following over $\mathbb{R}$.

Definition 4. The function $f: \mathbb{R} \rightarrow \mathbb{R}$ is 2π -periodic if $f(x) = f(x + 2\pi)$, $\forall x \in \mathbb{R}$; $\tilde{o} = \chi_{\{0\}}$.

Definition 5. Let $f, g \in C_{\mathcal{F}}(\mathbb{R})$. We say that f is fuzzy larger than g pointwise and denote it as $f \succsim g$ iff $f(x) \succsim g(x)$ iff $f_-^{(r)}(x) \geq g_-^{(r)}(x)$ and $f_+^{(r)}(x) \geq g_+^{(r)}(x)$, $\forall x \in \mathbb{R}$, $\forall r \in [0, 1]$, iff $f_-^{(r)} \geq g_-^{(r)}$, $f_+^{(r)} \geq g_+^{(r)}$, $\forall r \in [0, 1]$.

Let L be a map from $C_{\mathcal{F}}(\mathbb{R})$ to itself. We call it a fuzzy linear operator iff

$$L(c_1 \odot f_1 \oplus c_2 \odot f_2) = c_1 \odot L(f_1) \oplus c_2 \odot L(f_2),$$

for any $c_1, c_2 \in \mathbb{R}$, $f_1, f_2 \in C_{\mathcal{F}}(\mathbb{R})$. We say that L is a fuzzy positive linear operator iff for $f, g \in C_{\mathcal{F}}(\mathbb{R})$ with $f \lesssim g$ we obtain $L(f) \lesssim L(g)$, iff $(Lf)^{(r)}_- \geq (Lg)^{(r)}_-$ and $(Lf)^{(r)}_+ \geq (Lg)^{(r)}_+$ on $\mathbb{R}$ for all $r \in [0, 1]$.

From [1], p. 107, we have the following:

Lemma 2. Let $f: \mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$ be a 2π -periodic and fuzzy continuous function, i.e., $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Then, for all $\delta \in [0, \pi]$, we have

$$\omega_1^{(\mathcal{F})}(f|_{[0, 2\pi]}, \delta) \leq \omega_1^{(\mathcal{F})}(f, \delta) \leq 2\omega_1^{(\mathcal{F})}(f|_{[0, 2\pi]}, \delta).$$

From [1], p. 108, we have the following:

Lemma 3. Let $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$; then, f is fuzzy bounded and fuzzy uniformly continuous, i.e., $C_{2\pi}^{(\mathcal{F})}(\mathbb{R}) = {}_{2\pi}C_{\mathcal{F}}^U(\mathbb{R})$, the space of fuzzy uniformly continuous 2π -periodic functions.

From [1], p. 108, we have the following:

Proposition 1. Let $f: \mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy real-number-valued function. Assume that $\omega_1^{(\mathcal{F})}(f, \delta)$, $\omega_1(f_-^{(r)}, \delta)$, $\omega_1(f_+^{(r)}, \delta)$ are finite for $\delta > 0$, where ω_1 is the usual real modulus of continuity. Then it holds true that

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup_{r \in [0, 1]} \max\{\omega_1(f_-^{(r)}, \delta), \omega_1(f_+^{(r)}, \delta)\}.$$

Next we move to a more general setting. We need the following:

Definition 6. Let $f, g: U \rightarrow \mathbb{R}_{\mathcal{F}}$, $U \subseteq (M, d)$ metric space, be fuzzy real-number-valued functions. The distance between f and g is defined by

$$D^*(f, g) := \sup_{x \in U} D(f(x), g(x)).$$

Denote $[f]^r = [f_-^{(r)}, f_+^{(r)}]$ for $[f(x)]^r = [f_-^{(r)}(x), f_+^{(r)}(x)]$, $\forall x \in U$, $r \in [0, 1]$.

Definition 7. Let $L: C_{\mathcal{F}}(U) \hookrightarrow C_{\mathcal{F}}(U)$, where U is open or compact $\subseteq (M, d)$ metric space, such that

$$L(c_1 \odot f \oplus c_2 \odot g) = c_1 \odot L(f) \oplus c_2 \odot L(g), \quad \forall c_1, c_2 \in \mathbb{R}.$$

We call L a fuzzy linear operator.

Definition 8. Let $f, g: U \rightarrow \mathbb{R}_{\mathcal{F}}$, $U \subseteq (M, d)$ metric space. We denote $f \lesssim g$ iff $f(x) \lesssim g(x)$, $\forall x \in U$, iff $f_+^{(r)}(x) \geq g_+^{(r)}(x)$ and $f_-^{(r)}(x) \geq g_-^{(r)}(x)$, $\forall x \in U$, $\forall r \in [0, 1]$, iff $f_+^{(r)} \geq g_+^{(r)}$ and $f_-^{(r)} \geq g_-^{(r)}$, $\forall r \in [0, 1]$.

Definition 9. Let $L: C_{\mathcal{F}}(U) \hookrightarrow C_{\mathcal{F}}(U)$ be a fuzzy linear operator, U open or compact $\subseteq (M, d)$ metric space. We say that L is positive iff whenever $f, g \in C_{\mathcal{F}}(U)$ are such that $f \lesssim g$, then $L(f) \lesssim L(g)$, iff

$$(L(f))_+^{(r)} \geq (L(g))_+^{(r)}$$

and

$$(L(f))_-^{(r)} \geq (L(g))_-^{(r)}, \quad \forall r \in [0, 1].$$

Here we denote

$$[L(f)]^r = [(L(f))_-^{(r)}, (L(f))_+^{(r)}], \quad \forall r \in [0, 1].$$

Definition 10. Let $U \subseteq (M, d)$ a metric space, and let $f: U \rightarrow \mathbb{R}_{\mathcal{F}}$. We define the (first) fuzzy modulus of continuity of f as

$$\omega_1^{(\mathcal{F})}(f; \delta) := \sup_{\substack{x, y \in U \\ d(x, y) \leq \delta}} D(f(x), f(y))$$

for $0 < \delta \leq \text{diameter}(U)$. If $\delta > \text{diam}(U)$, then we define

$$\omega_1^{(\mathcal{F})}(f; \delta) := \omega_1^{(\mathcal{F})}(f; \text{diam}(U)).$$

Proposition 2 (see [1], p. 160). Let $U \subseteq (M, d)$ metric space and $f: U \rightarrow \mathbb{R}_{\mathcal{F}}$. Assume that

$$\omega_1^{(\mathcal{F})}(f, \delta), \quad \omega_1(f_-^{(r)}, \delta), \quad \omega_1(f_+^{(r)}, \delta)$$

are finite for any $\delta > 0$. Here ω_1 is the usual real modulus of continuity, i.e., for $g: U \rightarrow \mathbb{R}$, we define

$$\omega_1(g; \delta) := \sup_{\substack{x, y \in U \\ d(x, y) \leq \delta}} |g(x) - g(y)|,$$

etc. Then,

$$\omega_1^{(\mathcal{F})}(f; \delta) = \sup_{r \in [0, 1]} \max\{\omega_1(f_-^{(r)}, \delta), \omega_1(f_+^{(r)}, \delta)\}.$$

We need the following:

Definition 11. Let U open or compact $\subseteq (M, d)$ metric space and $f: U \rightarrow \mathbb{R}_{\mathcal{F}}$. We say that f is fuzzy continuous at $x_0 \in U$ iff whenever $x_n \rightarrow x_0$, then $D(f(x_n), f(x_0)) \rightarrow 0$. If f is continuous for every $x_0 \in U$, we then call f a fuzzy continuous real-number-valued function. We denote the related space as $C_{\mathcal{F}}(U)$.

Definition 12. Let $f: K \rightarrow \mathbb{R}_{\mathcal{F}}$, K open or compact $\subseteq (M, d)$ metric space. We call f a fuzzy uniformly continuous real-number-valued function iff $\forall \varepsilon > 0, \exists \delta > 0$: whenever $d(x, y) \leq \delta$, $x, y \in K$, implying that $D(f(x), f(y)) \leq \varepsilon$. We denote the related space as $C_{\mathcal{F}}^U(K)$.

Definition 13. Let $f: U \rightarrow \mathbb{R}_{\mathcal{F}}$, $U \subseteq (M, d)$ metric space. If $D(f(x), \tilde{0}) \leq M, \forall x \in U, M \geq 0$, we call f a fuzzy bounded real-number-valued function.

In particular, if $f \in C_{\mathcal{F}}([a, b])$, $[a, b] \subseteq \mathbb{R}$, then f is a fuzzy bounded function; additionally, $\omega_1^{(\mathcal{F})}(f; \delta) < \infty$ for any $0 < \delta \leq b - a$, etc. Furthermore, notice that $C_{\mathcal{F}}^U(K) = C_{\mathcal{F}}(K)$, for K compact $\subseteq (V, \|\cdot\|)$ real normed vector space.

We now use the following:

Proposition 3 (see [1], p. 161). Let $K \subseteq (V, \|\cdot\|)$ a real normed vector space and

$$\omega_1^{(\mathcal{F})}(f; \delta) = \sup_{\substack{x, y \in K \\ \|x - y\| \leq \delta}} D(f(x), f(y)), \quad \delta > 0,$$

the fuzzy modulus of continuity for $f: K \rightarrow \mathbb{R}_{\mathcal{F}}$. Then, the reasoning process is as follows:

(1) If $f \in C_{\mathcal{F}}^U(K)$, K open convex or compact convex $\subseteq (V, \|\cdot\|)$, then $\omega_1^{(\mathcal{F})}(f; \delta) < \infty, \forall \delta > 0$.

(2) Assume that K is open convex or compact convex $\subseteq (V, \|\cdot\|)$; then, $\omega_1^{(\mathcal{F})}(f; \delta)$ is continuous on $\mathbb{R}_+$ in δ for $f \in C_{\mathcal{F}}^U(K)$.

(3) Assume that K is convex; then,

$$\omega_1^{(\mathcal{F})}(f, t_1 + t_2) \leq \omega_1^{(\mathcal{F})}(f, t_1) + \omega_1^{(\mathcal{F})}(f, t_2), \quad t_1, t_2 \geq 0,$$

that is, the subadditivity property is true. Furthermore, it holds true that

$$\omega_1^{(\mathcal{F})}(f, n\delta) \leq n\omega_1^{(\mathcal{F})}(f, \delta),$$

and

$$\omega_1^{(\mathcal{F})}(f, \lambda\delta) \leq \lceil \lambda \rceil \omega_1^{(\mathcal{F})}(f, \delta) \leq (\lambda + 1)\omega_1^{(\mathcal{F})}(f, \delta),$$

where $n \in \mathbb{N}$, $\lambda > 0$, $\delta > 0$, $\lceil \cdot \rceil$ is the ceiling of the number.

(4) Clearly, in general, $\omega_1^{(\mathcal{F})}(f; \delta) \geq 0$ and is increasing in $\delta > 0$ and $\omega_1^{(\mathcal{F})}(f; 0) = 0$.

(5) Let K be open or compact $\subseteq (V, \|\cdot\|)$. Then, $\omega_1^{(\mathcal{F})}(f; \delta) \rightarrow 0$ as $\delta \downarrow 0$ iff $f \in C_{\mathcal{F}}^U(K)$.

(6) It holds true that

$$\omega_1^{(\mathcal{F})}(f \oplus g; \delta) \leq \omega_1^{(\mathcal{F})}(f; \delta) + \omega_1^{(\mathcal{F})}(g; \delta),$$

for $\delta > 0$, any $f, g: K \rightarrow \mathbb{R}_{\mathcal{F}}$, $K \subseteq (V, \|\cdot\|)$ is arbitrary.

We also need the following:

Lemma 4 (see [1], p. 161). Let K be a compact subset of the real normed vector space $(V, \|\cdot\|)$ and $f \in C_{\mathcal{F}}(K)$. Then, f is a fuzzy bounded function.

We use the following:

Proposition 4 (see [1], p. 162). Let U open or compact $\subseteq (M, d)$ metric space, $f \in C_{\mathcal{F}}(U)$. Then, $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0, 1]$ over U , respectively, in $\pm$.

For the reverse, we have the following:

Proposition 5 (see [1], p. 162). Let $f_{-}^{(r)}, f_{+}^{(r)}$ be equicontinuous with respect to $r \in [0, 1]$ on U -open or compact $\subseteq (M, d)$ -metric space, respectively, in $\pm$, then $f \in C_{\mathcal{F}}(U)$.

We mention the following:

Definition 14 (see [18]). Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists a $z \in \mathbb{R}_{\mathcal{F}}$, such that $x = y + z$, then, we call z the H -difference of x and y , denoted by $z := x - y$.

Definition 15 (see [18]). Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f: T \rightarrow \mathbb{R}_{\mathcal{F}}$ is differentiable at $x \in T$ if there exists $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \rightarrow 0^+} \frac{f(x) - f(x-h)}{h},$$

exist and are equal to $f'(x)$. We call f' the derivative of f at x . If f is differentiable at any $x \in T$, we call f differentiable, and it has a derivative over T , the function f' .

Similarly, we define higher-order fuzzy derivatives. Regarding functions of several variables, one can define them the same way as partial derivatives in the fuzzy sense.

Let Q be a compact convex subset of $\mathbb{R}^k$, $k > 1$ and $n \in \mathbb{N}$. By $C_{\mathcal{F}}^n(Q)$ we mean all the functions from Q into $\mathbb{R}_{\mathcal{F}}$ that are n times continuously differentiable in the fuzzy sense.

We use the following:

Terminology 1 (see [21]). Let $f: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$ be fuzzy differentiable. Let $t \in [a, b]$, $0 \leq r \leq 1$. Clearly,

$$[f(t)]^r = [(f(t))_{-}^{(r)}, (f(t))_{+}^{(r)}] \subseteq \mathbb{R}.$$

Then, $(f(t))_{\pm}^{(r)}$ are differentiable and

$$[f^r]' = [((f(t))_{-}^{(r)})', ((f(t))_{+}^{(r)})'],$$

i.e.,

$$(f')_{\pm}^{(r)} = (f_{\pm}^{(r)})', \text{ for any } r \in [0, 1].$$

We make the following remark:

Remark 1. (1) Let $f \in C_{\mathcal{F}}^n([a, b])$. Then, by Theorem 1 and Proposition 4, we obtain $f_{\pm}^{(r)} \in C^n([a, b])$ and

$$[f^{(i)}(t)]^r = [((f(t))_{-}^{(r)})^{(i)}, ((f(t))_{+}^{(r)})^{(i)}],$$

for $i = 0, 1, 2, \dots, n$, and, in particular, we can infer that

$$(f^{(i)})_{\pm}^{(r)} = (f_{\pm}^{(r)})^{(i)},$$

for any $r \in [0, 1]$.

(2) Let $f \in C_{\mathcal{F}}^n(Q)$ denote $f_{\alpha} := \frac{\partial^{\alpha} f}{\partial x^{\alpha}}$, where $\alpha := (\alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, k$ and $0 < |\alpha| := \sum_{i=1}^k \alpha_i \leq n$, $n > 1$. Then, by Theorem 1, we can infer that

$$(f_{\pm}^{(r)})_{\alpha} = (f_{\alpha})_{\pm}^{(r)},$$

for any $r \in [0, 1]$ and any $\alpha: |\alpha| \leq n$. Here $f_{\pm}^{(r)} \in C^n(Q)$.

3. Deterministic Multicomposite Neural Networks Essentials

Definition 16 ([4], p. 530). The general sigmoid activation functions we deal with here are described as follows: $h: \mathbb{R} \rightarrow [-1, 1]$, strictly increasing $h(0) = 0$, $h(-x) = -h(x)$, $x \in \mathbb{R}$, $h(+\infty) = 1$, $h(-\infty) = -1$. Furthermore, h is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

More precisely, we will deal with the following sigmoid activation functions, which possess all of the above properties: the algebraic activation function,

$$g_1(x) = \frac{x}{\sqrt[2m]{1 + x^{2m}}}, \quad m \in \mathbb{N}, x \in \mathbb{R}, \quad (1)$$

(see [3], pp. 2–3);

The normalized and parametrized arctangent

$$g_2(x) = \frac{2}{\pi} \arctan\left(\frac{x}{2} \lambda x\right) = \frac{2}{\pi} \int_0^{\frac{\pi \lambda x}{2}} \frac{dz}{1 + z^2}, \quad x \in \mathbb{R}, \lambda > 0, \quad (2)$$

(see [4], pp. 156–157);

The normalized and parametrized Gudermannian function

$$\begin{aligned} g_3(x) &= \frac{2}{\pi} gd(\lambda x) = \frac{4}{\pi} \arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right) \\ &= \frac{2}{\pi} \int_0^{\lambda x} \frac{dt}{\cosh t} = \frac{4}{\pi} \int_0^{\lambda x} \frac{dt}{e^t + e^{-t}}, \quad x \in \mathbb{R}, \lambda > 0, \end{aligned} \quad (3)$$

where

$$gd(x) := \int_0^x \frac{dt}{\cosh t} = 2 \arctan\left(\tanh\left(\frac{x}{2}\right)\right), \quad \forall x \in \mathbb{R},$$

is the Gudermannian function (see [4], pp. 198–199);

The parametrized error function

$$g_4(x) = erf \lambda x = \frac{2}{\pi} \int_0^{\lambda x} e^{-t^2} dt, \quad x \in \mathbb{R}, \lambda > 0, \quad (4)$$

(see [4], pp. 226–227);

The parametrized hyperbolic tangent function

$$\begin{aligned} g_5(x) &= \tanh \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}} = \frac{e^{2\lambda x} - 1}{e^{2\lambda x} + 1} \\ &= \frac{1 - e^{-2\lambda x}}{1 + e^{-2\lambda x}}, \quad x > 0, \lambda > 0, \end{aligned} \quad (5)$$

(see [4], p. 72);

For small $0 < \lambda < 1$, $\tanh \lambda x$ is expected to behave better than *ReLU* and *Leaky ReLU* activation functions);

The parametrized hyperbolic tangent-like activation function,

$$g_6(x) = \frac{A^{\lambda x} - A^{-\lambda x}}{A^{\lambda x} + A^{-\lambda x}}, \quad A > 1, \lambda > 0, x \in \mathbb{R} \quad (6)$$

(see [4], pp. 262–263);

And finally, the β -parametrized half-hyperbolic tangent function

$$g_7(x) = \frac{1 - e^{-\beta x}}{1 + e^{-\beta x}}, \quad \beta > 0, \forall x \in \mathbb{R}, \quad (7)$$

(see [4], pp. 509–510).

We denote them as $g_i(x)$, $x \in \mathbb{R}$, $i = 1, 2, \dots, 7$.

Here we deal with the specific sequences of activation functions $(h_i^{(q)})$, $q \in \mathbb{N}$, $i \in \mathbb{N}$, such that all $h_i^{(q)} \in G := \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$. That is, $h_i^{(q)}$ for different $i \in \mathbb{N}$ could be identical, $q \in \mathbb{N}$.

Therefore, in the sequences of activation functions $h_1^{(q)}, h_2^{(q)}, \dots$, we allow random repetitions. Thus, any two random $h_i^{(q)}$ activation functions are not necessarily different, but all belong to G . Call $H^{(q)} = \{h_1^{(q)}, h_2^{(q)}, \dots\}$, $q \in \mathbb{N}$, and by $\Omega := \{H^{(q)}, \text{ all } q \in \mathbb{N}\}$. From here, for simplicity, we can consider such a sequence $\{h_1, h_2, \dots\}$.

Notice here that $0 < h_i(1) \leq 1$, $i = 1, 2, \dots$. Any composition $h_1 \circ h_2 \circ h_3 \circ \dots \circ h_\lambda$ is meant to be $h_1|_{[-1,1]} \circ h_2|_{[-1,1]} \circ h_3|_{[-1,1]} \circ \dots \circ h_{\lambda-1}|_{[-1,1]} \circ h_\lambda$, $\lambda \in \mathbb{N}$, and for convenience, we denote it as $G_\lambda := h_1 \circ h_2 \circ h_3 \circ \dots \circ h_\lambda$. We have for any $\lambda \in \mathbb{N}$: $0 < h_\lambda(1) \leq 1$, hence $0 < h_{\lambda-1}(h_\lambda(1)) \leq h_{\lambda-1}(1) \leq 1$, and $0 < h_{\lambda-2}(h_{\lambda-1}(h_\lambda(1))) \leq h_{\lambda-2}(h_{\lambda-1}(1)) \leq h_{\lambda-2}(1) \leq 1$.

Inductively, we derive that $0 < G_\lambda(1) \leq 1$, $\forall \lambda \in \mathbb{N}$.

Clearly, it is $G_\lambda(0) = 0$, and G_λ is strictly increasing over $\mathbb{R}$. Furthermore, it holds true that

$$\begin{aligned} G_\lambda(-x) &= h_1(h_2(h_3(\dots(h_{\lambda-1}(h_\lambda(-x))))) = h_1(h_2(h_3(\dots(h_{\lambda-1}(-h_\lambda(x))))) \\ &= \dots = -h_1(h_2(h_3(\dots(h_{\lambda-1}(h_\lambda(x))))) = -G_\lambda(x), \quad x \in \mathbb{R}. \end{aligned}$$

Clearly, it holds true that $G_\lambda^{(2)} \in C(\mathbb{R})$.

We notice that

$$\begin{aligned} G_\lambda(+\infty) &= h_1(h_2(h_3(\dots(h_{\lambda-1}(h_\lambda(+\infty))))) = \\ &= h_1(h_2(h_3(\dots(h_{\lambda-1}(1))))) = G_{\lambda-1}(1), \end{aligned} \quad (8)$$

and

$$\begin{aligned} G_\lambda(-\infty) &= h_1(h_2(h_3(\dots(h_{\lambda-1}(h_\lambda(-\infty))))) = h_1(h_2(h_3(\dots(h_{\lambda-1}(-1))))) \\ &= -h_1(h_2(h_3(\dots(h_{\lambda-1}(1))))) = -G_{\lambda-1}(1). \end{aligned} \quad (9)$$

Consequently, it holds true that

$$-G_{\lambda-1}(1) \leq G_\lambda(x) \leq G_{\lambda-1}(1), \quad \forall x \in \mathbb{R}. \quad (10)$$

Thus, $y = \pm G_{\lambda-1}(1)$ are asymptotes of $G_\lambda(x)$, any $\lambda \in \mathbb{N}$.

Next, we act over $(-\infty, 0]$: let $\lambda, \mu \geq 0 : \lambda + \mu = 1$. Then, by the convexity of h_2 there, we have

$$h_2(\lambda x + \mu y) \leq \lambda h_2(x) + \mu h_2(y), \quad \forall x, y \in (-\infty, 0];$$

and

$$\begin{aligned} h_1(h_2(\lambda x + \mu y)) &\leq h_1(\lambda h_2(x) + \mu h_2(y)) \leq \\ &\leq \lambda(h_1 \circ h_2)(x) + \mu(h_1 \circ h_2)(y), \quad \forall x, y \in (-\infty, 0]. \end{aligned} \quad (11)$$

Thus, $h_1 \circ h_2$ is convex over $(-\infty, 0]$.

Now we work on $[0, +\infty)$: let $\lambda, \mu \geq 0 : \lambda + \mu = 1$. Then, by the concavity of h_2 there, we have

$$h_2(\lambda x + \mu y) \geq \lambda h_2(x) + \mu h_2(y), \quad \forall x, y \in [0, +\infty);$$

and

$$\begin{aligned} h_1(h_2(\lambda x + \mu y)) &\geq h_1(\lambda h_2(x) + \mu h_2(y)) \geq \\ &\geq \lambda(h_1 \circ h_2)(x) + \mu(h_1 \circ h_2)(y), \quad \forall x, y \in [0, +\infty). \end{aligned} \quad (12)$$

Thus, $h_1 \circ h_2$ is concave over $[0, +\infty)$.

Therefore, $G_2 = h_1 \circ h_2$ is a general sigmoid activation function with asymptotes $y = \pm h_1(1)$ and fulfilling the rest of conditions of Definition 16.

Arguing as above, $h_2 \circ h_3 : \mathbb{R} \rightarrow [-1, 1]$ fulfills Definition 16, and $h_1 \circ h_2 \circ h_3$ does the same with asymptotes $y = \pm h_1(h_2(1))$.

Inductively, we prove that G_λ fulfills Definition 16 with asymptotes $y = \pm G_{\lambda-1}(1)$.

We have established the following:

Theorem 1. Let $q, \lambda \in \mathbb{N}$. Then, $G_\lambda^{(q)} := h_1^{(q)} \circ h_2^{(q)} \circ h_3^{(q)} \circ \dots \circ h_\lambda^{(q)}$ fulfill all the properties of Definition 16 with asymptotes $y = \pm G_{\lambda-1}^{(q)}(1)$. That is, $G_\lambda^{(q)}$ are multicomposite general sigmoid activation functions from $\mathbb{R} \rightarrow [-1, 1]$.

Corollary 1. $\frac{G_\lambda^{(q)}}{G_{\lambda-1}^{(q)}(1)}$ fulfill Definition 16 with asymptotes $y = \pm 1$.

We call

$$\tilde{G}_\lambda^{(q)} := \frac{G_\lambda^{(q)}}{G_{\lambda-1}^{(q)}(1)}. \quad (13)$$

Remark 2. Next, we consider the functions

$$T_\lambda^{(q)}(x) := \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(x+1) - \tilde{G}_\lambda^{(q)}(x-1) \right) > 0, \quad \forall x \in \mathbb{R}, q, \lambda \in \mathbb{N}. \quad (14)$$

We observe that

$$\begin{aligned} T_\lambda^{(q)}(-x) &= \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(-x+1) - \tilde{G}_\lambda^{(q)}(-x-1) \right) = \\ &= \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(-(x-1)) - \tilde{G}_\lambda^{(q)}(-(x+1)) \right) = \frac{1}{4} \left(-\tilde{G}_\lambda^{(q)}(x-1) + \tilde{G}_\lambda^{(q)}(x+1) \right) = \\ &= \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(x+1) - \tilde{G}_\lambda^{(q)}(x-1) \right) = T_\lambda^{(q)}(x). \end{aligned} \quad (15)$$

That is, $T_\lambda^{(q)}$ are even functions,

$$T_\lambda^{(q)}(-x) = T_\lambda^{(q)}(x), \quad \forall x \in \mathbb{R}, q, \lambda \in \mathbb{N}. \quad (16)$$

We see that

$$T_\lambda^{(q)}(0) = \frac{\tilde{G}_\lambda^{(q)}(1)}{2}. \quad (17)$$

If we let $x > 1$, we can infer that

$$T_\lambda^{(q)'}(x) = \frac{1}{4} \left(\tilde{G}_\lambda^{(q)'}(x+1) - \tilde{G}_\lambda^{(q)'}(x-1) \right) < 0,$$

by $\tilde{G}_\lambda^{(q)'}(x)$ being strictly decreasing over $[0, +\infty)$.

Now, let $0 < x < 1$; then, $1-x > 0$ and $0 < 1-x < 1+x$. It holds true that $\tilde{G}_\lambda^{(q)'}(x-1) = \tilde{G}_\lambda^{(q)'}(1-x) > \tilde{G}_\lambda^{(q)'}(x+1)$, so that again, $T_\lambda^{(q)'}(x) < 0$. Consequently, $T_\lambda^{(q)}$ is strictly decreasing on $(0, +\infty)$.

Clearly, $T_\lambda^{(q)}$ is strictly increasing on $(-\infty, 0)$, and $T_\lambda^{(q)'}(0) = 0$.

Observe that

$$\lim_{x \rightarrow +\infty} T_\lambda^{(q)}(x) = \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(+\infty) - \tilde{G}_\lambda^{(q)}(+\infty) \right) = 0, \quad (18)$$

and

$$\lim_{x \rightarrow -\infty} T_\lambda^{(q)}(x) = \frac{1}{4} \left(\tilde{G}_\lambda^{(q)}(-\infty) - \tilde{G}_\lambda^{(q)}(-\infty) \right) = 0, \quad q, \lambda \in \mathbb{N}. \quad (19)$$

That is, the x -axis is the horizontal asymptote on $T_\lambda^{(q)}$.

As a result, $T_\lambda^{(q)}$ is a bell-shaped symmetric function with maximum

$$T_\lambda^{(q)}(0) = \frac{\tilde{G}_\lambda^{(q)}(1)}{2}. \quad (20)$$

We need the following:

Theorem 2. It holds true that

$$\sum_{i=-\infty}^{\infty} T_{\lambda}^{(q)}(x-i) = 1, \quad \forall x \in \mathbb{R}; \quad q, \lambda \in \mathbb{N}. \quad (21)$$

Proof. Being similar to [4], p. 286, the proof is omitted. $\square$

Theorem 3. We know that

$$\int_{-\infty}^{\infty} T_{\lambda}^{(q)}(x) dx = 1, \quad q, \lambda \in \mathbb{N}. \quad (22)$$

Proof. Being similar to [4], p. 287, it is omitted. $\square$

Thus, $T_{\lambda}^{(q)}(x)$ can serve as a density function in general.
We need the following:

Theorem 4. Let $0 < \alpha < 1$ and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then,

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} T_{\lambda}^{(q)}(nx-k) < \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2)\right), \quad (23)$$

and

$$\lim_{n \rightarrow +\infty} \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2)\right) = 0. \quad (24)$$

Proof. Having been presented in [4], pp. 531–532, the proof is omitted. $\square$

Denote as $\lfloor \cdot \rfloor$ the integral part of the number, and denote as $\lceil \cdot \rceil$ the ceiling of the number.

We also need the following:

Theorem 5. Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds true that

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx-k)} < \frac{1}{T_{\lambda}^{(q)}(1)}, \quad \forall x \in [a, b]; \quad q, \lambda \in \mathbb{N}. \quad (25)$$

Proof. Being similar to [4], p. 289, the proof is omitted. $\square$

Remark 3. We know that

$$\lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx-k) \neq 1; \quad q, \lambda \in \mathbb{N}, \quad (26)$$

for at least some $x \in [a, b]$.

See [4], p. 290, with the same reasoning.

Note 2. For large enough n , we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Furthermore, $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general, it holds true (by (21)) that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx-k) \leq 1. \quad (27)$$

We make the following remark:

Remark 4. We define

$$\widehat{Z}_\lambda^{(q)}(x_1, \dots, x_N) := \widehat{Z}_\lambda^{(q)}(x) := \prod_{i=1}^N T_\lambda^{(q)}(x_i), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N, \quad q, \lambda, N \in \mathbb{N}. \quad (28)$$

It has the following properties:

(i)

$$\widehat{Z}_\lambda^{(q)}(x) > 0, \quad \forall x \in \mathbb{R}^N, \quad (29)$$

(ii)

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \widehat{Z}_\lambda^{(q)}(x - k) &:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \widehat{Z}_\lambda^{(q)}(x_1 - k_1, \dots, x_N - k_N) = \\ &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \prod_{i=1}^N T_\lambda^{(q)}(x_i - k_i) = \prod_{i=1}^N \left(\sum_{k_i=-\infty}^{\infty} T_\lambda^{(q)}(x_i - k_i) \right) \stackrel{(21)}{=} 1. \end{aligned}$$

Hence,

$$\sum_{k=-\infty}^{\infty} \widehat{Z}_\lambda^{(q)}(x - k) = 1, \quad q, \lambda \in \mathbb{N}. \quad (30)$$

That is,

(iii)

$$\sum_{k=-\infty}^{\infty} \widehat{Z}_\lambda^{(q)}(nx - k) = 1, \quad \forall x \in \mathbb{R}^N; \quad q, \lambda, n \in \mathbb{N}. \quad (31)$$

Furthermore,

(iv)

$$\int_{\mathbb{R}^N} \widehat{Z}_\lambda^{(q)}(x) dx = \int_{\mathbb{R}^N} \left(\prod_{i=1}^N T_\lambda^{(q)}(x_i) \right) dx_1 \dots dx_N = \prod_{i=1}^N \left(\int_{-\infty}^{\infty} T_\lambda^{(q)}(x_i) dx_i \right) \stackrel{(22)}{=} 1; \quad (32)$$

thus,

$$\int_{\mathbb{R}^N} \widehat{Z}_\lambda^{(q)}(x) dx = 1, \quad (33)$$

that is, $\widehat{Z}_\lambda^{(q)}$ is a multivariate density function, $q, \lambda \in \mathbb{N}$.

Here, denote $x = (x_1, \dots, x_N)$, $\|x\|_\infty := \max\{|x_1|, \dots, |x_N|\}$, $x \in \mathbb{R}^N$; also set $\infty := (\infty, \dots, \infty)$, $-\infty := (-\infty, \dots, -\infty)$ upon the multivariate context, $0 < \beta < 1$,

(v) We have

$$\begin{aligned} \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \widehat{Z}_\lambda^{(q)}(nx - k) &= \sum_{k_1=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \left(\prod_{i=1}^N T_\lambda^{(q)}(nx_i - k_i) \right) = \\ &= \prod_{i=1}^N \left(\sum_{\substack{k_i=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} T_\lambda^{(q)}(nx_i - k_i) \right) \leq \quad (\text{for some } r_* \in \{1, \dots, N\}) \end{aligned}$$

$$\left(\prod_{i=1, i \neq r_*}^N \left(\sum_{k_i=-\infty}^{\infty} T_{\lambda}^{(q)}(nx_i - k_i) \right) \right) \left(\sum_{\left\{ \begin{array}{l} k_{r_*} = -\infty \\ \left| \frac{k_{r_*}}{n} - x_{r_*} \right| > \frac{1}{n^{\beta}} \end{array} \right\}} T_{\lambda}^{(q)}(nx_{r_*} - k_{r_*}) \right) = \quad (34)$$

$$\sum_{\left\{ \begin{array}{l} k_{r_*} = -\infty \\ \left| \frac{k_{r_*}}{n} - x_{r_*} \right| > \frac{1}{n^{\beta}} \end{array} \right\}} T_{\lambda}^{(q)}(nx_r - k_r) = \sum_{\left\{ \begin{array}{l} k_{r_*} = -\infty \\ |nx_{r_*} - k_{r_*}| > n^{1-\beta} \end{array} \right\}} T_{\lambda}^{(q)}(nx_{r_*} - k_{r_*}) \stackrel{(23)}{<} \\ 1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\beta} - 2).$$

That is,

$$\sum_{\left\{ \begin{array}{l} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{array} \right\}} \hat{Z}_{\lambda}^{(q)}(nx - k) < 1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\beta} - 2), \text{ as } n \rightarrow +\infty, \quad (35)$$

$$0 < \beta < 1, q, \lambda, n \in \mathbb{N} : n^{1-\beta} > 2, \forall x \in \prod_{i=1}^N [a_i, b_i].$$

4. Main Results Regarding Fuzziness

4.1. Univariate Results

We need to establish the following definition:

Definition 17. Let $f \in C([a, b])$ and $q, \lambda, n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce the multicomposite linear neural network operator

$${}^{(q)}_{\lambda} A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) T_{\lambda}^{(q)}(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k)}, \quad x \in [a, b]. \quad (36)$$

Notice that ${}^{(q)}_{\lambda} A_n(1) = 1$, for all $q, \lambda \in \mathbb{N}$, and it is a positive linear operator.

Here, clearly, ${}^{(q)}_{\lambda} A_n(f, x) \in C([a, b])$.

In this work, $0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$; where $\|\cdot\|_{\infty}$ is the supremum norm.

Definition 18. Let $f \in C_{\mathcal{F}}([a, b])$ (fuzzy continuous functions on $[a, b] \subset \mathbb{R}$), $q, \lambda, n \in \mathbb{N}$. We define the following Fuzzy Multicomposite Quasi-Interpolation Neural Network operator:

$${}^{(q)}_{\lambda} A_n^{\mathcal{F}}(f, x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot \frac{T_{\lambda}^{(q)}(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k)}, \quad (37)$$

$\forall x \in [a, b]$; see also (36).

By [4], Ch. 23, p. 589, we know that

$$\left({}^{(q)}_{\lambda} A_n^{\mathcal{F}}(f, x) \right)_{\pm}^{(r)} = {}^{(q)}_{\lambda} A_n(f_{\pm}^{(r)}, x), \quad (38)$$

respectively, $\forall r \in [0, 1], \forall x \in [a, b]$, all $q, \lambda \in \mathbb{N}$.

Therefore, we obtain

$$D\left({}^{(q)}A_n^{\mathcal{F}}(f, x), f(x)\right) = \sup_{r \in [0,1]} \max \left\{ \left| {}^{(q)}A_n\left(f_{-}^{(r)}, x\right) - f_{-}^{(r)}(x) \right|, \left| {}^{(q)}A_n\left(f_{+}^{(r)}, x\right) - f_{+}^{(r)}(x) \right| \right\}, \quad (39)$$

$\forall x \in [a, b]; q, \lambda \in \mathbb{N}$.

Here, clearly, $\left({}^{(q)}A_n^{\mathcal{F}}\right)_{n \in \mathbb{N}}$ is a fuzzy positive linear operator from $C_{\mathcal{F}}([a, b])$ into itself, $[a, b] \subseteq \mathbb{R}$.

By Theorem 7.7, ref. [1], p. 102, we have the following:

Theorem 6. Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then,

$$D^*\left({}^{(q)}A_n^{\mathcal{F}}f, f\right) \leq 2\omega_1^{(\mathcal{F})}(f, \delta_n), \quad (40)$$

where

$$\delta_n = \left(\left\| \left({}^{(q)}A_n\left((t-x)^2\right)\right)(x) \right\|_{\infty} \right)^{\frac{1}{2}}, \quad \forall f \in C_{\mathcal{F}}([a, b]). \quad (41)$$

In particular, we obtain

$$D^*\left({}^{(q)}A_n^{\mathcal{F}}f, f\right) \leq 2\omega_1^{(\mathcal{F})}\left(f, \frac{1}{\sqrt{T_{\lambda}^{(q)}(1)}} \sqrt{\frac{1}{n^{2\alpha}} + (b-a)^2 \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2)\right)}\right). \quad (42)$$

Here, clearly, we can derive ${}^{(q)}A_n^{\mathcal{F}}f \xrightarrow{D^*} f, \forall f \in C_{\mathcal{F}}([a, b])$, that is, ${}^{(q)}A_n^{\mathcal{F}} \rightarrow I$, the unit operator uniformly in the fuzzy sense, as $n \rightarrow +\infty$.

Proof. We estimate

$$\begin{aligned} {}^{(q)}A_n\left((t-x)^2\right)(x) &= \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\frac{k}{n} - x\right)^2 T_{\lambda}^{(q)}(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx - k)} \leq \\ &= \frac{1}{T_{\lambda}^{(q)}(1)} \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\frac{k}{n} - x\right)^2 T_{\lambda}^{(q)}(nx - k) \right) = \\ &= \frac{1}{T_{\lambda}^{(q)}(1)} \left[\sum_{\substack{k=\lceil na \rceil \\ : \left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\alpha}}} }^{\lfloor nb \rfloor} \left(\frac{k}{n} - x\right)^2 T_{\lambda}^{(q)}(nx - k) + \sum_{\substack{k=\lceil na \rceil \\ : \left|\frac{k}{n} - x\right| > \frac{1}{n^{\alpha}}} }^{\lfloor nb \rfloor} \left(\frac{k}{n} - x\right)^2 T_{\lambda}^{(q)}(nx - k) \right] \leq \\ &= \frac{1}{T_{\lambda}^{(q)}(1)} \left\{ \frac{1}{n^{2\alpha}} + (b-a)^2 \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2)\right) \right\}. \end{aligned} \quad (43)$$

Consequently, it holds that

$$\delta_n \leq \frac{1}{\sqrt{T_{\lambda}^{(q)}(1)}} \sqrt{\frac{1}{n^{2\alpha}} + (b-a)^2 \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2)\right)}, \quad (44)$$

proving (42); see also [1], p. 101. $\square$

Next, we apply the fuzzy periodic Korovkin-type inequality; see [1], p. 111, Theorem 8.12 there.

Theorem 7. Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then,

$$D^*\left({}^{(q)}A_n^{\mathcal{F}}f, f\right) \leq 2\omega_1^{(\mathcal{F})}(f, \varepsilon_n), \quad (45)$$

where

$$\varepsilon_n := \pi \left\| {}^{(q)}A_n \left(\sin^2 \left(\frac{t-x}{2} \right) \right) (x) \right\|_{\infty}^{\frac{1}{2}}, \quad \forall f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R}), \quad (46)$$

where $D^*, {}^{(q)}A_n^{\mathcal{F}}, {}^{(q)}A_n, \|\cdot\|_{\infty}$ are over $[a, b] \subset \mathbb{R}$.

In particular, we obtain

$$D^*\left({}^{(q)}A_n^{\mathcal{F}}f, f\right) \leq 2\omega_1^{(\mathcal{F})} \left(f, \frac{\pi}{\sqrt{T_{\lambda}^{(q)}(1)}} \sqrt{\omega_1 \left(\sin^2, \frac{1}{2n^{\alpha}} \right) + \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right)} \right). \quad (47)$$

Here, clearly, we derive ${}^{(q)}A_n^{\mathcal{F}}f \xrightarrow{D^*} f$, $\forall f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ on $[a, b] \subset \mathbb{R}$, that is, ${}^{(q)}A_n^{\mathcal{F}} \rightarrow I$, uniformly in the fuzzy sense over $[a, b]$, as $n \rightarrow +\infty$.

Proof. We estimate ($x \in [a, b]$)

$$\begin{aligned} \varepsilon_n &= \pi \left\| \left({}^{(q)}A_n \left(\sin^2 \left(\frac{t-x}{2} \right) \right) \right) (x) \right\|_{\infty}^{\frac{1}{2}} = \\ &\pi \left\| \left(\frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\sin^2 \left(\frac{\frac{k}{n}-x}{2} \right) \right) T_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx-k)} \right) \right\|_{\infty}^{\frac{1}{2}} \leq \\ &\frac{\pi}{\sqrt{T_{\lambda}^{(q)}(1)}} \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\sin^2 \left(\frac{\frac{k}{n}-x}{2} \right) \right) T_{\lambda}^{(q)}(nx-k) \right\|_{\infty}^{\frac{1}{2}} = \\ &\frac{\pi}{\sqrt{T_{\lambda}^{(q)}(1)}} \left\| \sum_{\substack{k=\lceil na \rceil \\ : \left| \frac{k}{n}-x \right| \leq \frac{1}{n^{\alpha}}} }^{\lfloor nb \rfloor} \left(\sin^2 \left(\frac{\frac{k}{n}-x}{2} \right) - \sin^2(0) \right) T_{\lambda}^{(q)}(nx-k) + \right. \\ &\quad \left. \sum_{\substack{k=\lceil na \rceil \\ : \left| \frac{k}{n}-x \right| > \frac{1}{n^{\alpha}}} }^{\lfloor nb \rfloor} \sin^2 \left(\frac{\frac{k}{n}-x}{2} \right) T_{\lambda}^{(q)}(nx-k) \right\|_{\infty}^{\frac{1}{2}} \leq \\ &\frac{\pi}{\sqrt{T_{\lambda}^{(q)}(1)}} \left\| \sum_{\substack{k=\lceil na \rceil \\ : \left| \frac{k}{n}-x \right| \leq \frac{1}{n^{\alpha}}} }^{\lfloor nb \rfloor} \omega_1 \left(\sin^2, \left| \frac{\frac{k}{n}-x}{2} \right| \right) T_{\lambda}^{(q)}(nx-k) + 1 \cdot \sum_{\substack{k=\lceil na \rceil \\ : \left| \frac{k}{n}-x \right| > \frac{1}{n^{\alpha}}} }^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx-k) \right\|_{\infty}^{\frac{1}{2}} \leq \\ &\frac{\pi}{\sqrt{T_{\lambda}^{(q)}(1)}} \sqrt{\omega_1 \left(\sin^2, \frac{1}{2n^{\alpha}} \right) + \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right)}. \end{aligned} \quad (48)$$

We have proven that

$$\varepsilon_n \leq \frac{\pi}{\sqrt{T_\lambda^{(q)}(1)}} \sqrt{\omega_1\left(\sin^2, \frac{1}{2n^\alpha}\right) + \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right)}. \quad (49)$$

□

Next, we give a fuzzy differentiation approximation result. It comes from [1], pp. 172–173 (Theorem 11.32 there).

Theorem 8. Here we restrict ${}^{(q)}_A_n^{\mathcal{F}}$ on the subspace $C_{\mathcal{F}}^N([a, b])$, and ${}^{(q)}_A_n$ is restricted to $C^N([a, b])$, where $n, N \in \mathbb{N}$. Let $f \in C_{\mathcal{F}}^N([a, b])$, such that $\omega_1^{(\mathcal{F})}(f^{(N)}, h) > 0$ for any $h > 0$. Then,

(1)

$$D\left({}^{(q)}_A_n^{\mathcal{F}} f, f\right)(x) \leq \sum_{k=1}^N \frac{\left|{}^{(q)}_A_n\left((\cdot - x)^k\right)(x)\right|}{k!} D\left(f^{(k)}(x), \tilde{o}\right) + \quad (50)$$

$$\frac{\left({}^{(q)}_A_n\left((\cdot - x)^{N+1}\right)(x)\right)^{\frac{N}{N+1}}}{N!} \left(1 + \frac{1}{(N+1)}\right) \omega_1^{(\mathcal{F})}\left(f^{(N)}, \left({}^{(q)}_A_n\left(|\cdot - x|^{N+1}\right)(x)\right)^{\frac{1}{N+1}}\right),$$

$\forall x \in [a, b]$,

and

(2)

$$D^*\left({}^{(q)}_A_n^{\mathcal{F}} f, f\right) \leq \sum_{k=1}^N \frac{\left\|{}^{(q)}_A_n\left((\cdot - x)^k\right)(x)\right\|_\infty}{k!} D^*\left(f^{(k)}, \tilde{o}\right) + \quad (51)$$

$$\frac{\left\|{}^{(q)}_A_n\left((\cdot - x)^{N+1}\right)(x)\right\|_\infty^{\frac{N}{N+1}}}{N!} \left(1 + \frac{1}{(N+1)}\right) \omega_1^{(\mathcal{F})}\left(f^{(N)}, \left\|{}^{(q)}_A_n\left(|\cdot - x|^{N+1}\right)(x)\right\|_\infty^{\frac{1}{N+1}}\right).$$

From this, a related result follows:

Theorem 9. Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. The rest are as in Theorem 8. Then,

(1)

$$D^*\left({}^{(q)}_A_n^{\mathcal{F}} f, f\right) \leq \frac{1}{T_\lambda^{(q)}(1)} \left(\sum_{k=1}^N \frac{\left[\frac{1}{n^{k\alpha}} + (b-a)^k \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right)\right]}{k!} \right) D^*\left(f^{(k)}, \tilde{o}\right) +$$

$$\frac{\left[\frac{1}{n^{(N+1)\alpha}} + (b-a)^{N+1} \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right)\right]^{\frac{N}{N+1}}}{\left(T_\lambda^{(q)}(1)\right)^{\frac{N}{N+1}} N!} \left(1 + \frac{1}{(N+1)}\right) \quad (52)$$

$$\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{\left(T_\lambda^{(q)}(1)\right)^{\frac{1}{N+1}}} \left[\frac{1}{n^{(N+1)\alpha}} + (b-a)^{N+1} \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2)\right)\right]^{\frac{1}{N+1}}\right).$$

As $n \rightarrow +\infty$, we obtain ${}^{(q)}_A_n^{\mathcal{F}} f \xrightarrow{D^*} f$, $\forall f \in C_{\mathcal{F}}^N([a, b])$. That is, ${}^{(q)}_A_n^{\mathcal{F}} \rightarrow I$, as $n \rightarrow +\infty$, fuzzy and uniformly, where I is the fuzzy unit operator.

(2) Furthermore, for particular $x \in [a, b]$ we have $f^{(k)}(x) = \tilde{o}$, $k = 1, \dots, N$. Thus,

$$D^* \left({}^{(q)}_A f, f \right) \leq \frac{\left[\frac{1}{n^{(N+1)\alpha}} + (b-a)^{N+1} \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2) \right) \right]^{\frac{N}{N+1}}}{\left(T_\lambda^{(q)}(1) \right)^{\frac{N}{N+1}} N!} \left(1 + \frac{1}{(N+1)} \right) \quad (53)$$

$$\omega_1^{(\mathcal{F})} \left(f^{(N)}, \frac{1}{\left(T_\lambda^{(q)}(1) \right)^{\frac{1}{N+1}}} \left[\frac{1}{n^{(N+1)\alpha}} + (b-a)^{N+1} \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2) \right) \right]^{\frac{1}{N+1}} \right).$$

Here, clearly, $\left({}^{(q)}_A f \right)(x) \xrightarrow{D} f(x)$, as $n \rightarrow +\infty$, at a much higher rate given fuzzy differentiability.

Proof. By Theorem 8, if we let $m \in \mathbb{N}$, we need to estimate

$$\begin{aligned} \left({}^{(q)}_A (|t-x|^m) \right)(x) &= \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left| \frac{k}{n} - x \right|^m T_\lambda^{(q)}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_\lambda^{(q)}(nx-k)} \leq \\ &= \frac{1}{T_\lambda^{(q)}(1)} \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left| \frac{k}{n} - x \right|^m T_\lambda^{(q)}(nx-k) \right) = \\ &= \frac{1}{T_\lambda^{(q)}(1)} \left[\sum_{\substack{k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left| \frac{k}{n} - x \right|^m T_\lambda^{(q)}(nx-k) + \sum_{\substack{k=\lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left| \frac{k}{n} - x \right|^m T_\lambda^{(q)}(nx-k) \right] \leq \quad (54) \\ &= \frac{1}{T_\lambda^{(q)}(1)} \left[\frac{1}{n^{m\alpha}} + (b-a)^m \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2) \right) \right]. \end{aligned}$$

Thus, it holds true that

$$\left\| \left({}^{(q)}_A (|\cdot - x|^m) \right)(x) \right\|_\infty \leq \frac{1}{T_\lambda^{(q)}(1)} \left[\frac{1}{n^{m\alpha}} + (b-a)^m \left(1 - \tilde{G}_\lambda^{(q)}(n^{1-\alpha} - 2) \right) \right], \quad (55)$$

$m \in \mathbb{N}$. $\square$

4.2. Multivariate Results

Here, we work on $\prod_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$, $d \in \mathbb{N}$. We denote

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil, \dots, \lceil na_d \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor, \dots, \lfloor nb_d \rfloor), \quad n \in \mathbb{N}. \end{aligned}$$

Definition 19. Let $f \in C\left(\prod_{i=1}^d [a_i, b_i]\right)$ and $q, \lambda, n \in \mathbb{N} : \lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \dots, d \in \mathbb{N}$ (also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, \dots, d$). We introduce the multicomposite multivariate linear neural network operator $(x := (x_1, \dots, x_d) \in \prod_{i=1}^d [a_i, b_i])$:

$${}^{(q)}_B(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \hat{Z}_\lambda^{(q)}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{Z}_\lambda^{(q)}(nx-k)} = \quad (56)$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_d=\lceil na_d \rceil}^{\lfloor nb_d \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_d}{n}\right) \left(\prod_{i=1}^d T_{\lambda}^{(q)}(nx_i - k_i)\right)}{\prod_{i=1}^d \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} T_{\lambda}^{(q)}(nx_i - k_i)\right)}.$$

Notice that ${}^{(q)}_{\lambda} B_n(1) = 1$, for all $q, \lambda \in \mathbb{N}$, and it is a positive linear operator.

Here, clearly, ${}^{(q)}_{\lambda} B_n(f, x) \in C\left(\prod_{i=1}^d [a_i, b_i]\right)$.

Again, here, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$.

Definition 20. Let $f \in C_{\mathcal{F}}\left(\prod_{i=1}^d [a_i, b_i]\right)$ (fuzzy continuous functions on $\prod_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$), $q, \lambda, n \in \mathbb{N}$. We define the following Fuzzy Multicomposite Multivariate Quasi-Interpolation linear Neural Network operator:

$$\begin{aligned} {}^{(q)}_{\lambda} B_n^{\mathcal{F}}(f, x) &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \odot \frac{\widehat{Z}_{\lambda}^{(q)}(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widehat{Z}_{\lambda}^{(q)}(nx - k)} = \\ &= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_d=\lceil na_d \rceil}^{\lfloor nb_d \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_d}{n}\right) \odot \left(\prod_{i=1}^d T_{\lambda}^{(q)}(nx_i - k_i)\right)}{\prod_{i=1}^d \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} T_{\lambda}^{(q)}(nx_i - k_i)\right)}, \end{aligned} \quad (57)$$

$\forall x \in \prod_{i=1}^d [a_i, b_i]$; see also (56).

As in [4], p. 620, we derive that

$$\left({}^{(q)}_{\lambda} B_n^{\mathcal{F}}(f, x)\right)_{\pm}^{(r)} = {}^{(q)}_{\lambda} B_n\left(f_{\pm}^{(r)}, x\right), \quad (58)$$

$\forall r \in [0, 1], \forall x \in \prod_{i=1}^d [a_i, b_i]$.

Therefore, we obtain

$$\begin{aligned} D\left({}^{(q)}_{\lambda} B_n^{\mathcal{F}}(f, x), f(x)\right) &= \\ \sup_{r \in [0, 1]} \max \left\{ \left| {}^{(q)}_{\lambda} B_n\left(f_{-}^{(r)}, x\right) - f_{-}^{(r)}(x) \right|, \left| {}^{(q)}_{\lambda} B_n\left(f_{+}^{(r)}, x\right) - f_{+}^{(r)}(x) \right| \right\}, \end{aligned} \quad (59)$$

$\forall x \in \prod_{i=1}^d [a_i, b_i]; q, \lambda \in \mathbb{N}$.

Here, clearly, $\left({}^{(q)}_{\lambda} B_n^{\mathcal{F}}\right)_{n \in \mathbb{N}}$ is a fuzzy positive linear operator from $C_{\mathcal{F}}\left(\prod_{i=1}^d [a_i, b_i]\right)$ to itself, $\prod_{i=1}^d [a_i, b_i] \subseteq \mathbb{R}^d$.

Some notation must be established for $f \in C_{\mathcal{F}}^N\left(\prod_{i=1}^d [a_i, b_i]\right)$, $N, d \in \mathbb{N}$. Denote $f_{\alpha} = \frac{\partial^{\alpha} f}{\partial x^{\alpha}}$, the partials, where $\alpha := (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, d$, and $0 < |\alpha| := \sum_{i=1}^d \alpha_i \leq N$.

Next we apply [1], p. 183, Theorem 11.43 there.

Theorem 10. Here, $f \in C_{\mathcal{F}}^N\left(\prod_{i=1}^d [a_i, b_i]\right)$, $N \in \mathbb{N}$, and $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then,

(1)

$$\begin{aligned}
& D\left(\left({}^{(q)}B_n^{\mathcal{F}}f\right)(x), f(x)\right) \leq \\
& \sum_{j=1}^N \left\{ \sum_{|\alpha|=j} \left[\frac{\left| \left({}^{(q)}B_n\left(\prod_{i=1}^d (z_i - x_i)^{\alpha_i}\right)\right)(x) \right|}{\prod_{i=1}^d \alpha_i!} D(f_\alpha(x), \tilde{o}) \right] \right\} + \\
& \frac{\left({}^{(q)}B_n\left(\|z - x\|_{l_1}^{N+1}\right)(x)\right)^{\frac{N}{N+1}}}{N!} \left[1 + \frac{1}{(N+1)}\right] \times \\
& \max_{|\alpha|=N} \omega_1^{(\mathcal{F})}\left(f_\alpha, \left({}^{(q)}B_n\left(\|z - x\|_{l_1}^{N+1}\right)(x)\right)^{\frac{1}{N+1}}\right),
\end{aligned} \tag{60}$$

$$\forall x \in \prod_{i=1}^d [a_i, b_i].$$

Furthermore, it holds true that

(2)

$$\begin{aligned}
& D^*\left({}^{(q)}B_n^{\mathcal{F}}f, f\right) \leq \\
& \sum_{j=1}^N \left\{ \sum_{|\alpha|=j} \left[\frac{\left\| \left({}^{(q)}B_n\left(\prod_{i=1}^d (z_i - x_i)^{\alpha_i}\right)\right)(x) \right\|_\infty}{\prod_{i=1}^d \alpha_i!} D^*(f_\alpha(x), \tilde{o}) \right] \right\} + \\
& \frac{\left\| \left({}^{(q)}B_n\left(\|z - x\|_{l_1}^{N+1}\right)(x) \right\|_\infty^{\frac{N}{N+1}}}{N!} \left[1 + \frac{1}{(N+1)}\right] \times \\
& \max_{|\alpha|=N} \omega_1^{(\mathcal{F})}\left(f_\alpha, \left\| \left({}^{(q)}B_n\left(\|z - x\|_{l_1}^{N+1}\right)(x) \right\|_\infty^{\frac{1}{N+1}}\right).
\end{aligned} \tag{61}$$

Next, to obtain a fuzzy multivariate Korovkin-type result, we use Theorem 11.44 of [1], p. 184.

Theorem 11. Here, all is the same as in Theorem 10. It holds true that $\left\| \left({}^{(q)}B_n\left(\|z - x\|_{l_1}^{N+1}\right)(x) \right\|_\infty \rightarrow 0$, as $n \rightarrow +\infty$. Thus, $D^*\left({}^{(q)}B_n^{\mathcal{F}}f, f\right) \rightarrow 0$, as $n \rightarrow +\infty$, $\forall f \in C_{\mathcal{F}}^N\left(\prod_{i=1}^d [a_i, b_i]\right)$ at a higher rate, i.e., ${}^{(q)}B_n^{\mathcal{F}} \rightarrow I$, the unit, as $n \rightarrow +\infty$, fuzzy and uniformly.

Proof. We know that

$$\left({}^{(q)}B_n\left(\|z - x\|_{l_1}^{N+1}\right)\right)(x) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x \right\|_{l_1}^{N+1} \widehat{Z}_\lambda^{(q)}(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widehat{Z}_\lambda^{(q)}(nx - k)}$$

By [4], p. 634, (25.23) there, we have

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \widehat{Z}_\lambda^{(q)}(nx - k)} < \frac{1}{\left(T_\lambda^{(q)}(1)\right)^d}, \quad \forall x \in \prod_{i=1}^d [a_i, b_i], \quad n \in \mathbb{N}. \tag{62}$$

$$\leq \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x \right\|_{l_1}^{N+1} \widehat{Z}_\lambda^{(q)}(nx - k)}{\left(T_\lambda^{(q)}(1)\right)^d} = \tag{63}$$

$$\frac{1}{\left(T_{\lambda}^{(q)}(1)\right)^d} \left\{ \sum_{\substack{k=\lceil na \rceil \\ \lceil \frac{k}{n} - x \rceil \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x \right\|_{l_1}^{N+1} \widehat{Z}_{\lambda}^{(q)}(nx - k) + \sum_{\substack{k=\lceil na \rceil \\ \lceil \frac{k}{n} - x \rceil > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x \right\|_{l_1}^{N+1} \widehat{Z}_{\lambda}^{(q)}(nx - k) \right\} \stackrel{\text{(by (31), (35))}}{\leq} \quad (64)$$

$$\frac{1}{\left(T_{\lambda}^{(q)}(1)\right)^d} \left(\frac{1}{n^{(N+1)\alpha}} + \left(\sum_{i=1}^d (b_i - a_i) \right)^{N+1} \left(1 - \widetilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) \right) \rightarrow 0,$$

as $n \rightarrow +\infty$.

That is,

$$\left\| \left({}_{\lambda}^{(q)} B_n \left(\|z - x\|_{l_1}^{N+1} \right) (x) \right) \right\|_{\infty} \leq \quad (65)$$

$$\frac{1}{\left(T_{\lambda}^{(q)}(1)\right)^d} \left(\frac{1}{n^{(N+1)\alpha}} + \left(\sum_{i=1}^d (b_i - a_i) \right)^{N+1} \left(1 - \widetilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) \right) \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

□

Furthermore, we derive the following"

Theorem 12. All is the same as in Theorem 10. Then,

$$D^* \left({}_{\lambda}^{(q)} B_n^{\mathcal{F}} f, f \right) \leq \sum_{j=1}^N \left\{ \sum_{|\alpha|=j} \left[\frac{1}{j!} \left[\left(\frac{1}{n^{(N+1)\alpha}} + \left(\sum_{i=1}^d (b_i - a_i) \right)^{N+1} \left(1 - \widetilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) \right)^{\frac{j}{N+1}} \right] D^*(f_{\alpha}, \widetilde{\partial}) \right] \right\} \quad (66)$$

$$+ \frac{\left[1 + \frac{1}{N+1} \right]}{N!} \left[\left(\frac{1}{n^{(N+1)\alpha}} + \left(\sum_{i=1}^d (b_i - a_i) \right)^{N+1} \left(1 - \widetilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) \right)^{\frac{N}{N+1}} \right] \frac{1}{\left(T_{\lambda}^{(q)}(1)\right)^d} \right]$$

$$\max_{|\alpha|=N} \omega_1^{(\mathcal{F})} \left(f_{\alpha}, \left[\left(\frac{1}{n^{(N+1)\alpha}} + \left(\sum_{i=1}^d (b_i - a_i) \right)^{N+1} \left(1 - \widetilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) \right)^{\frac{1}{N+1}} \right] \frac{1}{\left(T_{\lambda}^{(q)}(1)\right)^d} \right) \rightarrow 0,$$

as $n \rightarrow +\infty$.

Proof. By Hölder's inequality and the Riesz Representation Theorem, we infer that

$$\left\| \left({}_{\lambda}^{(q)} B_n \left(\|z - x\|_{l_1}^j \right) (x) \right) \right\|_{\infty} \leq \left\| \left({}_{\lambda}^{(q)} B_n \left(\|z - x\|_{l_1}^{N+1} \right) (x) \right) \right\|_{\infty}^{\frac{j}{N+1}},$$

for $j = 1, \dots, N$. Therefore, we obtain $\left\| \left({}^{(q)}_B B_n \left(\|z - x\|_{l_1}^j \right) (x) \right) \right\|_{\infty} \rightarrow 0, 1 \leq j \leq N$, as $n \rightarrow +\infty$. Notice that

$$\left({}^{(q)}_B B_n \left(\|z - x\|_{l_1}^j \right) (x) \right) = \sum_{|\alpha|=j} \frac{j!}{\prod_{i=1}^d \alpha_i!} \left({}^{(q)}_B B_n \left(\prod_{i=1}^d |z_i - x_i|^{\alpha_i} \right) \right) (x). \quad (67)$$

Hence, since in the last equality all parts are nonnegative, we have

$$\frac{j!}{\prod_{i=1}^d \alpha_i!} \left({}^{(q)}_B B_n \left(\prod_{i=1}^d |z_i - x_i|^{\alpha_i} \right) \right) (x) \leq \left({}^{(q)}_B B_n \left(\|z - x\|_{l_1}^j \right) (x) \right). \quad (68)$$

Consequently, we obtain the result that

$$\begin{aligned} \frac{\left\| \left({}^{(q)}_B B_n \left(\prod_{i=1}^d |z_i - x_i|^{\alpha_i} \right) \right) (x) \right\|_{\infty}}{\prod_{i=1}^d \alpha_i!} &\leq \frac{\left\| \left({}^{(q)}_B B_n \left(\|z - x\|_{l_1}^j \right) (x) \right) \right\|_{\infty}}{j!} \leq \\ &\frac{\left\| \left({}^{(q)}_B B_n \left(\|z - x\|_{l_1}^{N+1} \right) (x) \right) \right\|_{\infty}^{\frac{j}{N+1}}}{j!} \stackrel{(\text{by (65)})}{\leq} \\ &\frac{1}{j!} \left[\frac{\left(\frac{1}{n^{(N+1)\alpha}} + \left(\sum_{i=1}^d (b_i - a_i) \right)^{N+1} \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) \right)}{\left(T_{\lambda}^{(q)}(1) \right)^d} \right]^{\frac{j}{N+1}}, \end{aligned} \quad (69)$$

for $j = 1, \dots, N$. $\square$

Next, we give an $L_p, p \geq 1$, result.

Theorem 13. All is the same as in Theorem 10. Then,

$$\begin{aligned} &\left\| D^* \left(\left({}^{(q)}_B B_n^{\mathcal{F}} f \right) (x), f(x) \right) \right\|_p \leq \\ &\sum_{j=1}^N \left\{ \sum_{|\alpha|=j} \left[\frac{\left\| \left({}^{(q)}_B B_n \left(\prod_{i=1}^d (z_i - x_i)^{\alpha_i} \right) \right) (x) \right\|_{\infty}}{\prod_{i=1}^d \alpha_i!} \|D(f_{\alpha}(x), \tilde{d})\|_p \right] \right\} + \\ &\frac{\left\| \left({}^{(q)}_B B_n \left(\|z - x\|_{l_1}^{N+1} \right) (x) \right) \right\|_{\infty}^{\frac{N}{N+1}}}{N!} \left[1 + \frac{1}{N+1} \right] \times \\ &\max_{|\alpha|=N} \omega_1^{(\mathcal{F})} \left(f_{\alpha}, \left\| \left({}^{(q)}_B B_n \left(\|z - x\|_{l_1}^{N+1} \right) (x) \right) \right\|_{\infty}^{\frac{1}{N+1}} \right) \left(\prod_{i=1}^d (b_i - a_i) \right)^{\frac{1}{p}}. \end{aligned} \quad (70)$$

We continue with the following, $N = 1$ case.

Corollary 2. Here $f \in C_{\mathcal{F}}^1\left(\prod_{i=1}^d [a_i, b_i]\right)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $p \geq 1$. Then,

$$\begin{aligned} & \left\| D\left(\left({}^{(q)}B_n^{\mathcal{F}} f\right)(x), f(x)\right) \right\|_p \leq \\ & \sum_{j=1}^N \left[\left\| \left({}^{(q)}B_n(z_i - x_i)\right)(x) \right\|_{\infty} \left\| D\left(\frac{\partial f}{\partial x_i}(x), \tilde{o}\right) \right\|_p \right] + \\ & \frac{3}{2} \left\| \left({}^{(q)}B_n\left(\|z - x\|_{l_1}^2\right)(x) \right) \right\|_{\infty}^{\frac{1}{2}} \cdot \\ & \max_{i \in \{1, \dots, d\}} \omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \left\| \left({}^{(q)}B_n\left(\|z - x\|_{l_1}^2\right)(x) \right) \right\|_{\infty}^{\frac{1}{2}}\right) \left(\prod_{i=1}^d (b_i - a_i)\right)^{\frac{1}{p}}. \end{aligned} \quad (71)$$

Notice that both (70) and (71) give the fuzzy L_p convergence of $\left({}^{(q)}B_n^{\mathcal{F}} f\right)$ to f , as $n \rightarrow +\infty$.

It is also important, for the sake of completeness, to add the basic case.

Theorem 14. Let $\|\cdot\|$ a norm on $\prod_{i=1}^d [a_i, b_i]$ and $f \in C_{\mathcal{F}}\left(\prod_{i=1}^d [a_i, b_i]\right)$. Then,

$$(1) \quad D\left(\left({}^{(q)}B_n^{\mathcal{F}} f(x), f(x)\right) \leq 2\omega_1^{(\mathcal{F})}\left(f, \left(\left({}^{(q)}B_n\left(\|\cdot - x\|^2\right)\right)(x)\right)^{\frac{1}{2}}\right), \quad (72)$$

$$\forall x \in \prod_{i=1}^d [a_i, b_i],$$

and
(2)

$$D^*\left(\left({}^{(q)}B_n^{\mathcal{F}} f, f\right) \leq 2\omega_1^{(\mathcal{F})}\left(f, \left\| \left({}^{(q)}B_n\left(\|\cdot - x\|^2\right)\right)(x) \right\|_{\infty}^{\frac{1}{2}}\right). \quad (73)$$

Proof. Use Theorem 11.45, p. 185 of [1]. $\square$

Theorem 15. Let $f \in C_{\mathcal{F}}\left(\prod_{i=1}^d [a_i, b_i]\right)$, $p \geq 1$. Then,

$$\left\| D\left(\left({}^{(q)}B_n^{\mathcal{F}} f\right)(x), f(x)\right) \right\|_p \leq \quad (74)$$

$$2\omega_1^{(\mathcal{F})}\left(f, \left\| \left({}^{(q)}B_n\left(\|\cdot - x\|^2\right)\right)(x) \right\|_{\infty}^{\frac{1}{2}}\right) \left(\prod_{i=1}^d (b_i - a_i)\right)^{\frac{1}{p}},$$

where $\|\cdot\|$ is a norm on $\prod_{i=1}^d [a_i, b_i]$.

Proof. Use Theorem 11.46, p. 186 of [1]. $\square$

Remark 5. The fact $\left\| \left({}^{(q)}B_n\left(\|\cdot - x\|^2\right)\right)(x) \right\|_{\infty} \rightarrow 0$, as $n \rightarrow +\infty$, implies $\left({}^{(q)}B_n^{\mathcal{F}} f \xrightarrow{D^*} f\right)$, fuzzy uniformly, and $\left({}^{(q)}B_n^{\mathcal{F}} f \xrightarrow{D} f\right)$, fuzzy pointwise; also, it is fuzzy L_p convergent.

If $\|\cdot\| = \|\cdot\|_{l_1}$, then, by (65), we obtain

$$\left\| \left({}^{(q)}B_n\left(\|z - x\|_{l_1}^2\right)(x) \right) \right\|_{\infty} \leq \quad (75)$$

$$\frac{1}{\left(T_{\lambda}^{(q)}(1)\right)^d} \left(\frac{1}{n^{2\alpha}} + \left(\sum_{i=1}^d (b_i - a_i) \right)^2 \left(1 - \tilde{G}_{\lambda}^{(q)}(n^{1-\alpha} - 2) \right) \right) \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

By equivalence of norms in $\prod_{i=1}^d [a_i, b_i]$ it also holds that $\left\| \left(\tilde{G}_{\lambda}^{(q)} B_n \left(\|z - x\|^2 \right) (x) \right) \right\|_{\infty} \rightarrow 0$; hence, all the above convergences are true.

5. Conclusions

Our fuzzy neural network operators operate as fuzzy positive linear operators. They are constructed by stacking (multiply composing) seven standard activation functions. They apply to continuous functions that map real intervals (univariate case) or boxes (multivariate case) to fuzzy real numbers. The study quantifies how quickly these fuzzy network operators converge to the “fuzzy unit” operator. It establishes pointwise, uniform, and L_p convergence limits using inequalities based on fuzzy moduli of continuity. The most striking takeaway is that these complex fuzzy convergence behaviors are governed and guaranteed by the mathematical properties of their deterministic, real-valued counterparts. Because the seven foundational activation functions can be assembled into virtually limitless multicomposite combinations, the framework yields an infinite variety of fuzzy positive linear operators.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Anastassiou, G.A. *Fuzzy Mathematics: Approximation Theory*; Springer: Berlin/Heidelberg, Germany, New York, NY, USA, 2010.
2. Anastassiou, G.A. *Intelligent Systems II: Complete Approximation by Neural Network Operators*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2016.
3. Anastassiou, G.A. *Banach Space Valued Neural Network*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2023.
4. Anastassiou, G.A. *Parametrized, Deformed and General Neural Networks*; Springer, Berlin/Heidelberg, Germany; New York, NY, USA, 2023.
5. Chen, Z.; Cao, F. The Approximation Operators with Sigmoidal Functions. *Comput. Math. Appl.* **2009**, *58*, 758–765.
6. Gal, S. Chapter 13—Approximation Theory in Fuzzy Setting. In *Handbook of Analytic-Computational Methods in Applied Mathematics*; Anastassiou, G., Ed.; Chapman & Hall/CRC: Boca Raton, FL, USA; New York, NY, USA, 2000; pp. 617–666.
7. Wu, C.; Gong, Z. On Henstock integrals of interval-valued functions and fuzzy valued functions. *Fuzzy Sets Syst.* **2000**, *115*, 377–391.
8. Costarelli, D.; Spigler, R. Approximation results for neural network operators activated by sigmoidal functions. *Neural Netw.* **2013**, *44*, 101–106.
9. Costarelli, D.; Spigler, R. Multivariate neural network operators with sigmoidal activation functions. *Neural Netw.* **2013**, *48*, 72–77.
10. Haykin, I.S. *Neural Networks: A Comprehensive Foundation*, 2nd ed.; Prentice Hall: New York, NY, USA, 1998.
11. McCulloch, W.; Pitts, W. A logical calculus of the ideas immanent in nervous activity. *Bull. Math. Biophys.* **1943**, *7*, 115–133.
12. Mitchell, T.M. *Machine Learning*; WCB-McGraw-Hill: New York, NY, USA, 1997.
13. Yu, D.S.; Cao, F.L. Construction and approximation rate for feed-forward neural network operators with sigmoidal functions. *J. Comput. Appl. Math.* **2025**, *453*, 116150.
14. Cen, S.; Jin, B.; Quan, Q.; Zhou, Z. Hybrid neural-network FEM approximation of diffusion coefficient in elliptic and parabolic problems. *IMA J. Numer. Anal.* **2024**, *44*, 3059–3093.
15. Coroianu, L.; Costarelli, D.; Natale, M.; Pantiş, A. The approximation capabilities of Durrmeyer-type neural network operators. *J. Appl. Math. Comput.* **2024**, *70*, 4581–4599.
16. Warin, X. The GroupMax neural network approximation of convex functions. *IEEE Trans. Neural Netw. Learn. Syst.* **2024**, *35*, 11608–11612.

17. Upadhyay, D.; Joshi, H. Mathematical modeling of local calcium signaling in neurons using artificial neural networks. *Discret. Contin. Dyn. Syst. S* **2025**, *18*, 1392–1415.
18. Wu, C.; Gong, Z. On Henstock integral of fuzzy-number-valued functions (I). *Fuzzy Sets Syst.* **2001**, *120*, 523–532.
19. Goetschel, R., Jr.; Voxman, W. Elementary fuzzy calculus. *Fuzzy Sets Syst.* **1986**, *18*, 31–43.
20. Wu, C.; Ma, M. On embedding problem of fuzzy number spaces: Part 1. *Fuzzy Sets Syst.* **1991**, *44*, 33–38.
21. Kaleva, O. Fuzzy differential equations. *Fuzzy Sets Syst.* **1987**, *24*, 301–317.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.