

Article

Optimal Control of Stochastic Differential Delay Equations (SDDE) with Jumps and Markov Switching, and with an Application in Economics

Mariya Svishchuk ¹ and Anatoliy V. Swishchuk ^{2,*}¹ Department of Mathematics and Computing, Mount Royal University, 4825 Mount Royal Gate SW, Calgary, AB T3E 6K6, Canada; msvishchuk@mtroyal.ca² Department of Mathematics and Statistics, University of Calgary, 2500 University Drive NW, Calgary, AB T2N 1N4, Canada

* Correspondence: aswish@ucalgary.ca

Abstract

This paper considers stochastic optimal control of stochastic differential delay equations (SDDEs) with jumps and Markov switching, and its economical applications. The Hamilton–Jacobi–Bellman (HJB) equation and the inverse HJB-equation are derived by applying the Dynkin’s formula and solution of the Dirichlet–Poisson problem for those SDDEs. Three cases of modified Ramsey stochastic models in economics are studied; namely, a Ramsey diffusion model with jumps, a Ramsey diffusion model with Markov switching, and a Ramsey diffusion model with jumps and Markov switching. We also present numerical examples for all three cases. The contributions of the paper are four-fold: introducing and studying three new models for SDDEs: with jumps, Markov switching and both; studying optimal control for those models; applications of these results to economics with Ramsey’s stochastic models; and numerical examples for these different models.

Keywords: optimal control; SDDE; jumps; Markov switching; HJB equation; economics; Ramsey model

MSC: 60H20; 60G55; 60J27; 91B70; 49G20; 49K21; 34F05

1. Introduction

Stochastic optimal control is a subject of many research papers and books—see, e.g., [1–3]—with infinite dimensions’ stochastic optimal control [4] and jump-diffusion applied stochastic control [5]. Stochastic delay equations are also a subject of studies in many papers and books, including [6–9]. Delay differential equations driven by Lévy processes are considered in [10]. Averaged SDDEs with semi-Markov switching and stochastic optimal control for them and with application in economics were studied in [11].

Let us also mention some recent papers on SDDE. Stochastic delay differential equations with applications to disease modelling were considered in [12]. A comprehensive stability analysis of numerical simulation of fractional stochastic delay differential equations using spectral scheme was studied in [13]. The paper [14] considered solving some stochastic delay differential equations by Daubechies wavelet. General decay stability in mean square of stochastic delay differential equations was studied in [15]. The averaging principle for stochastic delay differential equations with Hadamard fractional integral



Academic Editors: Marcos Escobar-Anel and Quanxin Zhu

Received: 18 March 2026

Revised: 20 April 2026

Accepted: 1 May 2026

Published: 14 July 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

was investigated in [16]. Investigation of stability nonlinear stochastic differential equations with state-dependent delay was taken by [17]. The paper [18] was devoted to the controllability results of stochastic neutral functional differential equations with delay.

We mention, that no book or paper considered stochastic optimal control problems for SDDEs with jumps and regime-switching, or dealt with the Dirichlet–Poisson problem, the Dynkin formula, the HJB and inverse HJB-equations for such equations, and in particular with their applications in economics. These are exactly the subjects of investigation in the present paper, and those results are new and original. For stochastic optimal control and optimal performance definitions see Section 2.7. For definitions and basics of the theory related to the stochastic optimal control and stochastic optimal performance see [1,2]. For cases under consideration in the present work, equations with jumps and equations with jumps and Markov switching, we present related definitions for the cost/objective functionals and optimal performances.

In the earlier paper [19] we considered the following SDDE with control $u(t)$ (SDDE):

$$x(t) = x(0) + \int_0^t a(x(s-1), u(s))ds + \int_0^t b(x(s-1), u(s))dw(s), \quad (1)$$

Here: $x(t) = \phi(t)$, $t \in [-1, 0]$, is a given continuous process, $u(t)$ is a control process and $w(t)$ is a standard Wiener process.

The solution of the SDDE In the case is not Markov. However, we can make it Markovian by considering the pair $(x_t, x(t))$ where $x_t := x(t+s)$, $s \in [-1, 0]$, is the path of the process on the interval $[t-1, t]$ (see details in [20]). In this way, this pair is a strong Markov process to which one can apply the theoretical basics of the corresponding weak infinitesimal generator; see, e.g., [21] for more details. A solution of the Dirichlet–Poisson problem and a Dynkin formula for the SDDE are presented. We mention that these results are obtained from the Dynkin formulas and boundary value problems for multiplicative operator functionals of Markov processes [22]. Therefore, the Dynkin formula and the solution of the Dirichlet–Poisson problem are used in this case to obtain the Hamilton–Jacobi–Bellman (HJB) and the inverse HJB-equations. Complete proofs were given for two theorems about the stochastic optimal control for the diffusion model only in paper [19], namely, Theorem 1 (HJB-equation) and Theorem 2 (inverse of the HJB-equation). An application was given to a Ramsey model [23,24], a stochastic model in economics, that takes into account the delay and randomness in the production cycle.

The Ramsey–Cass–Koopmans model, (known also as the Ramsey growth model) is a neoclassical model of economic growth based primarily on the work of Frank P. Ramsey [24] with significant extensions by David Cass [25] and Tjalling Koopmans [26]. This model is very popular nowadays as well—see, e.g., [27–33].

Let us also mention some papers on the stochastic Ramsey’s model. A stochastic version of the Ramsey’s growth model was considered in [34]. The solution of the discrete-time stochastic Ramsey model was presented in [35]. Sustainability in the stochastic Ramsey model was investigated in [36]. Paper [37] studied the stochastic growth model and real business cycles. In the authors’ opinion, the paper [38] was one of the first ones to consider the stochastic Ramsey’s model.

The model in [19] was described by the equation

$$dK(t) = [AK(t-T) - u(K(t))C(t)]dt + \sigma(K(t-T))dw(t).$$

Here: K is the capital, C is the production rate, u is a control process, A is a positive constant, σ is the diffusion coefficient of the SDDE. The “initial capital”

$$K(t) = \phi(t), \quad t \in [-T, 0],$$

is a continuous bounded positive function. For this stochastic economic model the optimal control was found to be $u_{\min} = K(0) \cdot C(0)$, and the optimal performance was

$$\begin{aligned} J(K, u_{\min}) &= \frac{K^2(0)}{2} + \frac{K^2(0) \cdot C^2(0)}{2} + \int_{-T}^0 \phi^2(\theta) d\theta \\ &= \frac{K^2(0)}{2} (1 + C^2(0)) + \int_{-T}^0 \phi^2(\theta) d\theta. \end{aligned}$$

The delay T was normalized to $T = 1$ by time rescaling. This is one of our assumptions in the theoretical considerations that follow. However, the obtained results are valid for general delay $T > 0$.

In the current paper we consider stochastic optimal control of SDDEs with jumps and Markov switching, and present application to some three new Ramsey stochastic models in economics.

The novelty and contributions of the paper are four-fold: introducing and studying three new models for SDDEs: with jumps, Markov switching and both; studying optimal control for those models; applications of these results to economics with Ramsey's stochastic models; and numerical examples for these different models. We note that the paper [19] contains only diffusion model with delay without jumps and Markov switching. Also, the paper [11] contains only averaged diffusion model with semi-Markov switching, again without jumps and Markov switching. The applications in both papers are given to diffusion economic Ramsey models, without jumps and Markov switching. Therefore, the current paper contains new and original results compare with the mentioned former and latter papers.

The main type of SDDE under consideration is an equation with Markov switching and jumps:

$$\begin{aligned} x(t) &= x(0) + \int_0^t a(x(s-1), u(s), r(s)) ds \\ &+ \int_0^t b(x(s-1), u(s), r(s)) dw(s) + \int_0^t \int_{-1}^{+\infty} yx(s) \nu(dy, ds, u(s)), \end{aligned} \quad (2)$$

where $r(t)$ is a continuous-time finite-state Markov chain with state space $X = \{1, 2, \dots, N\}$ [39], and where $\nu(dy, dt, u)$ is a Poisson random measure with parameter $u\Pi(dy)dt$ [22] and u is a control state. We note that all sources of randomness, $w(t), r(t), \nu(dy, t, u)$, are independent.

The SDDE in (2) with jumps and Markov switching is very general and includes models with jumps only and Markov switching only; it has many applications, including economics, finance, and other fields; see, e.g., [40,41]. The states of the Markov chain $r(t)$ may describe different modes of economy, say, in economics, or different positions of traders in finance, to name a few.

From Equation (2) we can construct other SDDEs such as SDDEs with jumps only,

$$\begin{aligned} x(t) &= x(0) + \int_0^t a(x(s-1), u(s)) ds + \int_0^t b(x(s-1), u(s)) dw(s) \\ &+ \int_0^t \int_{-1}^{+\infty} yx(s) \nu(dy, ds, u(s)), \end{aligned} \quad (3)$$

and SDDEs with Markov switching only,

$$\begin{aligned} x(t) &= x(0) + \int_0^t a(x(s-1), u(s), r(s)) ds \\ &+ \int_0^t b(x(s-1), u(s), r(s)) dw(s). \end{aligned} \quad (4)$$

Of course, it is possible to consider the case when $\nu(dy, dt, u)$ also depends on the Markov chain $r(t)$, i.e., regime-switching. However, we do not consider this case in the present work, leaving it for future exploration.

The equations associated with (2)–(4) are some new versions of the Ramsey model in economics that have the following forms:

Ramsey model with jumps and Markov switching,

$$dK(t) = [A(r(t))K(t-T) - u(K(t))C(t)]dt + \sigma(K(t-T), r(t))dw(t) + \int_{-1}^{+\infty} yv(dy, dt, u(s)), \quad (5)$$

and associated Ramsey model with jumps only,

$$dK(t) = [AK(t-T) - u(K(t))C(t)]dt + \sigma(K(t-T))dw(t) + \int_{-1}^{+\infty} yv(dy, dt, u(s)), \quad (6)$$

and Ramsey model with Markov switching only

$$dK(t) = [A(r(t))K(t-T) - u(K(t))C(t)]dt + \sigma(K(t-T), r(t))dw(t). \quad (7)$$

We note that model (7) is, for example, a particular case of model (4) with

$$a(x, u, r) = A(r)K - u(K)C \quad b(x, u, r) = \sigma(K, r).$$

The paper is organized as follows. Controlled SDDEs with jumps and Markov switching are considered in Section 2. Hamilton-Jacobi-Bellman (HJB) equations for SDDEs with jumps and Markov switching and main results, Theorems 1–6, are studied in Section 3. Optimization problems for Ramsey's models are investigated in Section 4. Numerical examples are presented in Section 5. Conclusion is highlighted in Section 6. Section 7 is considered discussion and future work. Appendix A contains 10 figures to visualize the results. Appendix B gives the proofs for all Theorems 1–6.

2. Controlled Stochastic Differential Delay Equations with Jumps and Markov Switching

This section contains new three models of controlled SDDEs with jumps and Markov switching. It starts with introduction of some basic notations (Section 2.1), description of SDDEs with jumps and Markov switching (Section 2.2), SDDEs with jumps only (Section 2.3), SDDEs with Markov switching only, (Section 2.4), and weak infinitesimal operators for all three new models from Sections 2.2–2.4 (Section 2.5). The models are new and original compare with those introduced in [11,19], where we considered only diffusion models without jumps and Markov switching. Then Section 2.6 describes Dynkin's formula for SDDEs with jumps and Markov switching, highlighting those results for SDEs with jumps only (Section 2.6.1) and Markov switching only (Section 2.6.2). Section 2.7 are devoted to the solutions of Dirichlet-Poisson problem for SDDEs with jumps and Markov switching (Section 2.7.1), highlighting the results for SDDEs with jumps only (Section 2.7.2) and Markov switching only (Section 2.7.3). These results are new and original compare with those from [19] and [11]. Section 2.8 describes statement of the problem for optimal control and optimal performance.

2.1. Some Basic Notations

Let us remind some basic facts and notions from [20,42,43] necessary for subsequent exposition in this paper.

We consider the following objects in this paper: $x(t), t \in [-1, \infty)$ to be a stochastic process; $F_{uv}(x)$ is a minimal σ -algebra with respect to which $x(t)$ is measurable for every $t \in [u, v]$; $w(t), t \in [0, \infty)$ is a Wiener process with $w(0) = 0$, and with $F_{uv}(dw)$ be in a minimal Borel σ -algebra such that $w(t) - w(s)$ is measurable for all t, s with

$u \leq s \leq t \leq v$; $F_{uv}(dr)$ is a minimal Borel σ -algebra such that $r(t) - r(s)$ is measurable for all t, s with $u \leq s \leq t \leq v$, where $r(t)$ is a finite-state Markov chain with state space $X = \{1, 2, \dots, N\}$ [39]; $F_{uv}(dv)$ is a minimal Borel σ -algebra for which $v(du, t) - v(du, s)$ is measurable for all t, s with $u \leq s \leq t \leq v$, where $v(du, dt)$ is a Poisson random measure [22];

As for the control, $u(t) \in \mathcal{U}$, $t \in [-1, \infty)$ with values from $\mathcal{U} = \text{Borel set}$, and such that $u(t)$ is $F_{uv}(u)$ -adapted for all $t \in [u, v]$. See [20,42,43] for more details.

2.2. Controlled SDDE with Jumps and Markov Switching

Here, $\mathcal{D}$ the Banach space of all càdlàg functions defined on the interval $[-1, 0]$ and equipped with the Skorokhod topology; see [44] for definitions and further details.

Suppose that $a(\cdot, \cdot, r)$, $b(\cdot, \cdot, r)$ are continuous functionals defined on $\mathcal{D} \times \mathcal{U} \times X$.

Definition 1. A stochastic process $x(t)$ is called a solution of the stochastic differential delay equation with jumps and Markov switching

$$\begin{aligned} dx(t) &= a(x(t-1), u(t), r(t))dt + b(x(t-1), u(t), r(t))dw(t) \\ &+ \int_{-1}^{+\infty} yx(t)v(dy, dt), \quad t \in [0, \infty) \end{aligned} \quad (8)$$

provided

$$F_{-1t}(x) \vee F_{0t}(dw) \vee F_{0t}(u) \vee F_{0t}(dv) \vee F_{0t}(dr)$$

is independent of $F_{t\infty}(dw)$ for every $t \in [0, \infty)$. The notation $F_1 \vee F_2 \vee F_3 \vee F_4 \vee F_5$ is used for the minimal σ -algebra containing F_1, F_2, F_3, F_4 and F_5 , and $x(t)$ satisfies

$$\begin{aligned} x(t) &- x(s) = \int_s^t a(x(l-1), u(l), r(l))dl \\ &+ \int_s^t b(x(l-1), u(l), r(l))dw(l) + \int_s^t \int_{-1}^{+\infty} yx(l)v(dy, dl), \end{aligned} \quad (9)$$

where the middle integral is the Itô integral and the last one is the Poisson integral [22].

In the integral form Equation (8) has the form

$$\begin{aligned} x(t) &= x(0) + \int_0^t a(x(s-1), u(s), r(s))ds \\ &+ \int_0^t b(x(s-1), u(s), r(s))dw(s) + \int_0^t \int_{-1}^{+\infty} yx(s)v(dy, ds). \end{aligned} \quad (10)$$

The initial process $x(t) = \phi(t)$, $t \in [-1, 0]$, where $\phi \in \mathcal{D}$ is a given càdlàg function. In this way, the processes $\phi(t)$, $t \in [-1, 0]$, $w(t)$ and $u(t)$, $t \geq 0$, are defined on the probability space $(\Omega, \mathcal{F}, F_t, P)$ and $F_t := F_{-1t}(x) \vee F_{0t}(dw) \vee F_{0t}(u) \vee F_{0t}(dv) \vee F_{0t}(dr)$.

We suppose that following conditions are satisfied for Equation (9):

A.1 $a(\phi, u, r)$ and $b(\phi, u, r)$ are continuous real-valued functionals defined on $\mathcal{D} \times \mathcal{U} \times X$;

A.2 $\phi \in \mathcal{D}$ is a càdlàg function in the interval $[-1, 0]$, independent of $w(s)$, $s \geq 0$, and $E|\phi(t)|^4 < \infty$;

A.3 $\forall \phi, \psi \in \mathcal{D}$:

$$|a(\phi, u, r) - a(\psi, u, r)| + |b(\phi, u, r) - b(\psi, u, r)| \leq K \int_{-1}^0 |\phi(\theta) - \psi(\theta)|d\theta, \quad (11)$$

with $|a(\eta, u, r)| + |b(\eta, u, r)| \leq M$ for some $M, K > 0$ and all $\eta \in \mathcal{D}$, $u \in \mathcal{U}$, $r = r(0) \in X$.

A.4 $\int_{-1}^{+\infty} y^2 \Pi(dy) < +\infty$, where $\Pi(dy)$ is a measure associated with $v(dy, dt)$, i.e., $E v(dy, dt) = \Pi(dy)dt$.

Remark 1. We have the following existence of a solution statement (see also [19]): if the assumptions A.1–A.4 are satisfied, then the solution of the initial value problem (9) exists and the pair $(x_t, x(t))$

is a unique stochastic Markov process [20,42,43]. The solution $x(t)$ can be viewed at time $t \geq 0$ as an element x_t of the space $\mathcal{D}$, or as a point in $\mathbb{R}$.

2.3. Controlled SDDE with Jumps Only

In this case we do not have regime-switching, i.e. $r(t) = 0$, and functionals $a(\cdot, \cdot), b(\cdot, \cdot)$ are continuous and defined on $\mathcal{D} \times \mathcal{U}$ only.

Definition 2. The following stochastic process $x(t)$ is called a solution of the SDDE with jumps only

$$\begin{aligned} dx(t) &= a(x(t-1), u(t))dt + b(x(t-1), u(t))dw(t) \\ &+ \int_{-1}^{+\infty} yx(t)\nu(dy, dt), \quad t \in [0, \infty) \end{aligned} \quad (12)$$

provided

$$F_{-1t}(x) \vee F_{0t}(dw) \vee F_{0t}(u) \vee F_{0t}(d\nu)$$

is independent of $F_{t\infty}(dw)$ for every $t \in [0, \infty)$. Thus, $x(t)$ satisfies

$$\begin{aligned} x(t) - x(s) &= \int_s^t a(x(r-1), u(r))dr + \int_s^t b(x(r-1), u(r))dw(r) \\ &+ \int_s^t \int_{-1}^{+\infty} ux(l)\nu(du, dl), \end{aligned}$$

where the pre-last integral is the Itô integral and the last one is a Poisson integral [22].

Equation (12) is meant in the integral form

$$\begin{aligned} x(t) &= x(0) + \int_0^t a(x(s-1), u(s))ds + \int_0^t b(x(s-1), u(s))dw(s) \\ &+ \int_0^t \int_{-1}^{+\infty} yx(s)\nu(dy, ds), \end{aligned} \quad (13)$$

with the initial process $x(t) = \phi(t), t \in [-1, 0]$, where $\phi \in \mathcal{D}$ is a given càdlàg function. In this way, the processes $\phi(t), t \in [-1, 0], w(t)$ and $u(t), t \geq 0$, are defined on the probability space $(\Omega, \mathcal{F}, F_t, P)$ and $F_t := F_{-1t}(x) \vee F_{0t}(dw) \vee F_{0t}(u) \vee F_{0t}(d\nu)$.

We consider the following conditions to be satisfied for the Equation (12):

B.1 $a(\phi, u)$ and $b(\phi, u)$ are continuous real-valued functionals defined on $\mathcal{D} \times \mathcal{U}$;

B.2 $\phi \in \mathcal{D}$ is càdlàg with probability 1 in the interval $[-1, 0]$, independent of $w(s), s \geq 0$, and $E|\phi(t)|^4 < \infty$; here, $|\cdot|$ is the Skorokhod norm, [44].

B.3 $\forall \phi, \psi \in \mathcal{D}$:

$$|a(\phi, u) - a(\psi, u)| + |b(\phi, u) - b(\psi, u)| \leq K \int_{-1}^0 |\phi(\theta) - \psi(\theta)|d\theta,$$

with $|a(\eta, u)| + |b(\eta, u)| \leq M$ for some constants $M, K > 0$ and all $\eta \in \mathcal{D}, u \in \mathcal{U}$.

B.4 $\int_{-1}^{+\infty} y^2 \Pi(dy) < +\infty$, where $\Pi(dy)$ is a measure associated with $\nu(dy, dt)$, i.e., $E\nu(dy, dt) = \Pi(dy)dt$.

Remark 2. We have the following existence of a solution statement (see also [19]): if the assumptions B.1–B.4 are satisfied, then the solution of the initial value problem (11) exists and the pair $(x_t, x(t))$ is a unique stochastic Markov process [20,42,43].

2.4. Controlled SDDE with Markov Switching Only

In this case we do not have jumps, i.e., $\nu = 0$, and functionals $a(\cdot, \cdot, r), b(\cdot, \cdot, r)$ are continuous functionals defined on $\mathcal{D} \times \mathcal{U} \times X$.

Definition 3. The following stochastic process $x(t)$ is called a solution of the SDDE with Markov switching only

$$dx(t) = a(x(t-1), u(t), r(t))dt + b(x(t-1), u(t), r(t))dw(t), \quad t \in [0, \infty) \quad (14)$$

provided

$$F_{-1t}(x) \vee F_{0t}(dw) \vee F_{0t}(u) \vee F_{0t}(dr)$$

is independent of $F_{t\infty}(dw)$ for every $t \in [0, \infty)$. The process $x(t)$ satisfies the following SDDE

$$\begin{aligned} x(t) - x(s) &= \int_s^t a(x(l-1), u(l), r(l))dl \\ &+ \int_s^t b(x(l-1), u(l), r(l))dw(l), \end{aligned} \quad (15)$$

where the middle integral is the Itô integral and the last one is the Poisson integral [22].

Equation (15) in integral form has the following form

$$\begin{aligned} x(t) &= x(0) + \int_0^t a(x(s-1), u(s), r(s))ds \\ &+ \int_0^t b(x(s-1), u(s), r(s))dw(s). \end{aligned} \quad (16)$$

Here, the initial process $x(t) = \phi(t)$, $t \in [-1, 0]$, where $\phi \in \mathcal{D}$ is a given càdlàg function. Therefore, we assume that the processes $\phi(t)$, $t \in [-1, 0]$, $w(t)$ and $u(t)$, $t \geq 0$, are defined on the probability space $(\Omega, \mathcal{F}, F_t, P)$ and $F_t := F_{-1t}(x) \vee F_{0t}(dw) \vee F_{0t}(u) \vee F_{0t}(dr)$. See [19].

Let the following conditions be satisfied for Equation (14):

C.1 $a(\phi, u, r)$ and $b(\phi, u, r)$ are continuous real-valued functionals defined on $\mathcal{D} \times \mathcal{U} \times X$;

C.2 $\phi \in \mathcal{D}$ is càdlàg function in the interval $[-1, 0]$, independent of $w(s)$, $s \geq 0$, and $E|\phi(t)|^4 < \infty$;

C.3 $\forall \phi, \psi \in \mathcal{D}$:

$$|a(\phi, u, r) - a(\psi, u, r)| + |b(\phi, u, r) - b(\psi, u, r)| \leq K \int_{-1}^0 |\phi(\theta) - \psi(\theta)|d\theta,$$

with $|a(\eta, u, r)| + |b(\eta, u, r)| \leq M$ for some $M, K > 0$ and all $\eta \in \mathcal{D}$, $u \in \mathcal{U}$, $r = r(0) \in X$.

Remark 3. We have the following existence of a solution statement (see also [19]): if the assumptions C.1–C.3 are satisfied, then the solution of the initial value problem (16) exists and the pair $(x_t, x(t))$ is a unique stochastic Markov process [20,42,43]. The solution $x(t)$ can be viewed at time $t \geq 0$ as an element x_t of the space $\mathcal{D}$, or as a point in $\mathbb{R}$. See [19].

2.5. Weak Infinitesimal Operators of Markov Processes $(x_t, x(t))$, $(x_t, x(t), \nu(dy, t), r(t))$, $(x_t, x(t), \nu(dy, t))$ and $(x_t, x(t), r(t))$

It is well-known that the solution $x(t)$ of stochastic differential delay equations (SDDEs) (1) (see also in [19]) is not Markov. We can make it as a Markov process if we take the pair $(x_t, x(t)) := (x(t+s), x(t))$, $s \in [-1, 0]$. In what follows, $E_{x,x(0)}[\cdot] := E[\cdot | x_0 = x, x(0)]$ and $E_{x,x(0),u}[\cdot] := E[\cdot | x_0 = x, x(0), u(0) = u]$. We note that the following triplet $(x_t, x(t), r(t))$ is also Markov as long as $r(t)$ is a Markov chain.

Definition 4. A real valued functional $J(x_t, x(t), u(t))$ on $\mathcal{D}$ is said to be in the domain of A^u , the weak infinitesimal operator (w.i.o.) of the process $(x_t, x(t))$, if the limit

$$\begin{aligned} \lim_{t \rightarrow 0+} ((E_{x,x(0)}^u J(x_t, x(t), u(t)) - J(x, x(0), u)) / t) &:= q(x, x(0), u), \\ x = x_0 = \phi \in \mathcal{D}, u \in \mathcal{U}, \end{aligned}$$

exists pointwise on $\mathcal{D} \times \mathcal{U}$, and

$$\lim_{t \rightarrow 0+} \sup_{x,u} |E_{x,x(0)}^u q(x_t, x(t), u) - q(x, x(0), u)| = 0.$$

Here $x_t := x_t(\theta) = x(t + \theta)$, $\theta \in [-1, 0]$, is in $\mathcal{D}$ and $E_{x,x(0)}^u$ is the expectation under the conditional probability with respect to $x, x(0)$ and u . Therefore, we set $A^u J(x, x(0), u) := q(x, x(0), u)$.

We denote by A_G^u , for n open and bounded set $G \subset \mathbb{R}$, the w.i.o. of $\tilde{x}_t = x_t$, the process stopped at $\tau_G := \inf\{t : x(t) \notin G\}$.

We need one more condition to be satisfied, which is related to the w.i.o.:

A.5 The inequality (11) is satisfied for all initial functions $\psi, \phi \in \mathcal{D}$ and every $\rho > 0$ with $|\psi| \leq \rho, |\phi| \leq \rho$. Moreover, $\tilde{A}$ is the weak infinitesimal operator of the process $\tilde{x}(t)$ satisfying equation (1) with a, b replaced by $\tilde{a}, \tilde{b}$ and satisfying assumptions A.1–A.3. Let also the functions $\tilde{a} = a, \tilde{b} = b$ are defined on the bounded open set G .

We suppose that F is continuous and bounded on bounded sets.

Definition 5. Then the restriction of F to G is in $D(\tilde{A}_G^u)$, provided $F \in D(\tilde{A})$ and $\tilde{A}^u J = q$ is bounded on bounded sets, and we have

$$\begin{aligned} \tilde{A}_G^u J(x) &= L^u F(x(0)) = q(x(0), u) := \\ &= F'_{x(0)}(x(0))a(x(0), u) + F''_{x(0)x(0)}(x(0))\frac{1}{2}b^2(x(0), u) \end{aligned} \quad (17)$$

where $u = u(0)$ (see [43]).

Also let $F(\alpha, \beta), F'_\beta(\alpha, \beta), F''_{\beta\beta}(\alpha, \beta)$ be continuous and bounded in α, β . Then $J(x) \in D(\tilde{A}_G^u)$, and we have three cases for the generator of three processes (SDDE with jumps and Markov switching, SDDE with jumps only and SDEE with Markov switching only).

Generator for the SDDE with Markov switching and jumps is

$$\begin{aligned} \tilde{A}_G^r J(x(0), r) &:= q(x(0), r) = F(x(0), x(0), r) - F(x(-1), x(0), r) \\ &+ \int_{-1}^0 L^u F(x(s), x(0), r) ds + \\ &\int_{-1}^0 \int_{-1}^{+\infty} [F(x(s) + x(0)y, x(0), r) - F(x(-1), x(0)) \\ &- F_x(x(s), x(0), r)y] \Pi(dy) ds \\ &+ \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(s), x(0), j) ds. \end{aligned} \quad (18)$$

Here: the operator L^u is defined by (17). It is acted on F as a function of $x(0)$ only (see [43]), and $Q := (q_{ij})_{i,j=1,2,\dots,N}$ is the generator of Markov chain $r(t)$.

We note that 4th and 5th terms in (18) are bounded due to conditions A.1–A.5.

The generator for the SDDE with jumps only is (which follows from (18))

$$\begin{aligned} \tilde{A}_G J(x(0)) &= q(x(0)) = F(x(0), x(0)) - F(x(-1), x(0)) \\ &+ \int_{-1}^0 L^u F(x(s), x(0)) ds \\ &+ \int_{-1}^0 \int_{-1}^{+\infty} [F(x(s) + x(0)y, x(0)) \\ &- F(x(-1), x(0)) - F_x(x(s), x(0))y] \Pi(dy) ds. \end{aligned} \quad (19)$$

This follows from (18) with $r(t) = 0$. Here: the operator L^u is defined by (17) and it is acted on F as a function of $x(0)$ only (see [43]). We note that 4th term in (19) is bounded as long as F_{xx} is continuous and bounded, and conditions B.1–B.4 and A.5 hold.

The generator for the SDDE with Markov switching is (which follows from (18))

$$\begin{aligned}\tilde{A}_G^r J(x(0), r) &:= q(x(0), r) = F(x(0), x(0), r) - F(x(-1), x(0), r) \\ &+ \int_{-1}^0 L^u F(x(s), x(0), r) ds \\ &+ \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(s), x(0), j) ds.\end{aligned}$$

This follows from (18) with $v = 0$. Here the operator L^u is defined by (17) and acts on F as a function of $x(0)$ only (see [43]), and $Q := (q_{ij})_{i,j=1,2,\dots,N}$ is the generator of Markov chain $r(t)$. We note that 4th term in (20) is bounded as long as F is continuous and bounded, $q_{ij}, i, j \in X$, are finite, and conditions C1–C.3 and A.5 hold.

2.6. Dynkin's Formula for SDDEs with Jumps and Markov Switching

2.6.1. Dynkin's Formula for SDDEs with Jumps and Markov Switching

We obtain from (9) and (18) the following Itô formula for the functional $J(x_t, x(t), r(t))$:

$$\begin{aligned}J(x_t, x(t), r(t)) &= \int_{-1}^0 F(x(t+\theta), x(t), r(t)) d\theta : \\ J(x_t, x(t), r(t)) &= J(x, x(0), r) + \int_0^t F(x(s), x(s), r(s)) ds \\ &- \int_0^t F(x(s-1), x(s), r(s)) ds + \int_0^t \int_{-1}^0 L^u F(x(t+\theta), x(s), r(s)) d\theta ds \\ &+ \int_0^t \int_{-1}^0 \sigma(x(s-1)) F'_x(x(t+\theta), x(s), r(s)) d\theta dw(s) + \\ &\int_0^t \int_{-1}^0 \int_{-1}^{+\infty} [F(x(t+\theta) + x(0)y, x(s), r(s)) - F(x(-1), x(0), r) \\ &- F_x(x(t+\theta), x(s), r(s))y] \Pi(dy) ds d\theta \\ &+ \int_0^t \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(t+\theta), x(s), j) ds d\theta.\end{aligned}\tag{20}$$

The next results, including Lemma 1, follow from this Itô formula and from general Dynkin's formula for Markov processes, [21,22]. We note that the process $(x_t, x(t), r(t))$ is Markov, as is the process $(x_t, x(t), \nu(dy, t)r(t))$.

Lemma 1. Dynkin's formula for SDDEs with jumps and Markov switching.

We consider τ to be a stopping time for the strong Markov process $(x_t, x(t))$ such that $E_{x,x(0)}|\tau| < \infty$. We suppose that all conditions from Sections 2.1–2.5 are satisfied, including A.1–A.5. Then we have the following Dynkin's formula:

$$\begin{aligned}
& E_{x,x(0),r} J(x_\tau, x(\tau), r(\tau)) = J(x, x(0), r) \\
& + E_{x,x(0),r} \int_0^\tau F(x(s), x(s), r(s)) ds - \int_0^\tau F(x(s-1), x(s), r(s)) ds \\
& + E_{x,x(0),r} \int_0^\tau \int_{-1}^0 L^u F(\phi(\theta), x(s), r(s)) d\theta ds + \\
& E_{x,x(0),r} \int_0^\tau \int_{-1}^0 \int_{-1}^{+\infty} [F(x(t+\theta) + x(0)y, x(s), r(s)) \\
& - F(x(-1), x(0), r) - F_x(x(t+\theta), x(s), r(s))y] \Pi(dy) ds d\theta \\
& + E_{x,x(0),r} \int_0^\tau \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(t+\theta), x(s), j) ds d\theta dt \\
& = J(x, x(0)) + E_{x,x(0),r} \int_0^\tau \tilde{A}_G^r J(x_s, x(s), r(s)) ds
\end{aligned} \tag{21}$$

where $\tilde{A}_G^r$ is defined in (18) and L^u in (17).

2.6.2. Dynkin's Formula for SDDEs with Jumps Only

From (13) and (19) we obtain the Itô formula for the functional $J(x_t, x(t))$:

$$\begin{aligned}
J(x_t, x(t)) &= \int_{-1}^0 F(x(t+\theta), x(t)) d\theta : \\
J(x_t, x(t)) &= J(x, x(0)) + \int_0^t F(x(s), x(s)) ds \\
&- \int_0^t F(x(s-1), x(s)) ds + \int_0^t \int_{-1}^0 L^u F(x(t+\theta), x(s)) d\theta ds \\
&+ \int_0^t \int_{-1}^0 \sigma(x(s-1)) F'_x(x(t+\theta), x(s)) d\theta dw(s) \\
&+ \int_0^t \int_{-1}^0 \int_{-1}^{+\infty} [F(x(t+\theta) \\
&+ x(0)y, x(s)) - F(x(-1), x(0)) - F_x(x(t+\theta), x(s))y] \Pi(dy) ds d\theta,
\end{aligned}$$

where operator L^u is defined in (17).

Corollary 1. Dynkin's formula for SDDEs with jumps only.

Consider τ as a stopping time for the strong Markov process $(x_t, x(t))$ such that $E_{x,x(0)}|\tau| < \infty$. We have then the following Dynkin's formula (which follows from (21)):

$$\begin{aligned}
E_{x,x(0)} J(x_\tau, x(\tau)) &= J(x, x(0)) + E_{x,x(0)} \int_0^\tau F(x(s), x(s)) ds - \\
&- \int_0^\tau F(x(s-1), x(s)) ds + E_{x,x(0)} \int_0^\tau \int_{-1}^0 L^u F(\phi(\theta), x(s)) d\theta ds + \\
&E_{x,x(0)} \int_0^\tau \int_{-1}^0 \int_{-1}^{+\infty} [F(x(t+\theta) + x(0)y, x(s)) - F(x(-1), x(0)) - \\
&- F_x(x(t+\theta), x(s))y] \Pi(dy) ds d\theta \\
&= J(x, x(0)) + E_{x,x(0)} \int_0^\tau \tilde{A}_G J(x_s, x(s)) ds,
\end{aligned} \tag{22}$$

where $\tilde{A}_G$ is defined in (19) and L^u in (17).

Proof. Follows from Lemma 1 above. $\square$

Corollary 2. Dynkin's formula for SDDEs with Markov Switching Only.

From (16) and (20) we obtain the following Itô formula for the functional $J(x_t, x(t), r(t))$:

$$\begin{aligned}
J(x_t, x(t), r(t)) &= \int_{-1}^0 F(x(t+\theta), x(t), r(t)) d\theta : \\
J(x_t, x(t), r(t)) &= J(x, x(0), r) + \int_0^t F(x(s), x(s), r(s)) ds \\
&\quad - \int_0^t F(x(s-1), x(s), r(s)) ds + \int_0^t \int_{-1}^0 L^u F(x(t+\theta), x(s), r(s)) d\theta ds \\
&\quad + \int_0^t \int_{-1}^0 \sigma(x(s-1)) F'_x(x(t+\theta), x(s), r(s)) d\theta dw(s) + \\
&\quad + \int_0^t \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(t+\theta), x(s), j) ds d\theta.
\end{aligned}$$

We consider τ to be a stopping time for the strong Markov process $(x_t, x(t))$ such that $E_{x,x(0)}|\tau| < \infty$. Then we have the following Dynkin's formula which follows from (21):

$$\begin{aligned}
E_{x,x(0),r} J(x_\tau, x(\tau), r(\tau)) &= J(x, x(0), r) \\
&\quad + E_{x,x(0),r} \int_0^\tau F(x(s), x(s), r(s)) ds - \int_0^\tau F(x(s-1), x(s), r(s)) ds \\
&\quad + E_{x,x(0),r} \int_0^\tau \int_{-1}^0 L^u F(\phi(\theta), x(s), r(s)) d\theta ds + \\
&\quad + E_{x,x(0),r} \int_0^\tau \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(t+\theta), x(s), j) ds d\theta dt \\
&= J(x, x(0)) + E_{x,x(0),r} \int_0^\tau \tilde{A}_G^r J(x_s, x(s), r(s)) ds
\end{aligned} \tag{23}$$

where $\tilde{A}_G^r$ is defined in (18) and L^u in (17).

Proof. Follows from Lemma 1 above. $\square$

2.7. Solution of Dirichlet–Poisson Problem for SDDEs with Jumps and Markov Switching

2.7.1. Solution of Dirichlet–Poisson Problem for SDDEs with Jumps and Markov Switching

Let $\psi(x, x(0), r) \in C(\partial(H \times G \times X))$ be bounded and $F(x, x(0), u, r) \in \mathbf{C}(\mathcal{D} \times C \times \mathcal{U} \times X)$ be such that

$$E_{x,x(0)} \left[\int_{-1}^0 \int_0^{\tau_{H \times G}} |F(\phi(\theta), x(s), u(s), r(s))| ds d\theta \right] < \infty$$

for all $(x, x(0), r) \in H \times G \times X$.

Here: $\tau_{H \times G \times X} = \inf\{t : (x_t, x(t), r(t)) \notin H \times G \times X\}$, $\mathbf{C}$ is a set of continuous and bounded functions on $\mathcal{D} \times C \times \mathcal{U} \times X$, and $\partial(H \times G \times X)$ is a regular boundary of $H \times G \times X$, $H \in C, G \in \mathcal{D}$ (see, e.g., [21]).

Lemma 2. *Solution of Dirichlet–Poisson problem for SDDEs with jumps and Markov switching.*
Define

$$\begin{aligned}
J(x, x(0), u) &:= E_{x,x(0),r} \left[\int_{-1}^0 \int_0^{\tau_{H \times G}} F(\phi(\theta), x(s), u(s), r(s)) ds d\theta \right] \\
&\quad + E_{x,x(0),r} [\psi(x_{\tau_G}, x(\tau_G), r(\tau_G))], \quad (x, x(0), r) \in H \times G \times X.
\end{aligned}$$

Then (see also [22])

$$\tilde{A}_G^r J(x, x(0), u) = - \int_{-1}^0 F(\phi(\theta), x, u) d\theta, \quad \text{in } H \times G \quad \forall u \in \mathcal{U}$$

and

$$\lim_{t \uparrow \tau_G} J(x_t, x(t), u) = \psi(x(\tau_G), x(\tau_G)) \quad \forall (x, x(0)) \in H \times G,$$

where $\tilde{A}_G^r$ is generator defined above in (18) (see $\tilde{A}_G^r$).

Proof. Follows from the solution of the Dirichlet–Poisson problem for SDDEs in [19] with A^v replaced by $\tilde{A}_G^r$ in (18). We note that in this case, $J \in D(\tilde{A}_G^r)$ and 5th and 6th terms in (18) are bounded due to conditions from Sections 2.1–2.5, including A.1–A.5. $\square$

2.7.2. Dirichlet–Poisson Problem’s Solution for SDDEs with Jumps Only

Let us take $\psi(x, x(0)) \in C(\partial(H \times G))$ to be bounded and let $F(x, x(0), u) \in \mathbf{C}(\mathcal{D} \times C \times \mathcal{U})$ be such that

$$E_{x, x(0)} \left[\int_{-1}^0 \int_0^{\tau_{H \times G}} |F(\phi(\theta), x(s), u(s))| ds d\theta \right] < \infty$$

for all $(x, x(0)) \in H \times G$.

Here: $\tau_{H \times G} = \inf\{t : (x_t, x(t)) \notin H \times G\}$, $\mathbf{C}$ is a set of continuous and bounded functions on $\mathcal{D} \times C \times \mathcal{U}$, and $\partial(H \times G)$ is a regular boundary of $H \times G$, $H \in \mathcal{C}$, $G \in \mathcal{D}$ (see e.g., [21]).

Corollary 3. *Solution of Dirichlet–Poisson problem for SDDEs with jumps only.*

Define

$$J(x, x(0), u) := E_{x, x(0)} \left[\int_{-1}^0 \int_0^{\tau_{H \times G}} F(\phi(\theta), x(s), u(s)) ds d\theta \right] \\ + E_{x, x(0)} [\psi(x_{\tau_G}, x(\tau_G))], \quad (x, x(0)) \in H \times G.$$

Then (see also [22])

$$\tilde{A}_G J(x, x(0), u) = - \int_{-1}^0 F(\phi(\theta), x, u) d\theta, \quad \text{in } H \times G \quad \forall u \in \mathcal{U}$$

and

$$\lim_{t \uparrow \tau_G} J(x_t, x(t), u) = \psi(x(\tau_G), x(\tau_G)) \quad \forall (x, x(0)) \in H \times G,$$

where $\tilde{A}_G$ is generator defined above in (19) (see $\tilde{A}_G$).

Proof. Follows from Lemma 2 above. $\square$

2.7.3. Solution of Dirichlet–Poisson Problem for SDDEs with Markov Switching Only

We consider $\psi(x, x(0), r) \in C(\partial(H \times G \times X))$ to be bounded and let $F(x, x(0), u, r) \in \mathbf{C}(\mathcal{D} \times C \times \mathcal{U} \times X)$ be such that

$$E_{x, x(0)} \left[\int_{-1}^0 \int_0^{\tau_{H \times G}} |F(\phi(\theta), x(s), u(s), r(s))| ds d\theta \right] < \infty$$

for all $(x, x(0), r) \in H \times G \times X$.

Here: $\tau_{H \times G \times X} = \inf\{t : (x_t, x(t), r(t)) \notin H \times G \times X\}$, $\mathbf{C}$ is a set of continuous and bounded functions on $\mathcal{D} \times C \times \mathcal{U} \times X$, and $\partial(H \times G \times X)$ is a regular boundary of $H \times G \times X$, $H \in \mathcal{C}$, $G \in \mathcal{D}$ (see, e.g., [21]).

Corollary 4. *Solution of Dirichlet–Poisson problem for SDDEs with Markov Switching.*

Define

$$J(x, x(0), u) := E_{x, x(0), r} \left[\int_{-1}^0 \int_0^{\tau_{H \times G}} F(\phi(\theta), x(s), u(s), r(s)) ds d\theta \right] \\ + E_{x, x(0), r} [\psi(x_{\tau_G}, x(\tau_G), r(\tau_G))], \quad (x, x(0), r) \in H \times G \times X.$$

Then (see also [22])

$$\tilde{A}_G^r J(x, x(0), u) = - \int_{-1}^0 F(\phi(\theta), x, u) d\theta, \quad \text{in } H \times G \quad \forall u \in \mathcal{U}$$

and

$$\lim_{t \uparrow \tau_G} J(x_t, x(t), u) = \psi(x(\tau_G), x(\tau_G)) \quad \forall (x, x(0)) \in H \times G,$$

where $\tilde{A}_G^r$ is generator defined above in (20).

Proof. Follows from Lemma 2 above. $\square$

2.8. Optimal Control and Performance: Statements of the Problems

2.8.1. Statement of the Problem for SDDs with Jumps and Markov Switchings

Assume that the cost function is given in the form

$$J(x, x(0), u, r) := E_{x, x(0), r} \left[\int_{-1}^0 \int_0^{\tau_{H \times G \times X}} F(\phi(\theta), x(s), u(s), r(s)) ds d\theta \right. \\ \left. + \psi(x_{\tau_{H \times G \times X}}, x(\tau_{H \times G \times X}), r(\tau_{H \times G \times X})) \right]. \quad (24)$$

Here, F is bounded real and continuous function, ψ is a bounded real function, and $\tau_{H \times G \times X}$ is the exit time of the process $(x_t, x(t))$ defined in (12) from the fixed open set $H \times G \times X \subset \mathcal{D} \times \mathcal{C} \times X$. We note that $\tau_{H \times G}$ can be a fixed time t_0 . We assume that $E_{x, x(0), r} |\tau_{H \times G \times X}| < \infty, \forall (x, x(0), r) \in H \times G \times X$.

The problem is: for each $(x, x(0), r) \in H \times G \times X$ find number $J^*(x, x(0), r)$ and control $u^* = u^*(x, x(0), r, \omega)$ such that

$$J^*(x, x(0), r) := \inf_u \{J(x, x(0), u, r)\} = J(x, x(0), u^*, r).$$

Remark 4. The infimum here is taken over all F_t -adapted processes $u(t) \in \mathcal{U}$. Such a control u^* , if it exists, is called an optimal control and $J^*(x, x(0), r)$ is called the optimal performance.

2.8.2. Statement of the Problem for SDDs with Jumps Only

Assume that the cost function is given in the form

$$J(x, x(0), u) := E_{x, x(0)} \left[\int_{-1}^0 \int_0^{\tau_{H \times G}} F(\phi(\theta), x(s), u(s)) ds d\theta \right. \\ \left. + \psi(x_{\tau_{H \times G}}, x(\tau_{H \times G})) \right]. \quad (25)$$

Here, F is bounded real and continuous function, ψ is a bounded real function, and $\tau_{H \times G}$ is the exit time of the process $(x_t, x(t))$ defined in (11) from the fixed open set $H \times G \subset \mathcal{D} \times \mathcal{C}$. In particular, $\tau_{H \times G}$ can be a fixed time t_0 . We assume that $E_{x, x(0)} |\tau_{H \times G}| < \infty, \forall (x, x(0)) \in H \times G$.

The problem here is: for each $(x, x(0)) \in H \times G$ find number $J^*(x, x(0))$ and control $u^* = u^*(x, x(0), \omega)$ such that

$$J^*(x, x(0)) := \inf_u \{J(x, x(0), u)\} = J(x, x(0), u^*).$$

2.8.3. Statement of the Problem for SDDEs with Markov Switchings Only

We assume that the cost function is given in the form

$$J(x, x(0), u, r) := E_{x, x(0), r} \left[\int_{-1}^0 \int_0^{\tau_{H \times G \times X}} F(\phi(\theta), x(s), u(s), r(s)) ds d\theta \right. \\ \left. + \psi(x_{\tau_{H \times G \times X}}, x(\tau_{H \times G \times X}), r(\tau_{H \times G \times X})) \right]. \quad (26)$$

Here: F is bounded real and continuous function, ψ is a bounded real function, and $\tau_{H \times G \times X}$ is the exit time of the process $(x_t, x(t))$ defined in (14) from the fixed open set $H \times G \times X \subset \mathcal{D} \times C \times X$. In particular, $\tau_{H \times G}$ can be a fixed time t_0 . We assume that $E_{x, x(0), r} |\tau_{H \times G \times X}| < \infty, \forall (x, x(0), r) \in H \times G \times X$.

The problem in this case is: for each $(x, x(0), r) \in H \times G \times X$ find number $J^*(x, x(0), r)$ and control $u^* = u^*(x, x(0), r, \omega)$ such that

$$J^*(x, x(0), r) := \inf_u \{J(x, x(0), u, r)\} = J(x, x(0), u^*, r).$$

3. Hamilton–Jacobi–Bellman Equation for SDDEs with Jumps and Markov Switching

This Section 3 contains the descriptions of HJB and converse HJB equations for SDDEs with jumps and Markov switching (Theorems 1 and 2, Section 3.1), HJB and converse HJB equations for SDDEs with jumps only (Theorems 3 and 4, Section 3.2). HJB and converse HJB equations for SDDEs with Markov switching only (Theorems 5 and 6, Section 3.3). Derivations and proofs of those results are presented in Appendix B. We note, that these results are new and original and are different from those in [11,19], which considered only diffusion SDDEs without jumps and Markov switching.

3.1. Hamilton–Jacobi–Bellman Equation for SDDEs with Jumps and Markov Switching

Only Markov controls are considered here: $u(t) := u(x_t, x(t), r(t))$. We define the following operator for every $u \in \mathcal{U}$

$$(A^{u, r} J)(x, x(0), r) = F(x(0), x(0), u(0), r) - F(x(-1), x(0), u(0), r) + \\ + \int_{-1}^0 L^v F(\phi(\theta), x(0), u(0), r) d\theta + \int_{-1}^0 \int_{-1}^{+\infty} [F(x(s) + x(0)y, x(0), r) \\ - F(x(-1), x(0), r) - F_x(x(s), x(0), r)y] \Pi(dy) ds \\ + \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(s), x(0), j) ds, \quad (27)$$

where $u(0) := u(x, x(0), r)$; operator L^u is given by (17) (see L^u):

$$J(x, x(0)) := \int_{-1}^0 F(\phi(\theta), x(0), u(0)) d\theta$$

For each control u , the pair $(x_t, x(t), r(t))$, where $x(t)$ is the solution of Equation (9), is an Ito diffusion with the infinitesimal generator $(AJ)(x, x(0)) = (A_G^r J)(x, x(0), r)$ with A_G^r from (18).

Theorem 1. (HJB-equation for SDDE with jumps and Markov switchings). Define

$$J^*(x, x(0), r) = \inf \{J(x, x(0), u, r) : u := u(x, x(0), r) - \text{Markov control}\}.$$

Suppose that $J \in D(A^{u,r})$ and the optimal control u^* exists. Then

$$\inf_{u \in \mathcal{U}} \left[\int_{-1}^0 F(\phi(\theta), x, v, r) d\theta + (A^{u,r} J^*)(x, x(0), r) \right] = 0, \quad (28)$$

$$\forall (x, x(0), r) \in H \times G \times X,$$

and

$$J^*(x, x(0), r) = \psi(x, x(0), r), \quad \forall (x, x(0), r) \in \partial(H \times G \times X).$$

Here, functions F and ψ are given by (17) and operator $A^{u,r}$ is given by (27); $\partial(H \times G \times X)$ is the boundary of set $H \times G \times X$ for $(x_t, x(t), r(t))$.

The infimum in (28) is achieved when $u = u^*(x, x(0), r)$, where u^* is optimal. In other words,

$$\int_{-1}^0 F(\phi(\theta), x, u^*, r) d\theta + (A^{u^*,r} J^*)(x, x(0), r) = 0, \quad (29)$$

$$\forall (x, x(0)) \in H \times G \times X,$$

which is Equation (28).

Proof. See Appendix B. $\square$

Theorem 2. (Converse of the HJB-equation for SDDE with jumps and Markov switchings). Let g be a bounded function and $g \in D(A^{u,r})$. Suppose that for all $u \in \mathcal{U}$ the inequality

$$\int_{-1}^0 F(x, x(0), \phi(\theta), u, r) d\theta + (A^{u,r} g)(x, x(0), r) \geq 0,$$

$$(x, x(0)) \in H \times G \times X$$

with the following boundary condition

$$g(x, x(0), r) = \psi(x, x(0)), \quad (x, x(0), r) \in \partial H \times G \times X \quad (30)$$

are satisfied. Then $g(x, x(0), r) \leq J(x, x(0), u, r)$ for all Markov controls $u \in \mathcal{U}$ and for all $(x, x(0), r) \in H \times G \times X$.

Moreover, if for every $(x, x(0), r) \in H \times G \times X$ there exists u^0 such that

$$\int_{-1}^0 F(x, x(0), \phi(\theta), u^0, r) d\theta + (A^{u^0,r} g)(x, x(0), r) = 0, \quad (31)$$

then u^0 is a Markov control, $g(x, r) = J(x, x(0), u^0, r) = J^*(x, x(0), r)$, and therefore u^0 is an optimal control.

Proof. See Appendix B. $\square$

3.2. Hamilton–Jacobi–Bellman Equation for SDDEs with Jumps Only

Only Markov controls are considered here as well. $u(t) := u(x_t, x(t))$. Let us define the following operator for every $u \in \mathcal{U}$

$$\begin{aligned} (A^u J)(x, x(0)) &= F(x(0), x(0), u(0)) - F(x(-1), x(0), u(0)) + \\ &+ \int_{-1}^0 L^u F(\phi(\theta), x(0), u(0)) d\theta + \int_{-1}^0 \int_{-1}^{+\infty} [F(x(s) + x(0)y, x(0) \\ &- F(x(-1), x(0)) - F_x(x(s), x(0))y] \Pi(dy) ds, \end{aligned} \quad (32)$$

where $u(0) := u(x, x(0))$; operator L^u is given by (11):

$$J(x, x(0)) := \int_{-1}^0 F(\phi(\theta), x(0), u(0)) d\theta$$

For each control u , the pair $(x_t, x(t))$, where $x(t)$ is the solution of Equation (12), is an Ito diffusion with the infinitesimal generator $(AJ)(x, x(0)) = (A_G^u J)(x, x(0))$.

Theorem 3. (HJB-equation for SDDE with jumps only). Define

$$J^*(x, x(0)) = \inf\{J(x, x(0), u) : u := u(x, x(0)) - \text{Markov control}\}. \quad (33)$$

Suppose that $J \in D(A^v)$ and the optimal control u^* exists. Then

$$\inf_{v \in \mathcal{U}} \left[\int_{-1}^0 F(\phi(\theta), x, v) d\theta + (A^v J^*)(x, x(0)) \right] = 0, \quad \forall (x, x(0)) \in H \times G, \quad (34)$$

and

$$J^*(x, x(0)) = \psi(x, x(0)), \quad \forall (x, x(0)) \in \partial(H \times G).$$

Here: functions F and ψ are given by (17) and operator A^u is given by (32), $\partial(H \times G)$ is the boundary of set $H \times G$ for $(x_t, x(t))$. The infimum in (34) is achieved when $v = u^*(x, x(0))$, where u^* is optimal. In other words,

$$\int_{-1}^0 F(\phi(\theta), x, u^*) d\theta + (A^{u^*} J^*)(x, x(0)) = 0, \quad \forall (x, x(0)) \in H \times G, \quad (35)$$

which is Equation (34).

Proof. See Appendix B. $\square$

Theorem 4. (Converse of the HJB-equation for SDDE with jumps only). Let g be a bounded function and $g \in D(A^u)$. Suppose that for all $u \in \mathcal{U}$ the inequality

$$\int_{-1}^0 F(x, x(0), \phi(\theta), u) d\theta + (A^u g)(x, x(0)) \geq 0, \quad (x, x(0)) \in H \times G$$

and the boundary condition

$$g(x, x(0)) = \psi(x, x(0)), \quad (x, x(0)) \in \partial H \times G \quad (36)$$

are satisfied. Then $g(x, x(0)) \leq J(x, x(0), u)$ for all Markov controls $u \in \mathcal{U}$ and for all $(x, x(0)) \in H \times G$.

Moreover, if for every $(x, x(0)) \in H \times G$ there exists u^0 such that

$$\int_{-1}^0 F(x, x(0), \phi(\theta), u^0) d\theta + (A^{u^0} g)(x, x(0)) = 0, \quad (37)$$

then u^0 is a Markov control, $g(x) = J(x, x(0), u^0) = J^*(x, x(0))$, and therefore u^0 is an optimal control.

Proof. See Appendix B. $\square$

3.3. Hamilton–Jacobi–Bellman Equation for SDDEs with Markov Switching Only

We consider only Markov controls $u(t) := u(x_t, x(t), r(t))$. We define the following operator for every $u \in \mathcal{U}$

$$(A^{u,r}J)(x, x(0), r) = F(x(0), x(0), u(0), r) - F(x(-1), x(0), u(0), r) + \int_{-1}^0 L^u F(\phi(\theta), x(0), u(0), r) d\theta + \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(s), x(0), j) ds. \quad (38)$$

Here, $u(0) := u(x, x(0), r)$; operator L^u is given by (17):

$$J(x, x(0)) := \int_{-1}^0 F(\phi(\theta), x(0), u(0)) d\theta$$

For each control u , the pair $(x_t, x(t), r(t))$, where $x(t)$ is the solution of Equation (9), is an Ito diffusion with the infinitesimal generator $(AJ)(x, x(0)) = (A^{u,r}J)(x, x(0), r)$ with $A^{u,r}$ from (38).

Theorem 5. (HJB-equation for SDDE with Markov Switchings Only). Define

$$J^*(x, x(0), r) = \inf \{ J(x, x(0), u, r) : u := u(x, x(0), r) - \text{Markov control} \}.$$

Suppose that $J \in D(A^{u,r})$ and the optimal control u^* exists. Then

$$\inf_{v \in \mathcal{U}} \left[\int_{-1}^0 F(\phi(\theta), x, v, r) d\theta + (A^{v,r}J^*)(x, x(0), r) \right] = 0, \quad (39)$$

$$\forall (x, x(0), r) \in H \times G \times X,$$

and

$$J^*(x, x(0), r) = \psi(x, x(0), r), \quad \forall (x, x(0), r) \in \partial(H \times G \times X).$$

Here, functions F and ψ are given by (17) and operator $A^{u,r}$ is given by (38), $\partial(H \times G \times X)$ is the boundary of set $H \times G \times X$ for $(x_t, x(t), r(t))$.

The infimum in (39) is achieved for $u = u^*(x, x(0), r)$, with u^* being optimal, i.e.,

$$\int_{-1}^0 F(\phi(\theta), x, u^*, r) d\theta + (A^{u^*,r}J^*)(x, x(0), r) = 0, \quad (40)$$

$$\forall (x, x(0)) \in H \times G \times X,$$

which is Equation (39).

Proof. See Appendix B. $\square$

Theorem 6. (Converse of the HJB-equation for SDDE with jumps and Markov switchings Only.) Let g be a bounded function and $g \in D(A^{u,r})$. Suppose that for all $u \in \mathcal{U}$ the inequality

$$\int_{-1}^0 F(x, x(0), \phi(\theta), u, r) d\theta + (A^{u,r}g)(x, x(0), r) \geq 0,$$

$$(x, x(0)) \in H \times G \times X$$

with the following boundary condition

$$g(x, x(0), r) = \psi(x, x(0)), \quad (x, x(0), r) \in \partial H \times G \times X \quad (41)$$

are satisfied. Then $g(x, x(0), r) \leq J(x, x(0), u, r)$ for all Markov controls $u \in \mathcal{U}$ and for all $(x, x(0), r) \in H \times G \times X$.

Moreover, if for every $(x, x(0), r) \in H \times G \times X$ there exists u^0 such that

$$\int_{-1}^0 F(x, x(0), \phi(\theta), u^0, r) d\theta + (A^{u^0, r}g)(x, x(0), r) = 0, \quad (42)$$

then u^0 is a Markov control, $g(x, r) = J(x, x(0), u^0, r) = J^*(x, x(0), r)$, and therefore u^0 is an optimal control.

Proof. See Appendix B. $\square$

4. Optimizations of Economics Ramsey's Models

In this Section 4 we consider optimization results for economic Ramsay's models with jumps and Markov switching. The optimal controls and cost functions are found for economic Ramsay's models with jumps and Markov switching (Section 4.2.1), for economic Ramsay's models with jumps only (Section 4.2.2) and for economic Ramsay's models with Markov switching only (Section 4.2.3). These results are new and original and are different from those in [11,19] where we considered only diffusion economic Ramsay's models without jumps and Markov switching.

4.1. Descriptions of the Models

For the description and optimal control of the diffusion Ramsey model with delay we refer to [19], and for the description of the averaged diffusion Ramsey model with semi-Markov regime-switching we refer to [11].

In the current paper, as we mentioned in the Introduction, we consider three new Ramsey models in economics.

The first model is the Ramsey diffusion model with jumps and Markov switchings:

$$\begin{aligned} dK(t) &= [A(r(t))K(t-T) - u(K(t))C(t)] dt \\ &+ \sigma(K(t-T), r(t))dw(t) + \int_{-1}^{+\infty} yv(dy, dt). \end{aligned} \quad (43)$$

We suppose that functions $A(r) > 0$ and $\sigma(\cdot, r) > 0$ are bounded and continuous functions by r on X .

The second one is the Ramsey diffusion model with jumps:

$$\begin{aligned} dK(t) &= [AK(t-T) - u(K(t))C(t)] dt + \sigma(K(t-T))dw(t) \\ &+ \int_{-1}^{+\infty} yv(dy, dt). \end{aligned} \quad (44)$$

The jumps in the model describe sudden shocks in an economical evolutions, such as sudden change of the capital K or in the consumption C .

The third model is the Ramsey diffusion model with Markov switchings:

$$\begin{aligned} dK(t) &= [A(r(t))K(t-T) - u(K(t))C(t)] dt \\ &+ \sigma(K(t-T), r(t))dw(t). \end{aligned} \quad (45)$$

The regimes in the model describe different states of the economy. For all three models, the initial condition is

$$K(t) = \phi(t), \quad t \in [-1, 0]. \quad (46)$$

In the next section we present a numerical example with optimization calculations for both cases.

4.2. Optimization Derivations

Optimization calculations are provided for the Ramsey diffusion model with jumps and with Markov switching. The models are different because their cost functions are different—see (48), (50) and (52). The problem is to minimize the investment capital under the assumption of labor being constant.

4.2.1. Optimization Derivations for Ramsey Diffusion Model with Jumps and Markov Switching

Here, we consider the Ramsey diffusion model with jumps and Markov switching in (34) with boundary condition (35).

Let us choose the following cost function:

$$J(K, u, r) = \frac{A(r)K^2(0)}{2} + \int_{-1}^0 \phi^2(\theta) d\theta + \frac{u^2(0)}{2} + \frac{K^2(0)u^2(0)}{2} \int_{-1}^{+\infty} y^2 \Pi(dy). \quad (47)$$

We note that the cost function should be positive-definite, and the choice for J to be as in (48) was inspired by [19]. The new expression for the J , compared with J in [19], contains regime-switching coefficient $A(r)$ and the last jump term with measure $\Pi(dy)$ in (48).

The operator $A_G^u J$ has the following form taking into account (38)

$$\begin{aligned} A^{u,r} J &= \frac{A(r)K^2(0)}{2} + \phi^2(0) + \frac{u^2(0)}{2} + \frac{K^2(0)u^2(0)}{2} \int_{-1}^{+\infty} y^2 \Pi(dy) \\ &\quad - \left[\frac{A(r)K^2(0)}{2} + \phi^2(-1) + \frac{u^2(0)}{2} \right] \\ &\quad + \left[K(0) \cdot (A(r) \cdot K(0) - u(0) \cdot C(0)) + \frac{1}{2} \sigma^2(K(0)) \right] \\ &\quad + \frac{1}{2} \int_{-1}^{+\infty} [(K(0) + yK(0))^2 - K^2(0) - 2K(0)yK(0)] \Pi(dy) \\ &= \frac{A(r)K^2(0)}{2} + \phi^2(0) + \frac{u^2(0)}{2} + \frac{K^2(0)u^2(0)}{2} \int_{-1}^{+\infty} y^2 \Pi(dy) \\ &\quad - \left[\frac{A(r)K^2(0)}{2} + \phi^2(-1) + \frac{u^2(0)}{2} \right] \\ &\quad + \left[K(0) \cdot (A(r) \cdot K(0) - u(0) \cdot C(0)) + \frac{1}{2} \sigma^2(K(0)) \right] + \frac{1}{2} K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy), \end{aligned}$$

since

$$\begin{aligned} F(K(0), K(0), u(0), r) &= \frac{A(r)K^2(0)}{2} + \phi^2(0) + \frac{u^2(0)}{2}, \\ F(K(0), K(-1), u(0), r) &= \frac{A(r)K^2(0)}{2} + \phi^2(-1) + \frac{u^2(0)}{2}, \\ L^u J(K, u, r) &= K(0) \left(A(r) \cdot K(0) - u(0) \cdot C(0) + \frac{1}{2} \sigma^2(K(0)) \right). \end{aligned}$$

From Theorem 1 (28), we obtain the following HJB-equation

$$\begin{aligned} \inf_u &\left[\frac{A(r)K^2(0)}{2} + \int_{-1}^0 \phi^2(\theta) d\theta + \frac{u^2(0)}{2} + \frac{K^2(0)u^2(0)}{2} \int_{-1}^{+\infty} y^2 \Pi(dy) \right. \\ &\quad + \phi^2(0) - \phi^2(-1) + A(r) \cdot K^2(0) \\ &\quad \left. + u(0) \cdot K(0) \cdot C(0) + \frac{1}{2} \sigma^2(K(0)) + \frac{1}{2} K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy) \right] = 0, \end{aligned}$$

or equivalently,

$$\inf_u \left[u^2(0) \left(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy) \right) - 2K(0)C(0)u(0) + (2\phi^2(0) - 2\phi^2(-1) + 2 \int_{-1}^0 \phi^2(\theta) d\theta \right. \\ \left. + K^2(0)(1 + 2A(r)) + \sigma^2(K(0))) + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy) \right] = 0.$$

Let

$$4K^2(0)C^2(0) \geq 4(2\phi^2(0) - 2\phi^2(-1) + 2 \int_{-1}^0 \phi^2(\theta) d\theta \\ + K^2(0)(1 + 2A(r)) + \sigma^2(K(0))) + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy) \left(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy) \right).$$

Hence, the infimum is achieved when

$$u(0) = - \left(- \frac{2K(0) \cdot C(0)}{2(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))} \right) = \frac{K(0) \cdot C(0)}{(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))}.$$

Therefore $u_{\min} = \frac{K(0) \cdot C(0)}{(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))}$ and

$$J(K, u_{\min}, r) = \frac{A(r)K^2(0)}{2} + \frac{K^2(0) \cdot C^2(0)}{2(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))^2} \\ + \int_{-1}^0 \phi^2(\theta) d\theta + \frac{K^4(0) \cdot C^2(0)}{2(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))^2} \int_{-1}^{+\infty} y^2 \Pi(dy).$$

As long as the last expression is positive for all $r \in X$, and $A(r)$ is positive, then

$$J(K, u_{\min}) = \min_r J(K, u_{\min}, r) = \min_r \left[\frac{A(r)K^2(0)}{2} + \frac{K^2(0) \cdot C^2(0)}{2(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))^2} \right. \\ \left. + \int_{-1}^0 \phi^2(\theta) d\theta + \frac{K^4(0) \cdot C^2(0)}{2(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))^2} \int_{-1}^{+\infty} y^2 \Pi(dy) \right] \\ = K^2(0) \left[\frac{\min_r A(r)}{2} + \frac{C^2(0)}{2(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))^2} \right. \\ \left. + \int_{-1}^0 \phi^2(\theta) d\theta + \frac{K^2(0) \cdot C^2(0)}{2(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))^2} \int_{-1}^{+\infty} y^2 \Pi(dy) \right]. \quad (48)$$

4.2.2. Optimization Derivations for Ramsey Diffusion Model with Jumps

Let us consider the Ramsey diffusion model with jumps in (45), with boundary condition (47). In this case, the cost function is (here, $A(r) = 1$)

$$J(K, u) = \frac{K^2(0)}{2} + \int_{-1}^0 \phi^2(\theta) d\theta + \frac{u^2(0)}{2} + \frac{K^2(0)u^2(0)}{2} \int_{-1}^{+\infty} y^2 \Pi(dy). \quad (49)$$

From (49) it follows that $u_{\min} = \frac{K(0) \cdot C(0)}{(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))}$ and the cost function is

$$J(K, u_{\min}) = \\ = K^2(0) \left[\frac{1}{2} + \frac{C^2(0)}{2(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))^2} \right. \\ \left. + \int_{-1}^0 \phi^2(\theta) d\theta + \frac{K^2(0) \cdot C^2(0)}{2(2 + K^2(0) \int_{-1}^{+\infty} y^2 \Pi(dy))^2} \int_{-1}^{+\infty} y^2 \Pi(dy) \right]. \quad (50)$$

4.2.3. Optimization Derivations for Ramsey Diffusion Model with Markov Switching

Let us consider the Ramsey diffusion model with Markov switching in (46), with boundary condition (47). In this case, the cost function is (here, $\Pi(dy) = 0$)

$$J(K, u) = \frac{A(r)K^2(0)}{2} + \int_{-1}^0 \phi^2(\theta) d\theta + \frac{u^2(0)}{2}. \quad (51)$$

From (49) it follows that $u_{\min} = \frac{K(0) \cdot C(0)}{2}$ and the cost function is

$$\begin{aligned} J(K, u_{\min}) &= \\ &= K^2(0) \left[\frac{\min_r A(r)}{2} + \frac{C^2(0)}{4} + \int_{-1}^0 \phi^2(\theta) d\theta \right]. \end{aligned} \quad (52)$$

5. Numerical Examples

This Section 5 contains numerical examples for all three economic Ramsay's models with jumps and Markov switching (Section 5.3), economic Ramsay's models with jumps only (Section 5.1) and economic Ramsay's models with Markov switching only (Section 5.2). These numerical examples are new and original. We note that the paper [19] does not contain any numerical example and paper [11] contains numerical example only for averaged economic Ramsay's models with semi-Markov regime-switching.

Suppose that $A(0) = 2$, $A(1) = 3$, $C(0) = 1$, $K(0) = 1$, $\phi(0) = 1$. We also suppose that jumps has exponential distribution (here, density is $\lambda e^{-\lambda y}$) with parameter $\lambda = 1$. We take $\phi(t) = e^{at}$ and consider two cases, $a = 1$ and $a = -1$.

5.1. Numerical Example: Optimization Calculation for Ramsey Diffusion Model with Jumps Only

From Section 4.2.2 it follows that $\int_{-1}^{+\infty} y^2 \Pi(dy) = 2$. Then $u_{\min} = 1/4$ and, according to (51), when $a = 1$, then $J(K, u_{\min}) = 0.94$, and when $a = -1$, then $J(K, u_{\min}) = 3.71$.

For dependence J on jumps (on λ in our case, the rest of the parameters are fixed), see Appendix A, Figure A3 ($a = -\pm 1$). For dependence J on jumps ($K(0)$ and $C(0)$ in our case, the rest of the parameters are fixed), see Appendix A, Figure A7 ($a = -1$) and Figure A8 ($a = 1$).

5.2. Numerical Example: Optimization Calculation for Ramsey Diffusion Model with Markov Switching Only

Taking into account results of Section 4.2.3, we have that $u_{\min} = 1/2$ and, according to (53), when $a = 1$, then $J(K, u_{\min}) = 1.68$, and when $a = -1$, then $J(K, u_{\min}) = 4.45$.

For dependence J on Markov switching (on $A(r)$ in our case, the rest of the parameters are fixed), see Appendix A, Figure A4 ($a = \pm 1$). For dependence J on Markov switching ($K(0)$ and $C(0)$ in our case, the rest of the parameters are fixed), see Appendix A, Figure A9 ($a = -1$) and Figure A10 ($a = 1$).

5.3. Numerical Example: Optimization Calculation for Ramsey Diffusion Model with Jumps and Markov Switching

Taking into account results of section 4.2.1, we have that $u_{\min} = 1/4$ and, according to (49), when $a = 1$, then $J(K, u_{\min}) = 1.53$, and when $a = -1$, then $J(K, u_{\min}) = 4.3$.

For dependence J on jumps and Markov switching (on λ and $\min A(r)$ in our case, the rest of the parameters are fixed), see Appendix A, Figure A1 ($a = -1$) and Figure A2 ($a = 1$). For dependence J on jumps and Markov switching ($K(0)$ and $C(0)$ in our case, the rest of the parameters are fixed), see Appendix A, Figure A5 ($a = -1$) and Figure A6 ($a = 1$).

6. Conclusions

This paper considered optimal control of SDDEs with jumps and Markov switching, and its economical applications. We derived a Hamilton–Jacobi–Bellman (HJB) equation and the inverse HJB-equation by applying the Dynkin formula and solution of the Dirichlet–Poisson problem. Three cases of modified Ramsey stochastic models in economics have been studied; namely, a Ramsey diffusion model with jumps, a Ramsey diffusion model with Markov switching, and a Ramsey diffusion model with jumps and Markov switching. We have also presented numerical examples for all three cases. The contributions of the paper are four-fold: introducing and studying three new models for SDDEs—with jumps, Markov switching and both; studying optimal control for those models; applications of these results to economics with Ramsey’s stochastic models; numerical examples for these different models.

7. Discussion and Future Work

The future work will be devoted to the sensitivity analysis of the Markov transition matrix coefficients to demonstrate how persistence in certain regimes affects long-term optimal control. Another interesting topic would be considering “tempered fractional gradient descent” as a promising perspective or future extension, given its relevance to optimal control and stochastic systems. In the economic application, our choice of stochastic parameters was not based on real data, and was not justified. However, to explain how jumps and Markov switching affect the rate of capital accumulation differently than standard diffusion, we have presented visualizations of our results in Figures A1–A10. Thus, in our future work we will consider these models and their analysis based on real economics data.

Author Contributions: Conceptualization, A.V.S.; methodology, A.V.S.; software, M.S. and A.V.S.; validation, M.S.; formal analysis, A.V.S. and M.S.; investigation, A.V.S. and M.S.; resources, M.S. and A.V.S.; data curation, writing—original draft preparation, A.V.S. and M.S.; writing—review and editing, A.V.S. and M.S.; visualization, supervision, project administration, funding acquisition, A.V.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by NSERC grant number RT732266.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: All authors are thankful to the ICIAM 2023 organizing committee (Tokyo, Japan) for the opportunity to present these results in the Poster session (Poster24, 13102, C034). The second author thanks the NSERC for continuing support. We also thank to Sudeesha Nawodh (University of Calgary) for preparing all the Figures A1–A10. And finally, the authors thank three anonymous referees for their valuable comments and suggestions that definitely improved the paper.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Figures

The Figures below are visualization of the dependence of cost functions J on jumps and Markov switching, regarding different parameters of the models under consideration. The Figures are new and original. We note that [11,19] do not contain any visualizations of the their results.

Both jumps and Markov switching: J vs λ and $\min A(r)$ ($a=-1$)

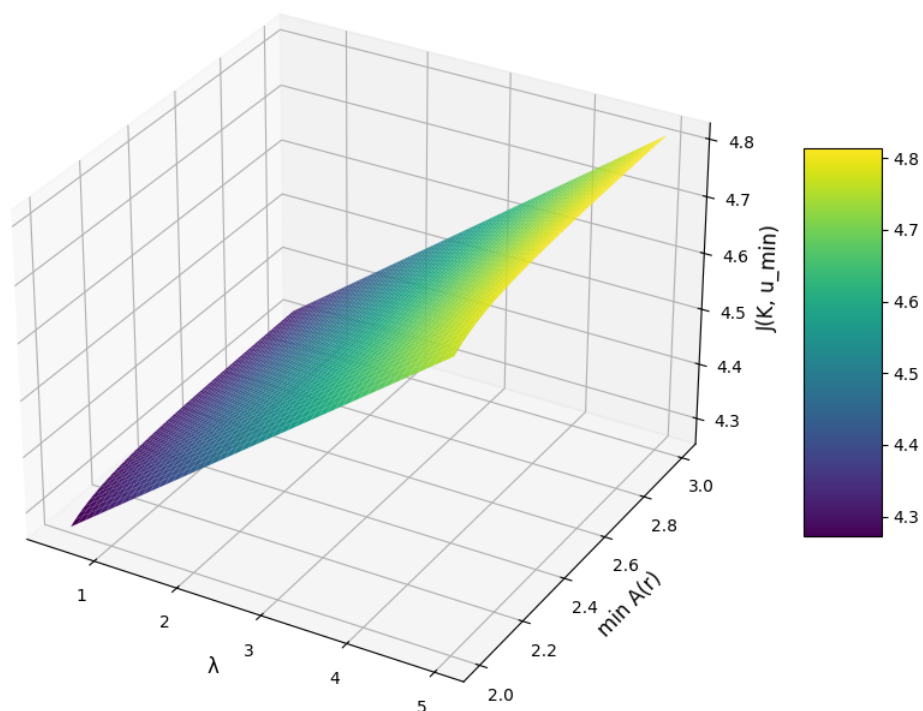


Figure A1. Dependence of cost function J on λ and $\min A(r)$: jumps and Markov switching case ($a = -1$).

Both jumps and Markov switching: J vs λ and $\min A(r)$ ($a=1$)

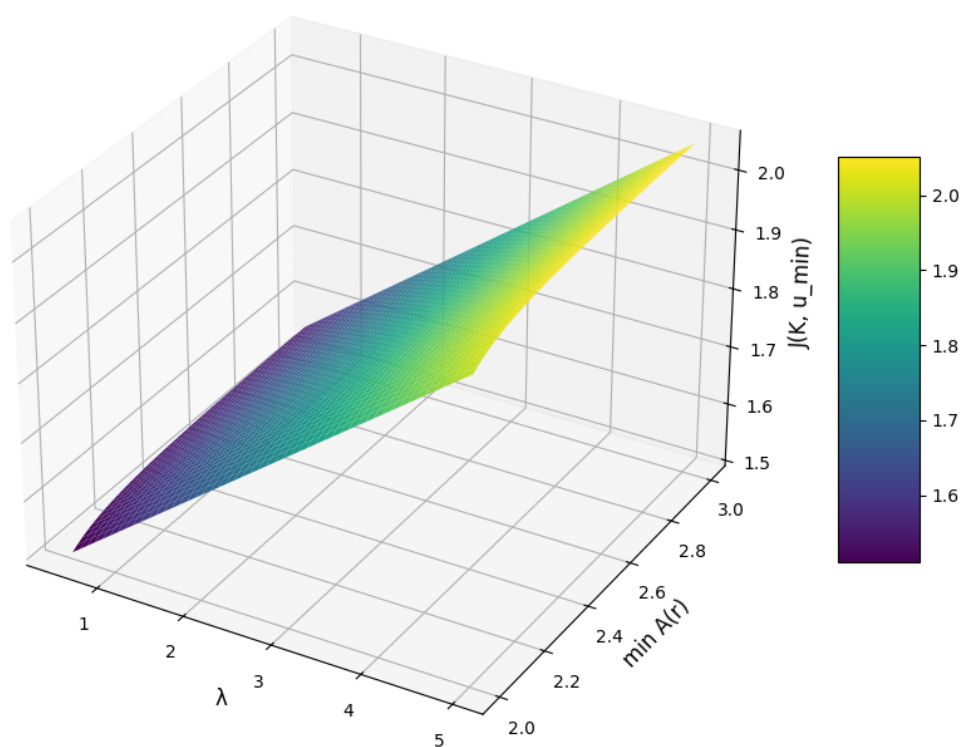


Figure A2. Dependence of cost function J on λ and $\min A(r)$: jumps and Markov switching case ($a = 1$).

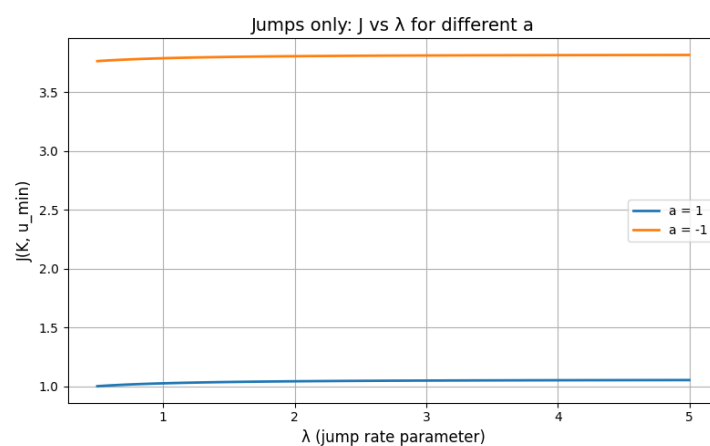


Figure A3. Dependence of cost function J on λ : jumps case ($a = \pm 1$).

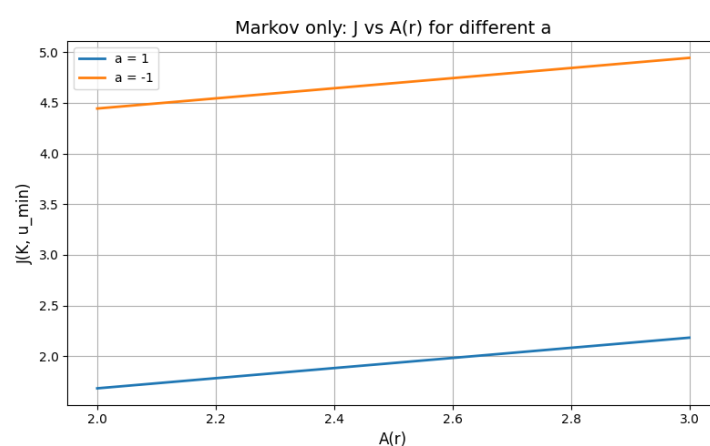


Figure A4. Dependence of cost function J on $A(r)$: Markov switching case ($a = \pm 1$).

Jumps and Markov: J vs $K(0)$ and $C(0)$ ($a=-1, \min A(r)=2, \lambda=1$)

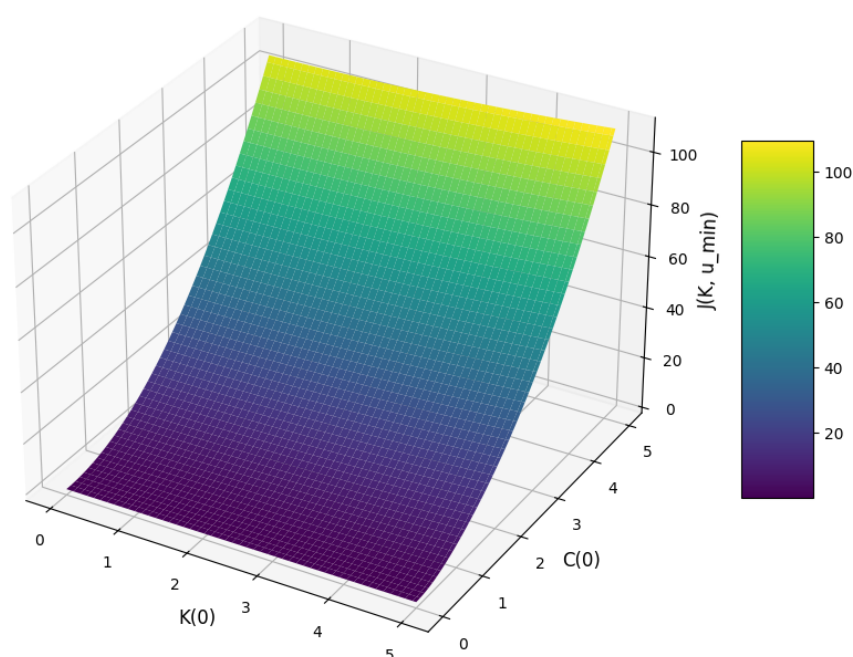


Figure A5. Dependence of cost function J on $K(0)$ and $C(0)$: jumps and Markov switching case ($a = -1$).

Jumps and Markov: J vs $K(0)$ and $C(0)$ ($a=1, \min A(r)=2, \lambda=1$)

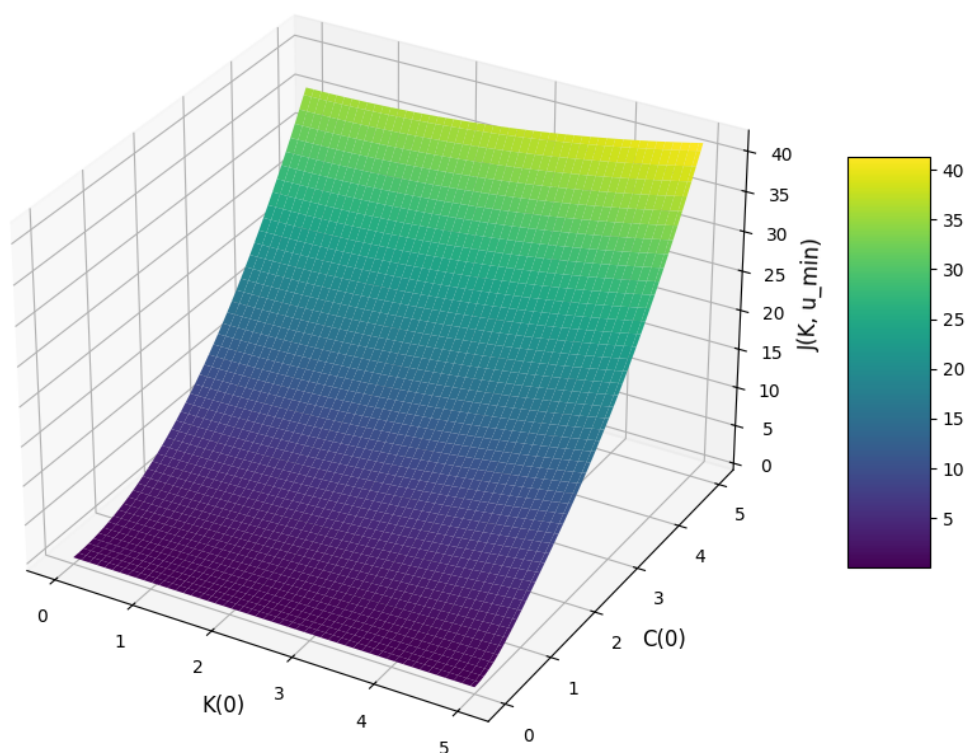


Figure A6. Dependence of cost function J on $K(0)$ and $C(0)$: jumps and Markov switching case ($a = 1$).

Jumps only: J vs $K(0)$ and $C(0)$ ($a=-1, \lambda=1$)

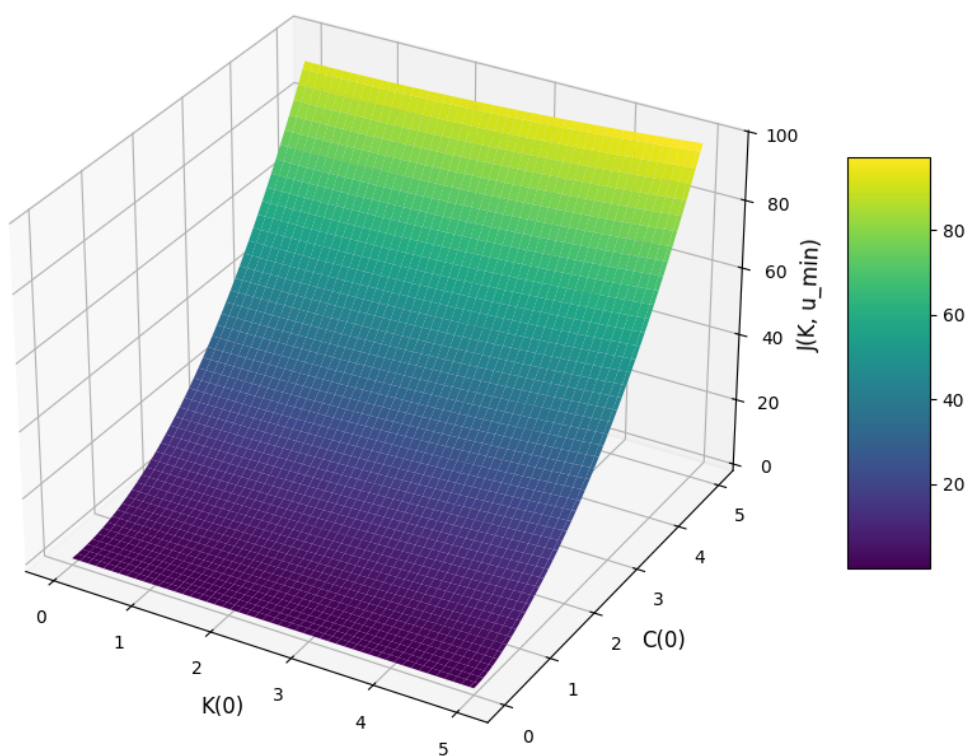


Figure A7. Dependence of cost function J on $K(0)$ and $C(0)$: jumps case ($a = -1$).

Jumps only: J vs $K(0)$ and $C(0)$ ($a=1, \lambda=1$)

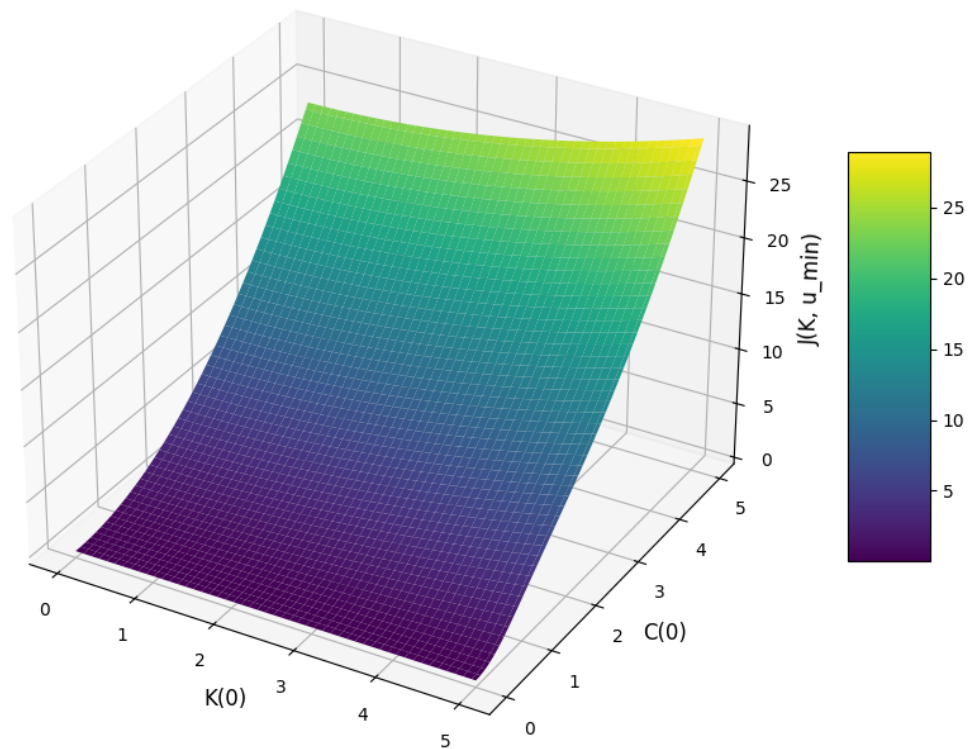


Figure A8. Dependence of cost function J on $K(0)$ and $C(0)$: jumps case ($a = 1$).

Markov only: J vs $K(0)$ and $C(0)$ ($a=-1, \min A(r)=2$)

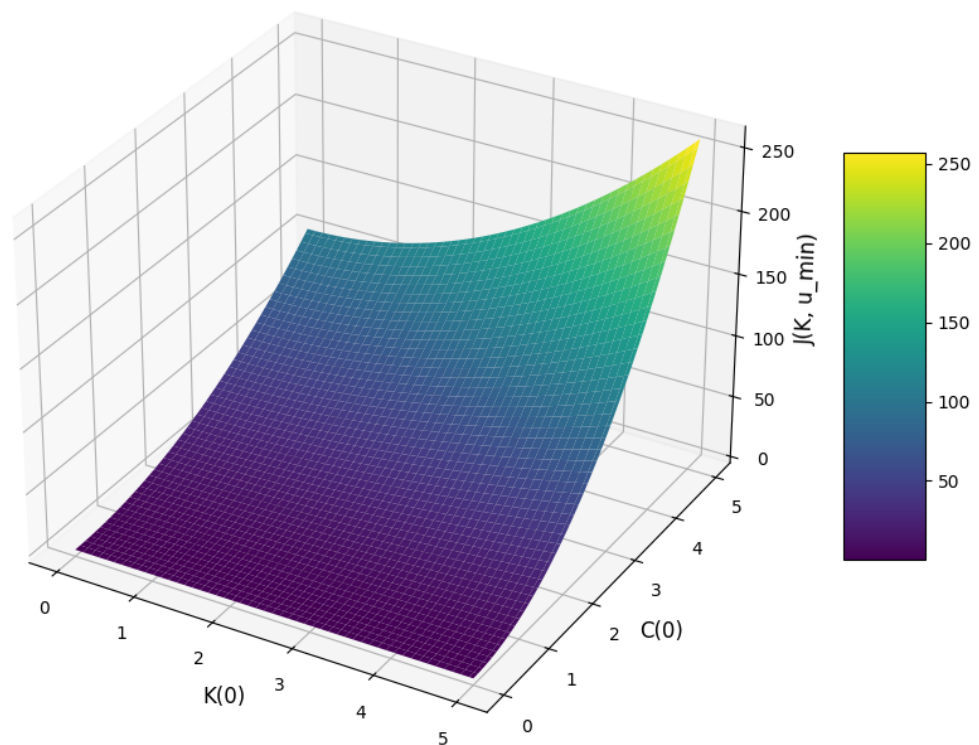
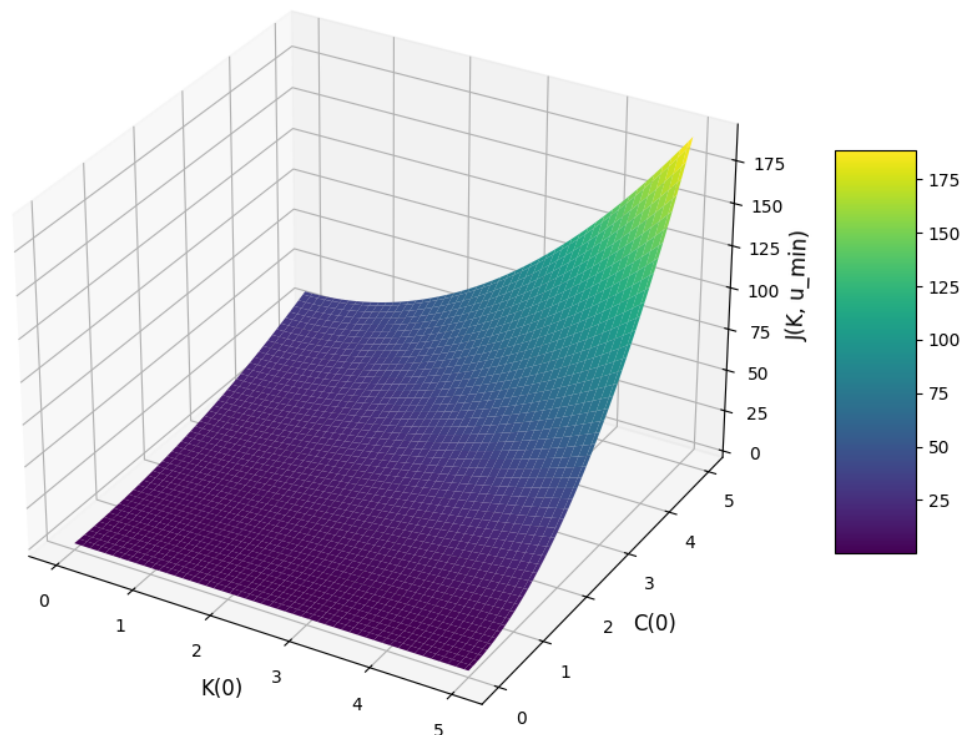


Figure A9. Dependence of cost function J on $K(0)$ and $C(0)$: Markov switching case ($a = -1$).

Markov only: J vs K(0) and C(0) (a=1,min A(r)=2)

**Figure A10.** Dependence of cost function J on $K(0)$ and $C(0)$: Markov switching case ($a = 1$).

Appendix B. Proofs

This Appendix B contains proofs for all Theorems 1–6 considered in Section 2.3. Those proofs are new and original and based on new weak infinitesimal operators (w.i.o.) considered in Section 2.5 for different SDEs considered in Sections 2.2–2.4. We note that paper [19] considered w.i.o. only for diffusion SDDE and paper [11] considered only w.i.o. for averaged diffusion SDDE with semi-Markov switching.

Proof of Theorem 1. Follows the ideas of Theorem 3.1 in [19] and Lemma 2 above with replacement operator A_G^u by $A^{u,r}$ from (27), i.e.,

$$\begin{aligned} (A^{u,r}J)(x, x(0), r) &= F(x(0), x(0), u(0), r) - F(x(-1), x(0), u(0), r) + \\ &+ \int_{-1}^0 L^v F(\phi(\theta), x(0), u(0), r) d\theta + \int_{-1}^0 \int_{-1}^{+\infty} [F(x(s) + x(0)y, x(0), r) \\ &- F(x(-1), x(0), r) - F_x(x(s), x(0), r)y] u(s) \Pi(dy) ds \\ &+ \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(s), x(0), j) ds, \end{aligned}$$

We note that

$$\begin{aligned} L^u F(x(0)) &= q(x(0), u) := \\ &= F'_{x(0)}(x(0))a(x(0), u) + F''_{x(0)x(0)}(x(0))\frac{1}{2}b^2(x(0), u). \end{aligned}$$

Let us show the details of this proof. We have to prove (28). We fix $(x, x(0), r(0)) \in H \times G \times X$, where $X = \{1, 2, \dots, N\}$ -state space of a Markov chain $r(t)$. Let $\alpha \leq \tau_{H \times G \times X}$ be a stopping time, where $\tau_{H \times G \times X} := \inf\{t : (x_t, x(t), r(t)) \notin H \times G \times X\}$. Using the strong Markov property of $(x_t, x(t), r(t))$ we obtain:

$$E_{x,x(0),r(0)}[J(x_\alpha, x(\alpha), u, r(\alpha))] = J(x, x(0), u, r(0)) - E_{x,x(0),r(0)}\left[\int_{-1}^0 \int_0^\alpha F(\phi(\theta), x(s), u(s), r(s)) ds d\theta\right].$$

From here we have:

$$J(x, x(0), u, r(0)) = E_{x,x(0),r(0)}\left[\int_{-1}^0 \int_0^\alpha F(\phi(\theta), x(s), u(s), r(s)) ds d\theta\right] + E_{x,x(0),r(0)}[J(x_\alpha, x(\alpha), u, r(\alpha))].$$

Consider $V \in H \times G \times X$ such that $V := \{(y, y(0), r(0)) \in H \times G \times X : |(y, y(0) - (x, x(0)))| < \epsilon, \forall r(0)\}$. We now let $\alpha = \tau_V$. If we suppose that the optimal control u^* exists, then we choose $\forall v \in \mathcal{U}$

$$u = \begin{cases} v(x, x(0), r(0)), & \text{if } (x, x(0), r(0)) \in V \\ u^*(x, x(0), r(0)), & \text{if } (x, x(0), r(0)) \in H \times G \times X \setminus V. \end{cases}$$

Thus, $J^*(x_\alpha, x(\alpha), r(\alpha)) := J^*(x_\alpha, x(\alpha), u^*, r(\alpha))$, and, taking into account the last two expressions above, we have

$$\begin{aligned} J^*(x, x(0), r(0)) \leq J^*(x, x(0), v, r(0)) &= E_{x,x(0),r(0)}\left[\int_{-1}^0 \int_0^\alpha F(\phi(\theta), x(s), u(s), r(s)) ds d\theta\right] \\ &+ E_{x,x(0),r(0)}[J(x_\alpha, x(\alpha), u, r(\alpha))]. \end{aligned}$$

According to Dynkin's formula (21) we obtain:

$$E_{x,x(0),r(0)}[J(x_\alpha, x(\alpha), u, r(\alpha))] = J(x, x(0), r(0)) + E_{x,x(0),r(0)}\left[\int_0^\alpha A^{u,r} J(x_s, x(s), v, r(s)) ds\right],$$

where $A^{u,r}$ is defined above or in (27). If we substitute the last expression into the previous inequality, then we will have

$$E_{x,x(0),r(0)}\left[\int_0^{-1} \int_0^\alpha F(x_s, x(s), \phi(\theta), v(s), r(s)) ds d\theta + \int_0^\alpha A^{v,r} J(x_s, x(s), v, r(s)) ds\right] \geq 0.$$

In this way,

$$\frac{E_{x,x(0),r(0)}\left[\int_0^{-1} \int_0^\alpha F(x_s, x(s), \phi(\theta), v(s), r(s)) ds d\theta + \int_0^\alpha A^{v,r} J(x_s, x(s), v, r(s)) ds\right]}{E_{x,x(0),r(0)}[\alpha]} \geq 0.$$

Finally, putting $\epsilon \rightarrow 0$ we obtain

$$\int_0^{-1} F(x, x(0), \phi(\theta), v, r(0)) d\theta + A^{v,r} J(x, x(0), v, r(0)) \geq 0,$$

which combined with (29) gives (28), and the proof is complete. We note that $r(0) = r$. $\square$

Proof of Theorem 2. Follows the ideas of Theorem 3.2 in [19] and Lemma 2 above with replacement A_G^u by $A^{u,r}$ from (27), i.e.,

$$\begin{aligned} (A^{u,r} J)(x, x(0), r) &= F(x(0), x(0), u(0), r) - F(x(-1), x(0), u(0), r) + \\ &+ \int_{-1}^0 L^v F(\phi(\theta), x(0), u(0), r) d\theta + \int_{-1}^0 \int_{-1}^{+\infty} [F(x(s) + x(0)y, x(0), r) \\ &- F(x(-1), x(0), r) - F_x(x(s), x(0), r)y] u(s) \Pi(dy) ds \\ &+ \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(s), x(0), j) ds, \end{aligned}$$

Let us show the details of the proof. We suppose that function g satisfies (30) and (31). If we take u to be a Markov control, then

$$A^{u,r}J \geq - \int_{-1}^0 F(x, x(0), \phi(\theta), u^0, r) d\theta$$

for all u in G . Applying Dynkin's formula (21), we have

$$\begin{aligned} E_{x,x(0),r(0)}[g(x_{\tau_r}, x(\tau_r), r(\tau_r))] &= g(x, x(0), r(0)) + E_{x,x(0),r(0)}[\int_0^{\tau_r} (A^{u,r}g)(x_s, x(s), r(s))ds] \\ &\geq g(x, x(0), r(0)) - E_{x,x(0),r(0)}[\int_0^{-1} \int_0^{\tau_r} F(\phi(\theta), x(s), u(s), r(s))d\theta ds]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} g(x, x(0), r(0)) &\leq E_{x,x(0),r(0)}[\int_0^{-1} \int_0^{\tau_r} F(\phi(\theta), x(s), u(s), r(s))d\theta ds + g(x_{\tau_r}, x(\tau_r), r(\tau_r))] \\ \rightarrow v \rightarrow +\infty \quad E_{x,x(0),r(0)}[\int_0^{-1} \int_0^{\tau_{H \times G \times X}} F(\phi(\theta), x(s), u(s), r(s))d\theta ds + \psi(x_{\tau_{H \times G \times X}}, x(\tau_{H \times G \times X}), r(\tau_{H \times G \times X}))], \end{aligned}$$

where $\tau_v := \min\{v, \tau_{H \times G \times X}, \inf\{t > 0 : |x_t| \geq v\}, \inf\{t > 0 : |x(t)| \geq v\}\}$, $\forall v < +\infty$. This proves the first statement of Theorem 2. Finally, if u^0 is such that (31) holds, then the above derivation gives the equality, and Theorem 2 is proved. $\square$

Proof of Theorem 3. Follows from the proof of Theorem 1 above with replacement operator $A^{u,r}$ in (27) by $A^{u,r}$ from (32), i.e.,

$$\begin{aligned} (A^u J)(x, x(0)) &= F(x(0), x(0), u(0)) - F(x(-1), x(0), u(0)) + \\ &+ \int_{-1}^0 L^u F(\phi(\theta), x(0), u(0)) d\theta + \int_{-1}^0 \int_{-1}^{+\infty} [F(x(s) + x(0)y, x(0) \\ &- F(x(-1), x(0)) - F_x(x(s), x(0))y]u(s)\Pi(dy)ds, \end{aligned}$$

In this case, $q_{ij} = 0, \forall i, j \in X$. $\square$

Proof of Theorem 4. Follows from the proof of Theorem 2 above with replacement $A^{u,r}$ in (27) by A^u from (32), i.e.,

$$\begin{aligned} (A^u J)(x, x(0)) &= F(x(0), x(0), u(0)) - F(x(-1), x(0), u(0)) + \\ &+ \int_{-1}^0 L^u F(\phi(\theta), x(0), u(0)) d\theta + \int_{-1}^0 \int_{-1}^{+\infty} [F(x(s) + x(0)y, x(0) \\ &- F(x(-1), x(0)) - F_x(x(s), x(0))y]u(s)\Pi(dy)ds. \end{aligned}$$

In this case, $q_{ij} = 0, \forall i, j \in X$. $\square$

Proof of Theorem 5. Follows from the proof of Theorem 1 above with replacement operator $A^{u,r}$ in (27) by $A^{u,r}$ from (38), i.e.,

$$\begin{aligned} (A^{u,r} J)(x, x(0), r) &= F(x(0), x(0), u(0), r) - F(x(-1), x(0), u(0), r) + \\ &+ \int_{-1}^0 L^u F(\phi(\theta), x(0), u(0), r) d\theta + \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(s), x(0), j)ds. \end{aligned}$$

In this case, measure $\nu(dy, dt) = 0$. $\square$

Proof of Theorem 6. Follows from the proof of Theorem 2 above with replacement $A^{v,r}$ in (27) by $A^{v,r}$ from (38), i.e.,

$$\begin{aligned} (A^{u,r} J)(x, x(0), r) &= F(x(0), x(0), u(0), r) - F(x(-1), x(0), u(0), r) + \\ &+ \int_{-1}^0 L^u F(\phi(\theta), x(0), u(0), r) d\theta + \int_{-1}^0 \sum_{j=1}^N q_{rj} F(x(s), x(0), j)ds. \end{aligned}$$

In this case, measure $\nu(dy, dt) = 0$. $\square$

References

1. Fleming, W.; Rishel, R. *Deterministic and Stochastic Optimal Control*; Springer: Berlin/Heidelberg, Germany, 1975.
2. Nisio, M. *Stochastic Control Theory*; Springer: Berlin/Heidelberg, Germany, 2015.
3. Stengel, R. *Stochastic Optimal Control: Theory and Applications*; Wiley: Hoboken, NJ, USA, 1986.
4. Fabbri, G.; Gozzi, F.; Swiech, A. *Stochastic Optimal Control in Infinite Dimensions*; Springer: Berlin/Heidelberg, Germany, 2012.
5. Oksendal, B.; Sulem, A. *Applied Stochastic Optimal Control of Jump Diffusions*, 3rd ed.; Springer: Berlin/Heidelberg, Germany, 2019.
6. Hale, J.K. Theory of Functional Differential Equations. *Appl. Math. Sci.* **1977**, *3*, 365.
7. Ivanov, A.F.; Kazmerchuk, Y.; Swishchuk, A.V. Theory, Stochastic Stability and Applications of Stochastic Delay Differential Equations: A Survey of Recent Results. *Differ. Equ. Dyn. Syst.* **2003**, *11*, 55–115.
8. Mohammed, S. Nonlinear flows of stochastic linear delay equations. *Stochastics* **1986**, *17*, 2007–2013. [[CrossRef](#)]
9. von Renesse, M.; Scheutzow, M. Existence and uniqueness of solutions of stochastic functional differential equations. *Random Oper. Stoch. Equ.* **2010**, *18*, 267–284. [[CrossRef](#)]
10. Reis, M.; Riedle, M.; van Gaans, O. Delay differential equations driven by Lévy processes: Stationarity and Feller properties. *Stoch. Process. Appl.* **2006**, *116*, 1409–1432. [[CrossRef](#)]
11. Swishchuk, M.; Swishchuk, A. Stochastic Optimal Control of Averaged SDDE with Semi-Markov Switching, and with Application in Economics. *Mathematics* **2025**, *13*, 1440. [[CrossRef](#)]
12. Babasola, O.; Omondi, E.; Oshinubi, K.; Imbusi, N. Stochastic Delay Differential Equations: A Comprehensive Approach for Understanding Biosystems with Application to Disease Modelling. *Appl. Math.* **2023**, *3*, 702–721. [[CrossRef](#)]
13. Li, S.; Khan, S.; Riaz, M.; AlQahtani, S.; Alamri, A. Numerical simulation of a fractional stochastic delay differential equations using spectral scheme: A comprehensive stability analysis. *Sci. Rep.* **2024**, *14*, 6930. [[CrossRef](#)] [[PubMed](#)]
14. Shariati, N.; Yaghouti, M.; Alipanah, A. On solving some stochastic delay differential equations by Daubechies wavelet. *J. Stat. Comput. Simul.* **2024**, *94*, 1445–1461.
15. Ngoc, P.; Tran, K. General decay stability in mean square of stochastic delay differential equations. *Syst. Control Lett.* **2026**, *212*, 106404. [[CrossRef](#)]
16. Fakhfaka, R.; Boulaaras, G.; Alomani, G.; Alzahrani, R.; Mahlouf, A. Averaging Principle for Stochastic Delay Differential Equations with Hadamard Fractional Integral. *J. Nonlinear Math. Phys.* **2026**, *33*, 27. [[CrossRef](#)]
17. Shaikhet, L. About Stability of Nonlinear Stochastic Differential Equations with State-Dependent Delay. *Symmetry* **2022**, *14*, 2307. [[CrossRef](#)]
18. Gou, H.; Shi, M. Controllability results to stochastic neutral functional differential equations with delay. *Evol. Equ. Control Theory* **2026**, *20*, 208–228. [[CrossRef](#)]
19. Ivanov, A.F.; Swishchuk, A.V. Optimal control of stochastic differential delay equations with application in economics. *Int. J. Qual. Theory Differ. Equ. Appl.* **2008**, *2*, 201–213.
20. Itô, K.; Nisio, M. On stationary solutions of a stochastic differential equation. *J. Math. Kyoto Univ.* **1964**, *4*, 1–70. [[CrossRef](#)]
21. Dynkin, E.B. *Markov Process*; Academic Press: New York, NY, USA, 1965; Volumes 1 and 2.
22. Skorokhod, A.V. *Asymptotic Methods in the Theory of Stochastic Differential Equations*; Naukova Dumka Publishers: Kyiv, Ukraine, 1989.
23. Gandolfo, G. *Economic Dynamics*; Springer: Berlin/Heidelberg, Germany, 1996; 610p.
24. Ramsey, F.P. A mathematical theory of savings. *Econ. J.* **1928**, *38*, 543–549. [[CrossRef](#)]
25. Cass, D. Optimum Growth in an Aggregative Model of Capital Accumulation. *Rev. Econ. Stud.* **1965**, *32*, 233–240 [[CrossRef](#)]
26. Koopmans, T.C. On the Concept of Optimal Economic Growth. In *The Economic Approach to Development Planning*; Rand McNally: Chicago, IL, USA, 1965; pp. 225–287.
27. Acemoglu, D. The Neoclassical Growth Model. In *Introduction to Modern Economic Growth*; Princeton University Press: Princeton, NJ, USA, 2009; pp. 287–326, ISBN 978-0-691-13292-1.
28. Barro, R.J.; Sala-i-Martin, X. Growth Models with Consumer Optimization. In *Economic Growth*, 2nd ed.; McGraw-Hill: New York, NY, USA, 2024; pp. 85–142, ISBN 978-0-262-02553-9.
29. Bénassy, J.-P. The Ramsey Model. In *Macroeconomic Theory*; Oxford University Press: New York, NY, USA, 2015; pp. 145–160, ISBN 978-0-19-538771-1.
30. Blanchard, O.J.; Fischer, S. Consumption and Investment: Basic Infinite Horizon Models. In *Lectures on Macroeconomics*; MIT Press: Cambridge, MA, USA, 1989; pp. 37–89, ISBN 978-0-262-02283-5.
31. Miao, J. Neoclassical Growth Models. *Economic Dynamics in Discrete Time*; MIT Press: Cambridge, MA, USA, 2014; pp. 353–364, ISBN 978-0-262-02761-8.
32. Novales, A.; Fernández, E.; Ruíz, J. *Optimal Growth: Continuous Time Analysis. Economic Growth: Theory and Numerical Solution Methods*; Springer: Berlin/Heidelberg, Germany, 2009; pp. 101–154, ISBN 978-3-540-68665-1.
33. Romer, D. Infinite-Horizon and Overlapping-Generations Models. In *Advanced Macroeconomics*, 4th ed.; McGraw-Hill: New York, NY, USA, 2011; pp. 49–77, ISBN 978-0-07-351137-5.

34. Zacarias-Espinoza, G.; Cruz-Suarez, H.; Lemus-Rodriguez, E. A Stochastic Version of the Ramsey's Growth Model. In *Proceedings of 5th the International Conference on Operations Research and Enterprise Systems (ICORES 2016)*; SciTePress: Setúbal, Portugal, 2016; pp. 323–329, ISBN 978-989-758-171-7.
35. Elbers, C. Solving The Discrete-Time Stochastic Ramsey Model. TI 2009-018/2, Tinbergen Institute Discussion Paper. February 2009. Available online: <https://papers.tinbergen.nl/09018.pdf> (accessed on 29 April 2026).
36. Rabi, B.; Hyeonju, K.; Mukul, M. Sustainability in the Stochastic Ramsey Model. *arXiv* **2015**, arXiv:1511.07419. [[CrossRef](#)]
37. Parker, J. Stochastic Growth Models and Real Business Cycles. In *Economics*; 314 Coursebook; Reed College: Portland, OR, USA, 2010.
38. Akian, M.; Bensoussan, A. On the Stochastic Ramsey Problem. In *Proceedings of the IEEE Conference on Decision and Control*; IEEE: Piscataway, NJ, USA, 1986; pp. 259–261.
39. Chung, K.L. *Markov Chains*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 1967.
40. Mamon, R.; Elliott, R. *Hidden Markov Models in Finance*; Springer: Berlin/Heidelberg, Germany, 2007.
41. Swishchuk, A.V.; Kazmerchuk, Y.I. Stability of stochastic Ito equations with delay, Poisson jumps and Markov switchings with applications to finance. *Theory Probab. Mathem. Statis.* **2021**, *64*, 45.
42. Fleming, W.; Nisio, M. On the existence of optimal stochastic control. *J. Math. Mech.* **1966**, *15*, 777–794.
43. Kushner, H. On the stability of processes defined by stochastic difference-differential equations. *J. Differ. Equ.* **1968**, *4*, 424–443. [[CrossRef](#)]
44. Skorokhod, A.V. *Studies in the Theory of Random Processes*; Addison-Wesley: Boston, MA, USA, 1965.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.