

Article

An Edge Operation That Increases the Hitting Time Index of Trees

José Luis Palacios ^{1,*} and Aleksandar Petojević ² ¹ Electrical and Computer Engineering Department, The University of New Mexico, Albuquerque, NM 87131, USA² Faculty of Education, University of Novi Sad, Podgorička 4, 25000 Sombor, Serbia; apetoje@uns.ac.rs

* Correspondence: jpalacios@unm.edu

Abstract

Suppose that G is an n -vertex tree with a path P_k attached to a vertex $b \in G$ and with another pendant vertex c that is also attached to b . If G^* is the tree constructed from G by moving the attachment point of the path from b to c , then under the hypothesis $n \geq 2k + 2$ we have $HT(G^*) > HT(G)$. This result allows us to exhibit families of trees with a monotonic behavior of the values of their HT index: the index increases as the number of pendant vertices decreases, thus giving support to the conjecture that the maximum HT index among n -vertex trees is attained by the n -path.

Keywords: expected hitting times; comb graphs; broom graphs**MSC:** 92E10

1. Introduction

In this article, a graph $G = (V, E)$ will be a finite simple connected undirected graph with vertex set $V = \{1, 2, \dots, n\}$, edge set E and vertex degrees $d_1, d_2, \dots, d_n$. For all graph theoretical concepts the reader can consult reference [1].

Mathematical chemistry uses these graphs to model molecules, identifying the vertices as the atoms and the edges as the atomic bonds between the vertices. Many descriptors, or indices, i.e., real-valued functions on the domain of all graphs, have been defined with the intention of describing the physico-chemical properties of the molecules and classifying them according to the values of their descriptors.

The simple random walk on G is the Markov chain $\{X_n, n \geq 0\}$ with state space V and with uniform transition probabilities from a given vertex i to any of its d_i neighboring vertices. The hitting time T_j of the vertex j is defined as the infimum of the number of transitions that the random walk needs to get to the vertex j :

$$T_j = \inf\{n \geq 0 : X_n = j\},$$

and its expectation when the walk is started at i is denoted by $E_i T_j$. Reader beware: $E_i T_i = 0$, and this should not be confused with the mean return time to the vertex i , $E_i T_i^+ = \frac{2|E|}{d_i}$, which involves a slightly different random variable $T_i^+ = \inf\{n \geq 1 : X_n = i\}$. The reader can consult [2] for all facts involving hitting times that will be used below.



Academic Editors: Qing-Wen Wang,
Askar Tuganbaev and Adolfo
Ballester-Bolinches

Received: 24 May 2026

Revised: 22 June 2026

Accepted: 9 July 2026

Published: 14 July 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and

conditions of the [Creative Commons](https://creativecommons.org/licenses/by/4.0/)[Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

We defined in [3] a new probabilistic index, the hitting time index $HT(G)$ of a graph G , as

$$HT(G) = \sum_{i < j} D(i, j), \quad (1)$$

where $D(i, j) = \max\{E_i T_j, E_j T_i\}$.

We showed that $D(i, j)$ is actually a distance on the set of all vertices of G , found some inequalities involving $HT(G)$ and other descriptors, and computed $HT(G)$ for some families of graphs.

When evaluating $HT(G)$, one way to do it is to begin by finding the transition probability P of the random walk on G , with entries $P(i, j) = \frac{1}{d_i}$ in case i and j are neighbors, and zero otherwise. Then we form the matrix W , whose rows are all equal to the stationary distribution $\pi = \frac{1}{2|E|}[d_1, d_2, \dots, d_n]$. Then we get the well-known fundamental matrix

$$Z = (I - P + W)^{-1},$$

where I is the $n \times n$ identity matrix. Then, the matrix E of expected hitting times is obtained with the matrix Z , whose entries are given by

$$E(i, j) = E_i T_j = (Z(j, j) - Z(i, j))/\pi_j.$$

The interested reader may consult reference [2] for a thorough discussion of the matrices Z and E . Finally, the HT index of the graph G is found by adding $\binom{n}{2}$ terms,

$$\sum_{i < j} \max\{E(i, j), E(j, i)\}.$$

A brief summary of previous results reading the HT index follows. In [3], specifying a bit more about what was said above, we showed inequalities relating the HT index to the random walk index, the Kirchhoff index and the Wiener index. We computed in closed form the HT index for some families of graphs and gave the conjecture that the graphs that maximize and minimize HT are the n -path P_n and the complete graph K_n , respectively. In [4] the authors closely studied the relationship between the HT index and the Kirchhoff index, gave some formulas for the HT index of bi-stars and broom graphs, and found upper and lower bounds for the HT index in terms of the maximum degree, the hyper-Wiener index, the Wiener index and the Harary index. In [5], new formulas were found, which were much simpler than those given in [4], for the HT index of broom graphs; some monotonicity results for certain broom graphs were also found. Even better, more compact formulas for the HT index of broom graphs, as well as wider monotonicity results, were found in [6]. In [7] some inequalities related to the HT index were refined, new results were found ($HT(T)$ is an integer when T is a tree, and the HT index is not monotonic under edge addition) and new closed-form expressions for some families of graphs with cutpoints were also found. Finally, in [8], a new relation between the HT index and the Kirchhoff index was given, and closed-form formulas for the HT index of some families of edge-transitive graphs were found.

In this article, we show the new result that the HT index increases when we subject the tree under consideration to a “straightening” procedure, whereby parts of the tree are turned into an increasingly larger path. This result supports the conjecture, weaker than the one formulated in [3], that the maximum value over all n -vertex trees is attained by the n -path P_n , and it allows us to establish some hierarchies of values of the HT index over some families of trees that extend a few results in [4–6], and yield other new results.

2. The Results

The following formula will be essential in our derivations. Its proof can be found in [9].

Lemma 1. *If G is a tree, for any pair of vertices a and b in G we have*

$$E_a T_b + E_b T_a = 2(n-1)d(a, b). \quad (2)$$

We will also be using the following (one-directional) refinement of (2) found in [10].

Lemma 2. *For any tree G and two vertices $a, b \in G$, let P be the unique path from a to b , $V(P)$ the set of vertices in P and $E(G_x)$ the connected component of $E - P$ containing x . Then,*

$$E_a T_b = d(a, b)^2 + 2 \sum_{x \in V(P)} |E(G_x)| d(x, b) \quad (3)$$

In what follows, we lay the groundwork to prove our main proposition, where we show that if we have a tree G with one pendant vertex c and one path P_k both incident on the same vertex b , and we construct a new tree G^* where we move the attachment point of P_k from b to c , thereby creating a P_{k+1} path incident at b , then $HT(G) < HT(G^*)$. In general, it is not true that the value of every term $D(x, y)$ in the summation defining $HT(G)$ is smaller than the value of the corresponding term $D^*(x, y)$ in $HT(G^*)$, though the increases in some terms supersede the decreases in other terms, and for most terms the equality $D(x, y) = D^*(x, y)$ holds.

We will consider an n -vertex tree G composed of a subtree G_1 , a path P_k with vertices $d_1, d_2, \dots, d_k$ attached to $b \in G_1$, and another pendant vertex c that is also attached to b . We create the new tree G^* by removing the edge (d_1, b) and replacing it with the edge (d_1, c) . In other words, we move the attachment point of P_k from b to c , thereby creating a P_{k+1} path incident at b . First we notice that all hitting times $E_i T_j$ and $E_i T_j^*$ are equal when $i, j \in G_1$ and when $i, j \in P_k$.

The reason is that for any $x \in G_1 - \{b\}$ and $y \in G_1$ we have $E_x T_y = E_x T_y^*$ because all parameters used in Formula (3) to compute hitting times, between the vertices mentioned, are identical for both trees G and G^* . Notice that b is not included among the starting points, but it is included in the target points.

But now, with the same choices for x and y we have, by (2):

$$E_y T_x = 2(n-1)d(x, y) - E_x T_y = 2(n-1)d(x, y) - E_x T_y^* = E_y T_x^*.$$

A similar argument shows that $E_x T_y = E_x T_y^*$ whenever $x, y \in P_k$, implying that

$$D(i, j) = D^*(i, j) \text{ whenever } (i, j) \in G_1 \cup P_k. \quad (4)$$

Now we need to consider all other missing choices for pairs (i, j) .

Case 1. For $x \in G_1$, we have

$$E_c T_x = 1 + E_b T_x, \text{ and } E_x T_c = E_x T_b + 2n - 3,$$

so that

$$D(x, c) = \max\{1 + E_b T_x, 2n - 3 + E_x T_b\}. \quad (5)$$

Also

$$E_x T_c^* = E_x T_b + E_b T_c^*$$

and since

$$E_{d_k} T_b^* = (k+1)^2 = E_{d_k} T_c^* + E_c T_b^* = k^2 + E_c T_b^*,$$

solving for $E_c T_b^*$ we get

$$E_c T_b^* = 2k + 1,$$

and using (2), we get

$$E_b T_c^* = 2n - 2k - 3.$$

Then we can write

$$E_x T_c^* = E_x T_b^* + E_b T_c^* = E_x T_b + 2n - 2k - 3,$$

and

$$E_c T_x^* = E_c T_b^* + E_b T_x^* = 2k + 1 + E_b T_x,$$

and therefore

$$D^*(x, c) = \max\{E_b T_x + 2k + 1, E_x T_b + 2n - 2k - 3\}. \quad (6)$$

From (5) and (6), we conclude that

$$D^*(c, x) \geq D(c, x) - 2k, \text{ for } x \in G_1. \quad (7)$$

Case 2. We will look at $x \in G_1$ and $d_i \in P_k$. We start by noticing that

$$E_{d_k} T_b = k^2 = E_{d_k} T_{d_i} + E_{d_i} T_b = (k-i)^2 + E_{d_i} T_b,$$

and solving for $E_{d_i} T_b$ we get

$$E_{d_i} T_b = (2k-i)i,$$

and also, applying (2), we have

$$E_b T_{d_i} = 2(n-1)i - (2k-i)i = (2n-2k+i-2)i.$$

Now, $E_x T_{d_i} = E_x T_b + E_b T_{d_i} = E_x T_b + (2n-2k+i-2)i$, and $E_{d_i} T_x = E_{d_i} T_b + E_b T_x = (2k-i)i + E_b T_x$, and therefore

$$D(x, d_i) = \max\{E_x T_b + (2n-2k+i-2)i, E_b T_x + (2k-i)i\}. \quad (8)$$

On the other hand

$$E_{d_k} T_b^* = (k+1)^2 = E_{d_k} T_{d_i}^* + E_{d_i} T_b^* = (k-i)^2 + E_{d_i} T_b^*.$$

Solving for $E_{d_i} T_b^*$ we get

$$E_{d_i} T_b^* = (2k-i+1)(i+1),$$

and using (2) we get

$$E_b T_{d_i}^* = (2n-2k+i-3)(i+1).$$

Then we can write

$$E_{d_i} T_x^* = E_{d_i} T_b^* + E_b T_x^* = (2k-i+1)(i+1) + E_b T_x,$$

and

$$E_x T_{d_i}^* = E_x T_b^* + E_b T_{d_i}^* = E_x T_b + (2n-2k+i-3)(i+1),$$

so that

$$D^*(x, d_i) = \max\{E_x T_b + (2n - 2k + i - 3)(i + 1), E_b T_x + (2k - i + 1)(i + 1)\}. \quad (9)$$

If $D(x, d_i) = E_b T_x + (2k - i)i$ then $D^*(x, d_i) \geq E_b T_x + (2k - i)i + 2k + 1 = D(x, d_i) + 2k + 1$. And if $D(x, d_i) = E_x T_b + (2n - 2k + i - 2)i$ then $D^*(x, d_i) \geq E_x T_b + 2n - 2k - 3 = D(x, d_i) + 2n - 2k - 3$. Then we can guarantee

$$D^*(x, d_i) \geq D(x, d_i) + 2k + 1, \quad (10)$$

for $x \in G_1, d_i \in P_k$, as long as $2n - 2k - 3 \geq 2k + 1$, i.e., as long as $n \geq 2k + 2$.

Case 3. Now we will look at the behavior of c with respect to P_k , with the help of the values for $E_b T_{d_i}$ and $E_{d_i} T_b$ found in case 2. First we find

$$E_c T_{d_i} = E_c T_b + E_b T_{d_i} = 1 + (2n - 2k + i - 2)i,$$

and

$$E_{d_i} T_c = E_{d_i} T_b + E_b T_c = (2k - i)i + 2n - 3.$$

On the other hand, it is clear by the way the graph G^* has been built that

$$E_{d_i} T_c^* = E_{d_i} T_b = (2k - i)i,$$

and then by (2) we have

$$E_c T_{d_i}^* = (2n - 2k + i - 2)i.$$

Now, for $i = 1$ it is clear that

$$D(c, d_1) = \max\{2n - 2k, 2n + 2k - 4\} = 2n + 2k - 4,$$

valid for all $k \geq 1$, and

$$D^*(c, d_1) = \max\{2k - 1, 2n - 2k - 1\} = 2n - 2k - 1,$$

whenever $n \geq 2k$. Therefore, we have that

$$D^*(c, d_1) = D(c, d_1) - 4k + 3, \text{ for } n \geq 2k. \quad (11)$$

For the rest of the values $i \geq 2$ we argue that if $D(c, d_i) = 1 + (2n - 2k + i - 2)i$, then

$$D^*(c, d_i) \geq D(c, d_i) - 1, \quad (12)$$

whereas if $D(c, d_i) = (2k - i)i + 2n - 3$, then

$$D^*(c, d_i) \geq D(c, d_i) - 2n + 3. \quad (13)$$

Therefore, (12) and (13) imply that we can guarantee

$$D^*(c, d_i) \geq D(c, d_i) - 2n + 3. \text{ for } 2 \leq i \leq k. \quad (14)$$

Now we are ready to prove our main result in the following

Proposition 1. Let G be an n -vertex tree composed of a subtree G_1 on $n - k - 1$ vertices, a path P_k with vertices $d_1, d_2, \dots, d_k$ attached to $b \in G_1$ and another pendant vertex c attached also to b . Let G^* be the tree constructed from G by removing the edge (d_1, b) and replacing it with the edge

(d_1, c) , or in other words, by moving the attachment point of the path from b to c . Then, under the hypothesis $n \geq 2k + 2$ we have

$$HT(G^*) > HT(G).$$

Proof. Putting together the conditions (4), (7), (10), (11) and (14), and noticing that all these hold simultaneously under the most restrictive hypothesis $n \geq 2k + 2$, we can claim

$$HT(G^*) \geq HT(G) + (2k + 1)k(n - k - 1) - 2k(n - k - 1) - 4k + 3 - (k - 1)(2n - 3), \quad (15)$$

for $n \geq 2k + 2$, on account of the addition of the contributions of

- (i) $n - k - 1$ pairs (c, j) with $j \in G_1$ in case 1, where each pair loses at most $2k$ (Equation (7));
- (ii) The $k(n - k - 1)$ pairs of vertices (i, j) with $i \in G_1, j \in P_k$ in case 2, where each pair gains at least $2k + 1$ (Equation (10));
- (iii) One pair (c, d_1) that loses $4k - 3$ (Equation (11));
- (iv) The $k - 1$ pairs of vertices (c, j) with $j \in \{d_2, \dots, d_k\}$ in case 3, where each pair loses at most $2n - 3$ (Equation (14)).

Now we can rewrite (15) as

$$HT(G^*) \geq HT(G) + F(n, k),$$

where $F(n, k)$ is the sum of all those summands in terms of n and k that can be regrouped as

$$F(n, k) = n(2k^2 - 3k + 2) - k^2(2k + 1).$$

We want to check that $F(n, k) > 0$ under the constraint given, which is equivalent to showing that

$$n > \frac{k^2(2k + 1)}{2k^2 - 3k + 2},$$

on account of the fact that $2k^2 - 3k + 2$ has no real roots. Since $n \geq 2k + 2$, it is enough to show that

$$2k + 2 > \frac{k^2(2k + 1)}{2k^2 - 3k + 2},$$

or

$$2k^3 - 3k^2 - 2k + 4 > 0.$$

But this is true for $k = 1$ by direct check, and it is true for $k \geq 2$ using elementary Calculus. $\square$

Note 1. The condition $n \geq 2k + 2$ is sufficient, but not necessary. See the note after example (ii) below.

Now we can prove a number of corollaries of the main result.

(i) **Comb graphs.** A $2n$ -vertex comb graph H_{2n} , $n \geq 3$, is constructed by adding to each vertex of an n -path P_n (including the endpoints) an additional pendant vertex. We can apply proposition 1 starting from the left repeatedly until we have an $n + 2$ -path attached to a vertex which cannot be used as the b vertex of the proposition because $k = n + 2$ and $2k + 2$ exceeds n . Then we start applying the proposition from the right until the graph turns into a $2n$ -path P_{2n} . The sequence of graphs thus generated is monotonic in the value of their HT index. See Figure 1 for an illustration of the straightening process applied to H_{10} : we show the progression from the initial graph marked (a), through the successive applications of Proposition 1, marked (b), (c), etc., indicating at each step which vertices are the ones responsible for the edge operation, namely vertices b , c and d_1 . We also show the

increasing values of the HT index in the progression of graphs. Similar labelings will be applied below to the rest of the examples.

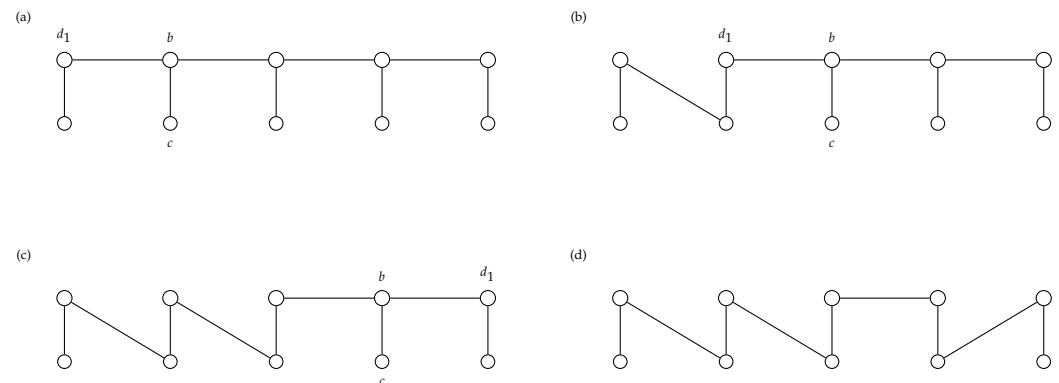


Figure 1. Straightening of H_{10} .

(ii) **Broom graphs.** The $B_{n,d}$ broom graph is constructed by adding $n - d$ pendant vertices to one end of a d -vertex path P_d . Starting from S_n , the n -star graph, the value of the HT index increases as the number of pendant vertices decreases to two (when we have an n -path P_n). We apply Proposition 1 to the n -star graph S_n , $n \geq 6$, which can be thought of as the broom $B_{n,2}$, until we get to a $B_{n,\lfloor \frac{n}{2} \rfloor + 1}$, and we get the sequence

$$HT(B_{n,2}) < HT(B_{n,3}) < \cdots < HT(B_{n,\lfloor \frac{n}{2} \rfloor + 1}).$$

This extends results that we found in [5,6].

Once we reach the broom $B_{n,\lfloor \frac{n}{2} \rfloor + 1}$, we cannot enlarge the path obtained so far without violating the condition $n \geq 2k + 2$, so we start applying the proposition choosing c and d_1 as two of the remaining pendant vertices. Eventually we get to $B_{n,n-1} = P_n$, though these graphs following $B_{n,\lfloor \frac{n}{2} \rfloor + 1}$ are no longer brooms. In Figure 2, we show the straightening process of S_6 , in which all intermediate graphs are brooms, and in Figure 3 we show the process for S_7 , in which the intermediate graphs are brooms only up to $B_{7,4}$.

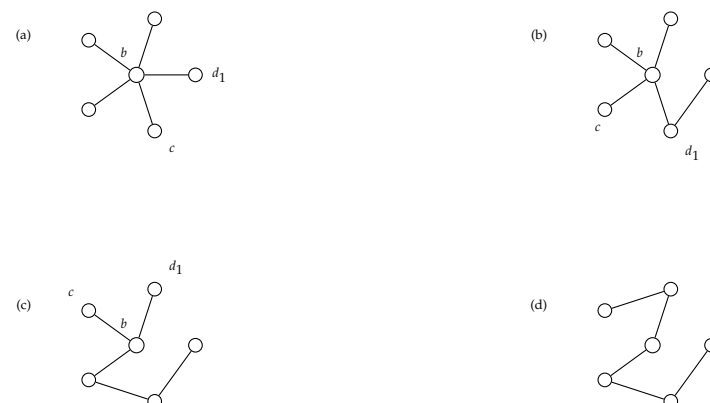


Figure 2. Straightening of S_6 .

Note 2. The straightening of S_7 serves to illustrate that the condition $n \geq 2k + 2$ is sufficient, but not necessary. When moving from the graph in step (c) to the next graph, instead of the procedure shown in Figure 3, we could have chosen to lengthen the current P_3 by taking d_1 to be the vertex that attaches the P_3 to b , and still we would have seen an increase in the value of the HT index (from 348 to 396) even though $n = 7 < 2(k) + 2 = 2(3) + 2 = 8$.

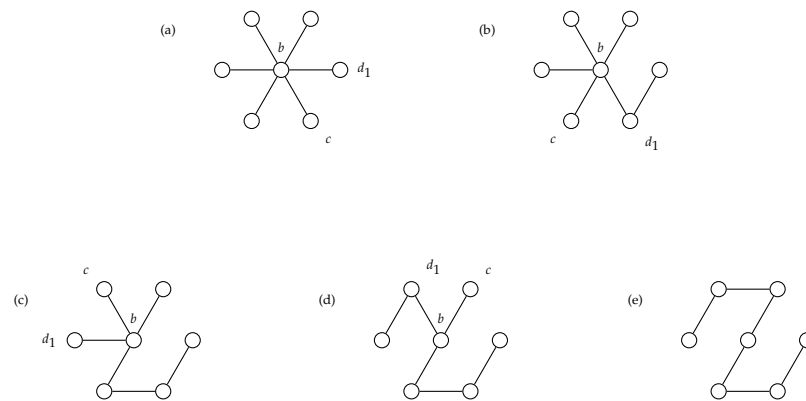


Figure 3. Straightening of S_7 .

(iii) **Double broom graphs.** A double broom graph $D(d, a, b)$ is constructed by adding to a $(d - 1)$ -path P_{d-1} a set of a pendant vertices on one end and b pendant vertices on the other end. We apply proposition 1 starting on the side with the least pendant vertices, say, $a \leq b$. Then the condition $n \geq 2k + 2$ holds for every $k \leq a$. That means that we can straighten the end with the a pendant vertices until we get a broom $B_{a+b+d-1, a+d-1}$. Along the way, we get a sequence of increasing values of some intermediate graphs which are not double brooms because not all the subgraphs incident at a are pendant vertices, but we do find that

$$HT(D(d, a, b)) < HT(D(d + a - 1, 1, b)) = HT(B_{a+b+d-1, a+d-1}).$$

In other words, the hitting time index of any double broom is bounded above by the hitting time index of the one-sided broom with the larger number of pendant vertices of the two endpoints of the double broom. In Figure 4 we show the straightening of the double broom $D(4, 3, 4)$ into the broom $B_{10,6}$.

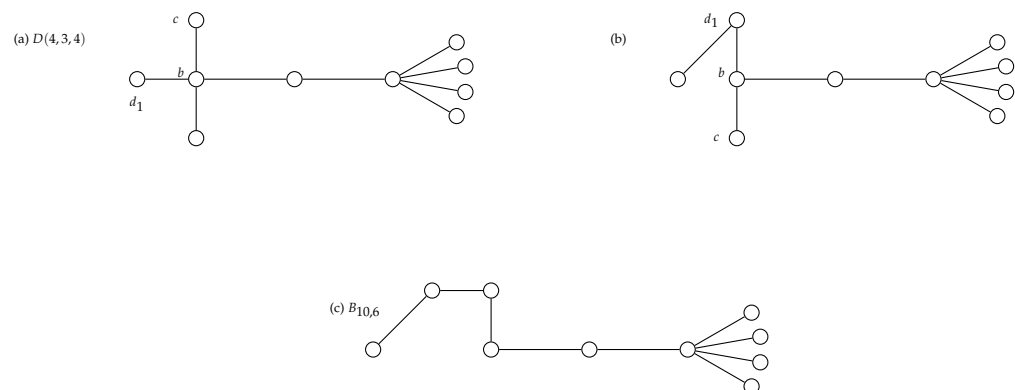


Figure 4. Straightening of $D(4, 3, 4)$.

3. Concluding Remarks

In this article, we have discussed an edge operation on trees that increases the value of the HT index. Namely, if we start with an n -vertex tree G with a path P_k attached to a vertex $b \in G$ and with another pendant vertex c that is also attached to b , and if G^* is the tree constructed from G by moving the attachment point of the path from b to c , then under the hypothesis $n \geq 2k + 2$ we find that $HT(G^*) > HT(G)$. We saw that the condition imposed on the length of the path P_k is sufficient but not necessary, and we showed several families of trees with a monotonic behavior of the values of their HT index: the index increases as the number of pendant vertices decreases, thus giving support to—though not

proving—the conjecture that the maximum *HT* index among n -vertex trees is attained by the n -path.

For future work, we plan to study a more general edge operation on n -vertex trees, where a pair of paths P_k and P_r , incident on the same vertex b , can be replaced by a path P_{k+r} incident on b , while the value of the *HT* is increased. Hopefully, the conditions on n, k and r will be such that we will be able to finally prove the maximality conjecture.

Author Contributions: Conceptualization, J.L.P.; Methodology, J.L.P.; Validation, J.L.P.; Formal analysis, J.L.P.; Resources, A.P.; Data curation, A.P.; Writing—original draft, J.L.P. and A.P.; Writing—review and editing, J.L.P. and A.P.; Visualization, A.P.; Supervision, J.L.P.; Project administration, J.L.P.; Funding acquisition, J.L.P. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the University of New Mexico, Electrical and Computer Engineering Department.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Buckley, F.; Harary, F. *Distance in Graphs*; Addison-Wesley: Redwood, CA, USA, 1990.
2. Grinstead, M.; Snell, J.L. *Introduction to Probability*, 2nd ed.; American Mathematical Society: Providence, RI, USA, 1997.
3. Palacios, J.L. A new probabilistic molecular index. *MATCH Commun. Math. Comput. Chem.* **2025**, *93*, 499–507.
4. He, M.; Mao, Y.; Furtula, B.; Zhang, X. Hitting time index of trees. *J. Math. Chem.* **2026**, *64*, 7. [[CrossRef](#)]
5. Palacios, J.L. On the hitting time index of broom graphs. *Mathematics* **2026**, *14*, 1508. [[CrossRef](#)]
6. Orlić, S.; Palacios, J.L.; Petojević, A. Hitting time index for broom graphs. *AppliedMath* **2026**, *6*, 93. [[CrossRef](#)]
7. Palacios, J.L.; Santamaría, A. Some new results for the hitting time index. *MATCH Commun. Math. Comput. Chem.* **2025**, *94*, 427–445. [[CrossRef](#)]
8. Palacios, J.L.; Santamaría, A. On the hitting time index of graphs. *Open J. Appl. Sci.* **2025**, *15*, 2464–2478. [[CrossRef](#)]
9. Chandra, A.K.; Raghavan, P.; Ruzzo, W.L.; Smolensky, R.; Tiwari, P. The electrical resistance of a graph captures its commute and cover times. In Proceedings of the Twenty First Annual ACM Symposium on Theory of Computing, Seattle, WA, USA, 15–17 May 1989; pp. 574–586.
10. Chen, H.; Zhang, F. The expected hitting times for graphs with cutpoints. *Stat. Prob. Lett.* **2004**, *66*, 9–17. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.