

Article

The Crossing Numbers of Join Products of All 5-Vertex Graphs with Discrete Graphs, Paths and Cycles

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Abstract

This paper investigates crossing numbers of join products involving all non-isomorphic graphs of order five and the graph families D_n , P_n , and C_n . By systematically examining all 34 graphs of order five, we establish a unified classification of 102 crossing number values together with the corresponding references and principal proof techniques. The collected results consolidate previously scattered contributions, clarify several inconsistencies in the literature, and identify the unique remaining unresolved case. For the graph H_5^7 , we verify the conjectured crossing numbers computationally for all feasible values $n \leq 5$, establish general upper bounds, and prove that the corresponding results for joins with paths and cycles follow directly from the discrete graph case. Consequently, determining the exact crossing numbers of H_5^7 is reduced to establishing the corresponding lower bounds for joins with discrete graphs. The resulting classification reveals that the crossing number values are governed by only a small number of recurring formula types, suggesting that the crossing behaviour of join products involving graphs of order five is determined primarily by a limited collection of fundamental structural patterns.

Keywords: crossingnumber; join products; five vertex graphs; discrete graphs; paths; cycles

MSC: 05C10

1. Introduction

Let G be a simple graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G , respectively. The crossing number of G , denoted by $cr(G)$, is defined as the minimum number of pairwise edge crossings in any drawing of G in the plane, where edges from $E(G)$ are represented as continuous arcs that do not pass through vertices from $V(G)$ represented by distinct points or intersect themselves. A drawing with a minimum number of crossings (an optimal drawing) is always a good drawing, meaning that no edge crosses itself, no two edges cross more than once, no two edges incident with the same vertex cross and no more than two edges cross at the same point. This concept plays a central role in both theoretical and applied graph theory, with significant implications for practical domains such as VLSI circuit design, network visualisation, and transportation systems, where minimising edge crossings can lead to improved readability, reduced interference, and lower manufacturing costs [1].

In graph theory, graph products (like the Cartesian product, strong product, and join) are operations that combine two graphs into a new one. Each has its own meaning and typical applications. The Cartesian product of two graphs $G \square H$ is the graph with



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vertex set $V(G) \times V(H)$ where two vertices (u, v) and (x, y) are adjacent if $u = x$ and $\{v, y\} \in E(H)$, or $v = y$ and $\{u, x\} \in E(G)$. The strong product of two graphs $G \boxtimes H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u, v) and (x, y) are adjacent if $u = x$ and $\{v, y\} \in E(H)$, or $v = y$ and $\{u, x\} \in E(G)$, or $\{u, x\} \in E(G)$ and $\{v, y\} \in E(H)$. In this article, we will focus on the join product $G + H$ of two graphs G and H which is formed by taking the disjoint union of G and H and adding edges connecting every vertex of G to every vertex of H . This operation typically results in a dense graph, often with a large number of edge crossings [2]. Let $|V(G)| = m$ and $|V(H)| = n$. Then the edge set of $G + H$ is the union of the edge-disjoint sets of the graphs G , H , and the complete bipartite graph $K_{m,n}$. Determining the crossing number $\text{cr}(K_{m,n})$ of the graph $K_{m,n}$ remains a fundamental open problem in graph theory. An important upper bound for this value is given by the Zarankiewicz number $Z(m, n)$, which is defined as

$$Z(m, n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

A good drawing of $K_{m,n}$ with $Z(m, n)$ crossings is shown in Figure 1. Zarankiewicz originally conjectured that this bound is tight, i.e., $\text{cr}(K_{m,n}) = Z(m, n)$, but the conjecture remains unresolved in the general case [3]. However, significant progress has been made by Kleitman [4], who proved the conjecture holds for all $m, n \geq 1$ if $\min\{m, n\} \leq 6$. Otherwise, $Z(m, n)$ is only known to coincide with $\text{cr}(K_{m,n})$ for the special cases when $7 \leq \min\{m, n\} \leq 8$ and $7 \leq \max\{m, n\} \leq 10$ [5]. The crossing numbers of families of graphs resulting from join products have, to date, only been computed exactly when at least one graph involved has at most six vertices [6]. Consequently, many of these values are expressed in terms of $Z(m, n)$.

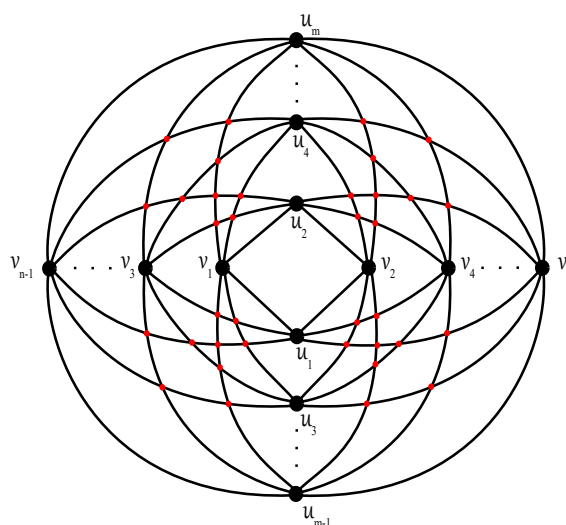


Figure 1. A good drawing of $K_{m,n}$ with $Z(m, n)$ crossings.

In particular, considerable attention has been devoted to the study of crossing numbers of join products of small graphs with the graph families D_n , P_n , and C_n . Let $n \geq 1$ be an integer. The graph D_n denotes the discrete (edgeless) graph on n vertices, i.e., $V(D_n) = \{v_1, v_2, \dots, v_n\}$ and $E(D_n) = \emptyset$. The graph P_n denotes the path on n vertices, obtained by adding edges between consecutive vertices of D_n . Finally, C_n denotes the cycle on n vertices, obtained from P_n by adding an edge joining its two end vertices.

The crossing numbers of join products of graphs with at most four vertices with D_n , P_n , and C_n have been comprehensively investigated by Klešć and Schrötter [7–9]. A survey by Clancy et al. [6] represents the most valuable reference on this topic, offering an extensive

and accurate catalogue of known crossing numbers for specific graphs and families of graphs. The principal aim of this survey was to consolidate existing results on crossing numbers from the literature, offering comprehensive references and appropriately crediting the original contributors. Notably, Chapter 4 of [6] is dedicated to examining the crossing numbers of join products involving simple graphs with up to six vertices.

Motivated by the overview of Clancy et al., we revisit all graphs of order five and provide an updated account of the corresponding join products. Many of these cases remained either unexplored or unclassified in terms of their crossing number characteristics as recently as six years ago. Specifically, seven crossing numbers for connected graphs were entirely absent from the literature, twenty-two values were marked with an asterisk indicating reliance on unverified sources, and for disconnected graphs, only preliminary data for two graphs were available at that time. In the current work, we analyse all 34 non-isomorphic graphs on five vertices—comprising 13 disconnected and 21 connected graphs—in join products with discrete graphs, paths, and cycles. Consequently, we consider 39 crossing number problems for disconnected graphs and 63 for connected graphs. The resulting crossing numbers are presented in structured tables, highlighting how increasing structural complexity affects the planarity and crossing behaviour of the investigated graph products. Besides consolidating previously scattered results, the paper provides new computational and theoretical results for the graph H_5^7 , the only remaining unresolved case in this classification.

2. All Non-Isomorphic Simple Five Vertex Graphs

There are exactly 34 non-isomorphic graphs on five vertices. The set includes all graphs that differ in edge presence or absence in such a way that they are not isomorphic to one another. Among these, all graphs with up to nine edges are planar—that is, their crossing number is zero. The only exception occurs on 10 edges, corresponding to the complete graph K_5 , which has the crossing number 1.

Before proceeding with the detailed analysis of these graphs—particularly in relation to product connections with discrete graphs, paths, and cycles—we first express the connected and disconnected graphs using a unified auxiliary notation. This facilitates the identification of structural connections among them. Throughout this work, we adopt the definitions and notation for crossing numbers of connected graphs as introduced by Klešč in [7]. For clarity and consistency, we introduce the following systematic labelling:

- Disconnected graphs are denoted by H_i^5 , where $i = 1, \dots, 13$;
- Connected graphs are denoted by G_i^5 , where $i = 1, \dots, 21$.

An overview of the labelling and drawing of particular graphs is provided in Table 1.

Table 1. Labelling and drawing of 34 non-isomorphic 5-vertex graphs.

All Non-Isomorphic Simple Graphs on Five Vertices				
Disconnected Graphs		Connected Graphs		
 H_1^5	 H_8^5	 G_1^5	 G_8^5	 G_{15}^5
 H_2^5	 H_9^5	 G_2^5	 G_9^5	 G_{16}^5
 H_3^5	 H_{10}^5	 G_3^5	 G_{10}^5	 G_{17}^5

Table 1. Cont.

All Non-Isomorphic Simple Graphs on Five Vertices				
Disconnected Graphs		Connected Graphs		
 H_4^5	 H_{11}^5	 G_4^5	 G_{11}^5	 G_{18}^5
 H_5^5	 H_{12}^5	 G_5^5	 G_{12}^5	 G_{19}^5
 H_6^5	 H_{13}^5	 G_6^5	 G_{13}^5	 G_{20}^5
 H_7^5		 G_7^5	 G_{14}^5	 G_{21}^5

3. Join Products of Disconnected Graphs on Five Vertices

Unlike for Cartesian products, the join product always results in a connected graph, even if one or both of the graphs involved is disconnected. Hence, we will also consider disconnected graphs in this section.

According to Klešč [7], the crossing numbers of the join product $P_m + C_n$ are uniquely determined when $m \geq 2$, $n \geq 3$, and $\min\{m, n\} \leq 6$. By extending the results from [10] for all disconnected subgraphs of the path P_5 in join products with paths and cycles, we obtain $\text{cr}(H_i^5 + D_n)$, $\text{cr}(H_i^5 + P_n)$, and $\text{cr}(H_i^5 + C_n)$ for $i = 1, \dots, 6$. In determining the crossing numbers of the graphs $H_{10}^5 + D_n$, $H_{10}^5 + P_n$, and $H_{10}^5 + C_n$ in [11], several parity symmetries were heavily utilized. The graph H_7^5 is obtained by removing one edge from the graph H_{10}^5 , and it is therefore assumed that their crossing numbers in the respective join products are the same. This graph represents the only case where the crossing number of join products has not yet been proven exactly. For this reason, a separate section will be devoted to it at the end of this paper.

The following idea has been employed in several papers. Let t be a vertex of D_n in the join product $G + D_n$. The *rotation* $\text{rot}_D(t)$ of a vertex t in the good drawing D of $G + D_n$ as the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t has been defined by Hernández-Vélez et al. [12] and Woodall [5]. This rotation is called a cyclic permutation. A similar concept is used also for P_n and C_n . The joint article by Klešč and Staš [13] is one of the first publications where properties of cyclic permutations were used. Its outcome includes $\text{cr}(H_i^5 + D_n)$, $\text{cr}(H_i^5 + P_n)$, and $\text{cr}(H_i^5 + C_n)$ for $i = 8, 9$. The graph H_{11}^5 holds the distinction of being the first disconnected graph containing a cycle as a subgraph that was investigated in [14].

Based on the idea of cyclic permutations and their properties, the result $\text{cr}(H_{12}^5 + D_n)$ was published by Klešč, Staš, and Petrillová [15], and later extended with results for P_n and C_n in [16] thanks to special drawings for even and odd n . A similar case occurred with the graph H_{13}^5 ; the crossing numbers of the join product with discrete graphs D_n in [17] were gradually supplemented with results for paths P_n [18] and cycles C_n [19] with a proof based on a special redrawing technique.

A corresponding summary of crossing numbers of join products of all 13 disconnected graphs on five vertices, together with discussed references, is listed in Table 2.

Table 2. Summary of crossing numbers of join products of all disconnected graphs on five vertices.

i	$\text{cr}(H_i^5 + D_n)$ $n \geq 1$	$\text{cr}(H_i^5 + P_n)$ $n \geq 2$	$\text{cr}(H_i^5 + C_n)$ $n \geq 3$	Refs.
1	$Z(5, n)$	$Z(5, n)$	$Z(5, n)$	[7,10]
2	$Z(5, n)$	$Z(5, n)$	$Z(5, n)$	[7,10]
3	$Z(5, n)$	$Z(5, n)$	$Z(5, n)$	[7,10]
4	$Z(5, n)$	$Z(5, n)$	$Z(5, n)$	[7,10]
5	$Z(5, n)$	$Z(5, n)$	$Z(5, n) + 1$	[7,10]
6	$Z(5, n)$	$Z(5, n)$	$Z(5, n)$	[7,10]
7	$\leq n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ ($n \geq 2$)	$\leq n^2 - 2n + \lfloor \frac{n}{2} \rfloor$	$\leq n^2 - 2n + \lfloor \frac{n}{2} \rfloor$	Section 5
8	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 1$	[13]
9	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 1$	[13]
10	$n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ ($n \geq 2$)	$n^2 - 2n + \lfloor \frac{n}{2} \rfloor$	$n^2 - 2n + \lfloor \frac{n}{2} \rfloor$	[11]
11	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 1$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 2$	[14]
12	$Z(5, n) + 2\lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2\lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2\lfloor \frac{n}{2} \rfloor + 2$ ($n \geq 4$)	[15,16]
13	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 1$ ($n \geq 3$)	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 3$	[17–19]

4. Join Products of Connected Graphs on Five Vertices

Again, according to Klešč [7], the crossing numbers of the join product $C_m + C_n$ are uniquely determined when $m, n \geq 3$ and $\min\{m, n\} \leq 6$. Based on this result, along with one mentioned in the previous section for $P_m + C_n$, we obtain direct outputs for two graphs G_1^5 and G_8^5 . Among other significant results of the previously mentioned publication [13] is the determination of $\text{cr}(G_i^5 + D_n)$, $\text{cr}(G_i^5 + P_n)$, and $\text{cr}(G_i^5 + C_n)$ for $i = 3, 4, 5, 7, 13$. It should be emphasised, however, that the priority for publishing the properties of graphs G_7^5 and G_{13}^5 belongs to Klešč and Schrötter [20].

Huang and Zhao [21], using the idea of a special decomposition technique, contributed to determining $\text{cr}(G_i^5 + D_n)$ for $i = 2, 6, 9, 11, 14$. These results were later supplemented, after a considerable time gap, with the crossing numbers of join products with P_n and C_n [22]. In the case of graphs G_{10}^5 and G_{17}^5 , the crossing numbers of the join product with discrete graphs D_n in [23] were gradually supplemented by results with paths P_n [24] and cycles C_n [25].

A significant milestone in graph theory was the determination of the crossing number of the complete tripartite graph G_{15}^5 in the join product with D_n [26]. The results for G_{12}^5 , G_{15}^5 , and G_{18}^5 with paths and cycles were extended by Staš and Timková [16], referring to the already proven value of $\text{cr}(G_{12}^5 + D_n)$ [15]. The crossing numbers of the join products $G_{18}^5 + D_n$ and $G_{18}^5 + P_n$ were also published in [27] and for $G_{18}^5 + C_n$ in [28], but only in Chinese, which makes independent verification difficult. The crossing number of graph G_{18}^5 in the join with D_n was eventually published in English in [17] as one of the consequences derived from the graph H_{13}^5 , and in the join with P_n and C_n in the already mentioned [16] as a finding implied by graph G_{15}^5 .

A similar approach was taken with graph G_{16}^5 , where the crossing numbers of the join product with discrete graphs D_n [17] were gradually completed with results involving paths P_n [18] and cycles C_n [19], all as an outcome resulting from graph H_{13}^5 . However,

it should be noted that the graph G_{16}^5 was firstly discussed with D_n , P_n , and C_n in [29], but the source is available only in Chinese, which complicates reliable verification of the stated results (for example, the case of the path P_n on two vertices appears to be incorrect). An equally important result by Ding and Huang [30] concerning $\text{cr}(G_{19}^5 + D_n)$ was later supplemented with outputs involving P_n and C_n [31]. In contrast to earlier publications available only in Chinese that discuss $\text{cr}(G_{19}^5 + P_n)$ in [32] and $\text{cr}(G_{19}^5 + C_n)$ in [33], the results in [31] are presented in English, along with a proposal to simplify certain parts of the proofs using software that generates all cyclic permutations from [34].

Thanks to one of the most significant publications on complete multipartite graphs [35], we know the exact value of the crossing number of the graph G_{20}^5 in the join product with D_n . This result was extended to $G_{20}^5 + P_n$ in [36], which was published in Chinese, and after a longer period to $G_{20}^5 + P_n$ and to $G_{20}^5 + C_n$ in English [37], which also includes an alternative proof for D_n based on a suitable classification of a large number of non-isomorphic drawings of graph G_{20}^5 . The non-planarity of the complete graph G_{21}^5 is well known. For quite some time, the value of $\text{cr}(G_{21}^5 + D_n)$ has been known thanks to Lu and Huang [38], but the results for paths and cycles were only available in the form of upper bounds. The crossing number of $G_{21}^5 + P_n$ was first published in [39], but with several issues in the main proof arising from an inconsistent treatment of certain subcases (e.g., incorrect drawings or the omission of some relevant configurations). Finally, a result was provided for P_n and C_n in [40], along with an alternative, simpler proof for D_n , thanks to the idea of utilising the existence or non-existence of a separating three-cycle C_3 in various drawings of K_5 .

A corresponding summary of crossing numbers of join products of all 21 connected graphs on five vertices, together with discussed references, is listed in Table 3. In Tables 2 and 3, we can observe five exceptions that occur precisely at the smallest possible values for the variable n . The first case of the graph H_7^5 is discussed separately in Section 5. In the case of the graph H_{10}^5 , $\text{cr}(H_{10}^5 + D_1) = 0$ due to a problem with a negative value for $n = 1$. In the remaining three cases, the exceptions are caused by the sparsity of the optimal drawings of the considered graphs $H_{12}^5 + C_3$, $H_{13}^5 + P_2$ and $G_{16}^5 + P_2$. The missing values are in the form: $\text{cr}(H_{12}^5 + C_3) = 7$, $\text{cr}(H_{13}^5 + P_2) = 3$ and $\text{cr}(G_{16}^5 + P_2) = 3$.

Table 3. Summary of crossing numbers of join products of all connected graphs on five vertices.

i	$\text{cr}(G_i^5 + D_n)$ $n \geq 1$	$\text{cr}(G_i^5 + P_n)$ $n \geq 2$	$\text{cr}(G_i^5 + C_n)$ $n \geq 3$	Refs.
1	$Z(5, n)$	$Z(5, n)$	$Z(5, n) + 1$	[7]
2	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor + 2$	[21,22]
3	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 1$	[13]
4	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 1$	[13]
5	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 2$	[13]
6	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor + 2$	[21,22]
7	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 1$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 2$	[20]
8	$Z(5, n)$	$Z(5, n) + 1$	$Z(5, n) + 2$	[7]
9	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor + 2$	[21,22]
10	$Z(5, n) + n$	$Z(5, n) + n + 1$	$Z(5, n) + n + 3$	[23–25]
11	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor + 1$	$Z(5, n) + 2 \lfloor \frac{n}{2} \rfloor + 3$	[21,22]

Table 3. Cont.

i	$\text{cr}(G_i^5 + D_n)$ $n \geq 1$	$\text{cr}(G_i^5 + P_n)$ $n \geq 2$	$\text{cr}(G_i^5 + C_n)$ $n \geq 3$	Refs.
12	$Z(5, n) + 2\lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2\lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2\lfloor \frac{n}{2} \rfloor + 2$	[15,16]
13	$Z(5, n) + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 1$	$Z(5, n) + \lfloor \frac{n}{2} \rfloor + 2$	[20]
14	$Z(5, n) + 2\lfloor \frac{n}{2} \rfloor$	$Z(5, n) + 2\lfloor \frac{n}{2} \rfloor + 1$	$Z(5, n) + 2\lfloor \frac{n}{2} \rfloor + 3$	[21,22]
15	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 2$	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 4$	[16,26]
16	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 1$ ($n \geq 3$)	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 3$	[17–19]
17	$Z(5, n) + n$	$Z(5, n) + n + 1$	$Z(5, n) + n + 3$	[23–25]
18	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 2$	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 4$	[16,17]
19	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor$	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 1$	$Z(5, n) + n + \lfloor \frac{n}{2} \rfloor + 4$	[30,31]
20	$Z(5, n) + 2n$	$Z(5, n) + 2n + 2$	$Z(5, n) + 2n + 5$	[35,37]
21	$Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$	$Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 4$	$Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 7$	[38,40]

5. The Last Unproven Graph H_7^5

As mentioned before, the graph H_7^5 can be obtained by removing one edge from the graph H_{10}^5 . Therefore, we assume that their crossing numbers in the respective join products are the same, i.e., the conjectures $\text{cr}(H_7^5 + D_n) = \text{cr}(H_7^5 + P_n) = \text{cr}(H_7^5 + C_n) = n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ for corresponding n need to be inspected.

Despite the apparent simplicity of H_7^5 , its sparse structure poses significant challenges for establishing the conjectured crossing numbers. For small values of n ($n \leq 5$), the conjectures can be verified computationally using the available software. We then derive general upper bounds corresponding to the proposed formulas.

Let us first consider the case $H_7^5 + D_n$ for $n = 1$. The result of the conjectured formula is -1 , resulting in an exception for the smallest possible value for the variable n . Because the graph $H_7^5 + D_1$ is planar, its crossing number is in fact equal to zero. Conjecture 1 is therefore exceptionally pronounced for n at least 2:

Conjecture 1. $\text{cr}(H_7^5 + D_n) = n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ for $n \geq 2$.

We are able to compute the exact values of crossing numbers of the join products of the graph H_7^5 with D_2 , D_3 , D_4 and D_5 using Chimani and Wiedera's algorithm with an online interface, Crossing Number Web Compute [41]. This algorithm can find the crossing numbers of small undirected graphs. It uses an Integer Linear Programming (ILP) formulation, based on Kuratowski subgraphs, and solves it via a hybrid algorithmic framework, branch-and-cut-and-price. The system also generates verifiable formal proofs, as described by Chimani and Wiedera [42]. Unfortunately, the capacity of the system is restricted, and in our case, it could be used for $n \leq 5$. The values obtained by Crossing Number Web Compute are summarized in Table 4. In each case, the computed crossing number coincides with the value predicted by Conjecture 1. As a result, Conjecture 1 holds for before mentioned graphs, and gives the following exact values summarised in Theorem 1:

Theorem 1. $\text{cr}(H_7^5 + D_2) = 1$, $\text{cr}(H_7^5 + D_3) = 4$, $\text{cr}(H_7^5 + D_4) = 10$, $\text{cr}(H_7^5 + D_5) = 17$.

Table 4. Computational verification of Conjecture 1 for small values of n .

n	Calculated Values of $\text{cr}(H_5^7 + D_n)$	Conjectured Value of $\text{cr}(H_5^7 + D_n)$
2	1	$2^2 - 2 \cdot 2 + \left\lfloor \frac{2}{2} \right\rfloor = 1$
3	4	$3^2 - 2 \cdot 3 + \left\lfloor \frac{3}{2} \right\rfloor = 4$
4	10	$4^2 - 2 \cdot 4 + \left\lfloor \frac{4}{2} \right\rfloor = 10$
5	17	$5^2 - 2 \cdot 5 + \left\lfloor \frac{5}{2} \right\rfloor = 17$

Let us concentrate now on an upper bound for crossing numbers of the join products of the graph H_7^5 with D_n as described in Lemma 1:

Lemma 1. $\text{cr}(H_7^5 + D_n) \leq n^2 - 2n + \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 2$.

Proof. Let the join product $H_7^5 + D_n$ consist of just one copy of H_7^5 and n vertices $t_1, t_2, \dots, t_n$. Here, each vertex t_i , $i = 1, 2, \dots, n$ is adjacent to every vertex of the graph H_7^5 . Let T^i , $1 \leq i \leq n$ denote the subgraph induced by five edges incident with the fixed vertex t_i . This means that the graph $T^1 \cup \dots \cup T^n$ is isomorphic to the complete bipartite graph $K_{5,n}$. We consider the cases of even and odd n separately. The constructions shown in Figures 2 and 3 are based on drawings introduced by Staš [11] for related join products. For the purposes of the present work, the drawings were modified to reflect the structure of H_7^5 , including the removal of one edge, adjustments to the labeling, and the highlighting of selected edges relevant to the proof.

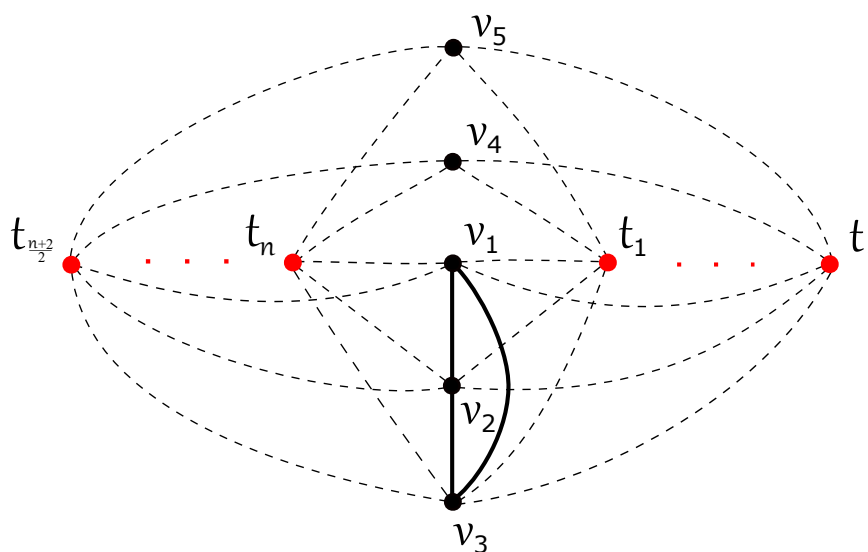


Figure 2. The good drawing of $H_7^5 + D_n$ for even $n \geq 2$ with $n^2 - 2n + \frac{n}{2}$ crossings, adapted from Staš [11].

Case 1: Let n be even.

In Figure 2, the edges of $K_{5,n}$ cross each other

$$4 \binom{\frac{n}{2}}{2} + 4 \binom{\frac{n}{2}}{2} = n(n-2)$$

times, and each subgraph T^i , $i = 1, \dots, \frac{n}{2}$ on the right side crosses the edge $\{v_1, v_3\}$ of the graph H_7^5 once, so $n^2 - 2n + \frac{n}{2}$ crossings appear among edges of $H_7^5 + D_n$ in this drawing. Thus, $\text{cr}(H_7^5 + D_n) \leq n^2 - 2n + \frac{n}{2}$ for n even.

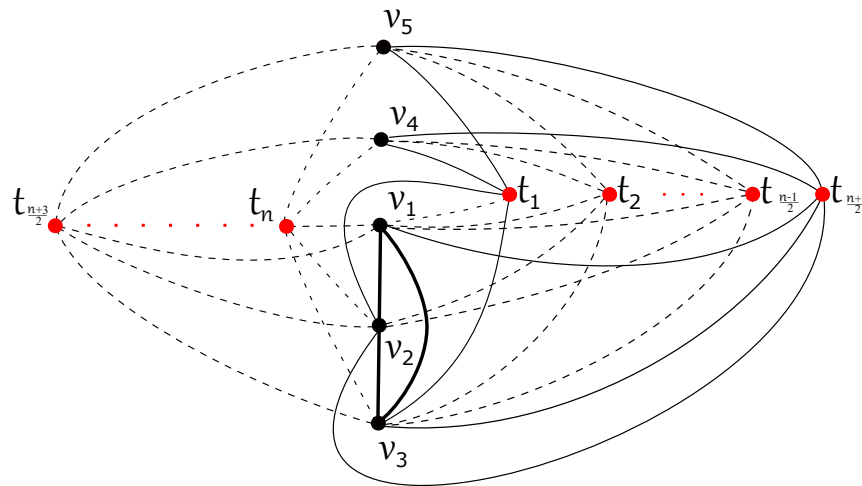


Figure 3. The good drawing of $H_7^5 + D_n$ for odd $n \geq 3$ with $n^2 - 2n + \frac{n-1}{2}$ crossings, adapted from Staš [11].

Case 2: Let n be odd.

In Figure 3, the subgraphs T^i , for $i = 2, 3, \dots, \frac{n-1}{2}$, cross each other $4\binom{\frac{n-3}{2}}{2}$ times and the subgraphs T^i , for $i = \frac{n+3}{2}, \frac{n+5}{2}, \dots, n$, cross each other $4\binom{\frac{n-1}{2}}{2}$ times. The edges of the mentioned $n-2$ subgraphs cross edges of subgraph T^1 and $T^{\frac{n+1}{2}}$ exactly $n-1+3(n-3)$ times. Considering that the subgraphs T^1 and $T^{\frac{n+1}{2}}$ cross each other 2 times, there are

$$4\binom{\frac{n-1}{2}}{2} + 4\binom{\frac{n-3}{2}}{2} + (n-1) + 3(n-3) + 2 = n^2 - 2n + 1$$

crossings on the edges of $K_{5,n}$. Since every T^i , $i = 2, \dots, \frac{n-1}{2}$ on the right side crosses the edge $\{v_1, v_3\}$ of the graph H_7^5 once, $n^2 - 2n + \frac{n-1}{2}$ crossings appear among the edges of $H_7^5 + D_n$ in this drawing.

Combining the results from Case 1 and Case 2, we obtain a common formula for any integer $n \geq 2$: $\text{cr}(H_7^5 + D_n) \leq n^2 - 2n + \lfloor \frac{n}{2} \rfloor$. $\square$

Let us continue with the crossing numbers of the join products of the graph H_7^5 with P_n and C_n .

Theorem 2. If $\text{cr}(H_7^5 + D_n) = n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ for $n \geq 2$, then $\text{cr}(H_7^5 + P_n) = \text{cr}(H_7^5 + D_n) = n^2 - 2n + \lfloor \frac{n}{2} \rfloor$.

Proof. Let $\text{cr}(H_7^5 + D_n) = n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ for arbitrary $n \geq 2$. We use the good drawings of the graph $H_7^5 + D_n$ with exactly $n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ crossings on Figure 2 and 3. There is an option to insert $n-1$ new edges $\{t_i, t_{i+1}\}$, $i = 1, \dots, n-1$, into these drawings without increasing the crossing number of the graph. That is, $\text{cr}(H_7^5 + P_n) = n^2 - 2n + \lfloor \frac{n}{2} \rfloor$. $\square$

Theorem 3. If $\text{cr}(H_7^5 + D_n) = n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ for $n \geq 3$, then $\text{cr}(H_7^5 + C_n) = \text{cr}(H_7^5 + D_n) = n^2 - 2n + \lfloor \frac{n}{2} \rfloor$.

Proof. Let $\text{cr}(H_7^5 + D_n) = n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ for arbitrary $n \geq 3$. We use the good drawings of the graph $H_7^5 + P_n$ with exactly $n^2 - 2n + \lfloor \frac{n}{2} \rfloor$ crossings from Theorem 2. The construction is based on the drawings shown in Figures 2 and 3, where it can be seen that the edge

$\{t_1, t_n\}$, which closes the path P_n into the cycle C_n , can be added without introducing any additional crossings. Consequently, $\text{cr}(H_7^5 + C_n) = n^2 - 2n + \left\lfloor \frac{n}{2} \right\rfloor$. $\square$

From Theorems 2 and 3, we obtain the following Corollaries 1 and 2, respectively.

Corollary 1. $\text{cr}(H_7^5 + P_2) = 1$, $\text{cr}(H_7^5 + P_3) = 4$, $\text{cr}(H_7^5 + P_4) = 10$, $\text{cr}(H_7^5 + P_5) = 17$.

Corollary 2. $\text{cr}(H_7^5 + C_3) = 4$, $\text{cr}(H_7^5 + C_4) = 10$, $\text{cr}(H_7^5 + C_5) = 17$.

The remaining step towards determining the exact crossing numbers of the join products involving H_5^7 is therefore to establish the corresponding lower bound for $H_5^7 + D_n$. Once this is achieved, Theorems 2 and 3 immediately imply the exact crossing numbers for the join products with paths and cycles. A promising direction appears to be a combination of cyclic permutation techniques, parity arguments, and suitable redrawing methods, which have proved successful for several related join products.

6. Conclusions

This paper establishes a unified classification of crossing numbers for join products involving all non-isomorphic graphs of order five and the graph families D_n , P_n , and C_n . The resulting collection of 102 crossing number values, together with the corresponding references and principal proof techniques, provides a consistent framework for further investigations of graph joins.

The graph H_5^7 remains the only unresolved member of this classification. For this graph, we verify the conjectured values computationally for all feasible values $n \leq 5$, establish general upper bounds, and prove that the corresponding results for joins with paths and cycles follow directly from the discrete graph case. Consequently, completing the classification now depends solely on establishing the matching lower bounds for $H_5^7 + D_n$.

Beyond consolidating existing results, the unified presentation reveals that the 102 crossing number values are described by only a small number of recurring formula types. This observation suggests that the crossing behaviour of join products involving graphs of order five is governed primarily by a limited collection of structural patterns rather than by the individual isomorphism classes of the underlying graphs. We hope that the framework developed here will facilitate not only the resolution of the remaining case H_5^7 , but also analogous classifications for graphs of higher order.

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